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On a nonlocal quasilinear parabolic model related to a current-carrying Stellarator

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1. Introduction

We study the existence of weak solutions of the quasilinear parabolic problem

$$(\mathscr{P})\quad \begin{cases} \beta(u)_t - \Delta u = G(u)(t,x) + J(u)(t,x) & \text{in } Q :=]0,T[\times\Omega,\\ u(t,x)=\gamma & \text{on } \Sigma :=]0,T[\times\partial\Omega,\\ \beta(u(0,x)) = \beta(u_0(x)) & \text{in } \Omega, \end{cases}$$

where Ω is an open, bounded, and regular set in $\mathbb{R}^2$ and $T>0$. Moreover, γ is a negative constant, and the nonlinear functions β, G, and J satisfy the following structural assumptions:

$$\beta(s) := \min(s,0) = -s_-, \tag{1}$$

$$G(u)(t,x) := \left[A - \lambda u_+(t,x)^2 + \int_{|u(t)>0|}^{|u(t)>u_+(t,x)|} g(u_+(t)_*(\sigma), |u_+(t)|_{L^\infty(\Omega)}, [u_+(t)]'_*(\sigma))\,\mathrm{d}\sigma\right]_+^{1/2}, \tag{2}$$

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$$J(u)(t,x) := j(u_+(t,x), |u_+(t)|_{L^\infty(\Omega)}, [u_+(t)]'_*(|u(t) > u(t,x)|)), \tag{3}$$

with $A > 0$, $\lambda > 0$, $u_+ = \max(u, 0)$, $|E|$ the Lebesgue measure of the set E. So, for example, $|u(t) > u_+(t,x)|$ denotes the measure of the set $\{y \in \Omega\colon u(t,y) > u_+(t,x)\}$, for a given $x \in \Omega$ and $t \in]0,T[$. The function $u_+(t)_* = [u_+(t)]_*$, defined on the interval $\Omega_* :=]0, |\Omega|[$, is the decreasing rearrangement of the function $u_+(t) : \Omega \to [0, +\infty)$; the latter is defined by $u_+(t)(x) = [u(t,x)]_+$, for $x \in \Omega$ and a fixed $t \in]0,T[$ (see, e.g. [10]). By $[u_+(t)]'_*$ we denote the (weak) derivative of the decreasing rearrangement. We assume that

$$g, j : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R} \quad \text{are bounded continuous functions,} \tag{4}$$

and we fix a constant C_0 such that

$$\max(|G(u)(t,x)|, |J(u)(t,x)|) \leqslant \frac{C_0}{2} \tag{5}$$

for all admissible functions u, for all $t \in]0,T[$ and $x \in \Omega$.

The above formulation is related to a problem arising in the study of the magnetic confinement of a plasma in a Stellarator device, when the plasma is assumed to be a perfect conductor but with a non-zero net current inside each flux magnetic surface (in contrast with ideal Stellarators).

Taking into account Ohm's and Faraday's laws, the associated Grads–Hafranov equation, obtained after an averaging process from the three-dimensional physical problem, can be formulated as a two-dimensional inverse problem of the form

$$\beta(u)_t - \Delta u = F(u) + F(u)F'(u) + \lambda u_+ \quad \text{in } Q, \tag{6}$$

where $F : \mathbb{R} \to \mathbb{R}_+$ is an *unknown* function satisfying $F(s) = \sqrt{A}$ (a given positive constant) for any $s \leqslant 0$ (the set $\{u < 0\}$ corresponds to the vacuum region, separating the plasma from the walls of the device; (see, e.g. [2,6]). The case of an ideal Stellarator, with zero net current within each flux magnetic surface, has been studied recently in [6]. In practice, however, this ideal condition does not hold, and a known current arises in the interior of each magnetic surface (see [4] for a physical modelling and [8] for a mathematical treatment, both for the associated stationary problem). Using the change of variables introduced in [8], the condition of a non-zero current inside each magnetic surface can be expressed in terms of a family of integrals, involving a given function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$:

$$\int_{\{u(t)>s\}} [F(u(t))F'(u(t)) + \lambda u_+(t)]\,\mathrm{d}x$$

$$= h(s_+, |u_+(t)|_{L^\infty(\Omega)}), \quad \forall s \in \left[\operatorname*{ess\,inf}_{\Omega} u(t), \operatorname*{ess\,sup}_{\Omega} u(t)\right] \tag{7}$$

and for any $t \in [0,T]$; this is known as the *current-carrying condition*. The present paper generalizes the results of [6], which is concerned with the special case $h \equiv 0$.

We point out that the physical model involves some weight functions a and b, which here are assumed to be equal to one, and the diffusion operator is a certain elliptic, second-order operator with variable coefficients. Our problem $(\mathscr{P})$ thus contains some simplifications. A more general framework will be considered in [7].

