

ON THE INVISCID AND NON-RESISTIVE LIMIT  
FOR THE EQUATIONS OF INCOMPRESSIBLE  
MAGNETOHYDRODYNAMICS

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We prove the convergence of the solutions for the incompressible homogeneous magnetohydrodynamics (MHD) system to the solutions to ideal MHD one in the inviscid and non-resistive limit, detailing the explicit convergence rates. For this study we consider a fluid occupying the whole space  $\mathbb{R}^3$  and we assume that the viscosity effects in this fluid can be described by two different operators: the usual Laplacian operator affected by the inverse of the Reynolds number or by a viscosity operator introduced by S. I. Braginskii in 1965.

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## 1. Introduction

The magnetohydrodynamic (MHD) equations govern the macroscopic behavior of electrically conducting fluids submitted to the influence of magnetic fields and they are obtained by coupling the Navier–Stokes system for the fluid flow and the quasi-stationary Maxwell’s equations (see, for instance, Cowling<sup>9</sup> or Strohmer<sup>24</sup>).

The nondimensional form of the one-fluid homogeneous incompressible MHD system in the whole space  $\mathbb{R}^3$  is given by

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{F}(\mathbf{u}) + \nabla p + \mathbf{S} \nabla \left( \frac{1}{2} \mathbf{B}^2 \right) - \mathbf{S}(\mathbf{B} \cdot \nabla) \mathbf{B} = \mathbf{f} & \text{in } (0, T) \times \mathbb{R}^3, \\ \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} + \nu_m \operatorname{rot}(\operatorname{rot} \mathbf{B}) = \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{u} = \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{B} = \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \mathbf{u}(\mathbf{0}, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) & \text{on } \mathbb{R}^3, \\ \mathbf{B}(\mathbf{0}, \mathbf{x}) = \mathbf{B}_0(\mathbf{x}) & \text{on } \mathbb{R}^3, \end{array} \right. \quad (1.1)$$

for a fixed  $T > 0$ , where  $\mathbf{u} = (\mathbf{u}_i)$ ,  $i = 1, 2, 3$  is the fluid velocity,  $\mathbf{B} = (\mathbf{B}_i)_i$ ,  $i = 1, 2, 3$ , the magnetic field,  $p$  the pressure of the fluid and  $\mathbf{f} \in \mathbb{R}^3$  represents a (nondimensional) volume density force. The two dimensionless numbers appearing in (1.1) are the magnetic viscosity  $\nu_m := \frac{1}{Rm}$  (with  $Rm := L_* U_* \sigma \mu$  being the magnetic Reynolds number, which is proportional to the magnetic permeability  $\mu$  and the electric conductivity of the fluid  $\sigma$ , where  $L_*$  and  $U_*$  the characteristic magnitudes for length and velocity respectively) and the constant  $S$  which is proportional to  $\mu$ . We have denote by  $\mathbf{F}(\mathbf{u})$  the viscosity forces acting on the flow which will be specified later.

If in the above system we neglect the viscosity forces in the fluid and assume it to be perfectly conductor, i.e. non-resistive, we are under the hypothesis of ideal magnetohydrodynamics whose mathematical equations are given by

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla) \mathbf{u}^0 + \frac{S}{2} \nabla (\mathbf{B}^0)^2 - \mathbf{S}(\mathbf{B}^0 \cdot \nabla) \mathbf{B}^0 + \nabla p^0 = \mathbf{f} & \text{in } (0, T) \times \mathbb{R}^3, \\ \frac{\partial \mathbf{B}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla) \mathbf{B}^0 - (\mathbf{B}^0 \cdot \nabla) \mathbf{u}^0 = \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{u}^0 = \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{B}^0 = \mathbf{0} & \text{in } (0, T) \times \mathbb{R}^3, \\ \mathbf{u}^0(\mathbf{0}, \mathbf{x}) = \mathbf{u}_0^0(\mathbf{x}) & \text{on } \mathbb{R}^3, \\ \mathbf{B}^0(\mathbf{0}, \mathbf{x}) = \mathbf{B}_0^0(\mathbf{x}) & \text{on } \mathbb{R}^3. \end{array} \right. \quad (1.2)$$

The MHD systems appear in diverse areas of interest, being the base of many complex models in astrophysics and nuclear fusion theory (see, for example, Biskamp<sup>2</sup> and Freidberg<sup>13</sup>). We are especially interested in this last application, where the incompressible ideal MHD plays an important role, providing the simplest model for the detection and description of equilibrium and stability properties for a magnetically confined plasma in fusion reactors as Tokamaks or Stellarators (see, e.g., Braginskii,<sup>3</sup> Freidberg<sup>13</sup> or Hazeltine and Meiss<sup>15</sup>). In these reactors, there

are gases in plasma state, whose behavior is modelled by the ideal MHD model, coexisting with other gases that are not under the hypothesis of ideality.

In the present paper we are concerned with the transition from MHD to ideal MHD in the whole space  $\mathbb{R}^3$ . Mathematically, this study is equivalent to establish the convergence of the solutions of the MHD system (1.1) to solutions of the ideal one (1.2), in the limit to infinity of the electric conductivity  $\sigma$  and to zero of the viscosity operator  $\mathbf{F}$  (the rigorous meaning of this limit will be cleared up once we have detailed  $\mathbf{F}$ ). We point out the analogy between this convergence and the passing from the Navier–Stokes equations to the Euler’s one in fluid dynamics, for which an extensive bibliography can be found since the pioneering works of Kato<sup>17</sup> or Swann.<sup>26</sup> In the study that we develop here, we will assume that the viscosity effects in the fluid are described by the usual Laplacian operator affected by the inverse of the Reynold’s number  $Re$ . But we shall also consider the case where viscosity is described by the Braginskii’s operator, introduced by Braginskii in 1965,<sup>3</sup> which is widely used in the theory of fusion plasmas by magnetic confinement (see, e.g., Hazeltine and Meiss<sup>15</sup>).

We have organized this article as follows: in Sec. 2 we introduce the functional framework in which we shall work and present the operators that we shall use for describing the viscosity forces in the fluid; in Sec. 3 we give a local in time existence and uniqueness result for the MHD systems which we shall use for the setting of the convergence of the solutions. Finally, in Sec. 4 we state the convergence for both viscosity operators in the norms of the Lebesgue space  $L^2$  and in the Sobolev spaces  $W^{2,s}$  for  $s > 3/2$ , giving explicit convergence rates. In this direction, we must mention the existence of two related papers due to Wu<sup>31</sup> and Díaz,<sup>11</sup> where the authors study, by using independent arguments, the convergence of the solution of the MHD system, with  $\mathbf{F} = \frac{1}{Re} \Delta$ , in the limit  $Re, Rm \rightarrow +\infty$  and  $S \rightarrow 0$ . This last limit, which is explicitly used in their proofs, is related with the degree of capacity of fluids flows to be shaped by using magnetic fields.

## 2. The Viscosity Operators

Due to the homogeneous incompressible character of the fluids that we are dealing with, the natural framework in which we shall work is that of the solenoidal vector fields of  $L^2(\mathbb{R}^3)^3$ . We shall denote by  $H^s$  the vectorial  $L^2$ -type Sobolev spaces  $H^s(\mathbb{R}^3)^3 = W^{s,2}(\mathbb{R}^3)^3$ , with  $s \geq 0$  (not necessarily an integer) and we shall use  $(\cdot, \cdot)$  and  $(\cdot, \cdot)_s$  for the inner product in  $L^2$  and  $H^s$  respectively;  $\|\cdot\|_0$  and  $\|\cdot\|_s$  will be used for the associated norms in those spaces (throughout this paper we shall use this notation indiscriminately for scalars and vectors).

Let  $P$  be the Leray operator, which maps vector-valued function into divergence free vectors, given by  $P_{jk} = \delta_{jk} - R_j R_k$ ,  $j, k = 1, 2, 3$ , where  $R_j$  are the Riesz transforms. It is well known (see, e.g., Constantin and Foias,<sup>8</sup> Kato and Ponce<sup>19</sup> or Cannone<sup>6</sup>) that this projection is a bounded operator acting on  $L^2(\mathbb{R}^3)^3$  and  $H^s$ ,  $s > 0$ . Throughout this paper we shall denote by  $H_\sigma^s$  the image by  $P$  of the  $H^s$

spaces with  $s > 0$  and by  $H_\sigma = PL^2(\mathbb{R}^3)^3$ . That is,

$$H_\sigma^s = \{\mathbf{u} \in \mathbf{H}^s : \operatorname{div} \mathbf{u} = 0\}.$$

Next, let us describe the viscosity force  $\mathbf{F}$  that appears in (1.1). As we mentioned in the Introduction, we shall consider viscosity effects described by two different operators. Usually, in an incompressible liquid these effects are assumed to be given by

$$\mathbf{F}(\mathbf{u}) = \frac{1}{\operatorname{Re}} \Delta \mathbf{u}, \tag{2.3}$$

where  $\operatorname{Re} := (\mathbf{L}_* \mathbf{U}_*)/\nu$  is the Reynolds number and  $\nu$  the kinematic viscosity of the fluid. In this paper we shall also contemplate the situation in which viscosity is described by the Braginskii operator  $\mathbf{F}(\mathbf{u}) = \hat{\mathbf{V}}\mathbf{u}$  (see Braginskii<sup>3</sup>), that appears very often in the study of fusion plasmas by magnetic confinement (see, for instance, Freidberg,<sup>13</sup> Hazeltine and Meiss<sup>15</sup>). This operator is given by

$$(\hat{\mathbf{V}}\mathbf{u})_i = \frac{-\partial \pi_{ij}}{\partial x_j}, \quad i, j = 1, 2, 3 \tag{2.4}$$

