

## Analysis of a Degenerate Obstacle Problem on an Unbounded Set Arising in the Environment\*

J. I. Díaz and C. Faghloumi

Departamento de Matemática Aplicada, Universidad Complutense de Madrid,  
28040 Madrid, Spain  
{J.I.Diaz,Chakib\_Faghloumi}@mat.ucm.es

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**Abstract.** We study a class of optimization dynamics problems related to investment under uncertainty. The general model problem is reformulated in terms of an obstacle problem associated to a second-order elliptic operator which is not in divergence form. The spatial domain is unbounded and no boundary conditions are a priori specified. By using the special structure of the differential operator and the spatial domain, and some approximating arguments, we show the existence and uniqueness of a solution of the problem. We also study the regularity of the solution and give some estimates on the location of the coincidence set.

**Key Words.** Elliptic free boundary, Degenerate operator, Unbounded domain, Environmental policy.

**AMS Classification.** 35J85, 35J70, 35R35, 91B76, 93E20.

### 1. Introduction

In this paper we study a class of optimization dynamics problems displaying irreversibility that are inspired by questions on the economics of environment management.

Much of the discussion concerning biodiversity as well as other aspects of environmental policy focuses on the irreversibility of certain actions that alter the environment. Economists have recently started to study the effect of the ability to delay an irreversible

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investment expenditure on the decision by firms to invest (see, e.g. [23] and [14]). In this paper we apply similar reasoning to studying the decision to proceed with projects that alter the environment in an irreversible way. The principal lesson is that, in analogy to the case of irreversible investments by firms, the decision to start a project when the present value of the expected loss it will cause in the environment equals its expected benefits, is incorrect, even in the absence of risk aversion (see [25] for a recent mathematical treatment). We point out that our study covers many different concrete examples. For instance, the Clean Air Act calls for reductions in overall emissions of sulfur dioxide ( $\text{SO}_2$ ) but to minimize the cost of these reductions it gives utilities a choice. The utility must decide whether to maintain flexibility by relying on allowances or invest in scrubbers. The numerical analysis of the model was treated in [18] (see also Chapter 12 of [14]). The results of the present paper complement the above references by giving a rigorous mathematical treatment of the model even under a larger generality on the utility function.

This paper is organized as follows: In Section 2 we present the modeling of the problems under consideration. We introduce a *utility function*  $J$  associated to some diffusion process  $X_t$  and  $Y_t$  which starts initially from a point of the set  $\Omega = \{(x, y) \in \mathbb{R}^2: x > 0, y > 0\}$ . The optimal value function given by maximizing the utility function among the temporal horizon  $T$ ,  $v(x, y) \doteq \max_T \{J(x, y, T)\}$ , is characterized as a solution of the *obstacle problem*

$$\min\{-\mathcal{L}v + \mu v - f(x), v - y/\mu\} = 0 \quad \text{in } \Omega, \quad (1)$$

where  $\mathcal{L}$  is a certain linear second-order partial differential operator of elliptic type, and  $\mu$  and  $f(x)$  are given in the definition of  $J$ . In Section 3 we show the existence and uniqueness of a solution to problem (1) when it is formulated in a weak form (but allowing consideration of a larger class of data than  $f(x)$  and the obstacle function). Notice that the uniqueness of solutions can be assured in spite of the absence of a boundary condition. This is possible thanks to the special structure of the coefficients of operator  $\mathcal{L}$ . Finally, in Section 4 we study conditions on the data,  $f(x)$  and the obstacle function, in order to assure that the *coincidence set* (the set of points  $(x, y) \in \Omega$  where  $u$  coincides with the obstacle) is not empty. We also get some estimates on the location of this set. Some special cases are illustrated at the end of the paper.

## 2. Modeling

We consider the environment and an alternative project such that its benefits at instant  $t$  are given by  $X_t$  and  $Y_t$ , respectively. We suppose that the benefits of the environment are completely destroyed when the alternative project is started. The benefits  $X_t$  and  $Y_t$  are given by some diffusion process of the type

$$dX_t = X_t \sigma_1 (dW_t + \varrho dZ_t), \quad X_0 = x, \quad (2)$$

$$dY_t = Y_t \sigma_2 (\varrho dW_t + dZ_t), \quad Y_0 = y, \quad (3)$$

where  $W_t$  and  $Z_t$  are Brownian motions defined on a probability space  $\{\Omega, \mathcal{F}, \mathcal{P}\}$  of correlation  $\varrho \in (-1, 1)$ , and for some nonnegative values  $x$  and  $y$ . We point out that the

special case of  $\varrho = 1$  and  $\sigma_1 = \sigma_2$  was previously considered in [24]. By introducing the Brownian motions  $B_t^i$ ,  $i = 1, 2$ , defined by

$$\sigma_1(1 + 3\varrho^2)^{1/2} B_t^1 \doteq \sigma_1(W_t + \varrho Z_t),$$

$$\sigma_2(1 + 3\varrho^2)^{1/2} B_t^2 \doteq \sigma_2(\varrho W_t + Z_t),$$

we can simplify the modeling process. Notice that the correlation between  $B_t^1$  and  $B_t^2$  is given by  $\bar{\varrho} = \varrho(3 + \varrho^2)/(1 + 3\varrho^2)$ . In that case the diffusion process becomes

$$dX_t = X_t \sigma_1 (1 + 3\varrho^2)^{1/2} dB_t^1, \quad X_0 = x, \tag{4}$$

$$dY_t = Y_t \sigma_2 (1 + 3\varrho^2)^{1/2} dB_t^2, \quad Y_0 = y. \tag{5}$$

We now consider a performance given by the *utility function*  $J$  defined by

$$J(x, y; T) \doteq E \left[ \int_0^T f(X_s) e^{-\mu s} ds + \int_T^\infty Y_s e^{-\mu s} ds \right], \tag{6}$$

where data  $f$  and  $\mu$  are some given nonnegative function and constant, respectively.

Our initial goal is to study the optimal value function  $v$  given as the maximum of the utility function among the temporal horizon  $T$ :

$$v(x, y) \doteq \max_T \{J(x, y, T)\}.$$

Here the pair  $(x, y)$  belongs to the open and unbounded set  $\Omega = \{(x, y) \in \mathbb{R}^2: x > 0, y > 0\}$ .

