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On a stochastic parabolic PDE arising in Climatology

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Abstract. We study the existence and uniqueness of solutions of a nonlinear stochastic pde proposed by R. North and R. F. Cahalan in 1982 for the modeling of non-deterministic variability (as, for instance, the volcano actions) in the framework of energy balance climate models. The more delicate point concerns the uniqueness of solutions due to the presence of a multivalued graph β in the right hand side of the equation. In contrast with the deterministic case, it is possible to prove the uniqueness of a suitable weak solution associated to each given monotone (univalued and discontinuous) section b of the maximal monotone graph β. We get some stability results when the white noise converges to zero.

Sobre una ecuación estocástica en derivadas parciales de tipo parabólico que surge en Climatología

Resumen. Estudiamos la existencia y unicidad de soluciones de una ecuación estocástica en derivadas parciales de tipo parabólico propuesta por R. North y R. F. Cahalan en 1982 para la modelización de variabilidad no determinista (como es el caso, por ejemplo, de la acción de volcanes) en el marco de los modelos de balance de energía. El punto más delicado se refiere a la unicidad de soluciones debido a la presencia de un grafo multívoco β en el término de la derecha de la ecuación. En contraste con el caso determinista, es posible mostrar la unicidad de una cierta noción de solución débil asociada a cada sección monotona (unívoca y discontinua) b del grafo máximal monótono β. En esta nota se dan unos resultados de estabilidad cuando el ruido blanco converge a cero.

1. Introduction

This note deals with the nonlinear stochastic pde

$$(\mathrm{E}_{\beta,\varepsilon})\begin{cases} u_t - u_{xx} + Bu \in QS(x)\beta(u) + f(x,t) + \varepsilon\dot{\mathcal{W}}, & (x,t) \in (-1,1)\times\mathbb{R}_+,\\ u_x(-1,t) = u_x(1,t) = 0, & t \in \mathbb{R}_+,\\ u(x,0) = u_0(x), & x \in (-1,1),\end{cases}$$

where B and ε are positive constants,

(H_β) β is a *bounded* maximal monotone graph of $\mathbb{R}^2$, i.e. $m \leq z \leq M$, $\forall z \in \beta(s)$, $\forall s \in \mathbb{R}$.

(H_s) $S : \mathcal{M} \to \mathbb{R}$, $S \in L^\infty(-1,1)$, $S_1 \geq S(x) \geq S_0 > 0$ a.e. $x \in (-1,1)$,

$u_0 \in C([-1,1])$, $f \in L^\infty((-1,1)\times\mathbb{R}_+)$ and the term $\mathcal{W}$ denotes a space-time white noise.

This kind of problems where proposed by R. North and R. F. Cahalan in 1982 ([5]) for the modeling of non-deterministic variability (as, for instance, the volcano actions) in the context of energy balance climate

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models. We recall that the distribution of temperature $u(x,t)$ is expressed pointwise after a standard average process, where the spatial variable x is given by $x = \sin\theta$ and θ is the latitude. Notice that, for simplicity, we are replacing the natural degenerate diffusion term $((1-x^2)u_x)_x$ by the usual $1d$-Laplacian operator and that the absence of boundary conditions for the degenerate diffusion is corrected by adding Neumann type boundary conditions since in the degenerate model the meridional heat flux $(1-x^2)u_x$ vanishes at the poles $x = \pm 1$. The balance of energies leads to the problem

$$\begin{cases} u_t - u_{xx} = R_a - R_e, & (x,t) \in (-1,1)\times\mathbb{R}_+, \\ u_x(-1,t) = u_x(1,t) = 0, & t \in \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in (-1,1), \end{cases}$$

where the terms R_a and R_e must be specified by means of constitutive laws (see, e.g., [5], [1] and [2]). The *absorbed energy* R_a depends, in a fundamental way, on the planetary *coalbedo* β representing the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflection is greater than over oceans, therefore, the coalbedo is smaller. So, there is a sharp transition between zones of high and low coalbedo. In the energy balance climate models, a main change of the coalbedo occurs in a neighborhood of a critical temperature for which ice become white, usually taken as $u = -10°$C. In the so called *Budyko model* the different values of the coalbedo are modeled by means of a discontinuous function of the temperature. As usual in pde, this function can be understood in the more general context of the maximal monotone graphs of $\mathbb{R}^2$. In particular, we assume that

$$\beta(u) = \begin{cases} m, & \text{if } u < -10, \\ [m,M], & \text{if } u = -10, \\ M, & \text{if } u > -10, \end{cases} \tag{1}$$

where $m = \beta_i$ and $M = \beta_w$ represent the coalbedo in the ice-covered zone and the free-ice zone, respectively and $0 < \beta_i < \beta_w < 1$ (the value of these constants has been estimated by observation from satellites). In contrast to the above assumption, in the so called *Sellers model* β is assumed to be a more regular function (at least, Lipschitz continuous) picewise constant function far from a neighborhood of $u = -10$. In both models, the whole absorbed energy is given by $R_a = QS(x)\beta(u)$ where $S(x)$ is the *insolation function* and Q is the so-called *solar constant*.