(summation under repeated index is understood), where  $\pi_{ij}$  is the stress tensor defined by mean of five viscosity coefficients  $\mu_\alpha$ ,  $\alpha = 0, 1, \dots, 4$ , as:

$$\pi_{ij} = \sum_{\alpha=0}^4 \gamma_\alpha \mu_\alpha \mathbf{W}_{\alpha ij} (\gamma_\alpha = -1 \text{ if } \alpha = 0, 1, 2 \text{ and } \gamma_\alpha = 1 \text{ for } \alpha = 3, 4),$$

with  $W_{\alpha ij} := \sum_{k,l=0}^3 A_{\alpha ij,kl}(\mathbf{h}) W_{kl}$  and  $W_{kl} = \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - \frac{2}{3} \delta_{kl} \nabla \cdot \mathbf{u}$ . The coefficients  $A_{\alpha ij,kl}$  are polynomials in  $\mathbf{h} := \mathbf{B}/|\mathbf{B}|$  and are given on p. 250 of Braginskii,<sup>3</sup> while the viscosity coefficients  $\mu_\alpha$  ( $\alpha = 0, 1, \dots, 4$ ) are positive and just depend on  $|\mathbf{B}|$ . Nevertheless, in order to avoid this pathological situation, we shall use here the approach already followed by Spada and Wobig<sup>23</sup> (see also Ref. 3 for a slightly different approach) where the coefficients in  $\hat{\mathbf{V}}$  result to be independent of the magnetic field (i.e.  $A_{\alpha ij,kl}$  are constants). In particular, we shall assume that the viscosity coefficients  $\mu_\alpha$  are approximated by positive constants.

It is well known (see, e.g., Constantin and Foias<sup>8</sup>) that, in the absence of boundaries, the Laplacian operator  $-\Delta$  and the Stokes's one  $-P\Delta$  acting on the  $H^s$ -spaces are identical (i.e.  $-\Delta$  and the projection  $P$  commute). It is easy to check, by using the Fourier transform, that, under the above hypothesis on  $\mu_\alpha$  and  $A_{\alpha ij,kl}$ , this commutation also holds for the Braginskii operator  $\hat{\mathbf{V}}$ . Furthermore, we have the following property:

**Lemma 2.1.**  $(-\hat{\mathbf{V}}, H_\sigma^2)$  is a continuous and  $m$ -dissipative operator acting on  $H_\sigma$ .

**Proof.** In order to prove that  $(-\hat{\mathbf{V}}, H_\sigma^2)$  is a  $m$ -dissipative operator in  $H_\sigma$  we use the bilinear form  $\mathcal{E} : \mathbf{H}^1(\mathbb{R}^3)^3 \times \mathbf{H}^1(\mathbb{R}^3)^3 \rightarrow \mathbb{R}$  introduced by Spada and Wobig,<sup>23</sup>

$$\mathcal{E}(\mathbf{u}, \mathbf{v}) = \sum_{\alpha=0}^4 \gamma_\alpha \mu_\alpha \int_{\mathbb{R}^3} (A_{\alpha ijkl} + A_{\alpha ijlk}) \frac{\partial u_k}{\partial x_l} \frac{\partial v_i}{\partial x_j} \mathbf{d}\mathbf{x}.$$

This form is continuous and coercive and verifies (see Spada and Wobig<sup>23</sup>)

$$\mathcal{E}(\mathbf{u}, \mathbf{v}) = (-\hat{\mathbf{V}}\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_\sigma^1.$$

Then,  $(-\hat{\mathbf{V}}, H_\sigma^2)$  is dissipative in  $H_\sigma$ ; in particular (see Ref. 23)

$$(-\hat{\mathbf{V}}\mathbf{v}, \mathbf{v}) = \mathcal{E}(\mathbf{v}, \mathbf{v}) \geq \min(\mu_\alpha) \|\nabla \mathbf{v}\|_0^2 \geq 0.$$

Thus, if we prove there exists  $\lambda_0 > 0$  such that,  $\forall \mathbf{g} \in \mathbf{H}_\sigma^2$  there exists  $\mathbf{v} \in \mathbf{H}_\sigma^2 = \mathbf{D}(\hat{\mathbf{V}})$  being the unique solution to  $\mathbf{v} - \lambda_0 \hat{\mathbf{V}}\mathbf{v} = \mathbf{g}$ , we would have that  $(-\hat{\mathbf{V}}, H_\sigma^2)$  is  $m$ -dissipative (see, e.g., Cazenave and Haraux<sup>7</sup>). Indeed, we consider the restriction of  $\mathcal{E}$  to the divergence-free space  $H_\sigma^1 \times H_\sigma^1$  and we define a new bilinear form  $a : H_\sigma^1 \times H_\sigma^1 \rightarrow \mathbb{R}$

$$a(\mathbf{v}, \mathbf{w}) := \mathcal{E}(\mathbf{v}, \mathbf{w}) + (\mathbf{v}, \mathbf{w}).$$

This bilinear form is again continuous and coercive, hence we can appeal to Lax-Milgram's theorem to conclude that there exists a unique  $\mathbf{v} \in L^2(\mathbb{R}^3)^3$  such that

$$\mathbf{v} - \hat{\mathbf{V}}\mathbf{v} = \mathbf{g}.$$

Moreover, by elliptic-regularity results (see, for example, Ref. 14) and using that the projection  $P$  commutes with Braginskii's operator, we deduce that  $\mathbf{v} \in \mathbf{H}_\sigma^2$ .

For the continuity of  $\hat{\mathbf{V}}$  in  $H_\sigma$ , we use that  $\forall \mathbf{v} \in \mathbf{H}_\sigma^2$ ,

$$(\hat{\mathbf{V}}\mathbf{v})_i = - \sum_{\alpha=0}^4 \gamma_\alpha \mu_\alpha \int_{\mathbb{R}^3} A_{\alpha ijkl} \left( \frac{\partial^2 v_k}{\partial x_j \partial x_l} + \frac{\partial^2 v_l}{\partial x_j \partial x_k} \right) \mathbf{d}\mathbf{x}, \quad i = 1, 2, 3.$$

Hence, as  $|\gamma_\alpha| = 1$ ,

$$\begin{aligned} \|(\hat{\mathbf{V}}\mathbf{v})_i\|_0^2 &= \int_{\mathbb{R}^3} \left\{ \sum_{k,l,j=1}^3 \left[ - \sum_{\alpha=0}^4 \gamma_\alpha \mu_\alpha A_{\alpha ijkl} \left( \frac{\partial^2 v_k}{\partial x_j \partial x_l} + \frac{\partial^2 v_l}{\partial x_j \partial x_k} \right) \right] \right\}^2 \mathbf{d}\mathbf{x} \\ &\leq \left( 4 \left( \max_{\alpha,j,k,l,i} |A_{\alpha ijkl}| \right) \left( \max_{\alpha} \mu_\alpha \right) \right)^2 \int_{\mathbb{R}^3} \left\{ \sum_{k,l,j=1}^3 \left| \frac{\partial^2 v_k}{\partial x_j \partial x_l} + \frac{\partial^2 v_l}{\partial x_j \partial x_k} \right| \right\}^2 \mathbf{d}\mathbf{x} \\ &\leq k' \left( \max_{\alpha} \mu_\alpha \right)^2 \|\mathbf{v}\|_2^2, \end{aligned} \tag{2.5}$$

with  $k' := (4 \max_{\alpha,j,k,l,i} |A_{\alpha ijkl}|)^2$ . □

**Remark 2.1.** From the above proof, it follows that  $\hat{\mathbf{V}}$  can also be considered as a continuous operator acting on  $H_\sigma^m$  into  $H_\sigma^{m-2}$ , i.e. there exists  $k'' > 0$  constant such that

$$\|\hat{\mathbf{V}}\mathbf{v}\|_{m-2} \leq k'' \left( \max_{\alpha=0,\dots,4} \mu_\alpha \right) \|\mathbf{v}\|_m, \quad \mathbf{v} \in \mathbf{H}_\sigma^m(\mathbb{R}^3)^3, m \geq 2. \tag{2.6}$$

### 3. Existence Results

The existence and uniqueness of solutions for both the MHD and ideal MHD systems have already been proved in previous papers, when the viscosity term in (1.1) is taken to be (2.3). In particular, we shall mention the works by Duvaut and Lions<sup>12</sup> and Sermange and Temam<sup>28</sup> for the MHD system in a bounded domain or in the whole spaces  $\mathbb{R}^n$ ,  $n = 2, 3$ , with periodic boundary conditions; for the ideal case, we mention the articles by Schmidt<sup>21</sup> and Secchi<sup>22</sup> where the existence and uniqueness of local strong solutions is proved in a bounded domain, and the paper by Sulem<sup>25</sup> for the whole 3D-space with analytic initial data. Concerning the viscosity operator of Braginskii (2.4), Spada and Wobig<sup>23</sup> proved the existence and uniqueness of a weak solution for the stationary MHD system in a bounded domain.

In this paper, we give a local existence and uniqueness result in the spaces  $H_\sigma^s$  above introduced, for the ideal MHD and MHD systems, for both (2.3) and (2.4) viscosity operators (for which, from now on, we shall use the notation  $\mathbf{F}$ ), whose proof lies on some results for evolution abstracts equation due to Kato (see Refs. 16 and 19). In order to apply this theory to the MHD systems we shall introduce some operators defined on the divergence-free spaces  $H_\sigma$ . We start with the linear operator  $\mathcal{A}_1$  defined on  $H_\sigma \times H_\sigma$  as

$$\mathcal{A}_1 = \begin{pmatrix} -\mathbf{F} & 0 \\ 0 & -\frac{1}{Rm} \Delta \end{pmatrix}, \tag{3.7}$$

with domain  $D(\mathcal{A}_1) = H_\sigma^2 \times H_\sigma^2$  (when  $\mathbf{F} = \frac{1}{Re} \Delta$  this is a well-known result and in the case of  $\mathbf{F} = \hat{\mathbf{V}}$  it can be easily checked by using standard elliptic regularity arguments (see, e.g., Ref. 14)).