The main goal of this paper is to prove the existence of a weak solution for $(\mathscr{P})$. The organization of the rest of the paper is as follows: In Section 2 we explain how (6) and (7) lead to non-local terms of the kind involved in $(\mathscr{P})$. In Section 3 we introduce the notion of weak solution and prove some a priori estimates for any weak solution. Finally, in Section 4, we state and prove the existence result. For the proof, we shall use a Galerkin method as in [6]. Notice that the equation in $(\mathscr{P})$ is elliptic–parabolic, depending on the sign of u. So, we start by approximating $(\mathscr{P})$ by a family of regularized problems $(\mathscr{P})_\alpha$ (obtained by approximating β by suitable strictly increasing functions β_α). Next, we approximate $(\mathscr{P})_\alpha$ by a sequence of finite-dimensional problems $(\mathscr{P})_{\alpha,m}$ and prove their solvability. Using a priori estimates, we pass to the limit, first in m, then in α.

2. On the non-local terms obtained from the inverse problem

The main goal of this section is to show how the family of conditions (7) allows us to write the unknown function F in terms of a non-local expression in u. In order to do so, we shall apply some technical results about decreasing and relative rearrangements.

Given a function $b \in L^1(0,T;L^1(\Omega))$, we define the function

$$w(t,\sigma) = \int_{\{u(t)>u(t)_*(\sigma)\}} b(t)\,\mathrm{d}x + \int_0^{\sigma-|u(t)>u(t)_*(\sigma)|} (b(t)|_{\{u(t)=u(t)_*(\sigma)\}})_*(s)\,\mathrm{d}s$$

for $\sigma \in]0,|\Omega|[$, $t\in]0,T[$. The *relative rearrangement of* $b(t)$ *with respect to* $u(t)$ is defined as the (weak) derivative $(\partial w/\partial\sigma)(t,\cdot)$, and we denote it by $b(t)_{*u(t)}$ (see, e.g. [10]). This function and the decreasing rearrangement have many useful properties, some of which will be used here (see, for instance, [8,10,11], or [12]).

Assuming u to be regular enough, we can apply Theorem 1.1 of [12] in order to obtain the derivative with respect to s of relation (7), for a fixed t. In fact,

$$\mu'(s)[F(u(t))F'(u(t)) + \lambda u_+(t)]_{*u(t)}(\mu(s)) = h'_s(s_+, u_+(t)_*(0)),$$

with $\mu(s) = |u(t) > s|$ the distribution function of $u(t)$ and h'_s the derivative of h with respect to its first variable (we used here that $|u_+(t)|_{L^\infty(\Omega)} = u_+(t)_*(0)$). Now, from Lemma 9 of [8], we get

$$\mu'(s)[F(u(t)_*)F'(u(t)_*) + \lambda u_+(t)_*](\mu(s)) = h'_s(s_+, u_+(t)_*(0)).$$

We also assume that $u(t)$ has no *flat region* (i.e., $|\{\nabla u(t) = 0\}| = 0$ for any fixed t). From this, $u(t)_*(\mu(s)) = u(t)_*(|u(t) > s|) = s$, and so we deduce that $[\mu'(s)]^{-1} = [u_+(t)]'_*(|u(t) > s|)$ for $s \geqslant 0$ (see, e.g. [8, Lemma 2]). Thus, the last relation can be written as

$$F(s)F'(s) + \lambda s_+ = h'_s(s_+, u_+(t)_*(0))[u_+(t)]'_*(|u(t) > s|), \tag{8}$$

and so, $(\frac{1}{2}F^2)'(s) = -\lambda s_+ + h'_s(s_+, u_+(t)_*(0))[u_+(t)]'_*(|u(t) > s|)$. (Note that (8) holds also for $s < 0$, since F is constant on $\mathbb{R}^-$). Integrating the last equality on $(0,\sigma_+)$

with $\sigma \in [\operatorname{essinf}_\Omega u(t), \operatorname{essup}_\Omega u(t)]$, we obtain that

$$F(\sigma) = \left[A - 2\lambda \int_0^{\sigma_+} s\,\mathrm{d}s + 2\int_0^{\sigma_+} h_s'(s_+, u_+(t)_*(0))[u_+(t)]_*'(|u(t) > s|)\,\mathrm{d}s \right]_+^{1/2}$$

$$= \left[A - \lambda\sigma_+^2 + 2\int_{|u(t)>0|}^{|u(t)>\sigma_+|} h_s'(u_+(t)_*(r), u_+(t)_*(0))([u_+(t)]_*'(r))^2\,\mathrm{d}r \right]_+^{1/2},$$

where we used the change of variable $s = u_+(t)_*(r)$ (note that $F(\sigma) = F(\sigma_+)$).

Taking $\sigma = u(t,x)$, we get

$$F(u(t,x)) = \left[A - \lambda u_+(t,x)^2 \right.$$

$$\left. + 2\int_{|u(t)>0|}^{|u(t)>u_+(t,x)|} h_s'(u_+(t)_*(s), u_+(t)_*(0))([u_+(t)]_*'(s))^2\,\mathrm{d}s \right]_+^{1/2},$$

which is a non-local expression in u, to be substituted for the first term on the right-hand side of (6).