It is well known (see, for instance [6]) that the optimal value function satisfies the *obstacle problem*

$$\min\{-\mathcal{L}v + \mu v - f(x), v - h\} = 0 \quad \text{in } \Omega, \tag{7}$$

where, as usual,  $\mathcal{L}$  is the differential operator associated with the multidiffusion  $(X_t, Y_t)$ , that is,

$$\mathcal{L}v \doteq \frac{1}{2}(\sigma_1^2(1 + 3\varrho^2)x^2 v_{xx} + \sigma_2^2(1 + 3\varrho^2)y^2 v_{yy}) + \sigma_1 \sigma_2 \varrho(3 + \varrho^2)xy v_{xy}$$

and

$$h(x, y) \doteq E \left[ \int_0^\infty Y_s e^{-\mu s} ds \right] = \frac{y}{\mu}.$$

One of the main goals of this work is to study the existence and properties of the optimal value function  $v: \Omega \rightarrow \mathbb{R}$  by means of the nonlinear elliptic obstacle problem (7). In fact, this problem generates a *free boundary* given as the boundary of the *coincidence set* (the set of points  $(x, y) \in \Omega$  where  $u$  coincides with the obstacle  $h$ ). Sometimes this free boundary can be obtained as the graph of a function  $g: [0, \infty) \rightarrow \mathbb{R}^+$  leading to the so-called “strong formulation”:

$$\begin{cases} -\mathcal{L}v + \mu v \geq f(x) & \text{and } v(x, y) = h(y) & \text{if } x \geq g(y), \\ -\mathcal{L}v + \mu v = f(x) & \text{and } v(x, y) \geq h(y) & \text{if } x \leq g(y) \end{cases} \tag{8}$$

(we point out that function  $g$  is also a priori unknown). Another general formulation of the nonlinear elliptic obstacle problem (7) can be obtained in terms of the multivalued operators. It is well known (see, e.g. [8] for a general reference) that the obstacle problem can be written as

$$-\mathcal{L}v + \mu v + \gamma(v - h) \ni f(x) \quad \text{in } \Omega, \tag{9}$$

where  $\gamma$  is the maximal monotone graph of  $\mathbb{R}^2$  given by

$$\gamma(u) = \begin{cases} \{0\} & \text{if } u > 0, \\ (-\infty, 0] & \text{if } u = 0, \\ \emptyset & \text{if } u < 0, \end{cases}$$

where  $\emptyset$  denotes the empty set.

Since operator  $\mathcal{L}(v)$  is not, a priori, in divergence form, several authors use the notion of *viscosity solutions* introduced by Crandall and Lions (see, e.g. the exposition made in [11]). Nevertheless, here we follow a different approach motivated by the linearity and special properties of operator  $\mathcal{L}(v)$ . So, an important ingredient of our analysis is the fact that operator  $\mathcal{L}(v)$  can be written with its principal part in *divergence form*. To do that we introduce the notation

$$\mathbf{A}(x, y) = \begin{pmatrix} k_1 x^2 & k_3 xy \\ k_3 xy & k_2 y^2 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(x, y) = \begin{pmatrix} (2k_1 + k_3)x \\ (2k_2 + k_3)y \end{pmatrix}$$

with

$$k_1 = \frac{1}{2}\sigma_1^2(1 + 3\rho^2), \quad k_2 = \frac{1}{2}\sigma_2^2(1 + 3\rho^2), \quad k_3 = \frac{1}{2}\sigma_1\sigma_2\rho(3 + \rho^2).$$

Then it is easy to see that

$$\mathcal{L}(v) = \operatorname{div}(\mathbf{A}\nabla v) + \mathbf{b} \cdot \nabla v.$$

A controversial fact, arising with problem (7) as well as with other problems given in terms of operators of a similar nature to operator  $\mathcal{L}(v)$  (see, e.g. [7]), deals with the absence of boundary conditions and whether it becomes natural or not to prescribe some values of  $v$  or  $\nabla v$  on the boundary of  $\Omega$ . Here we adopt a concrete philosophy which consists in assuming that operator  $\mathcal{L}(v)$  already prescribes some “natural” boundary conditions once

$$(\mathbf{A}\nabla u) \cdot \nu = 0 \quad \text{in } \partial\Omega, \tag{10}$$

where  $\nu$  denotes the unit outnormal vector (notice that those boundary conditions are automatically satisfied due to the special form of the matrix  $\mathbf{A}(x, y)$ ).

Given a general obstacle function  $h = h(x, y)$  it is also useful to introduce the following change of variable  $u = v - h$ . Then the problem under consideration becomes

$$(\mathcal{P}) \quad \begin{cases} -\operatorname{div}(\mathbf{A}\nabla u) + \mathbf{b} \cdot \nabla u + \mu u + \gamma(u) \ni G & \text{in } \Omega, \\ (\mathbf{A}\nabla u) \cdot \nu = 0 & \text{in } \partial\Omega, \end{cases}$$

where  $G(x, y) = f(x) - h(x, y)$ . In fact, generally, we can assume that  $G(x, y)$  is a given function on  $\Omega$ .

Notice that since the domain  $\Omega$  is unbounded, the “natural” boundary conditions (10) must be completed by assuming some growth conditions on the solution  $v$ , for  $(x, y)$  large. As usual, this is implied by the type of growth conditions on the datum  $G(x, y)$ . Through this paper we always assume that

$$\text{there exists } m_1 > 1 \text{ and } m_2 > 1 \text{ such that } w^{1/2}G \in L^2(\Omega), \tag{11}$$

where the *weight*  $w$  is defined by

$$w(x, y) = (1 + x^2)^{-m_1}(1 + y^2)^{-m_1}, \tag{12}$$

in other words,

$$\int_{\Omega} (1 + x^2)^{-m_1}(1 + y^2)^{-m_1}G^2(x, y) \, dx \, dy < \infty.$$

We end this section by pointing out that when dealing with semilinear problems posed on unbounded domains, different authors (see, e.g., [10]) use a different weight ( $w(x, y) = e^{-\lambda|(x,y)|^p}$ , for some  $p > 0$ ), motivated by the transformation  $(x, y) = (\log x', \log y')$ . Nevertheless, in our case this transformation has no sense on the boundary of  $\Omega$ .