Let  $\Phi = (\mathbf{u}, \mathbf{B}) \in \mathbf{H}_\sigma^r \times \mathbf{H}_\sigma^r$ ,  $r > 3/2$ , be fixed. We define another linear operator on  $H_\sigma \times H_\sigma$  by

$$\mathcal{A}_2(\Phi) = \begin{pmatrix} P(\mathbf{u} \cdot \nabla) & -SP(\mathbf{B} \cdot \nabla) \\ -P(\mathbf{B} \cdot \nabla) & P(\mathbf{u} \cdot \nabla) \end{pmatrix} \tag{3.8}$$

with  $D(\mathcal{A}_2(\Phi)) = H_\sigma^1 \times H_\sigma^1$  (notice that, from Sobolev's imbedding  $H_\sigma^r \hookrightarrow (L^\infty(\mathbb{R}^3))^3$  when  $r > 3/2$ , and so  $(\mathbf{u} \cdot \nabla)\mathbf{v} \in (L^2(\mathbb{R}^3))^3$  for every  $\mathbf{v} \in \mathbf{H}_\sigma^1$ ).

Finally, let us denote by  $(\mathcal{A}(\Phi), D(\mathcal{A}(\Phi)))$  the operator sum of the two above-mentioned, i.e.

$$\mathcal{A}(\Phi) = \mathcal{A}_1 + \mathcal{A}_2(\Phi), \quad D(\mathcal{A}(\Phi)) = H_\sigma^2 \times H_\sigma^2.$$

We get that

**Lemma 3.1.** *Let  $\Phi \in \mathbf{H}_\sigma^r \times \mathbf{H}_\sigma^r$ ,  $r > 3/2$ , be fixed. Then,  $(-\mathcal{A}(\Phi), D(\mathcal{A}(\Phi)))$  generates a contraction semigroup in  $H_\sigma \times H_\sigma$ .*

**Proof.** Let us first note that, as we are working with solenoidal vector fields, the following identity holds:

$$((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{v}) = 0, \quad \forall \mathbf{u} \in \mathbf{H}_\sigma^r, \quad \forall \mathbf{v} \in \mathbf{H}_\sigma^1.$$

Thus, given  $\Phi \in \mathbf{H}_\sigma^r \times \mathbf{H}_\sigma^r$ ,

$$(\mathcal{A}_2(\Phi)\Psi, \Psi) = 0, \quad \forall \Psi \in \mathbf{H}_\sigma^1 \times \mathbf{H}_\sigma^1$$

and so  $(-\mathcal{A}_2(\Phi), H_\sigma^1 \times H_\sigma^1)$  is dissipative. Moreover, using that  $P$  is a unitary operator on the  $H^s$ -spaces and from Sobolev's imbedding, it follows the existence of a positive constant  $c$  such that

$$\begin{aligned} \|\mathcal{A}_2(\Phi)\Psi\|_0 &= \|P(\mathbf{u} \cdot \nabla)\mathbf{v} - \mathbf{SP}(\mathbf{B} \cdot \nabla)\mathbf{C}\|_0 + \|P(\mathbf{u} \cdot \nabla)\mathbf{C} - \mathbf{P}(\mathbf{B} \cdot \nabla)\mathbf{v}\|_0 \\ &\leq c\|\mathbf{v}\|_1(\|\mathbf{u}\|_r + \|\mathbf{B}\|_r) + c\|\mathbf{C}\|_1(\|\mathbf{u}\|_r + S\|\mathbf{B}\|_r) \\ &\leq c \max(1, S)\|\Phi\|_r\|\Psi\|_2 = k\|\Psi\|_2 \quad \forall \Psi = (\mathbf{v}, \mathbf{C}) \in \mathbf{H}_\sigma^2 \times \mathbf{H}_\sigma^2 \end{aligned}$$

which allows us to state that  $(\mathcal{A}_2(\Phi), D(\mathcal{A}_2(\Phi)))$  is  $\mathcal{A}_1$ -bounded with relative bound being 0.

Therefore, if we prove that  $(\mathcal{A}_1, D(\mathcal{A}_1))$  generates a contraction semigroup in  $H_\sigma \times H_\sigma$ , we can appeal to the theory of perturbed operators (see, for example, Ref. 18 or 27) and conclude that  $(-\mathcal{A}(\Phi), H_\sigma^2 \times H_\sigma^2)$  is the infinitesimal generator of a contraction semigroup in  $H_\sigma \times H_\sigma$ . But when  $\mathbf{F} = \frac{1}{Re} \Delta$ , it is a well-known result (see, for instance, Ref. 7) that  $(-\mathcal{A}_1, D(\mathcal{A}_1))$  generates a contraction semigroup on  $H_\sigma \times H_\sigma$  given by  $S(t) = (e^{\frac{1}{Re}\Delta t}, e^{\frac{1}{Rm}\Delta t})$  where  $e^{\Delta t}$  is the heat semigroup. In case that  $\mathbf{F} = \hat{\mathbf{V}}$ , we use that  $(-\hat{\mathbf{V}}, H_\sigma^2)$  is an  $m$ -dissipative operator on  $H_\sigma$  (see Lemma 2.1) and from Hille-Yosida-Phillips's theorem (see, e.g., Ref. 7), we get that this operator also generates a contraction semigroup on  $H_\sigma$ .  $\square$

Let  $S$  be the isometric isomorphism  $S := (I - \Delta)^{1/2} : H^r \rightarrow H^{r-1}$ ,  $r \geq 1$ , that appears when defining the Sobolev spaces of exponent 2 via the Fourier transform (see, e.g., Ref. 20). We can consider the restriction of this operator to the divergence-free spaces, that we shall also denote by  $S$ , i.e.

$$S^r = (I - \Delta)^{r/2} : H_\sigma^r \rightarrow H_\sigma, \quad r \geq 1$$

that it is also an isometry (note that the expression of  $S^r$  in the Fourier variables is  $(1 + |\xi|^2)^{r/2}$  which commute with the unitary projection  $P$ ). We denote by  $\Lambda^r$  the product operator of components  $\Lambda^r = (S^r, S^r)$ , i.e.  $\Lambda^r : H_\sigma^r \times H_\sigma^r \rightarrow H_\sigma \times H_\sigma$ , which verifies the following lemma:

**Lemma 3.2.** *Let  $\Phi \in \mathbf{H}_\sigma^r \times \mathbf{H}_\sigma^r$ ,  $r > 5/2$  be fixed. There exists a positive constant  $\lambda$ , just depending on  $\Phi$ , such that*

$$\|[\Lambda^r, \mathcal{A}(\Phi)]\Lambda^{-r}\|_{\mathcal{L}(H_\sigma \times H_\sigma)} \leq \lambda$$

where  $[\cdot, \cdot]$  denotes the commutator operator.

**Remark 3.1.** The above lemma is formally equivalent (see Ref. 16 or 19) to the existence of a bounded linear operator  $B(\Phi) \in \mathcal{L}(H_\sigma \times H_\sigma)$  such that  $\Lambda^s \mathcal{A}(\Phi) \Lambda^{-s} = \mathcal{A}(\Phi) + B(\Phi)$ . We also note that the same result also holds when we consider not the operator  $\mathcal{A}(\Phi)$  but  $\mathcal{A}_2(\Phi)$ .

**Proof.** We recall the estimate

$$\|[(I - \Delta)^{r/2}, (\mathbf{u} \cdot \nabla)](I - \Delta)^{-r/2}\|_0 \leq c \|\nabla \mathbf{u}\|_{r-1}, \quad \text{for a fixed } \mathbf{u} \in \mathbf{H}_\sigma^r$$

obtained in Ref. 16 for the study of the Navier–Stokes equations. Due to the structure of the operator  $\mathcal{A}(\Phi)$  it suffices to show that an estimate of type

$$\|[S^r, \mathbf{F}]S^{-r}\|_0 \leq \lambda$$

holds for the viscosity operator  $\mathbf{F}$ , where  $\lambda$  is a positive constant. For this purpose we use that their expressions in the Fourier variables are  $\mathcal{F}(\mathbf{F})(\xi) = p(\xi)$  for a.e.  $\xi \in \mathbb{R}^3$  (we have denoted by  $\mathcal{F}$  the Fourier transform), where  $p(\xi) = |\xi|^2$  when  $\mathbf{F} = \Delta$  and  $p$  is a suitable more complicated second-order polynomial in the variable  $\xi$  in the case of the Braginskii operator (2.4). Thus, by using Parseval's identity, we obtain that

$$\begin{aligned} \|[(I - \Delta)^{r/2}, \mathbf{F}](I - \Delta)^{-r/2} \mathbf{v}\|_0 &= \|\mathcal{F}([(I - \Delta)^{r/2}, \mathbf{F}](I - \Delta)^{-r/2} \mathbf{v})\|_0 \\ &= \|[(1 + |\xi|^2)^{r/2} p(\xi)](1 + |\xi|^2)^{-r/2} \hat{\mathbf{v}} - p(\xi)(1 + |\xi|^2)^{r/2}(1 + |\xi|^2)^{-r/2} \hat{\mathbf{v}}\|_0 = 0 \end{aligned}$$

which gives the desired result.  $\square$

Finally, we can consider the operator  $\mathcal{A}$  mapping  $H_\sigma^r \times H_\sigma^r$ ,  $r > 3/2$ , into the set of generators of contraction semigroups on  $H_\sigma \times H_\sigma$ ,  $(-\mathcal{A}(\Phi), D(\mathcal{A}(\Phi)))$ . The restriction of this operator to the space  $H_\sigma^s \times H_\sigma^s$  with  $s > 5/2$  satisfies the following Lipschitz property:

**Lemma 3.3.** *There exists a positive constant  $L$  such that:*

$$\|\mathcal{A}(\Phi) - \mathcal{A}(\Psi)\|_{\mathcal{L}(H_\sigma^s \times H_\sigma^s, H_\sigma \times H_\sigma)} \leq L \|\Phi - \Psi\|_0,$$

for every  $\Phi, \Psi \in \mathbf{H}_\sigma^s \times \mathbf{H}_\sigma^s$  with  $s > 5/2$ .