Also, setting $s = u(t,x)$ in (8), we obtain

$$F(u(t,x))F'(u(t,x)) + \lambda u_+(t,x)$$
$$= h_s'(u_+(t,x), u_+(t)_*(0))[u_+(t)]_*'(|u(t) > u(t,x)|), \tag{9}$$

another non-local expression in u, which coincides with the sum of the second and third terms on the right-hand side of (6). Note that for any $s \in [\operatorname{essinf}_\Omega u(t), \operatorname{esssup}_\Omega u(t)]$,

$$\int_{\{u(t)>s\}} h_s'(u_+(t,x), u_+(t)_*(0))[u_+(t)]_*'(|u(t) > u(t,x)|)\,\mathrm{d}x = h(s_+, u_+(t)_*(0)),$$

by means of (7). Thus, (6) is transformed into a non-local equation like the one in $(\mathscr{P})$. After a truncation argument (as used in [8]), we are led to the assumption of globally bounded functions g and j as in (4). The justification of the truncation argument (passing to the limit) is the main goal of [7].

3. On the notion of weak solution and some a priori estimates

Definition 3.1. Assume that $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$. We say that a function u is a weak solution of $(\mathscr{P})$ if the function $w = u - \gamma$ satisfies the following conditions: $w \in L^2(0,T; H_0^1(\Omega))$, $\beta(w+\gamma)_t \in L^2(0,T; H^{-1}(\Omega))$, $\beta(w+\gamma)_t - \Delta w = G(w+\gamma) + J(w+\gamma)$ in $\mathscr{D}'(\Omega)$ for a.e. $t \in]0,T[$, and $\beta(w+\gamma)|_{t=0} = \beta(u_0)$.

We note that if u is a weak solution of $(\mathscr{P})$, then $\beta(u) \in C([0,T]; L^2(\Omega))$. We have

Lemma 3.2. *If u is any weak solution of $(\mathscr{P})$, then*

$$|\beta(u(t)) - \beta(\gamma)|_{L^\infty(\Omega)} \leqslant C_0 t + |\beta(u_0) - \beta(\gamma)|_{L^\infty(\Omega)} \quad \forall t \in [0,T].$$

Proof. For any integer $m \geqslant 2$, we define $g_m(\sigma) = |\sigma|^{m-2}\sigma$ and denote by T_k the truncation operator given by $T_k(\sigma) = \sigma$ if $|\sigma| \leqslant k$ and $k\,\mathrm{sign}(\sigma)$ otherwise. Then we have $w_{m,k} := g_m \circ T_k(\beta(u) - \beta(\gamma)) \in L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$ and

$$\langle \beta(w(t)+\gamma)_t, w_{m,k}(t)\rangle + \int_\Omega \nabla w(t)\cdot\nabla w_{m,k}(t)\,\mathrm{d}x$$
$$= \int_\Omega [G(w(t)+\gamma) + J(w(t)+\gamma)]w_{m,k}(t)\,\mathrm{d}x, \tag{10}$$

where $w = u - \gamma$ and $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. By the *integration by parts formula* (see, e.g. [1]), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} y_{m,k}(t) = \langle \beta(w(t)+\gamma)_t, w_{m,k}(t)\rangle, \tag{11}$$

where $y_{m,k}(t) = \int_\Omega \mathrm{d}x \int_0^{\beta(w(t)+\gamma)-\beta(\gamma)} g_m \circ T_k(\sigma)\,\mathrm{d}\sigma$. Since $\int_\Omega \nabla w(t)\cdot\nabla w_{m,k}(t)\,\mathrm{d}x \geqslant 0$, and using (5), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} y_{mk}(t) \leqslant C_0 \int_\Omega |w_{m,k}(t)|\mathrm{d}x \leqslant C_0|\Omega|^{1/m}\left(\int_\Omega |w_{m,k}(t)|^{m/(m-1)}\,\mathrm{d}x\right)^{1-1/m}, \tag{12}$$

and so,

$$\int_\Omega |w_{m,k}(t)|^{m/(m-1)}\,\mathrm{d}x = m y_{m,k}(t) - mk^{m-1}\int_\Omega (|\beta(w(t)+\gamma) - \beta(\gamma)| - k)_+\,\mathrm{d}x. \tag{13}$$

Then, relation (12) leads to $y'_{m,k}(t) \leqslant C_0|\Omega|^{1/m} m^{1-(1/m)} y_{m,k}(t)^{1-(1/m)}$, from which we conclude that $y_{m,k}^{1/m}(t) \leqslant C_0|\Omega|^{1/m} m^{-1/m} t + y_{m,k}^{1/m}(0)$. Using (13) and letting $k \to +\infty$ and $m \to +\infty$, we get the result. □