### 3. Existence and Uniqueness of Solution

In order to introduce the weak formulation of  $(P)$  we define the Hilbert spaces

$$L_w^2(\Omega) = \{v: w^{1/2}v \in L^2(\Omega)\},$$

equipped with the norm

$$\|u\|_{L_w^2(\Omega)} = \|w^{1/2}u\|_{L^2(\Omega)},$$

and

$$H_w^1(\Omega, \mathbf{A}) = \{v: v \in L_w^2(\Omega), xv_x \in L_w^2(\Omega), yv_y \in L_w^2(\Omega)\},$$

equipped with the norm

$$\|u\|_{H_w^1(\Omega, \mathbf{A})}^2 = \|u\|_{L_w^2(\Omega)}^2 + \|xu_x\|_{L_w^2(\Omega)}^2 + \|yu_y\|_{L_w^2(\Omega)}^2.$$

If we consider the convex subset  $\mathcal{K}$  defined by

$$\mathcal{K} = \{v \in H_w^1(\Omega, \mathbf{A}), v \geq 0 \text{ on } \Omega\},$$

then it is easy to see that any solution of  $(P)$  satisfies the problem under a weak formulation which can be defined in the following terms:

$$(\mathcal{WP}) \quad \begin{cases} u \in \mathcal{K}, \\ a(u, v - u) \geq L(v - u), \end{cases} \quad \text{for any } v \in \mathcal{K},$$

where

$$a(u, v) = \int_{\Omega} \{(\mathbf{A}\nabla u \cdot \nabla v)w + (\mathbf{A}\nabla u \cdot \nabla w + w\mathbf{b} \cdot \nabla u)v + \mu uvw\} \, dx \, dy$$

and

$$L(v) = \int_{\Omega} G_{vw} dx dy.$$

Reciprocally, it can be shown that the twice differentiable solution of  $(\mathcal{WP})$  also satisfies  $(\mathcal{P})$  (see, for instance, the exposition made in [9]).

Our existence result needs a technical condition on the different constants of type  $k$  and  $m$  assuming, on the operator  $\mathcal{L}(v)$  and on the growth of  $G$ ,

**Theorem 1.** *Assume (11) and that*

$$\mu > \mu_0 := 2m_1(m_1 - 1)k_1 + 2m_2(m_2 - 1)k_2 + (4m_1m_2 - m_1 - m_2)|k_3|. \quad (13)$$

*Then there exist a unique weak solution  $u \in \mathcal{K}$  of problem  $(\mathcal{WP})$ .*

*Proof.* We start by introducing a change of variables leading the unbounded set  $\Omega$  in a bounded open set. Let  $\mathcal{F}$  be the transformation given by

$$\begin{aligned} \mathcal{F}: \Omega &\rightarrow \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \\ (x, y) &\rightarrow (\alpha, \beta) = \mathcal{F}(x, y) = (\arctan x, \arctan y). \end{aligned}$$

For the sake of the notation we recall that

$$\mathcal{F}(\Omega) = \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right).$$

Notice that assumption (11) is equivalent to

$$\text{there exist } \hat{m}_1, \hat{m}_2 > 0 \text{ such that } \omega^{1/2} \hat{G} \in L^2(\mathcal{F}(\Omega)), \quad (14)$$

where  $\hat{m}_1 = m_1 - 1$ ,  $\hat{m}_2 = m_2 - 1$ ,  $\omega(\alpha, \beta) = \cos^{2\hat{m}_1}(\alpha) \cos^{2\hat{m}_2}(\beta)$  and  $\hat{G}(\alpha, \beta) = G(\mathcal{F}^{-1}(\alpha, \beta))$ . In other words,

$$\int_{\mathcal{F}(\Omega)} \omega(\alpha, \beta) \hat{G}^2(\alpha, \beta) d\alpha d\beta < \infty. \quad (15)$$

We also introduce the notation

$$\mathbf{S} = \begin{pmatrix} k_1 \sin^2 \alpha \cos^2 \alpha & k_3 \sin \alpha \cos \alpha \sin \beta \cos \beta \\ k_3 \sin \alpha \cos \alpha \sin \beta \cos \beta & k_2 \sin^2 \beta \cos^2 \beta \end{pmatrix},$$

$$\mathbf{p} = \begin{pmatrix} (2k_1 \cos^2 \alpha + k_3[\cos^2 \beta - \sin^2 \beta]) \sin \alpha \cos \alpha \\ (2k_2 \cos^2(\beta) + k_3[\cos^2 \alpha - \sin^2 \alpha]) \sin \beta \cos \beta \end{pmatrix}.$$

The model becomes

$$(\hat{\mathcal{P}}) \quad \begin{cases} -\operatorname{div}_{(\alpha, \beta)}(\mathbf{S}\nabla u) + \mathbf{p} \cdot \nabla u + \mu u + \gamma(u) \ni \hat{G} & \text{in } \mathcal{F}(\Omega), \\ (\mathbf{S}\nabla u) \cdot \vec{\nu} = 0 & \text{in } \partial\mathcal{F}(\Omega), \end{cases}$$

where now  $\tilde{\nu}$  denotes the unit outnormal vector to  $\partial\mathcal{F}(\Omega)$ . The weak formulation of  $(\hat{\mathcal{P}})$  starts, again, by defining the Hilbert spaces associated to the problem  $L_\omega^2(\mathcal{F}(\Omega)) = \{v: \omega^{1/2}v \in L^2(\mathcal{F}(\Omega))\}$  equipped with the norm:

$$\|u\|_{L_\omega^2(\mathcal{F}(\Omega))} = \|\omega^{1/2}u\|_{L^2(\mathcal{F}(\Omega))},$$

and  $H_\omega^1(\mathcal{F}(\Omega), \mathbf{S}) = \{v: v \in L_\omega^2(\mathcal{F}(\Omega)), (\sin 2\alpha)v_\alpha \in L_\omega^2(\mathcal{F}(\Omega)), (\sin 2\beta)v_\beta \in L_\omega^2(\mathcal{F}(\Omega))\}$  equipped with the norm:

$$\|u\|_{H_\omega^1(\mathcal{F}(\Omega), \mathbf{S})}^2 = \|u\|_{L_\omega^2(\mathcal{F}(\Omega))}^2 + \|(\sin 2\alpha)v_\alpha\|_{L_\omega^2(\mathcal{F}(\Omega))}^2 + \|(\sin 2\beta)v_\beta\|_{L_\omega^2(\mathcal{F}(\Omega))}^2.$$