**Proof.** We first note that, as we are assuming  $s > 5/2$ ,  $H_\sigma^s \times H_\sigma^s \hookrightarrow H_\sigma^2 \times H_\sigma^2$  and hence the above estimate makes sense. Due to the structure of the operator  $\mathcal{A}$  (notice that  $\mathcal{A}_1$  in the definition of  $\mathcal{A}(\Phi)$  does not depend on  $\Psi$ ), the proof of this estimate reduces to show that

$$\|(\mathbf{u} \cdot \nabla) \mathbf{w} - (\mathbf{v} \cdot \nabla) \mathbf{w}\|_0 \leq L \|\mathbf{u} - \mathbf{v}\|_0 \|\mathbf{w}\|_s \quad \text{for every } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_\sigma^s.$$

But this is a straightforward consequence of Hölder's inequality and the Sobolev imbedding  $H^{s-1} \hookrightarrow L^\infty(\mathbb{R}^3)$  for  $s > 5/2$ . Indeed,

$$\|(\mathbf{u} \cdot \nabla) \mathbf{w} - (\mathbf{v} \cdot \nabla) \mathbf{w}\|_0 \leq \|\mathbf{u} - \mathbf{v}\|_0 \|\nabla \mathbf{w}\|_{L^\infty} \leq c \|\mathbf{u} - \mathbf{v}\|_0 \|\mathbf{w}\|_s$$

where  $c$  is the constant appearing in the Sobolev imbedding.  $\square$

**Remark 3.2.** From the above lemma, we can deduce that for a given  $\Phi \in \mathbf{H}_\sigma^r \times \mathbf{H}_\sigma^r$ ,  $r > 3/2$ , the restriction of the operator  $(\mathcal{A}(\Phi), D(\mathcal{A}(\Phi)))$  to the spaces  $H_\sigma^s \times H_\sigma^s$  with  $s > 5/2$ , is a linear bounded operator. That is,  $\mathcal{A}(\Phi)|_{H_\sigma^s \times H_\sigma^s} \in \mathcal{L}(H_\sigma^s \times H_\sigma^s, H_\sigma \times H_\sigma)$ .

We are now in a position to state the existence and uniqueness result for the MHD systems. We begin with the viscous case:

**Theorem 3.1.** *Let  $T > 0$  and  $(\mathbf{u}_0, \mathbf{B}_0) \in H^s(\mathbb{R}^3)^3 \times H^s(\mathbb{R}^3)^3$ ,  $s > 5/2$ , such that  $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{B}_0 = \mathbf{0}$ . Let us assume that  $\mathbf{f} \in \mathcal{C}([0, T]; \mathbf{L}^2(\mathbb{R}^3)) \cap \mathbf{L}^1(0, T; \mathbf{H}_\sigma^s(\mathbb{R}^3))$ . Then, there exists a couple  $(\mathbf{u}, \mathbf{B})$  being the unique solution to (1.1), with  $\mathbf{F}$  given by (2.3) or (2.4), satisfying*

$$(\mathbf{u}, \mathbf{B}) \in \mathcal{C}([0, T']; (\mathbf{H}^s(\mathbb{R}^3)^3)^2) \cap \mathcal{C}^1([0, T']; \mathbf{H}^{s-2}(\mathbb{R}^3)^3 \times \mathbf{H}^{s-2}(\mathbb{R}^3)^3)$$

where  $T' \in (0, T]$  just depends on the initial data and  $\mathbf{f}$ . Moreover, there exists a constant  $C > 0$ , independent of the viscosities, such that

$$\|\mathbf{u}(t)\|_s + \|\mathbf{B}(t)\|_s \leq C, \quad \forall t \in [0, T']. \quad (3.9)$$

**Proof.** In order to prove this theorem, we project the MHD system into the space of solenoidal vector fields of  $L^2(\mathbb{R}^3)^3$  by means of the Leray operator. Then, using the operators that we have introduced all along this section, we can write the projected system as the Cauchy problem

$$\begin{cases} \frac{d\Phi}{dt} + \mathcal{A}(\Phi)\Phi = \mathbf{G}(t) \\ \Phi(0) = (\mathbf{u}_0, \mathbf{B}_0), \end{cases} \quad (3.10)$$

where  $\Phi$  takes value on the Hilbert space  $H_\sigma \times H_\sigma$  and  $\mathbf{G}(t) := (\mathbf{f}_0^{(t)})$ . Lemmas 3.1–3.3 allow us to appeal to the theory developed in Kato<sup>16</sup> (see also Ref. 19) for abstract evolution equations and conclude the existence of a unique  $\Phi = (\mathbf{u}, \mathbf{B})$ , solution of (3.10) satisfying

$$\Phi \in \mathcal{C}([0, T^*]; H_\sigma^s \times H_\sigma^s) \cap \mathcal{C}^1([0, T^*]; H_\sigma \times H_\sigma) \quad (3.11)$$

for some  $T^* \in (0, T]$  which just depends on initial data (we note that the bounds in Lemmas 3.2 and 3.3 do not depend on the viscosities  $\nu_m$ ,  $\text{Re}$  or  $\mu_\alpha$ ).

These results by Kato<sup>16</sup> prove the existence and uniqueness of local in time solutions for quasi-linear equations of type (3.10), where  $\Phi(t)$  takes values in a Banach space  $X$ ;  $\mathcal{A}(\Phi)$  is a linear operator on  $X$  and there exists a subspace  $Y \hookrightarrow X$  dense, such that for every  $\Phi \in Y$ ,  $(-\mathcal{A}(\Phi), D(\mathcal{A}(\Phi)))$  generates a  $\mathcal{C}^0$  semigroup on  $X$ . Roughly speaking, the proof of this result lies in showing the existence of a fixed point for the mapping  $\Psi \rightarrow \Phi$  where  $\Phi$  solves  $\frac{d\Phi}{dt} + \mathcal{A}(\Psi)\Phi = \mathbf{G}(t)$  with  $\Phi(0) = (\mathbf{u}_0, \mathbf{B}_0)$ , by means of the contraction mapping theorem. To this end, the operator  $\mathcal{A}$  must satisfy a Lipschitz condition similar to that of Lemma 3.3. In the case of the MHD system, we have chosen  $H_\sigma \times H_\sigma$  to play the role of  $X$  and space  $Y$  is taken to be  $H_\sigma^s \times H_\sigma^s$ , where  $s > 5/2$  is the lowest exponent for which Lemma 3.3 holds.

The additional regularity on  $\Phi$ , i.e.  $\Phi \in C^1([0, T^*], \mathbf{H}_\sigma^{s-2} \times \mathbf{H}_\sigma^{s-2})$ , is a consequence of

$$\frac{d\Phi}{dt}(t) = -\mathcal{A}(\Phi(t))\Phi(t) + F(t) \in H_\sigma^{s-2} \times H_\sigma^{s-2}, \quad \forall t \in [0, T^*] \quad (3.12)$$

being a continuous function on  $[0, T^*]$ . Indeed, from the regularity in (3.11) we know that  $\mathcal{A}_1\Phi \in C([0, T^*]; H_\sigma^{s-2} \times H_\sigma^{s-2})$ ; moreover,  $H_\sigma^{s-1}$  is a Banach algebra for  $s > 5/2$  (see, e.g., Ref. 1) and so  $\mathcal{A}_2(\Phi)\Phi \in C([0, T^*]; H_\sigma^{s-1} \times H_\sigma^{s-1})$ .