The Earth's surface and atmosphere, warmed by the Sun, emit part of the absorbed solar flux as an infrared long-wave radiation. This energy R_e is represented, following the proposal by Budyko, by $R_e = Bu - f(x,t)$. Here, B and f are obtained, again, by observation and depend on the *greenhouse effect*.

The main goal of this note is to present the mathematical analysis of the model ([5] was limited to the application of the Fourier method to the linear case $\beta = 0$). We recall that in the deterministic case ($\varepsilon = 0$) the existence of solutions was given in [1] (see [2] for the generalization to bidimensional models). When β is as the Sellers coalbedo this solution is unique, nevertheless, if β is multivalued it was shown there that there is lack of uniqueness of solutions except in the class of the, so called, *non degenerate solutions*. As we shall specify later, a curious fact is produced for problem $(E_{\beta,\varepsilon})$: the presence of a stochastic perturbation produces the uniqueness of the solutions associated to any given monotone (univalued and discontinuous) section b of the maximal monotone graph β, (*i.e.* a function such that $b(r) \in \beta(r)$, for any $r \in \mathbb{R}$).

In this note we use some previous results due to I. Gyöngy and E. Pardoux ([3],[4]) in order study the stability of solutions when $\varepsilon \to 0$. We shall show that the associated solution $u^{\varepsilon,b}$ converges to a *solution* u^b of the deterministic problem and we characterize the limit for the case of the two distinguished sections of β.

2. The Seller colbedo case

In this section we consider the formulation corresponding to the Sellers coalbedo function. In fact, it is useful to start with a bounded truncation of the complete deterministic source function

$$F(x,t,r) = QS(x)\beta(r) - BT_n(r) + f(x,t)$$

where $n \in \mathbb{N}$ and $T_n(r) = \min\{|r|, n\}sign(r)$. We rewrite the problem as

$$(\mathrm{E}_{F,\varepsilon}) \begin{cases} u_t = u_{xx} + F(x,t,u) + \varepsilon \dot{\mathcal{W}}, & \text{in }]-1,1[\times\mathbb{R}_+, \\ u_x(-1,t) = u_x(1,t) = 0, & t \in \mathbb{R}_+, \\ u(\cdot,0) = u_0(\cdot), & \text{on } [-1,1]. \end{cases}$$

Notice that F is a bounded Lipschitz function and so, in particular,

$$|F(x,t,r) - F(x,t,\widehat{r})| \leq \mathrm{K}|r - \widehat{r}|, \tag{2}$$

for some positve constant K and for $x, y \in [-1,1]$, $t \in \mathbb{R}_+$, $r, \widehat{r} \in \mathbb{R}$.

We recall that the notion of *weak solution* corresponds to $\mathcal{B}([-1,1]) \bigotimes \mathcal{P}$ measurable and continuous random field $\{u^\varepsilon(x,t)\}_{(x,t)\in[-1,1]\times\overline{\mathbb{R}}_+}$ such that

$$\int_{-1}^1 u^\varepsilon(y,t)\varphi(y)dy = \int_{-1}^1 u_0(y)\varphi(y)dy + \int_0^t \int_{-1}^1 [u^\varepsilon(y,s)\varphi''(y) + F(u^\varepsilon)(y,s)\varphi(y)]\,dyds \\ + \varepsilon \int_0^t \int_{-1}^1 \varphi(y)\mathcal{W}(dy,ds), \quad t \geq 0, \mathbb{P} - a.s., \tag{3}$$

for $\varphi \in \mathcal{C}^2(]-1,1[) \cap \mathcal{C}^1([-1,1])$, $\varphi'(-1) = \varphi'(1) = 0$. Here $F(u)(y,s) \doteq F(y,s,u(y,s))$, $\mathcal{W}$ is *a space-time white noise* on a filtered probability space $(\mathcal{O}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathcal{B}([-1,1])$ is the collection of all the *Borel* sets of $[-1,1]$ and $\mathcal{P}$ is the σ -algebra of the progressively measurable subsets of $\mathcal{O} \times \mathbb{R}_+$.