We also introduce the notation

$$\begin{aligned} \mathcal{F}(H_w^1(\Omega, \mathbf{A})) &= \{v(\alpha, \beta): v(\alpha, \beta) \in H_w^1(\Omega, \mathbf{A})\} \quad \text{and} \\ \hat{\mathcal{K}} &= \{v \in H_\omega^1(\mathcal{F}(\Omega), \mathbf{S}), v \geq 0\}. \end{aligned}$$

We have that  $\mathcal{F}(H_w^1(\Omega, \mathbf{A})) = H_\omega^1(\mathcal{F}(\Omega), \mathbf{S})$  and  $\mathcal{F}(\mathcal{K}) = \hat{\mathcal{K}}$ . So, the weak formulation of  $(\hat{\mathcal{P}})$  can be stated in the following terms:

$$(\mathcal{WP}) \quad \begin{cases} u \in \hat{\mathcal{K}}, \\ \hat{a}(u, v - u) \geq \hat{L}(u, v - u), \quad \forall v \in \hat{\mathcal{K}}, \end{cases}$$

where

$$\hat{a}(u, v) = \int_{\mathcal{F}(\Omega)} (\omega \nabla v \cdot \mathbf{S} \nabla u + (\nabla \omega \mathbf{S} + \omega \mathbf{p}) \cdot \nabla uv + \mu \omega uv) \, d\alpha \, d\beta$$

and

$$\hat{L}(v) = \int_{\mathcal{F}(\Omega)} (\omega \hat{G}v) \, d\alpha \, d\beta.$$

The following two lemmata lead to the conclusion of Theorem 1.

**Lemma 1.** *Under assumption (13)  $u$  is a solution of  $(\mathcal{WP})$  if and only if  $u \circ \mathcal{F}$  is a solution of  $(\mathcal{WP})$ .*

*Proof.* We have

$$\begin{aligned} \hat{a}(u, v) &= \int_{\mathcal{F}(\Omega)} \{\omega \nabla v \cdot \mathbf{S} \nabla u + (\nabla \omega \mathbf{S} + \omega \mathbf{p}) \cdot \nabla uv + \mu \omega uv\} \, d\alpha \, d\beta \\ &= \int_{\Omega} \{w \nabla v(\mathcal{F}) \cdot \mathbf{A} \nabla u(\mathcal{F}) + (\nabla w \mathbf{A} + w \mathbf{b}) \cdot \nabla u(\mathcal{F})v(\mathcal{F}) \\ &\quad + \mu w u(\mathcal{F})v(\mathcal{F})\} \, dx \, dy \\ &= a(u \circ \mathcal{F}, v \circ \mathcal{F}) \end{aligned}$$

and

$$\begin{aligned}\hat{L}(v) &= \int_{\mathcal{F}(\Omega)} \omega \hat{G} v \, d\alpha \, d\beta \\ &= \int_{\Omega} G w v \circ \mathcal{F} \, dx \, dy \\ &= L(v \circ \mathcal{F}).\end{aligned}$$

Taking into consideration that  $\mathcal{F}$  is a diffeomorphism we get the conclusion.  $\square$

**Lemma 2.** *Assume that (14) and (13) hold. Then there exist a unique solution  $u$  of problem  $(\mathcal{W}\hat{\mathcal{P}})$ .*

*Proof.* It is easy to see that the bilinear form,  $\hat{a}$ , is continuous since by the Hölder inequality

$$|\hat{a}(u, v)| \leq \xi \|u\|_{H_w^1(\mathcal{F}(\Omega), \mathcal{S})} \|v\|_{H_w^1(\mathcal{F}(\Omega), \mathcal{S})},$$

where

$$\xi = \left(\frac{5}{4} + \hat{m}_1\right)k_1 + \left(\frac{5}{4} + \hat{m}_2\right)k_2 + \left(\frac{9}{4} + \hat{m}_1 + \hat{m}_2\right)|k_3| + \mu.$$

Moreover,  $\hat{a}$  is coercive. Indeed,

$$\begin{aligned}\hat{a}(u, u) &= \int_{\mathcal{F}(\Omega)} \omega \nabla v \cdot \mathbf{S} \nabla u + \omega (\mathbf{p} \cdot \nabla u) v + \mu \omega u v \\ &\geq \int_{\mathcal{F}(\Omega)} \frac{1}{4} \sigma_1^2 (-\varrho^3 + 3\varrho^2 - 3\varrho + 1) \omega (\sin^2 2\alpha) (u_\alpha)^2 \\ &\quad + \int_{\mathcal{F}(\Omega)} \frac{1}{4} \sigma_2^2 (-\varrho^3 + 3\varrho^2 - 3\varrho + 1) \omega (\sin^2 2\beta) (u_\beta)^2 \\ &\quad + \int_{\mathcal{F}(\Omega)} (\mu - k_1 l_1 - k_2 l_2 + k_3 l_3) \omega u^2,\end{aligned}$$

where

$$\begin{aligned}l_1 &= 2(\hat{m}_1^2 + 5\hat{m}_1 + 4) \cos^4 \alpha - 2(2\hat{m}_1^2 + 6\hat{m}_1 + 3) \cos^2 \alpha + 2\hat{m}_1(\hat{m}_1 + 1), \\ l_2 &= 2(\hat{m}_2^2 + 5\hat{m}_2 + 4) \cos^4 \beta - 2(2\hat{m}_2^2 + 6\hat{m}_2 + 3) \cos^2 \beta + 2\hat{m}_2(\hat{m}_2 + 1), \\ l_3 &= (4\hat{m}_1 \hat{m}_2 + 3\hat{m}_1 + 6\hat{m}_2 + 4) \cos^2 \alpha + (4\hat{m}_1 \hat{m}_2 + 6\hat{m}_1 + 3\hat{m}_2 + 4) \cos^2 \beta \\ &\quad - (4\hat{m}_1 \hat{m}_2 + 5\hat{m}_1 + 5\hat{m}_2 + 8) \cos^2 \alpha \cos^2 \beta + (2 + \hat{m}_1 + \hat{m}_2) \sin 2\alpha \sin 2\beta \\ &\quad - (4\hat{m}_1 \hat{m}_2 + 3\hat{m}_1 + 3\hat{m}_2 + 2).\end{aligned}$$