Finally, let us prove (3.9). Given  $t \in [0, T^*]$ , we multiply the equation in (3.10) by  $\Phi(t) = (\mathbf{u}(t), \mathbf{B}(t))$  in  $H^s \times H^s$ :

$$\frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_s^2 + (\mathcal{A}_1\Phi, \Phi)_s = -(\mathcal{A}_2(\Phi)\Phi, \Phi)_s(t) + (\mathbf{G}, \Phi)_s(t). \quad (3.13)$$

By using the estimate

$$|((\mathbf{v} \cdot \nabla)\mathbf{w}, \mathbf{w})_r| \leq c\|\mathbf{v}\|_r\|\mathbf{w}\|_r^2 \text{ for all } \mathbf{v}, \mathbf{w} \in \mathbf{H}_\sigma^r \text{ with } r > \frac{5}{2}, \quad (3.14)$$

with  $c > 0$  a constant (proved in Ref. 8) and since the operator  $\mathcal{A}_1$  is dissipative, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_s^2 \leq c\|\Phi\|_s^3 + \|\mathbf{f}(t)\|_s\|\Phi\|_s.$$

Using Young's inequality we find that  $\|\Phi(t)\|_s^2$  satisfies

$$\frac{d}{dt} \|\Phi(t)\|_s \leq c\|\Phi\|_s + \|\mathbf{f}(t)\|_s, \quad \|\Phi(0)\|_s = \|(\mathbf{u}_0, \mathbf{B}_0)\|_s, \quad (3.15)$$

and thus,  $\|\Phi(t)\|_s \leq \varphi(t)$ , where  $\varphi$  is the solution to the scalar Cauchy problem

$$\frac{d\varphi}{dt}(t) = c'\varphi^2(t) + \|\mathbf{f}(t)\|_s, \quad \varphi(0) = \|(\mathbf{u}_0, \mathbf{B}_0)\|_s. \quad (3.16)$$

Since  $\mathbf{f} \in \mathbf{L}^1(0, T; \mathbf{H}_\sigma^s)$ , it is clear that (3.16) admits a unique solution  $\varphi$  defined in an interval  $[0, T']$ ,  $T' > 0$  and that  $T', \varphi$  are independent of the viscosities  $\text{Re}^{-1}$ ,  $\text{Rm}^{-1}$  and  $\mu_\alpha$ . □

For the ideal case we have:

**Theorem 3.2.** *Let  $T > 0$ ,  $(\mathbf{u}_0^0, \mathbf{B}_0^0) \in H^s(\mathbb{R}^3)^3 \times H^s(\mathbb{R}^3)^3$ ,  $s > 5/2$ , such that  $\nabla \cdot \mathbf{u}_0^0 = \nabla \cdot \mathbf{B}_0^0 = \mathbf{0}$ , and assume  $\mathbf{f} \in C([0, T]; \mathbf{L}^2(\mathbb{R}^3)) \cap \mathbf{L}^1(0, T; \mathbf{H}_\sigma^s(\mathbb{R}^3))$ . Then, there exists  $T_0 \in (0, T]$ , just depending in the initial data, and a unique solution  $(\mathbf{u}^0, \mathbf{B}^0)$  of (1.2) verifying*

$$(\mathbf{u}^0, \mathbf{B}^0) \in C([0, T_0]; \mathbf{H}^s(\mathbb{R}^3)^3 \times \mathbf{H}^s(\mathbb{R}^3)^3) \cap C^1([0, T_0]; (\mathbf{H}^{s-1}(\mathbb{R}^3)^3)^2)$$

and such that,  $\forall T^* < T_0$  the following estimate holds

$$\int_0^{T^*} (\|\nabla \mathbf{u}^0(t)\|_{L^\infty} + \|\nabla \mathbf{B}^0(t)\|_{L^\infty}) dt < \infty. \quad (3.17)$$

Moreover, if  $T_0 < \infty$ , then at least one of the above integrals blows-up for some  $T^* \leq T_0$ .

**Proof.** As in the previous theorem, we consider the projected ideal MHD system into the divergence-free spaces. The resulting system can be regarded as the initial value problem

$$\begin{cases} \frac{d\Phi^0}{dt} + \mathcal{A}_2(\Phi^0)\Phi^0 = \mathbf{G}(t) \\ \Phi^0(0) = (\mathbf{u}_0^0, \mathbf{B}_0^0), \end{cases} \quad (3.18)$$

in  $H_\sigma \times H_\sigma$ , with  $\Phi^0 = (\mathbf{u}^0, \mathbf{B}^0)$ . Again, we appeal to the existence and uniqueness results of Kato<sup>16</sup> which apply in this case, provided that, for a fixed  $\Phi \in \mathbf{H}_\sigma^s \times \mathbf{H}_\sigma^s$ , the operator  $(-\mathcal{A}_2(\Phi), D(\mathcal{A}_2(\Phi)))$  generates a  $C^0$ -semigroup on  $H_\sigma \times H_\sigma$ . We remark that this is the only point that remains to be proved for the  $\mathcal{A}_2$  operator (Lemmas 3.2 and 3.3 hold if we replace  $\mathcal{A}$  by  $\mathcal{A}_2$ ). Due to the structure of  $\mathcal{A}_2$ , it suffices to show that, for a fixed  $\mathbf{v} \in \mathbf{H}_\sigma^s$ , the operators  $(P(\mathbf{v} \cdot \nabla), H_\sigma)$  and  $(-P(\mathbf{v} \cdot \nabla), H_\sigma)$  are infinitesimal generators of  $C^0$ -semigroups on  $H_\sigma$ .

Let  $\mathbf{v} \in \mathbf{H}_\sigma^s$  be fixed. It is a well-known result (see, e.g., Refs. 10 and 30) that the transport operator  $(\mathbf{v} \cdot \nabla)$  with domain  $H^1 \hookrightarrow L^2(\mathbb{R}^3)^3$  generates a  $C^0$ -isometries group on  $L^2(\mathbb{R}^3)^3$  (note that the fixed vector  $\mathbf{v}$  is solenoidal). Since  $H_\sigma^1$  is a closed subset of  $H^1$  and  $P$  is unitary, the operator  $P(\mathbf{v} \cdot \nabla)$  acting on  $H_\sigma^1$  is also a generator of a  $C^0$ -isometries group on  $H_\sigma$ . Furthermore, this result also holds for  $(-P(\mathbf{v} \cdot \nabla), H_\sigma^1)$ , which is possible since we are dealing with the generator of a group, and so we can conclude the desired result.

Then, by Kato<sup>16</sup> (Theorems 6 and 7), there exists  $T_0 \in (0, T]$  and a unique  $\Phi^0 = (\mathbf{u}^0, \mathbf{B}^0)$  solution of (3.18) such that

$$\Phi^0 \in C([0, T_0]; H_\sigma^s \times H_\sigma^s) \cap C^1([0, T_0]; H_\sigma \times H_\sigma).$$

Furthermore, we can use that  $H^{s-1} \times H^{s-1}$  is a Banach algebra for  $s > 5/2$  and hence, by the above regularity, we get that

$$\frac{d\Phi^0}{dt} = -\mathcal{A}_2(\Phi(t))\Phi(t) + F(t) \in C([0, T_0]; H_\sigma^{s-1} \times H_\sigma^{s-1}).$$

Finally, the Beale–Kato–Majda's type condition (3.17) for the ideal MHD equation is a result due to Caffisch, Klapper and Steele.<sup>5</sup> □

#### 4. Convergence of the MHD Solutions to the Ideal MHD Ones

In this section we prove the convergence of the solutions  $(\mathbf{u}, \mathbf{B})$  of the MHD for a viscous and resistive fluid found in Theorem 3.1, to the solutions  $(\mathbf{u}_0, \mathbf{B}_0)$  of the ideal MHD equations (1.2). We shall start by establishing this convergence in the  $L^2$ -norm:

**Theorem 4.1.** *Let  $(\mathbf{u}_0, \mathbf{B}_0) \in H_\sigma^s \times H_\sigma^s$ ,  $s > 5/2$ . Assume that  $(\mathbf{u}, \mathbf{B})$  and  $(\mathbf{u}^0, \mathbf{B}^0)$  are the solutions to (1.1) with Braginskii viscosity (2.4) and to (1.2) respectively,*

both with initial data  $(\mathbf{u}_0, \mathbf{B}_0)$ , given in Theorems 3.1 and 3.2. Then,

$$\begin{aligned} & \| \mathbf{u}(t) - \mathbf{u}^0(t) \|_0^2 + S \| \mathbf{B}(t) - \mathbf{B}^0(t) \|_0^2 \\ & \leq \exp \left( t + 2S' \int_0^t \eta(s) ds \right) \int_0^t \left[ k(\mu^*)^2 \| \mathbf{u}^0 \|_2^2 + \frac{1}{Rm^2} \| \nabla \mathbf{B}^0 \|_0^2 \right] ds, \end{aligned} \quad (4.19)$$

where  $k$  is a positive constant,  $\mu^* := \max_\alpha \mu_\alpha$ ,  $S' = \max(1, S)$  and  $\eta(t) = \| \nabla \mathbf{u}^0(t) \|_{L^\infty} + \| \nabla \mathbf{B}^0(t) \|_{L^\infty}$ . Furthermore,  $\forall t \leq T'$ , with  $T' < +\infty$  satisfying (3.17) we have

$$(\mathbf{u}(t), \mathbf{B}(t)) \longrightarrow (\mathbf{u}^0(t), \mathbf{B}^0(t)) \text{ in } L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$$

when  $Rm \rightarrow +\infty$  and  $\mu_\alpha \rightarrow 0$ ,  $\alpha = 0, 1, \dots, 4$ , with convergence rates  $o(\mu^*) + o(\frac{1}{Rm})$ .