We point out that the above statement remains true if we replace β by any nondecreasing Lipschitz function, as in particular $\beta_\alpha(s) = \alpha s_+ - s_-$, for a given $\alpha > 0$. In addition, we have

Lemma 3.3. *Let $\beta_\alpha(s) = \alpha s_+ - s_-$, with $0 \leqslant \alpha \leqslant 1$, and let u be a weak solution of $(\mathscr{P})$, but with β replaced by β_α. Then, for any $t \in [0,T]$,*

$$|u_+(t)|_{L^\infty(\Omega)} \leqslant \frac{1}{4\pi} C_0|\Omega|. \tag{14}$$

Proof. First, let $\alpha = 0$. We have $(\partial u_-/\partial t) \in L^2(0,T;H^{-1}(\Omega))$ and $\langle (\partial u_-/\partial t)(t), (u_+(t) - \theta)_+\rangle = 0$ for all $\theta > 0$ and a.e. $t \in]0,T[$. Thus, from the equation satisfied by u,

$$\int_{\{u_+(t)>\theta\}} |\nabla u_+(t)|^2\,\mathrm{d}x = \int_\Omega [G(u(t)) + J(u(t))](u_+(t) - \theta)_+\,\mathrm{d}x.$$

Differentiating with respect to θ and using (5), we get

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{u_+(t)>\theta\}} |\nabla u_+(t)|^2\,\mathrm{d}x \leqslant C_0|u_+(t) > \theta|.$$

Then, the conclusion holds by standard arguments (see, e.g. [8]). Now assume that $0 < \alpha \leqslant 1$. We argue as in [10]. Taking $(u_+(t) - \theta)_+$ as a test function and differentiating with respect to θ, we get

$$\alpha \int_{\{u_+(t)>\theta\}} \frac{\partial}{\partial t} u_+(t)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\{u_+(t)>\theta\}} |\nabla u_+(t)|^2\,\mathrm{d}x$$
$$= \int_{\{u_+(t)>\theta\}} G(u(t))\,\mathrm{d}x + \int_{\{u_+(t)>\theta\}} J(u(t))\,\mathrm{d}x,$$

for a.e. $\theta > 0$. Then arguing as in [5], we obtain that

$$-4\pi s \frac{\partial}{\partial s} u_+(t)_*(s) \leqslant C_0 s - \alpha \int_0^s \frac{\partial}{\partial t} u_+(t)_*(\sigma)\,\mathrm{d}\sigma, \tag{15}$$

for all $s \in]0, |\Omega|[$. If we introduce $K(t,s) = \int_0^s u_+(t)_*(\sigma)\,\mathrm{d}\sigma$, then relation (15) leads to

$$\begin{cases} \alpha \dfrac{\partial K}{\partial t}(t,s) - 4\pi s \dfrac{\partial^2 K}{\partial s^2}(t,s) \leqslant C_0 s, \\ K(t,0) = 0, \quad \dfrac{\partial K}{\partial s}(t, |\Omega|) = 0. \end{cases}$$

We now define a function $\hat{K}(s)$, satisfying

$$C_0 s = -4\pi s \frac{\mathrm{d}^2 \hat{K}}{\mathrm{d}s^2}, \quad \hat{K}(0) = 0, \quad \frac{\mathrm{d}\hat{K}}{\mathrm{d}s}(|\Omega|) = 0,$$

that is,

$$\hat{K}(s) = -\frac{C_0}{4\pi} s^2 + \frac{C_0}{4\pi} |\Omega|.$$

Then, from the comparison principle (see [5]), we deduce that $\hat{K}(t,s) \leqslant \hat{K}(s)$ for any $s \in [0, |\Omega|]$ and any $t \in [0,T]$. In particular, we get (14). □

Lemma 3.4. *Assume the conditions of Lemma* 3.3. *Then, for any* $t \in [0,T]$,

$$\int_0^t \int_\Omega |\nabla u(\sigma,x)|^2\,\mathrm{d}x\,\mathrm{d}\sigma + \int_\Omega \mathrm{d}x \int_0^{u_0(x)} \beta_\alpha(\sigma+\gamma)\,\mathrm{d}\sigma$$
$$\leqslant \int_\Omega (u_0(x) - \gamma)\beta_\alpha(u_0(x))\,\mathrm{d}x + C_0 \int_0^t \int_\Omega |u(\sigma,x) - \gamma|\,\mathrm{d}x\,\mathrm{d}\sigma.$$

Proof. By the *integration by parts formula* we have

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + \int_\Omega |\nabla w(t)|^2\,\mathrm{d}x = \int_\Omega [G(w(t)+\gamma) + J(w(t)+\gamma)]w(t)\,\mathrm{d}x, \tag{16}$$

where $w = u - \gamma$ and

$$y(t) = \int_\Omega w(t)\beta_\alpha(w(t)+\gamma)\,\mathrm{d}x - \int_\Omega \mathrm{d}x \int_0^{w(t,x)} \beta_\alpha(\sigma+\gamma)\,\mathrm{d}\sigma.$$

Integrating relation (16) with respect to t, dropping some non-negative terms ($y(t) \geqslant 0$ since β_α is nondecreasing), and using (5), we get the result. □

4. On the existence of a weak solution

The main result of this paper is the following:

Theorem 4.1. *Assume that* $u_0-\gamma\in H_0^1(\Omega)\cap L^\infty(\Omega)$. *Then there exists a weak solution* u *of* $(\mathscr{P})$. *Moreover,* $u\in L^\infty(Q)$, $u-\gamma\in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$, *and* $\beta(u)_t\in L^2(Q)$.