So we have

$$\begin{aligned}k_1 l_1 &\leq 2\hat{m}_1(\hat{m}_1 + 1)k_1, & k_2 l_2 &\leq 2\hat{m}_2(\hat{m}_2 + 1)k_2, \\ k_3 l_3 &\geq -(4\hat{m}_1 \hat{m}_2 + 3\hat{m}_1 + 3\hat{m}_2 + 2)|k_3|\end{aligned}$$



and

$$-\varrho^3 + 3\varrho^2 - 3\varrho + 1 > 0, \quad \forall \varrho \in (-1, 1).$$

Then, from assumption (13), we have

$$\hat{a}(u, u) \geq \eta \|u\|_{H_\omega^1(\mathcal{F}(\Omega), \mathbf{S})}^2,$$

where

$$\eta = \min\{\frac{1}{4}\sigma_1^2(-\varrho^3 + 3\varrho^2 - 3\varrho + 1), \frac{1}{4}\sigma_2^2(-\varrho^3 + 3\varrho^2 - 3\varrho + 1), \mu - \mu_0\}.$$

On the other hand, the linear form  $\hat{L}$  is continuous. Indeed, thanks to assumption (14)

$$\begin{aligned} |\hat{L}(v)| &= \int_0^{\pi/2} \int_0^{\pi/2} \omega \hat{G} v \, dy \, dz \\ &\leq \|\hat{G}\|_{L_\omega^2(\mathcal{F}(\Omega))} \|v\|_{L_\omega^2(\mathcal{F}(\Omega))}, \end{aligned}$$

since  $\hat{G} \in L_\omega^2(\mathcal{F}(\Omega))$ . Finally  $\hat{\mathcal{K}}$  is a convex and closed subset of  $H_\omega^1(\mathcal{F}(\Omega), \mathbf{S})$ . So, in virtue of the Stampacchia Theorem (see, e.g. [9]) we conclude that problem  $(\mathcal{W}\hat{\mathcal{P}})$  has a unique solution  $u \in \hat{\mathcal{K}}$ , such that

$$\|u\|_{H_\omega^1(\mathcal{F}(\Omega), \mathbf{S})} \leq \frac{\xi}{\eta} \|\hat{G}\|_{L_\omega^2(\mathcal{F}(\Omega))},$$

which ends the proof of the lemma and of Theorem 1. □

Our next result shows that the above weak solution satisfies, locally, problem (1) in almost every point of the domain  $\Omega$ . This is reduced to showing that the expression  $\mathcal{L}(v) = \operatorname{div}(\mathbf{A}\nabla v) + \mathbf{b} \cdot \nabla v$  makes sense in almost every point, a property which is clearly implied by showing that  $u \in H_{\text{loc}}^2(\Omega)$ .

**Proposition 1.** *Assume the conditions of Theorem 1 and let  $u$  be the weak solution of  $(\mathcal{P})$ . Then  $u \in H_{\text{loc}}^2(\Omega)$ .*

*Proof.* It is clear that if  $u$  is the solution of  $(\mathcal{W}\mathcal{P})$ , then  $u$  satisfies the problem in a local way. More exactly,  $u$  is a *weak local solution* of  $(\mathcal{P})$  in the sense of Brezis (see [8]), i.e.

$$a(u, \eta(v - u)) \geq L(\eta(v - u)), \quad \forall v \in \mathcal{K} \quad \text{and} \quad \forall \eta \in \mathcal{D}(\Omega).$$

Given  $\varepsilon > 0$  small enough, we consider the auxiliary local problem

$$(\mathcal{P}_*) \quad \begin{cases} -\mathcal{L}u_* + \mu u_* + \gamma(u_*) \ni G & \text{in } \Omega'_\varepsilon, \\ (\mathbf{A} \cdot \nabla u_*) \cdot \nu = (\mathbf{A} \cdot \nabla u) \cdot \nu & \text{in } \partial\Omega'_\varepsilon, \end{cases}$$

where  $\Omega'_\varepsilon = \{(x, y) \in \Omega: d((x, y), \Omega') < \varepsilon\}$  and  $\Omega'$  is a bounded subset of  $\Omega$  such that  $\Omega' \subset\subset \Omega$ . We assume that  $d = \text{dist}(\partial\Omega, \Omega'_\varepsilon) > 0$ . Then we have

$$\begin{aligned} \xi \cdot \mathbf{A}\xi &= k_1 x^2 \xi_1 \xi_1 + k_2 y^2 \xi_2 \xi_2 + k_3 x y \xi_1 \xi_2 + k_3 x y \xi_2 \xi_1 \geq cd |\xi|^2, \\ \forall \xi &\in \mathbb{R}^2 - \{0\}, \end{aligned}$$

for some  $c > 0$  depending only on  $\sigma_1, \sigma_2$  and  $\varrho$ . We define

$$H^{1,+}(\Omega'_\varepsilon) = \{v \in H^1(\Omega'_\varepsilon): v \geq 0 \text{ on } \Omega'_\varepsilon\}.$$

Thanks to the regularity results of [8], the (unique) weak solution  $u_* \in H^{1,+}(\Omega'_\varepsilon)$  of problem  $(\mathcal{P}_*)$ ,  $u_* \in H^2(\Omega'_\varepsilon)$ . So,  $u_*$  is a *strong local solution* of  $(\mathcal{P}_*)$  in the sense that

$$\begin{aligned} \langle -\mathcal{L}u_* + \mu u_*, \eta(v - u_*) \rangle_{L^2(\Omega'_\varepsilon)} &\geq \langle G, \eta(v - u_*) \rangle_{L^2(\Omega'_\varepsilon)}, \\ \forall \eta &\in \mathcal{D}(\Omega'_\varepsilon) \quad \text{and} \quad \forall v \in H^{1,+}(\Omega'_\varepsilon). \end{aligned}$$