**Proof.** Let us denote by  $(\mathbf{v}, \mathbf{C})$  the difference between the solutions of the MHD and ideal MHD systems with initial data  $(\mathbf{u}_0, \mathbf{B}_0)$  stated in Theorems 3.1 and 3.2, i.e.  $(\mathbf{v}, \mathbf{C}) = (\mathbf{u}, \mathbf{B}) - (\mathbf{u}^0, \mathbf{B}^0)$ ; this couple satisfies

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{u}^0 + (\mathbf{u} \cdot \nabla) \mathbf{v} - \mathbf{S}(\mathbf{C} \cdot \nabla) \mathbf{B}^0 - \mathbf{S}(\mathbf{B} \cdot \nabla) \mathbf{C} \\ \quad + \frac{S}{2} \nabla(\mathbf{B}^2 - \mathbf{B}^0) - \hat{\mathbf{V}} \mathbf{u} + \nabla(\mathbf{p} - \mathbf{p}^0) = \mathbf{0} \\ \frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B}^0 + (\mathbf{u} \cdot \nabla) \mathbf{C} - (\mathbf{C} \cdot \nabla) \mathbf{u}^0 - (\mathbf{B} \cdot \nabla) \mathbf{v} - \frac{1}{Rm} \Delta \mathbf{B} = \mathbf{0} \\ \operatorname{div} \mathbf{v} = \mathbf{0}, \quad \operatorname{div} \mathbf{C} = \mathbf{0} \\ \mathbf{v}|_{t=0} = \mathbf{C}|_{t=0} = \mathbf{0}. \end{cases} \quad (4.20)$$

Let us fix  $t \in (0, T']$  and multiply in  $L^2(\mathbb{R}^3)^3$  the first two equations in (4.20) by  $\mathbf{v}(t)$  and  $\mathbf{B}(t)$  respectively. Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| \mathbf{v}(t) \|_0^2 &= (\hat{\mathbf{V}} \mathbf{u}(t), \mathbf{v}(t)) - ((\mathbf{v} \cdot \nabla) \mathbf{u}^0, \mathbf{v})(t) \\ &\quad + S((\mathbf{C} \cdot \nabla) \mathbf{B}^0, \mathbf{v})(t) + S((\mathbf{B} \cdot \nabla) \mathbf{C}, \mathbf{v})(t) \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| \mathbf{C}(t) \|_0^2 &= \frac{1}{Rm} (\Delta \mathbf{B}(t), \mathbf{C}(t)) - ((\mathbf{v} \cdot \nabla) \mathbf{B}^0, \mathbf{C})(t) \\ &\quad + ((\mathbf{C} \cdot \nabla) \mathbf{u}^0, \mathbf{C})(t) + ((\mathbf{B} \cdot \nabla) \mathbf{v}, \mathbf{C})(t), \end{aligned} \quad (4.22)$$

where we have used that the terms  $(\nabla(p - p^0), \mathbf{v}(t))$ ,  $((\mathbf{u}(t) \cdot \nabla) \mathbf{v}(t), \mathbf{v}(t))$ , and  $-\frac{S}{2} (\nabla(\mathbf{B}^2 - (\mathbf{B}^0)^2)(t), \mathbf{v}(t))$  vanish since  $\mathbf{v}(t)$  and  $\mathbf{u}(t)$  are divergence-free (this can be easily seen by integrating by parts). Similarly,  $-((\mathbf{u} \cdot \nabla) \mathbf{C}, \mathbf{C}(t))$  vanish too.

We shall estimate the integrals in (4.21) and (4.22), in order to derive an energy inequality for  $\mathbf{v}(t)$  and  $\mathbf{C}(t)$  in  $L^2(\mathbb{R}^3)^3$ . Let us start with the terms containing the

viscosity operators; we know (Lemma 2.1) that these operators are dissipative in  $L^2$  and continuous from  $H^2$  to  $L^2$ , hence we can appeal to Hölder's and Young's inequalities to obtain

$$\begin{aligned} (\hat{\mathbf{V}} \mathbf{u}(t), \mathbf{v}(t)) &= (\hat{\mathbf{V}} \mathbf{v}(t), \mathbf{v}(t)) + (\hat{\mathbf{V}} \mathbf{u}^0(t), \mathbf{v}(t)) \\ &\leq - \left( \min_\alpha \mu_\alpha \right) \| \nabla \mathbf{v}(t) \|_0^2 + k \left( \max_\alpha \mu_\alpha \right) \| \mathbf{u}^0(t) \|_2 \| \mathbf{v}(t) \|_0 \\ &\leq - \left( \min_\alpha \mu_\alpha \right) \| \nabla \mathbf{v}(t) \|_0^2 + \frac{k}{2} \left( \max_\alpha \mu_\alpha \right)^2 \| \mathbf{u}^0(t) \|_2^2 \\ &\quad + \frac{1}{2} \| \mathbf{v}(t) \|_0^2 \end{aligned} \quad (4.23)$$

in the case of the fluid viscosity (here,  $k$  is the continuity constant in Lemma 2.1). Similarly, for the magnetic viscosity term we have

$$\frac{1}{Rm} (\Delta \mathbf{B}(t), \mathbf{C}(t)) \leq - \frac{1}{Rm} \| \nabla \mathbf{C}(t) \|_0^2 + \frac{1}{2Rm^2} \| \Delta \mathbf{B}^0(t) \|_0^2 + \frac{1}{2} \| \mathbf{C}(t) \|_0^2. \quad (4.24)$$

We proceed now to estimate the transport terms in (4.21) and (4.22). For this purpose, we apply again Cauchy-Schwarz's and Young's inequalities and we arrive at

$$\begin{aligned} \text{(i)} \quad & |((\mathbf{v} \cdot \nabla) \mathbf{u}^0, \mathbf{v})| \leq \| \nabla \mathbf{u}^0(t) \|_\infty \| \mathbf{v}(t) \|_0^2, \\ \text{(ii)} \quad & S |((\mathbf{C} \cdot \nabla) \mathbf{B}^0, \mathbf{v})| \leq S \| \nabla \mathbf{B}^0(t) \|_\infty \| \mathbf{v}(t) \|_0 \| \mathbf{C}(t) \|_0 \\ & \leq \frac{S}{2} \| \nabla \mathbf{B}^0(t) \|_\infty (\| \mathbf{v}(t) \|_0^2 + \| \mathbf{C}(t) \|_0^2), \\ \text{(iii)} \quad & |((\mathbf{v} \cdot \nabla) \mathbf{B}^0, \mathbf{C})(t)| \leq \| \nabla \mathbf{B}^0(t) \|_\infty \| \mathbf{v}(t) \|_0 \| \mathbf{C}(t) \|_0 \\ & \leq \frac{1}{2} \| \nabla \mathbf{B}^0(t) \|_\infty (\| \mathbf{v}(t) \|_0^2 + \| \mathbf{C}(t) \|_0^2), \\ \text{(iv)} \quad & |((\mathbf{C} \cdot \nabla) \mathbf{u}^0, \mathbf{C})(t)| \leq \| \nabla \mathbf{u}^0(t) \|_\infty \| \mathbf{C}(t) \|_0. \end{aligned} \quad (4.25)$$

Putting together the preceding estimates and (4.23), (4.24) we obtain:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\| \mathbf{v}(t) \|_0^2 + S \| \mathbf{C}(t) \|_0^2) + \left( \min_\alpha \mu_\alpha \right) \| \nabla \mathbf{v} \|_0^2 + \frac{S}{Rm} \| \nabla \mathbf{C}(t) \|_0^2 \\ & \leq \frac{1}{2} \left( k \left( \max_\alpha \mu_\alpha \right)^2 \| \mathbf{u}^0 \|_2^2 + \frac{S}{Rm^2} \| \Delta \mathbf{B}^0 \|_0^2 \right) + \frac{1}{2} (\| \mathbf{v}(t) \|_0^2 + S \| \mathbf{C}(t) \|_0^2) \\ & \quad + \| \nabla \mathbf{u}^0(t) \|_\infty (\| \mathbf{v}(t) \|_0^2 + S \| \mathbf{C}(t) \|_0^2) + S \| \nabla \mathbf{B}^0(t) \|_\infty (\| \mathbf{v}(t) \|_0^2 + \| \mathbf{C}(t) \|_0^2) \\ & \quad + S((\mathbf{B} \cdot \nabla) \mathbf{C}, \mathbf{v}(t)) + S((\mathbf{B} \cdot \nabla) \mathbf{v}, \mathbf{C}(t)). \end{aligned} \quad (4.26)$$

(We have used here that  $S$  is a positive constant.)

Let us set  $S' = \max(1, S)$ . By dropping the positive terms on the left-hand side of (4.26) and using the identity  $((\mathbf{B} \cdot \nabla) \mathbf{C}, \mathbf{v}(t)) = -((\mathbf{B} \cdot \nabla) \mathbf{v}, \mathbf{C}(t))$  (see,

e.g., Temam<sup>29</sup>), we can write

$$\begin{aligned} \frac{d}{dt}(\|\mathbf{v}(t)\|_0^2 + S\|\mathbf{C}(t)\|_0^2) &\leq \left( k \left( \max_{\alpha} \mu_{\alpha} \right)^2 \|\mathbf{u}^0(t)\|_2^2 + \frac{S}{Rm^2} \|\Delta \mathbf{B}^0(t)\|_0^2 \right) \\ &\quad + \hat{\eta}(t)(\|\mathbf{v}(t)\|_0^2 + S\|\mathbf{C}(t)\|_0^2), \end{aligned}$$

where  $\hat{\eta}(t) := 1 + 2\|\nabla \mathbf{u}^0(t)\|_{\infty} + 2S'\|\nabla \mathbf{B}^0(t)\|_{\infty}$ . We note that  $\eta$  is an integrable function on  $[0, t]$  for  $t \leq T'$  satisfying (3.17). Thus, we can apply Gronwall's inequality to

$$\frac{d}{dt}\varphi(t) \leq \left( k \left( \max_{\alpha} \mu_{\alpha} \right)^2 \|\mathbf{u}^0(t)\|_2^2 + \frac{S}{Rm^2} \|\Delta \mathbf{B}^0(t)\|_0^2 \right) + \hat{\eta}(t)\varphi(t)$$

where  $\varphi(t) := \|\mathbf{v}(t)\|_0^2 + S\|\mathbf{C}(t)\|_0^2$  and hence we obtain (4.19). The convergence in the statement of the theorem follows from this inequality.  $\square$

When the viscosity in (1.1) is given by (2.3) we arrive at an analogous result:

**Theorem 4.2.** *Let  $(\mathbf{u}_0, \mathbf{B}_0) \in H_{\sigma}^s \times H_{\sigma}^s$ ,  $s > 5/2$ . Assume that  $\mathbf{F} = \text{Re}^{-1}\Delta$  in (1.1) and that  $(\mathbf{u}, \mathbf{B})$  and  $(\mathbf{u}^0, \mathbf{B}^0)$  are the solutions to the MHD and ideal MHD systems respectively, with initial data  $(\mathbf{u}_0, \mathbf{B}_0)$ , given by Theorems 3.1 and 3.2. Then,*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^0(t)\|_0^2 + S\|\mathbf{B} - \mathbf{B}^0(t)\|_0^2 &\leq \exp\left(t + 2S' \int_0^t \eta(s) ds\right) \int_0^t \left[ \frac{1}{\text{Re}^2} \|\nabla \mathbf{u}^0\|_0^2 + \frac{1}{Rm^2} \|\nabla \mathbf{B}^0\|_0^2 \right] ds, \quad (4.27) \end{aligned}$$

where  $S' = \max(1, S)$  and  $\eta(t) = \|\nabla \mathbf{u}^0(t)\|_{L^{\infty}} + \|\nabla \mathbf{B}^0(t)\|_{L^{\infty}}$ . In particular,  $\forall t \leq T'$  with  $T' < +\infty$  verifying (3.17) we conclude that

$$(\mathbf{u}(t), \mathbf{B}(t)) \rightarrow (\mathbf{u}^0(t), \mathbf{B}^0(t)) \text{ in } L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3$$

as  $\text{Re}, Rm \rightarrow +\infty$  with a convergence rate  $o(\frac{1}{\text{Re}}) + o(\frac{1}{Rm})$ .