Theorem 1 ([6], [3]) *Under the above conditions there exists a unique weak solution of* $(\mathrm{E}_{F,\varepsilon})$. *Moreover, if* $\varepsilon \leq \varepsilon'$ *the comparison* $u^\varepsilon \leq u^{\varepsilon'}$ $\mathbb{P} - a.s.$ *holds.* ■

The following result supply some information on the stability of the solutions when $\varepsilon \to 0$

Theorem 2 *Under the above assumptions, if* u^ε *and* $u^{\varepsilon'}$ *are solutions of* $(\mathrm{E}_{F,\varepsilon})$ *and* $(\mathrm{E}_{F,\varepsilon'})$, *respectively, one has*

$$\sup_x \mathbb{E}\left[\left(u^\varepsilon(x,t) - u^{\varepsilon'}(x,t)\right)^2\right] \leq |\varepsilon - \varepsilon'|^2 \left[\sqrt{\frac{t}{2\pi}} + \mathrm{K}^2 \int_0^t \sqrt{\frac{s}{2\pi(t-s)}} \exp\left(\mathrm{K}^2\sqrt{\frac{2s}{\pi}}\right) ds\right], \tag{4}$$

for any $t \geq 0$.

Proof. Let $\mathrm{G}(x,y,t)$ be the *fundamental solution* of heat equation on $[-1,1] \times \overline{\mathbb{R}}_+$ with homogeneous Neumann's boundary conditions. A simple probabilistic interpretation of $\mathrm{G}(x,y,t)$ involving a suitable Brownian motion, $\{B(t)\}_{t\geq 0}$, on the complete probability space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$, shows

$$\int_D \mathrm{G}(x,y,t)dx \leq \mathbb{P}\left(\{B(t) \in D\}\right), \quad \text{for all } D \in \mathcal{B}([-1,1]).$$

Then it follows

$$\mathrm{G}(x,y,t) \leq \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right), \quad x,y \in [-1,1],\ t \in \mathbb{R}_+,$$

whence inequality

$$\int_0^t \int_{-1}^1 \mathrm{G}^2(x,y,t-s)dyds < \int_0^t \frac{ds}{2\sqrt{2\pi(t-s)}} < \infty \tag{5}$$

holds. It is a simple exercise to verify that

$$\int_{-1}^{1} u^{\varepsilon}(y,t)\psi(y,t)dy = \int_{-1}^{1} u_0(y)\psi(y,0)dy$$

$$+\int_0^t\int_{-1}^{1}\left[u^{\varepsilon}(y,s)\left(\psi_{yy}(y,s)+\psi_s(y,s)\right)+F(u^{\varepsilon})(y,s)\psi(y,s)\right]dyds+\varepsilon\int_0^t\int_{-1}^{1}\psi(y,s)\mathcal{W}(dy,ds),$$

for every $\psi \in \mathcal{C}^{2,1}_{x,t}(]-1,1[\times\mathbb{R}_+)\cap\mathcal{C}^{1,0}_{x,t}([-1,1]\times\mathbb{R}_+)\cap\mathcal{C}([-1,1]\times\overline{\mathbb{R}}_+)$, $\psi_x(-1,t)=\psi_x(1,t)=0$, $t\in\overline{\mathbb{R}}_+$. Now, for any fixed t we define

$$\psi(y,s)=\mathrm{G}(\varphi,y,t-s)\doteq\int_{-1}^{1}\mathrm{G}(y,z,t-s)\varphi(z)dz,$$

where φ is as in (3). Since by construction

$$\mathrm{G}(\varphi,y,t)=\varphi(y)+\int_0^t\mathrm{G}(\varphi,y,s)\varphi''(y)ds$$

we deduce

$$\psi(y,t)=\varphi(y)\quad\text{and}\quad\psi_s+\psi_{yy}=0.$$

So, the solution u^{ε} satisfies

$$\int_{-1}^{1}u^{\varepsilon}(y,t)\varphi(y)dy=\int_{-1}^{1}u_0(x)\mathrm{G}(\varphi,y,t)dy+\int_0^t\int_{-1}^{1}F(u^{\varepsilon})(y,s)\mathrm{G}(\varphi,y,t-s)dyds$$
$$+\,\varepsilon\int_0^t\int_{-1}^{1}\mathrm{G}(\varphi,y,t-s)\mathcal{W}(dy,ds).$$