We will start by proving a similar result for $(\mathscr{P})_\alpha$, that is, problem $(\mathscr{P})$ with β replaced by β_α (as defined in Lemma 3.3), for $0<\alpha\leqslant 1$:

Theorem 4.2. *Assume that* $u_0-\gamma\in H_0^1(\Omega)\cap L^\infty(\Omega)$ *and* $0<\alpha\leqslant 1$. *Then there exists a weak solution* u^α *of* $(\mathscr{P})_\alpha$. *Moreover,* $u^\alpha\in L^2(0,T;H^2(\Omega))\cap L^\infty(Q)$ *and* $u_t^\alpha\in L^2(Q)$.

The proof of Theorem 4.2 will be obtained by means of a Galerkin method as in [6]. First, we shall find solutions w_m of some auxiliary, finite-dimensional problems $(\mathscr{P})_{\alpha,m}$.

4.1. *On the finite-dimensional problems* $(\mathscr{P})_{\alpha,m}$

Let $(\lambda_k,\varphi_k)_{k\geqslant 1}$ be the eigenvalues and eigenfunctions associated with $-\Delta$ on Ω with zero boundary conditions, i.e., $-\Delta\varphi_k=\lambda_k\varphi_k$ and $\varphi_k\in H_0^1(\Omega)$. For $m\geqslant 1$, we denote by V_m the vector space spanned by $\{\varphi_1,\dots,\varphi_m\}$. For all $v\in V_m$ we shall use the decomposition $v=\sum_{i=1}^m v^i\varphi_i$.

For a fixed α with $0<\alpha\leqslant 1$, we consider the following finite-dimensional approximations to problem $(\mathscr{P})_\alpha$: To find $w_m\in L^1(0,T;V_m)$, $w_m(t)=\sum_{i=1}^m w_m^i(t)\varphi_i$, satisfying

$$(\mathscr{P})_{\alpha,m}\begin{cases}\displaystyle\int_\Omega\left(\frac{\partial}{\partial t}\beta_\alpha(w_m(t)+\gamma)\right)\varphi_k\,\mathrm{d}x+\int_\Omega\nabla w_m(t)\cdot\nabla\varphi_k\,\mathrm{d}x\\ \displaystyle\quad=\int_\Omega G(w_m(t)+\gamma)\varphi_k\,\mathrm{d}x+\int_\Omega J(w_m(t)+\gamma)\varphi_k\,\mathrm{d}x,\ \ k=1,\dots,m,\\ w_m(0)=P_m(u_0-\gamma),\end{cases}$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m.

Theorem 4.3. *For each* $m\geqslant 1$ *there exists a solution* w_m *of problem* $(\mathscr{P})_{\alpha,m}$. *Furthermore, there exists a number* k_0 *such that* $w_m\not\equiv 0$ *for all* $m\geqslant k_0$.

Problem $(\mathscr{P})_{\alpha,m}$ can be written as a nonlinear differential system for the functions $w_m^1(t),\dots,w_m^m(t)$. Indeed, these functions satisfy

$$\sum_{i=1}^m a_{ik}(w_m(t))\frac{\mathrm{d}}{\mathrm{d}t}w_m^i(t)+\sum_{i=1}^m b_{ik}w_m^i(t)=\hat{\mathscr{I}}_k(w_m(t)),$$

$$w_m^k(0)=\text{the } k\text{th component of } P_m(u_0-\gamma),\quad k=1,\dots,m, \tag{17}$$

where $a_{ik}(v):=\int_\Omega\beta'_\alpha(v+\gamma)\varphi_i\varphi_k\,\mathrm{d}x$, $b_{ik}:=\int_\Omega\nabla\varphi_i\cdot\nabla\varphi_k\,\mathrm{d}x$, and $\hat{\mathscr{I}}_k(v):=\int_\Omega G(v+\gamma)\varphi_k\,\mathrm{d}x+\int_\Omega J(v+\gamma)\varphi_k\,\mathrm{d}x$, for $v\in V_m$ and $i,k=1,\dots,m$.