On the other hand, it is clear that  $u \in H^{1,+}(\Omega'_\varepsilon)$ . So, taking  $\eta^* = \eta w \in \mathcal{D}(\Omega)$  such that  $\eta^* \in \mathcal{D}(\Omega')$ , we have

$$a(u - u_*, \eta^*(u - u_*)) \leq 0.$$

However, in general,

$$a(v, \eta v) = \int_{\Omega'_\varepsilon} \eta \nabla v \cdot \mathbf{A} \nabla v + \{(\mu - k_1 - k_2 - k_3)\eta - \text{div}(\mathbf{A} \nabla \eta) - \mathbf{b} \cdot \nabla \eta\} v^2.$$

So, if we take  $\eta^*$  such that

$$\begin{aligned} (\mu - k_1 - k_2 - k_3)\eta^* - \text{div}(\mathbf{A} \nabla \eta^*) - \mathbf{b} \cdot \nabla \eta^* &\geq 0, \\ \eta^* &= 1 \quad \text{in } \Omega' \quad \text{and} \quad 0 \leq \eta^* \leq 1, \end{aligned}$$

then we have that

$$\begin{aligned} 0 &\geq a(u - u_*, \eta^*(u - u_*)) \\ &\geq \int_{\Omega'} cd \nabla(u - u_*) \mathbf{A} \nabla(u - u_*) + (\mu - k_1 - k_2 - k_3)(u - u_*)^2 \\ &\geq \min\{cd, \mu - k_1 - k_2 - k_3\} |v|_{H^1(\Omega')}^2 \end{aligned}$$

and from the coercivity of  $a$  we deduce that  $u = u_*$  on  $\Omega'$ , and so  $u \in H^2(\Omega')$ .  $\square$

In order to study the free boundary it is useful to get some  $L^\infty(\Omega)$  estimates on the solution.

**Proposition 2.** *Assume the conditions of Theorem 1. Let  $u$  be the weak solution of  $(\mathcal{P})$  and assume, additionally, that*

$$w_0 G \in L^\infty(\Omega)$$

with

$$w_0(x, y) = (1 + x^2)^{-(m_1-1)/2} (1 + y^2)^{-(m_2-1)/2}. \quad (16)$$

Then we have

$$0 \leq w_0 u \leq \frac{1}{\mu - \mu_1} \|G^+ w\|_{L^\infty(\Omega)},$$

where

$$\mu_1 = k_1(m_1 - 1)(m_1 - 3)^+ + k_2(m_2 - 1)(m_2 - 3)^+ + 2|k_3|(m_1 m_2 - m_1 - m_2).$$

*Proof.* Let  $k = (1/(\mu - \mu_1)) \|G^+ w_0\|_{L^\infty(\Omega)}$  and  $v_0 = u - (u - kw_0^{-1})^+$ . So, we have  $v_0 \in H_w^1(\Omega, \mathbb{A})$ . Moreover

$$a(u, (u - kw_0^{-1})^+) \leq \int_{\Omega} w G(u - kw_0^{-1})^+$$

and

$$\begin{aligned} a(u - kw_0^{-1}, (u - kw_0^{-1})^+) &\leq -k(-\mathcal{L}(w_0^{-1}) + \mu w_0^{-1}, w(u - kw_0^{-1})^+)_{L^2(\Omega)} \\ &\quad + \int_{\Omega} w G(u - kw_0^{-1})^+ \\ &\leq \int_{\Omega} w (G - kw_0^{-1} M)(u - kw_0^{-1})^+, \end{aligned}$$

where

$$\begin{aligned} M(x, y) &= \mu - k_1(m_1 - 1) \left( \frac{x^2}{1 + x^2} \right) \left( 1 + (m_1 - 3) \left( \frac{x^2}{1 + x^2} \right) \right) \\ &\quad - k_2(m_2 - 1) \left( \frac{y^2}{1 + y^2} \right) \left( 1 + (m_2 - 3) \left( \frac{y^2}{1 + y^2} \right) \right) \\ &\quad - 2k_3(m_1 - 1)(m_2 - 1) \left( \frac{x^2 y^2}{(1 + x^2)(1 + y^2)} \right). \end{aligned}$$

So, for all  $(x, y) \in \Omega$  we have  $M(x, y) \leq \mu - \mu_1$ , and since  $\mu - \mu_1 > 0$  then

$$\begin{aligned} a(u - kw_0^{-1}, (u - kw_0^{-1})^+) &\leq \int_{\Omega} w (G - (\mu - \mu_1)kw_0^{-1})(u - kw_0^{-1})^+ \\ &\leq \int_{\Omega} w (G - \|G^+ w_0\|_{L^\infty} w_0^{-1})(u - kw_0^{-1})^+ \leq 0. \end{aligned}$$

From the coercivity of  $a$ ,  $(u - kw_0^{-1})^+ \equiv 0$  on  $\Omega$  which leads to the conclusion.  $\square$

#### 4. Location of the Coincidence Set

In this section we are interested in studying the *coincidence set*. This is defined as the set of points  $(x, y) \in \Omega$  where the solution coincides with the obstacle. In the case of the formulation  $u = v - h$  the “new obstacle” is reduced to the zero function, so the coincidence set is the set of points where  $u$  vanishes. We also get some estimates on the location of this set. Given a general function  $\Psi: \Omega \rightarrow \mathbb{R}^+$  we define the *null* and *positive* sets associated to  $\Psi$  as the sets

$$N(\Psi) \doteq \text{null set of } \Psi \doteq \{(x, y) \in \bar{\Omega}: \Psi(x, y) = 0\},$$

$$S^+(\Psi) \doteq \text{positive set of } \Psi \doteq \overline{\{(x, y) \in \Omega: \Psi(x, y) > 0\}}.$$

Notice that  $\Omega = N(u) \cup S^+(u)$  but we do not know, a priori, if the coincidence set  $N(u)$  is empty or not. Obviously this depends of the nature of the data  $G(x, y)$ . Our following result shows that this set is not empty if  $G(x, y)$  is “negative enough” in some “big enough” part of  $\Omega$ . In fact, in this case, we obtain some estimates on the location of  $N(u)$ .