Since $\mathbb{E}[(u^{\varepsilon}(x,t))^2]$ is bounded in $[-1,1]$ (see [3, Proposition 3.1]), Fubini's Theorem implies that $(u^{\varepsilon}(x,t))^2$ is integrable with respect to x, $a.e.\ \omega\in\mathcal{O}$. Then, if φ approaches a delta function, as t goes to 0, it follows

$$\begin{aligned}u^{\varepsilon}(x,t)&=\int_{-1}^{1}\mathrm{G}(x,y,t)u_0(x)dy+\int_0^t\int_{-1}^{1}\mathrm{G}(x,y,t-s)F(u^{\varepsilon})(y,s)dyds\\&+\varepsilon\int_0^t\int_{-1}^{1}\mathrm{G}(x,y,t-s)\mathcal{W}(dy,ds),\quad \mathbb{P}-a.s.\ (x,t).\end{aligned}\tag{6}$$

We note that since $\mathcal{W}:\ \mathcal{B}([-1,1]\times\mathbb{R}_+)\to\mathbb{H}$ and the Gaussian space $\mathbb{H}$ is contained in $L^2(\mathcal{O},\mathcal{F},\mathbb{P})$, the estimate (5) gives a sense to the stochastic integral

$$\mathcal{W}\left(\mathrm{G}(x,\cdot,t-\cdot)\right)=\int_0^t\int_{-1}^{1}\mathrm{G}(x,y,t-s)\mathcal{W}(dy,ds).$$

Next, for $v(x,t)=u^{\varepsilon}(x,t)-u^{\varepsilon'}(x,t)$ we define $\mathrm{V}(x,t)=\mathbb{E}[(v(x,t))^2]$ and $\mathrm{V}(t)=\sup_x\mathrm{V}(x,t)$. Then, from (6) we get that

$$\begin{aligned}\mathrm{V}(x,t)&=\int_0^t\int_{-1}^{1}\mathbb{E}\left[\left(F(u^{\varepsilon})(y,s)-F(u^{\varepsilon'})(y,s)+\varepsilon-\varepsilon'\right)^2\right]\mathrm{G}^2(x,y,t-s)dyds\\&\le\mathrm{K}^2\int_0^t\int_{-1}^{1}\mathrm{V}(y,s)\mathrm{G}^2(x,y,t-s)dyds+|\varepsilon-\varepsilon'|^2\int_0^t\int_{-1}^{1}\mathrm{G}^2(x,y,t-s)dyds,\end{aligned}$$

(see (2)) and

$$V(t) \leq K^2 \int_0^t V(s) \left(\int_{-1}^1 G^2(x,y,t-s)dy \right) ds + |\varepsilon - \varepsilon'|^2 \int_0^t \int_{-1}^1 G^2(x,y,t-s)dyds$$
$$\leq \frac{K^2}{2\sqrt{2\pi}} \int_0^t V(s) \frac{ds}{\sqrt{t-s}} + \frac{|\varepsilon-\varepsilon'|^2}{2}\sqrt{2\pi} \int_0^t \frac{ds}{\sqrt{t-s}}$$

due to (5). Then, applying Gronwall inequality we get the result. ■

Remark 1 Notice that (4) implies the uniqueness of the solution of problem $(E_{F,\varepsilon})$. ■

Since we are dealing here with the case β Lipschitz continuous, using that the solution of $(E_{F,0})$ is bounded due to the assumptions on the data (see [1]) and the representation (6) we get

Corollary 1 *We have $u^\varepsilon \searrow u^0$, as $\varepsilon \to 0$, at least in $C\left([0,\infty[: L^2\left(-1,1: L^2(\mathcal{O},\mathcal{F},\mathbb{P})\right)\right)$, with the convergence rate given by (4) for $\varepsilon' = 0$, where u^0 is the unique solution of the (deterministic) limit problem $(E_{F,0})$.* ■

3. The multivalued Budyko coalbedo case

Consider now a maximal monotone graph β satisfying (H_β) and multivalued at $r = -10$. Given any (univalued and discontinuous) section b of β, we rewrite the energy balance model in terms of $(E_{F_b,\varepsilon})$ with

$$F_b(x,t,r) = QS(x)b(r) - Br + f(x,t)$$

(notice that now there is no truncation in the above definition). We start by mentioning that the results of [3] still hold. Indeed, F_b is a locally bounded and $\mathcal{B}([-1,1]) \bigotimes \mathcal{B}(\mathbb{R}_+) \bigotimes \mathcal{B}(\mathbb{R})$ measurable function verifying the *one side linear growth* condition $rF_b(x,t,r) \leq c(1+r^2)$ for a constant c independent of (x,t,r). Then, by using suitable approximations of F_b it was proved in [3, Theorem 5.1 and Theorem 5.2]) the existence of a unique continuous and $\mathcal{B}([-1,1]) \bigotimes \mathcal{P}$ measurable solution of the relative problem $(E_{F_b,\varepsilon})$.