To prove the existence of a solution of this initial-value problem, we need the following

Lemma 4.4. *The functions* $\hat{\mathscr{I}}_k : V_m \to \mathbb{R}$, $k = 1, \ldots, m$, *are continuous.*

Proof. This is a consequence of assumption (4) and of the following Lemma 4.5 (see [8] for details). □

Lemma 4.5. *Let* $(v_n)_{n \geqslant 1}$ *be a sequence in* $V_m \setminus \{0\}$ *and let* v *be in* $V_m \setminus \{0\}$, *such that* $v_n \to v$ *in* V_m. *Then one has that*

$$v_n \underset{n\to\infty}{\longrightarrow} v \quad \textit{strongly in } \mathscr{C}^k(\bar{\Omega}) \ \forall k \in \mathbb{N} \cup \{0\},$$

$$v_{n*} \underset{n\to\infty}{\longrightarrow} v_* \quad \textit{strongly in } \mathscr{C}(\bar{\Omega}_*),$$

$$v'_{n+*} \underset{n\to\infty}{\longrightarrow} v'_{+*} \quad \textit{strongly in } L^q(\Omega_*) \ \forall 1 \leqslant q < \infty,$$

$$v'_{n+*}(|v_n > v_n(\cdot)|) \underset{n\to\infty}{\longrightarrow} v'_{+*}(|v > v(\cdot)|) \quad \textit{strongly in } L^q(\Omega) \ \forall 1 \leqslant q < \infty.$$

Proof. See [8, Lemma 22]. Regarding the notation, recall that $\Omega_* :=]0, |\Omega|[$. Moreover, given a (measurable) function v on Ω and a function ϕ on $\bar{\Omega}_*$, we denote by $\phi(|v > v(\cdot)|)$ the function $x \mapsto \phi(|v > v(x)|)$, defined on Ω. □

Next, we will establish some a priori estimates.

Lemma 4.6. *Let* $w_m(t) = \sum_{i=1}^m w_m^i(t)\varphi_i$, *where* $(w_m^1(t), \ldots, w_m^m(t))$ *is a (local) solution of the initial-value problem* (17), *for some* $m \geqslant 1$. *Then we have*

$$\int_0^t |\nabla w_m(\sigma)|^2_{L^2(\Omega)}\, \mathrm{d}\sigma + 2\int_\Omega \mathrm{d}x \int_0^{w_m(0)} \beta_\alpha(\sigma + \gamma)\, \mathrm{d}\sigma$$

$$\leqslant 2\int_\Omega w_m(0)\beta_\alpha(w_m(0) + \gamma)\, \mathrm{d}x + \frac{C_0^2|\Omega|t}{\lambda_1}$$

and

$$\int_0^t |w'_m(s)|^2_{L^2(\Omega)}\, \mathrm{d}s \leqslant \frac{1}{\alpha}|\nabla w_m(0)|^2_{L^2(\Omega)} + \frac{1}{\alpha^2}C_0^2|\Omega|,$$

for all t *in the solution's interval of existence.*

Proof. The first statement follows from an estimate similar to the one in Lemma 3.4 and from Poincaré's inequality. To prove the second statement, we multiply the kth equation in (17) by $(\mathrm{d}/\mathrm{d}t)w_m^k(t)$. Summing over k, we get

$$\int_\Omega \beta'_\alpha(w_m(t) + \gamma))|w'_m(t)|^2\, \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w_m(t)|^2\, \mathrm{d}x$$

$$= \int_\Omega [G(w_m(t) + \gamma) + J(w_m(t) + \gamma)]w'_m(t)\, \mathrm{d}x.$$

Since $\beta'_\alpha \geqslant \alpha$, it follows that

$$\alpha|w'_m(t)|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla w_m(t)|^2_{L^2(\Omega)} \leqslant C_0|w'_m(t)|_{L^1(\Omega)}.$$

Applying Hölder's and Young's inequalities, we get

$$\alpha|w'_m(t)|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla w_m(t)|^2_{L^2(\Omega)} \leqslant \frac{\alpha}{2}|w'_m(t)|^2_{L^2(\Omega)} + \frac{C_0^2|\Omega|}{2\alpha}.$$

Integrating, we have

$$\int_0^t |w'_m(s)|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{1}{\alpha}|\nabla w_m(t)|^2_{L^2(\Omega)} \leqslant \frac{1}{\alpha}|\nabla w_m(0)|^2_{L^2(\Omega)} + \frac{1}{\alpha^2}C_0^2|\Omega|,$$

which proves the assertion. □

Proof of Theorem 4.3. Since $\{\varphi_1,\dots,\varphi_m\}$ is a basis for V_m and since $\beta_\alpha \in W^{1,\infty}(\mathbb{R})$ with $0<\alpha \leqslant \beta'_\alpha \leqslant 1$, the matrix of coefficients $a_{ik}(w_m(t))$ in (17) is invertible. So, by the Cauchy–Peano theorem, the initial-value problem (17) has a maximal solution $(w_m^1(t),\dots,w_m^m(t))$, defined on some interval $[0,T_m]$. From the a priori estimates given in Lemma 4.6, we have that, in fact, $T_m = T$; that is, $w_m(t)=\sum_{i=1}^m w_m^i(t)\varphi_i$ is a solution of $(\mathscr{P})_{\alpha,m}$. To finish the proof, we observe that there exists a number k_0 with $\int_\Omega \varphi_{k_0}(x)\,\mathrm{d}x \neq 0$, since $(\varphi_k)_{k\geqslant 1}$ is complete in $L^2(\Omega)$. Now suppose that $m \geqslant k_0$ and $w_m \equiv 0$. Then $G(w_m+\gamma)$ and $J(w_m+\gamma)$ are constants, and the k_0th equation in (17) implies that $\int_\Omega \varphi_{k_0}(x)\,\mathrm{d}x = 0$. Thus, $w_m \not\equiv 0$ if $m \geqslant k_0$. □