Our method of proof is inspired by the *method of local supersolutions* introduced, in a systematic way, by the first author (see, e.g. [13]). One of the main difficulties in applying such a method to our problem comes from the fact that the coefficients of the operator  $\mathcal{L}u$  depend strongly of the points  $(x, y) \in \Omega$ . This is solved by introducing an ad hoc distance  $\tilde{d}$  over the points of  $\Omega$  given by

$$\tilde{d}((x_0, y_0), (x, y)) = \sqrt{\left(\log\left(\frac{x}{x_0}\right)\right)^2 + \left(\log\left(\frac{y}{y_0}\right)\right)^2}.$$

Motivated by the special structure of the coefficients of  $\mathcal{L}u$  we introduce the auxiliary function

$$\varepsilon(x, y) = \frac{4k(k_1 + k_2 + |k_3|)e^2(m_1 + m_2 - 2)^2}{w_0(x, y)},$$

where  $w_0(x, y)$  is given by (16). We have

**Theorem 2.** *Let  $u$  be the solution of problem  $\mathcal{P}$  and assume that the set*

$$S^+(G + \varepsilon) \doteq \overline{\{(x, y) \in \Omega: G(x, y) \leq -\varepsilon(x, y)\}}$$

*is not empty. Then the coincidence set  $N(u)$  of  $u$  satisfies*

$$N(u) \supset \left\{ (x, y) \in S^+(G + \varepsilon) \text{ such that } \tilde{d}((x, y), \partial\Omega \cup \partial S^+(G + \varepsilon)) \geq \frac{1}{m_1 + m_2 - 2} \right\}.$$

*Proof.* Consider the set

$$\Omega_1 \doteq \left\{ (x, y) \in \Omega: G \leq -\varepsilon(x, y), \tilde{d}((x, y), \partial\Omega \cup \partial S^+(G + \varepsilon)) \geq \frac{1}{m_1 + m_2 - 2} \right\}$$

and let  $(x_0, y_0) \in \Omega_1$ . Define

$$\tilde{B}_R(x_0, y_0) \doteq \left\{ (x, y) \in \Omega: \left( \log \left( \frac{x}{x_0} \right) \right)^2 + \left( \log \left( \frac{y}{y_0} \right) \right)^2 < R^2 \right\}.$$

We obtain our conclusions by constructing a *local supersolution*  $\bar{u}(x, y: x_0, y_0)$  defined in  $\tilde{B}_R(x_0, y_0)$ , for each  $(x_0, y_0) \in \Omega_1$ . This is done by means of a “radially symmetric” function

$$\bar{u}(x, y: x_0, y_0) = \eta(r),$$

with  $\eta > 0, \eta' > 0$  and  $\eta'' > 0, \forall r > 0$ , to be determined and where  $r = \tilde{d}((x, y); (x_0, y_0))$ . We have

$$\begin{aligned} \mathcal{L}\eta = & \left\{ k_1 \frac{(\log(x) - \log(x_0))^2}{r^2} + k_2 \frac{(\log(y) - \log(y_0))^2}{r^2} \right. \\ & \left. + k_3 \frac{2(\log(x) - \log(x_0))(\log(y) - \log(y_0))}{r^2} \right\} \eta'' \\ & + \left\{ k_1 + k_2 - \left( k_1 \frac{(\log(x) - \log(x_0))^2}{r^2} + k_2 \frac{(\log(y) - \log(y_0))^2}{r^2} \right. \right. \\ & \left. \left. + k_3 \frac{2(\log(x) - \log(x_0))(\log(y) - \log(y_0))}{r^2} \right) \right\} \frac{\eta'}{r}. \end{aligned}$$

So,

$$\mathcal{L}\eta \leq \Lambda \left( \eta'' + \frac{\eta'}{r} \right),$$

where  $\Lambda = k_1 + k_2 + |k_3|$ . On the other hand, it is easy to see that, given  $\varepsilon_0 > 0$ , the solution of the problem

$$\begin{cases} -\Delta \left( \eta'' - \frac{\eta'}{r} \right) + \mathcal{B}(\eta) \ni -\varepsilon_0 & \text{in } (0, R), \\ \eta(0) = \eta'(0) = 0 \end{cases}$$

is given by

$$\eta(r) = \frac{\varepsilon_0}{4\Lambda} r^2.$$

The conditions

$$\varepsilon_0 \leq -G \quad \text{in } B_R(X_0) \quad \text{and} \quad \eta(R) \geq u|_{\partial B_R(X_0)},$$

or equivalently

$$\eta(R) = \frac{\varepsilon_0}{4\Lambda} R^2 \geq \sup_{\partial B_R} K w^{-l}(x, y) \quad \text{and} \quad \varepsilon_0 = \inf_{B_R(X_0)} \{\varepsilon(x, y)\},$$

imply that  $\eta$  is a supersolution of  $(P)$ . The above conditions hold and consequently  $\eta$  is a supersolution of problem  $(P)$  in  $\tilde{B}_R(x_0, y_0)$ . Finally, the choice of  $\varepsilon(x, y)$  and  $R = 1/(m_1 + m_2 - 2)$  leads to obtaining that  $u(x, y) = 0$  for every  $(x, y) \in \Omega_1$ .  $\square$

## 5. Some Relevant Particular Cases

In this section we study three particular cases corresponding to  $G(x, y) = f(x) - H(y)$  and when  $f$  or  $H$  are assumed to be linear or constant functions.

### 5.1. First Case: $f(x) = x$ and $H(y) = c$

If we return to the original modeling of the problem, this case represents a scenario in which the *alternative project* has a constant benefit. So, the optimal value function becomes

$$u(x) = \max_T \left\{ E \left[ \int_0^T X_s e^{-\mu s} ds + \frac{c}{\mu} e^{-\mu T} \right] \right\},$$

which is explicitly given by

$$u(x) = \begin{cases} \frac{c^{1-\gamma}}{\mu(1-\gamma)^{1-\gamma}|\gamma|^\gamma} x^\gamma + \frac{1}{\mu}x & \text{if } x \geq x_0, \\ \frac{c}{\mu} & \text{if } x \leq x_0, \end{cases}$$

where

$$\gamma = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{4\mu}{k_1}} \right)$$

and

$$x_0 = \frac{\gamma}{\gamma - 1} c.$$

So, we have  $N(u - c/\mu) = [0, x_0]$ . If  $c$  is large enough, then  $\Omega_1 = [1, x_1 - 1]$  where  $x_1 + (4k_1 e^2/\mu)(1 + x_1^2)^{1/2} = c$ . It is easy to prove that  $x_0 \geq x_1 - 1$ . In consequence  $\Omega_1 \subset N(u - c/\mu)$ .