Concerning the stability of the solutions when $\varepsilon \to 0$ we have:

Theorem 3 *There is a unique continuous and $\mathcal{B}([-1,1]) \bigotimes \mathcal{P}$ measurable solution $u^{\varepsilon,b}$ of $(E_{F_b,\varepsilon})$. It converges to a solution u^b of $(E_{F_b,0})$, as $\varepsilon \to 0$ and u^b is given by*

$$u^b(x,t) = \int_{-1}^1 u_0(x)G(x,y,t)dy + \int_0^t \int_{-1}^1 F_b(u^b)(y,s)G(x,y,t-s)dyds, \quad (x,t) \in [-1,1] \times \overline{\mathbb{R}}_+. \quad (7)$$

PROOF. We shall use the classical *theory of constructible solutions* introduced following ideas by N. V. Krylov. In short, if $h(x,t,r)$ is a $\mathcal{B}([-1,1]) \bigotimes \mathcal{B}(\mathbb{R}_+) \bigotimes \mathcal{B}(\mathbb{R})$ measurable and bounded function, we construct the smooth approximation

$$h_n(x,t,r) = n \int_{\mathbb{R}} h(x,t,r)\rho(n(r-z))dz$$

where $\rho \in C_c^\infty(\mathbb{R})$ is a nonnegative function with $\int_{\mathbb{R}} \rho(z)dz = 1$, and

$$h_n(x,t,r) = n \int_{\mathbb{R}} h(x,t,r)\rho(n(r-z))dz$$

where $\rho \in C_c^\infty(\mathbb{R})$ is a nonnegative function with $\int_{\mathbb{R}} \rho(z)dz = 1$, and

$$\left\{ \widetilde{h}_{n,k} \doteq \inf_{j=n,\ldots,k} h_j, \quad n \leq k, \qquad H_n \doteq \sup_{j=n,\ldots,\infty} h_j. \right.$$

Then $\widetilde{h}_{n,k}$ is Lipschitz continuous in r, uniformly with respect to (x,t), and

$$\widetilde{h}_{n,k} \searrow H_n, \quad \text{as } k \to \infty, \quad \text{and} \quad H_n \nearrow h, \quad \text{as } n \to \infty.$$

Now, if we apply the above procedure to $h(x,t,r) = F_b(x,t,r)$, from [3, Corollary 3.4 and Corollary 3.5] the solution $u^{\varepsilon,b}_{n,k}$ of $(\mathrm{E}_{\widetilde{h}_{n,k}})$ goes to the solution $u^{\varepsilon,b}_n$ of $(\mathrm{E}_{H_n,\varepsilon})$, as $k \to \infty$, with

$$u^{\varepsilon,b}_n \leq u^{\varepsilon,b}_{n,k} \leq u^{\varepsilon,b}_{n,k'} \quad \text{if } k \leq k'.$$

Moreover, similar comparison arguments show that

$$u^{\varepsilon,b}_{n,k} \geq u^{\varepsilon,b}_{m,k}, \quad n \leq m \leq k, \quad \text{and} \quad u^{\varepsilon,b}_n \nearrow u^{\varepsilon,b}, \quad \text{as } n \to \infty,$$

where $u^{\varepsilon,b}$ is the unique continuous and $\mathcal{B}([-1,1]) \bigotimes \mathcal{P}$ measurable solution of $(\mathrm{E}_{F_b,\varepsilon})$. Finally, from the convergence $u^{\varepsilon,b}_{n,k} \searrow u^b_{n,k}$, as $\varepsilon \to 0$ we conclude the result. ■

If $b = b^+$ (resp. b^-) denotes the section of β of maximum (resp. minimum) norm then by using the method of super and subsolutions, that $b^+(r) \geq z,\ z \in \beta(r)$, and [2, Lemme 3] we get

Corollary 2 *u^{b^+} (resp. u^{b^-}) is the maximal (resp. minimal) solution of the set of solutions of the deterministic problem.* ■

Remark 2 Obviously, if there is a *nondegenerate solution* u (in the sense of [2]) of the deterministic problem then *every* solution of the stochastic problem $u^{\varepsilon,b}$ converges to u. ■

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