Corollary 4.7. *For $m \geqslant 1$, let w_m be a solution of $(\mathscr{P})_{\alpha,m}$. Then we have*:
(a) $(w_m)_{m\geqslant 1}$ *is bounded in* $L^2(0,T;H_0^1(\Omega))$.
(b) $((\partial w_m/\partial t))_{m\geqslant 1}$ *is bounded in* $L^2(Q)$.
(c) $(w_m)_{m\geqslant 1}$ *is bounded in* $Y := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

Proof. (a) and (b) follow from the estimates in Lemma 4.6. To prove (c), we note that w_m satisfies

$$P_m(\beta_\alpha(w_m(t)+\gamma)_t) - \Delta w_m(t) = P_m(G(w_m(t)+\gamma) + J(w_m(t)+\gamma)), \tag{18}$$

for a.e. $t \in]0,T[$, where P_m is the orthogonal projection from $L^2(\Omega)$ onto V_m. From (b) and the fact that $\alpha \leqslant \beta'_\alpha \leqslant 1$, we get that $\beta_\alpha(w_m+\gamma)_t$ is bounded in $L^2(Q)$. From (18), we then infer that Δw_m remains in a bounded set of $L^2(Q)$. The rest is standard. □

4.2. Passing to the limit as $m \to \infty$

Proof of Theorem 4.2. For a fixed α with $0<\alpha\leqslant 1$, let $(w_m)_{m\geqslant 1}$ be a sequence of solutions of $(\mathscr{P})_{\alpha,m}$. By Corollary 4.7, $(w_m)_{m\geqslant 1}$ has a subsequence (still denoted by $(w_m)_{m\geqslant 1}$) that converges, weakly in Y, to a function $w^\alpha \in Y$. So, by compactness results (see [9,14]), we get (again for some subsequence)

$$w_m \to w^\alpha \text{ strongly in } L^2(0,T;H_0^1(\Omega)) \cap L^2(0,T;W^{1,p}(\Omega)) \quad \text{for all } p \geqslant 1.$$

As G and J are bounded, there exist $G_\alpha, J_\alpha \in L^\infty(Q)$ such that $G(w_m+\gamma) \rightharpoonup G_\alpha$ and $J(w_m+\gamma) \rightharpoonup J_\alpha$ weakly-star in $L^\infty(Q)$. Thus, w^α is a solution of the *limit problem*

$$\begin{cases} \beta_\alpha(w+\gamma)_t - \Delta w^\alpha = G_\alpha + J_\alpha, \\ w^\alpha(0) = u_0 - \gamma \text{ and } w^\alpha \in Y. \end{cases}$$

To verify that $G_\alpha = G(w^\alpha+\gamma)$ and $J_\alpha = J(w^\alpha+\gamma)$, we argue as in the proof of the continuity of the maps $\hat{\mathscr{I}}_k$ (see Lemma 4.4) and use the fact that $w_m \to w^\alpha$ strongly in $L^2(0,T;W^{1,p}(\Omega))$ for all $p \geqslant 1$ and Lemma 7 of [8]. So, $u^\alpha = w^\alpha + \gamma$ is a weak solution of $(\mathscr{P})_\alpha$; moreover, $u^\alpha - \gamma \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $u_t^\alpha \in L^2(Q)$. Finally, by Lemmas 3.2 and 3.3, $u^\alpha \in L^\infty(Q)$. □

4.3. *Passing to the limit as $\alpha \to 0$*

Let $(u^\alpha)_{0<\alpha\leqslant 1}$ be a family of solutions of $(\mathscr{P})_\alpha$, according to Theorem 4.2, and let $w^\alpha = u^\alpha - \gamma$.

Lemma 4.8. *We have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_-^\alpha(t)|^2\,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial t}u_-^\alpha(t)\Delta u^\alpha(t)\,\mathrm{d}x.$$

Moreover, the family $(u_-^\alpha)_{0<\alpha\leqslant 1}$ is bounded in $L^\infty(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega))$.