**Proof.** Let  $(\mathbf{v}, \mathbf{C}) = (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)$  as before. The proof of this theorem is analogous to the previous one, being the dissipative term in (4.21) the only difference. To estimate this term we proceed as in (4.24), i.e.

$$\begin{aligned} \frac{1}{\text{Re}}(\Delta \mathbf{u}(t), \mathbf{v}(t)) &\leq -\frac{1}{\text{Re}}\|\nabla \mathbf{v}(t)\|_0^2 + \frac{1}{\text{Re}}(\Delta \mathbf{u}^0, \mathbf{v}) \\ &\leq -\frac{1}{\text{Re}}\|\nabla \mathbf{v}(t)\|_0^2 + \frac{1}{2\text{Re}}\|\Delta \mathbf{u}^0\|_0^2 + \frac{1}{2}\|\mathbf{v}(t)\|_0^2, \quad (4.28) \end{aligned}$$

and hence, repeating the calculus as in Theorem 4.2 we obtain the following energy estimate:

$$\begin{aligned} \frac{d}{dt}(\|\mathbf{v}(t)\|_0^2 + S\|\mathbf{C}(t)\|_0^2) &\leq \left( \frac{1}{\text{Re}}\|\Delta \mathbf{u}^0\|_0^2 + \frac{S}{Rm^2}\|\Delta \mathbf{B}^0\|_0^2 \right) \\ &\quad + \hat{\eta}(t)(\|\mathbf{v}(t)\|_0^2 + S\|\mathbf{C}(t)\|_0^2), \end{aligned}$$

with  $\hat{\eta}$  as in the preceding proof. Then, we can apply Gronwall's inequality to  $\|\mathbf{v}(t)\|_0^2 + S\|\mathbf{C}(t)\|_0^2$  which yields to (4.27).  $\square$

Finally, we study the convergence in the  $H^s$  norm. We have:

**Theorem 4.3.** *Let us assume  $(\mathbf{u}_0, \mathbf{B}_0) \in H_{\sigma}^s \times H_{\sigma}^s$  with  $s > 9/2$ . Then, if  $(\mathbf{u}^0, \mathbf{B}^0)$  and  $(\mathbf{u}, \mathbf{B})$  are as in Theorem 4.2, the following inequality holds*

$$\begin{aligned} \|(\mathbf{u} - \mathbf{u}^0)(t)\|_{s-2}^2 + S\|(\mathbf{B} - \mathbf{B}^0)(t)\|_{s-2}^2 &\leq \exp\left(2 \int_0^t \xi(\tau) d\tau\right) \int_0^t \left[ k(\mu^*)^2 \|\mathbf{u}^0\|_s^2 + \frac{S}{Rm^2} \|\mathbf{B}^0\|_s^2 \right] d\tau, \quad (4.29) \end{aligned}$$

where  $k$  is a positive constant,  $\mu^* := \max_{\alpha} \mu_{\alpha}$  and  $\xi \in \mathcal{C}([0, T'])$  does not depend on  $Rm$  or  $\mu_{\alpha}$ . Moreover,  $\forall t \leq T'$  with  $T' < +\infty$  satisfying (3.17) we have

$$(\mathbf{u}(t), \mathbf{B}(t)) \rightarrow (\mathbf{u}^0(t), \mathbf{B}^0(t)) \text{ in } H^{s-2}(\mathbb{R}^3)^3 \times H^{s-2}(\mathbb{R}^3)^3\text{-strong}$$

as  $Rm, \text{Re} \rightarrow +\infty$  with a convergence rate of  $o(\frac{1}{\text{Re}}) + o(\frac{1}{Rm})$ .

**Proof.** As in the previous cases, we consider the couple  $(\mathbf{v}, \mathbf{C}) := (\mathbf{u}, \mathbf{B}) - (\mathbf{u}^0, \mathbf{B}^0)$  which satisfies the system (4.20). Multiplying (4.20) by  $\mathbf{v}(t)$  and  $\mathbf{C}(t)$  with the inner product of  $H^{s-2}$ , for  $t \in (0, T']$  fixed, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{s-2}^2 &= (\hat{V}\mathbf{u}, \mathbf{v})_{s-2} + S((\mathbf{C} \cdot \nabla)\mathbf{B}^0, \mathbf{v})_{s-2} + S((\mathbf{B} \cdot \nabla)\mathbf{C}, \mathbf{v})_{s-2} \\ &\quad - ((\mathbf{v} \cdot \nabla)\mathbf{u}^0, \mathbf{v})_{s-2} - ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{v})_{s-2}, \quad (4.30) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{C}(t)\|_{s-2}^2 &= \frac{1}{Rm}(\Delta \mathbf{B}, \mathbf{C})_{s-2} + ((\mathbf{C} \cdot \nabla)\mathbf{u}^0, \mathbf{C})_{s-2} + ((\mathbf{B} \cdot \nabla)\mathbf{v}, \mathbf{C})_{s-2} \\ &\quad - ((\mathbf{v} \cdot \nabla)\mathbf{B}^0, \mathbf{C})_{s-2} - ((\mathbf{u} \cdot \nabla)\mathbf{C}, \mathbf{C})_{s-2}. \quad (4.31) \end{aligned}$$

As before, we have used that  $\nabla \cdot \mathbf{v}(t) = \nabla \cdot \mathbf{C}(t) = \mathbf{0}$  and so the terms  $(\nabla(p - p^0), \mathbf{v}(t))_{s-2}$  and  $-\frac{S}{2}(\nabla(\mathbf{B}^2 - (\mathbf{B}^0)^2)(t), \mathbf{v}(t))_{s-2}$  vanish. Nevertheless, this is no longer true for the products  $((\mathbf{u}(t) \cdot \nabla)\mathbf{v}(t), \mathbf{v}(t))_{s-2}$  and  $((\mathbf{u} \cdot \nabla)\mathbf{C}, \mathbf{C}(t))_{s-2}$ .

The viscosity operators  $-\hat{V}$  and  $-\Delta$  are dissipative in the space  $H^{s-2}$ . This is well known for the Laplacian, while for the Braginskii's operator it comes from the fact that  $-\hat{V}$  being dissipative in  $L^2(\mathbb{R}^3)^3$  and commuting with  $S^{s-2} = (1 - \Delta)^{\frac{s-2}{2}}$ . Thus,

$$\begin{aligned} (\hat{V}\mathbf{u}, \mathbf{v})_{s-2}(t) &= (\hat{V}\mathbf{v}(t), \mathbf{v}(t))_{s-2} + (\hat{V}\mathbf{u}^0(t), \mathbf{v}(t))_{s-2} \\ &\leq \|\hat{V}\mathbf{u}^0(t)\|_{s-2} \|\mathbf{v}(t)\|_{s-2} \\ &\leq k'' \left( \max_{\alpha=0, \dots, 4} \mu_{\alpha} \right) \|\mathbf{u}^0(t)\|_s \|\mathbf{v}(t)\|_{s-2} \\ &\leq \frac{k''}{2} \left( \max_{\alpha=0, \dots, 4} \mu_{\alpha} \right)^2 \|\mathbf{u}^0(t)\|_s^2 + \frac{1}{2} \|\mathbf{v}(t)\|_{s-2}^2, \end{aligned}$$



where we have also used that  $\hat{V}$  is a continuous operator from  $H_\sigma^s$  into  $H_\sigma^{s-2}$  being  $k''$  the continuity constant (see Lemma 2.1). Also,

$$\begin{aligned} \frac{1}{Rm}(\Delta \mathbf{B}(t), \mathbf{C}(t))_{s-2} &= \frac{1}{Rm}(\Delta \mathbf{C}(t), \mathbf{C}(t))_{s-2} + \frac{1}{Rm}(\Delta \mathbf{B}^0(t), \mathbf{C}(t))_{s-2} \\ &\leq \frac{-1}{Rm} \|\nabla \mathbf{C}(t)\|_{s-2}^2 + \frac{1}{2Rm^2} \|\mathbf{B}^0(t)\|_s^2 + \frac{1}{2} \|\mathbf{C}(t)\|_{s-2}^2. \end{aligned}$$