### 5.2. Second Case: $f(x) = c$ and $H(y) = y$

Now we study the case when the benefit of the environment is constant. Then the optimal value function becomes

$$\begin{aligned} u(y) &= \max_T \left\{ E \left[ \int_0^T c e^{-\mu s} ds + \int_T^\infty Y_s e^{-\mu s} ds \right] \right\} \\ &= \max_T \left\{ E \left[ \frac{c}{\mu} (1 - e^{-\mu T}) + \int_0^\infty Y_s e^{-\mu s} ds - \int_0^T Y_s e^{-\mu s} ds \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{c}{\mu} + E \left[ \int_0^\infty Y_s e^{-\mu s} ds \right] - \min_T \left\{ E \left[ \int_0^T Y_s e^{-\mu s} ds + \frac{c}{\mu} e^{-\mu T} \right] \right\} \\
&= \frac{y+c}{\mu} - \min_T \left\{ E \left[ \int_0^T Y_s e^{-\mu s} ds + \frac{c}{\mu} e^{-\mu T} \right] \right\},
\end{aligned}$$

which is explicitly given by

$$u(y) = \begin{cases} \frac{(\delta-1)^{\delta-1}}{\mu c^{\delta-1} \delta^\delta} y^\delta + \frac{c}{\mu} & \text{if } y \leq y_0, \\ \frac{y}{\mu} & \text{if } y \geq y_0, \end{cases}$$

where

$$\delta = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4\mu}{k_2}} \right)$$

and

$$y_0 = \frac{\delta}{\delta-1} c.$$

So,  $N(u - y/\mu) = [y_0, \infty[$ . If  $c$  is large enough, then

$$\Omega_1 = [y_1 + 2, \infty], \quad \text{where } y_1 - \frac{4k_2 e^2}{\mu} (1 + y_1^2)^{1/2} = c,$$

and, as in the first case, it is easy to see that  $y_0 \leq y_1 + 2$ . So  $\Omega_1 \subset N(u - y/\mu)$ .

### 5.3. Third Case: $f(x) = x$ and $H(y) = y$

A kind of mixed situation is considered here. Then

$$\begin{aligned}
u(x, y) &= \max_T \left\{ E \left[ \int_0^T X_s e^{-\mu s} ds + \int_T^\infty Y_s e^{-\mu s} ds \right] \right\} \\
&= \max_T \left\{ E \left[ \int_0^T X_s e^{-\mu s} ds + \int_0^T c e^{-\mu s} ds \right. \right. \\
&\quad \left. \left. + \int_T^\infty c e^{-\mu s} ds + \int_T^\infty Y_s e^{-\mu s} ds - \frac{c}{\mu} \right] \right\} \\
&= \max_T \left\{ E \left[ \int_0^T X_s e^{-\mu s} + \frac{c}{\mu} e^{-\mu T} \right] + E \left[ \int_0^T c e^{-\mu s} + \int_T^\infty Y_s e^{-\mu s} \right] \right\} - \frac{c}{\mu} \\
&\leq \bar{u},
\end{aligned}$$

where  $\bar{u}$  is defined by

$$\begin{aligned}
\bar{u}(x, y) &= \max_T \left\{ E \left[ \int_0^T X_s e^{-\mu s} ds + \frac{c}{\mu} e^{-\mu T} \right] \right\} \\
&\quad + \max_T \left\{ E \left[ \int_0^T c e^{-\mu s} ds + \int_T^\infty Y_s e^{-\mu s} ds \right] \right\} - \frac{c}{\mu}.
\end{aligned}$$

So, from the representation formulae of Sections 5.1 and 5.2, we have

$$\bar{u}(x, y) = \begin{cases} \frac{y}{\mu} & \text{if } (x, y) \in \Omega_{1,c}, \\ \frac{(\delta - 1)^{\delta-1}}{\mu c^{\delta-1} \delta^{\delta}} y^{\delta} + \frac{c}{\mu} & \text{if } (x, y) \in \Omega_{2,c}, \\ \frac{c^{1-\gamma}}{\mu(1-\gamma)^{1-\gamma} |\gamma|^{\gamma}} x^{\gamma} + \frac{1}{\mu} x + \frac{(\delta - 1)^{\delta-1}}{\mu c^{\delta-1} \delta^{\delta}} y^{\delta} & \text{if } (x, y) \in \Omega_{3,c}, \\ \frac{c^{1-\gamma}}{\mu(1-\gamma)^{1-\gamma} |\gamma|^{\gamma}} x^{\gamma} + \frac{1}{\mu} x + \frac{y-c}{\mu} & \text{if } (x, y) \in \Omega_{4,c}, \end{cases}$$

where

$$\begin{aligned} \Omega_{1,c} &= \left\{ (x, y) \in \Omega: x \leq \frac{\gamma}{\gamma-1} c \quad \text{and} \quad y \geq \frac{\delta}{\delta-1} c \right\}, \\ \Omega_{2,c} &= \left\{ (x, y) \in \Omega: x \leq \frac{\gamma}{\gamma-1} c \quad \text{and} \quad y \leq \frac{\delta}{\delta-1} c \right\}, \\ \Omega_{3,c} &= \left\{ (x, y) \in \Omega: x \geq \frac{\gamma}{\gamma-1} c \quad \text{and} \quad y \leq \frac{\delta}{\delta-1} c \right\}, \\ \Omega_{4,c} &= \left\{ (x, y) \in \Omega: x \geq \frac{\gamma}{\gamma-1} c \quad \text{and} \quad y \geq \frac{\delta}{\delta-1} c \right\}. \end{aligned}$$

Since for each  $c \in \mathbb{R}^+$  we have  $\Omega_{1,c} \subset N(u - y/\mu)$ , we deduce that  $\{(x, y) \in \Omega: y \geq \alpha_0 x\} \subset N(u - y/\mu)$ , where

$$\alpha_0 = \frac{(1 + \sqrt{1 + 4\mu/k_1})(1 + \sqrt{1 + 4\mu/k_2})}{(\sqrt{1 + 4\mu/k_1} - 1)(\sqrt{1 + 4\mu/k_2} - 1)}.$$

Moreover, we can show that  $G(x, \alpha_0 x) + \varepsilon(x, \alpha_0 x) \geq 0$  for each  $x \in \mathbb{R}$ . In conclusion we get that  $\Omega_1 \subset N(u - y/\mu)$ .

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