Proof. The desired equality follows from an integration by parts. To obtain an estimate for $(\partial/\partial t)u_-^\alpha$, we multiply the differential equation in $(\mathscr{P})_\alpha$ by $(\partial/\partial t)u_-^\alpha$ and integrate over Ω. Using the equality stated in the lemma and (5), we find that

$$\left|\frac{\partial}{\partial t}u_-^\alpha(t)\right|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_-^\alpha(t)|^2\,\mathrm{d}x \leqslant C_0\left|\frac{\partial}{\partial t}u_-^\alpha(t)\right|_{L^1(\Omega)}.$$

From this we deduce, after integration with respect to t and a simple estimate, that

$$\int_0^t \left|\frac{\partial}{\partial t}u_-^\alpha(\sigma)\right|^2_{L^2(\Omega)}\mathrm{d}\sigma + |\nabla u_-^\alpha(t)|^2_{L^2(\Omega)} \leqslant |\nabla(u_0)_-|^2_{L^2(\Omega)} + T|\Omega|C_0^2.$$

The last assertion of the lemma now follows. □

With similar reasoning as in the proof of Corollary 4.7(c), we infer

Lemma 4.9. *The family $(u^\alpha)_{0<\alpha\leqslant 1}$ is bounded in $L^2(0,T;H^2(\Omega))$.*

Proof of Theorem 4.1. Due to the boundedness properties of $u^\alpha = w^\alpha + \gamma$ (see Lemmas 3.2–3.4), there exists $w \in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$ such that $w^\alpha \rightharpoonup w$ weakly in $L^2(0,T;H_0^1(\Omega))$ and weakly-star in $L^\infty(Q)$. Also, there exists $z \in L^2(Q)$ such that $(w^\alpha+\gamma)_- \to z$ strongly in $L^2(Q)$. To verify that $z = (w+\gamma)_-$, consider the maximal-monotone operator $A : L^2(Q) \to L^2(Q)$, defined by $Av := -(v+\gamma)_- = \min(0, v+\gamma)$ for any $v \in L^2(Q)$. By the previous arguments, $w^\alpha + \gamma \rightharpoonup w + \gamma$ weakly in $L^2(Q)$ and $-Aw^\alpha \to z$ strongly in $L^2(Q)$. Thus, by the theory of maximal-monotone operators (see [3]), we conclude that $z = -Aw^\alpha$, that is, $z = (w+\gamma)_-$. As a consequence of Lemmas 4.8 and 4.9, we have $(w+\gamma)_- \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $w \in L^2(0,T;H^2(\Omega))$. To identify the nonlinear terms in the equations, after passing to the limit as $\alpha \to 0$, we apply a compactness result due to Rakotoson and Temam (see [13]). We deduce that $w^\alpha \to w$ strongly in $L^2(Q)$. Then, from Lemma 4.8 and the boundedness of w^α in $L^\infty(Q)$, we have

$$\lim_{\alpha \searrow 0} \int_Q w^\alpha \frac{\partial}{\partial t}(w^\alpha + \gamma)_- \, \mathrm{d}x \, \mathrm{d}t = \int_Q w \frac{\partial}{\partial t}(w+\gamma)_- \, \mathrm{d}x \, \mathrm{d}t$$

and

$$\lim_{\alpha \searrow 0} \alpha \int_Q w^\alpha \frac{\partial}{\partial t}(w^\alpha + \gamma)_+ \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Furthermore, $(G+J)(w^\alpha+\gamma)$ converges, weakly in $L^2(Q)$, to some function h; thus,

$$\lim_{\alpha \searrow 0} \int_Q w^\alpha (G+J)(w^\alpha+\gamma) = \int_Q wh.$$

Multiplying the differential equation in $(\mathscr{P})_\alpha$ by w^α, integrating over Q, and letting $\alpha \to 0$, we deduce

$$\lim_{\alpha \searrow 0} \int_Q |\nabla w^\alpha|^2 \, \mathrm{d}x \, \mathrm{d}t = \int_Q w \frac{\partial}{\partial t}(w+\gamma)_- \, \mathrm{d}x \, \mathrm{d}t + \int_Q hw \, \mathrm{d}x \, \mathrm{d}t = \int_Q |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

From the weak convergence of w^α to w in $L^2(0,T;H_0^1(\Omega))$ and from the last equality, we get that $w^\alpha \to w$ strongly in $L^2(0,T;H_0^1(\Omega))$. Thus, we may assume that $w^\alpha(t) \to w(t)$ strongly in $H_0^1(\Omega)$ for a.e. $t \in]0,T[$. In fact, as w^α remains in a bounded set of $L^2(0,T;H^2(\Omega))$, Gagliardo–Nirenberg interpolation shows that

$$w^\alpha(t) \to w(t) \text{ strongly in } W^{1,p}(\Omega) \quad \text{for a.e. } t \in]0,T[\text{ and } 1 \leqslant p < \infty.$$

Finally, we argue as in the proof of Theorem 4.2 to show that $h = G(w+\gamma) + J(w+\gamma)$. We conclude that $u = w+\gamma$ is a weak solution of $(\mathscr{P})$ with the properties claimed in the theorem. □

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