From the choice of  $s$  it follows that  $H^{s-2}$  is a Banach algebra (note that  $s-2 > 3/2$ ), hence

$$\begin{aligned} |((\mathbf{v} \cdot \nabla) \mathbf{u}^0, \mathbf{v})_{s-2}(t)| &\leq \|\nabla \mathbf{u}^0(t)\|_{s-2} \|\mathbf{v}(t)\|_{s-2}^2, \\ |((\mathbf{C} \cdot \nabla) \mathbf{u}^0, \mathbf{C})_{s-2}(t)| &\leq \|\nabla \mathbf{u}^0(t)\|_{s-2} \|\mathbf{C}(t)\|_{s-2}^2. \end{aligned}$$

Analogously, Young's inequality yields to

$$\begin{aligned} |((\mathbf{C} \cdot \nabla) \mathbf{B}^0, \mathbf{v})_{s-2}(t)| &\leq \|\nabla \mathbf{B}^0(t)\|_{s-2} \|\mathbf{C}(t)\|_{s-2} \|\mathbf{v}(t)\|_{s-2} \\ &\leq \frac{1}{2} \|\nabla \mathbf{B}^0(t)\|_{s-2} (\|\mathbf{C}(t)\|_{s-2}^2 + \|\mathbf{v}(t)\|_{s-2}^2), \\ |((\mathbf{C} \cdot \nabla) \mathbf{B}^0, \mathbf{v})_{s-2}(t)| &\leq \frac{1}{2} \|\nabla \mathbf{B}^0(t)\|_{s-2} (\|\mathbf{C}(t)\|_{s-2}^2 + \|\mathbf{v}(t)\|_{s-2}^2). \end{aligned}$$

For the remaining terms in (4.30) and (4.31) we use the estimate (3.14) to get

$$\begin{aligned} |((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{v})_{s-2}(t)| &\leq c \|\mathbf{u}(t)\|_{s-2} \|\mathbf{v}(t)\|_{s-2}^2, \\ |((\mathbf{u} \cdot \nabla) \mathbf{C}, \mathbf{C})_{s-2}(t)| &\leq c \|\mathbf{u}(t)\|_{s-2} \|\mathbf{C}(t)\|_{s-2}^2, \end{aligned}$$

and also,

$$\begin{aligned} &((\mathbf{B} \cdot \nabla) \mathbf{C}, \mathbf{v})_{s-2}(t) + ((\mathbf{B} \cdot \nabla) \mathbf{v}, \mathbf{C})_{s-2}(t) \\ &\leq ((\mathbf{B} \cdot \nabla)(\mathbf{v} + \mathbf{C}), \mathbf{v} + \mathbf{C})_{s-2}(t) - ((\mathbf{B} \cdot \nabla) \mathbf{v}, \mathbf{v})_{s-2}(t) - ((\mathbf{B} \cdot \nabla) \mathbf{C}, \mathbf{C})_{s-2}(t) \\ &\leq 3c \|\mathbf{B}(t)\|_{s-2} (\|\mathbf{v}(t)\|_{s-2}^2 + \|\mathbf{C}(t)\|_{s-2}^2), \end{aligned}$$

where  $c$  is the constant in (3.14).

Collecting all the estimates that we have obtained for the terms appearing in (4.30) and (4.31) we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (S \|\mathbf{C}(t)\|_{s-2}^2 + \|\mathbf{v}\|_{s-2}^2) &\leq \frac{1}{2} k'' \left( \max_{\alpha=0, \dots, 4} \mu_\alpha \right)^2 \|\mathbf{u}^0(t)\|_s^2 + \frac{S}{2Rm^2} \|\mathbf{B}^0(t)\|_s^2 \\ &\quad + S c (\|\mathbf{v}(t)\|_{s-2}^2 + \|\mathbf{C}(t)\|_{s-2}^2) (\|\mathbf{u}(t)\|_{s-2} + 3 \|\mathbf{B}(t)\|_{s-2}) \\ &\quad + (\|\nabla \mathbf{u}^0(t)\|_{s-2} + S' \|\nabla \mathbf{B}^0(t)\|_{s-2}) (\|\mathbf{v}(t)\|_{s-2}^2 + \|\mathbf{C}(t)\|_{s-2}^2) \\ &\quad + \frac{1}{2} (\|\mathbf{v}(t)\|_{s-2}^2 + S \|\mathbf{C}(t)\|_{s-2}^2) \\ &\leq \frac{1}{2} k(\mu^*)^2 \|\mathbf{u}^0(t)\|_s^2 + \frac{S}{2Rm^2} \|\mathbf{B}^0(t)\|_s^2 + (\|\mathbf{v}\|_{m-2}^2 + S \|\mathbf{C}\|_{m-2}^2) \xi(t), \end{aligned}$$

where  $\xi(t) := K(\|\mathbf{u}(t)\|_{s-2} + \|\mathbf{B}(t)\|_{s-2} + \|\nabla \mathbf{u}^0(t)\|_{s-2} + \|\nabla \mathbf{B}^0(t)\|_{s-2} + 1/2)$  and  $K := \max(1, S, 3c)$  is a positive constant that just depend on  $S$  and on the exponent  $s$ . Moreover, from the regularity of  $(\mathbf{u}, \mathbf{B})$  and  $(\mathbf{u}^0, \mathbf{B}^0)$ , it follows that  $\xi : (0, T') \rightarrow \mathbb{R}$  is continuous and uniformly bounded with respect to  $\mu_\alpha$  and  $Rm$ ,  $\alpha = 0, \dots, 4$ , in  $[0, T']$  (see Theorem 3.1, (3.9)).

Thus, we can apply Gronwall's inequality to  $\|\mathbf{v}\|_{s-2}^2 + S \|\mathbf{C}(t)\|_{s-2}^2$  and using that  $(\mathbf{v}(0), \mathbf{C}(0)) = (\mathbf{0}, \mathbf{0})$ , we get

$$\begin{aligned} &\|\mathbf{v}(t)\|_{s-2}^2 + S \|\mathbf{C}(t)\|_{s-2}^2 \\ &\leq \exp \left( 2 \int_0^t \xi(\tau) d\tau \right) \int_0^t \left[ k \left( \max_{\alpha=0, \dots, 4} \mu_\alpha \right)^2 \|\mathbf{u}^0\|_s^2 + \frac{S}{Rm^2} \|\mathbf{B}^0\|_s^2 \right] d\tau, \end{aligned}$$

which concludes the proof of the theorem.  $\square$

Arguing in a similar way, for  $\mathbf{F} = \frac{1}{\text{Re}} \Delta$ , we obtain

**Theorem 4.4.** Assume  $(\mathbf{u}_0, \mathbf{B}_0) \in H_\sigma^s \times H_\sigma^s$  with  $s > 9/2$ , and let  $(\mathbf{u}^0, \mathbf{B}^0)$  and  $(\mathbf{u}, \mathbf{B})$  as in Theorem 4.2. Then

$$\begin{aligned} &\|(\mathbf{u} - \mathbf{u}^0)(t)\|_{s-2}^2 + S \|(\mathbf{B} - \mathbf{B}^0)(t)\|_{s-2}^2 \\ &\leq \exp \left( 2 \int_0^t \xi''(\tau) d\tau \right) \int_0^t \left[ \frac{1}{\text{Re}^2} \|\nabla \mathbf{u}^0\|_{s-2}^2 + \frac{S}{Rm^2} \|\nabla \mathbf{B}^0\|_{s-2}^2 \right] d\tau, \end{aligned} \quad (4.32)$$

where  $\xi'' \in \mathcal{C}([0, T'])$  does not depend on  $Rm$  or  $\text{Re}$ . Moreover,  $\forall t \leq T'$  with  $T' < +\infty$  verifying (3.17), (4.32) yields to

$$(\mathbf{u}(t), \mathbf{B}(t)) \rightarrow (\mathbf{u}^0(t), \mathbf{B}^0(t)) \text{ in } H^{s-2}(\mathbb{R}^3)^3 \times H^{s-2}(\mathbb{R}^3)^3$$

as  $\text{Re}, Rm \rightarrow +\infty$  with a convergence rate  $o(\frac{1}{\text{Re}}) + o(\frac{1}{Rm})$ .

**Remark 4.1.** In order to simplify the exposition, we have considered the same initial data for both, the MHD and the ideal MHD systems. Nevertheless, the convergences and convergence rates obtained in Theorems 4.1 (resp. 4.2) and 4.3 (resp. 4.4) remain true if we consider different initial data  $(\mathbf{u}_0^{\text{vis}}, \mathbf{B}_0^{\text{vis}})$  and  $(\mathbf{u}_0^0, \mathbf{B}_0^0)$  for (1.1) and (1.2) respectively, verifying

$$\|\mathbf{u}_0^{\text{vis}} - \mathbf{u}_0^0\|_0 + \|\mathbf{B}_0^{\text{vis}} - \mathbf{B}_0^0\|_0 \rightarrow 0 \text{ as } o(\mu^*) + o\left(\frac{1}{Rm}\right)$$

when  $Rm \rightarrow \infty$  and  $\mu_\alpha \rightarrow 0$ ,  $\alpha = 0, 1, \dots, 4$  (resp. as  $o(\frac{1}{\text{Re}}) + o(\frac{1}{Rm})$  when  $\text{Re}, Rm \rightarrow +\infty$ ), for the first case, or

$$\|\mathbf{u}_0^{\text{vis}} - \mathbf{u}_0^0\|_{s-2} + \|\mathbf{B}_0^{\text{vis}} - \mathbf{B}_0^0\|_{s-2} \rightarrow 0 \text{ as } o(\mu^*) + o\left(\frac{1}{Rm}\right)$$

in the second case.

**Remark 4.2.** The same remark holds if we consider different force terms  $\mathbf{f}$  for the MHD system and for the ideal MHD one.

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