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POSITIVITY FOR LARGE TIME OF SOLUTIONS OF THE HEAT EQUATION: THE PARABOLIC ANTIMAXIMUM PRINCIPLE

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Dedicated to Professor Mark Vishik in occasion of his 80th anniversary

Abstract. We study ~~here~~ the positivity, for large time, of the solutions to the heat equation $\mathcal{Q}_a(f, u^0)$:

$$\mathcal{Q}_a(f,u^0)\begin{cases} \partial_t u - \Delta u = au + f(t,x), & \text{in } Q =]0,\infty[\times\Omega, \\ u(t,x) = 0, & (t,x) \in]0,\infty[\times\partial\Omega, \\ u(0,x) = u^0(x), & x \in \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $a \in \mathbb{R}$. We obtain some sufficient conditions for having a finite time $t_p > 0$ (depending on a and on the data u^0 and f which are not necessarily of the same sign) such that $u(t,x) > 0\ \forall t > t_p,\ a.e.\, x \in \Omega$.

1. Introduction. We start by studying the following linear parabolic problem

$$\mathcal{P}_a(f,u^0)\begin{cases} \partial_t u - \Delta u = au + f(t,x), & \text{in } Q =]0,\infty[\times\Omega, \\ u(t,x) = 0, & (t,x) \in]0,\infty[\times\partial\Omega, \\ u(0,x) = u^0(x), & x \in \Omega. \end{cases}$$

We are concerned here with the positivity, for large t, of solutions of the heat equation $\mathcal{P}_a(f,u^0)$ where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $a \in \mathbb{R}$.

We seek sufficient conditions for having a finite time $t_p > 0$, depending on a and on the data $u^0$and f (which are not necessarily both positive or both negative) such that $u(t,x) > 0\ \forall t > t_p$, $a.e.\ x \in \Omega$.

We are motivated by the so-called *"antimaximum principle"* introduced in 1979 for the elliptic problem (Clément et Peletier [4]). This principle can be stated as

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follows: given Ω (open bounded and smooth enough set in $\mathbb{R}^N$) and $h \in L^p(\Omega)$, $p > N$, we consider the following boundary value problem

$$E(h,a)\begin{cases} -\Delta u = au + h(x) & x \in \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases}$$

If $h \leq 0$ *a.e.* in Ω, there exists $\delta(h) > 0$ such that $\forall a \in]\lambda_1, \lambda_1 + \delta[$ (with $\lambda_1 + \delta < \lambda_2$) any solution u to $E(h,a)$ is positive, *a.e.* in Ω; (here $\lambda_1 < \lambda_2$ are the first and second eigenvalues of the Dirichlet Laplacian operator on Ω).

Our main result shows that in the parabolic case the positivity of the solution can be obtained, for large time and under suitable assumptions, even if $f(t,x)$ is "very negative".

In a second part of the paper, we extend some of the conclusions to the solutions of the quasi-linear problem

$$\mathcal{QP}_a(f,u^0)\begin{cases} \partial_t u - \Delta\phi(u) = g(u) + f(t,x), & \text{in } Q, \\ \phi(u) = 0, & \text{on }]0,\infty[\times\partial\Omega, \\ u(0,x) = u^0(x), & x \in \Omega, \end{cases}$$

where $\phi \in C^1(\mathbb{R}), \phi$ is (strictly) increasing with $\phi(0) = 0$ and g is Lipschitz continuous.

Notations From now on, $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_k \leq ...$ are the eigenvalues of the Dirichlet Laplacian defined on Ω (each eigenvalue being repeated according to its algebraic multiplicity). We denote by $\{\varphi_k\}_{k\in\mathbb{N}}$ the associated orthonormal system of eigenfunctions in $L^2(\Omega)$:

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

We choose $\varphi_1 > 0$ in Ω. Recall that $\varphi_1 \in L^\infty(\Omega)$. For $p \geq 1$, we introduce the space $L^p_{\varphi_1}(\Omega) := \{g : \Omega \to \mathbb{R} \mid \int_\Omega |g|^p \varphi_1 dx < \infty\}$. We decompose $L^2_{\varphi_1}(\Omega)$: for any $g \in L^2_{\varphi_1}(\Omega)$, $g = g_1\varphi_1 + g^\perp$, with $g_1 := \int_\Omega g\varphi_1 dx$ and $\int_\Omega g^\perp \varphi_1 dx = 0$.

2. **Linear Case.** In order to fix ideas, we consider here the linear problem $\mathcal{P}_a(f,u^0)$. Let us first recall that the parabolic (strong) Maximum Principle can be used for example when $u^0 \in L^1(\Omega)$ and $f \in L^1_{loc}(]0,\infty[; L^1(\Omega))$, in order to show that

$$u^0 \geq 0,\ f \geq 0,\ (u^0, f) \not\equiv (0,0),\ \Longrightarrow u(t,x) > 0\ \forall t > 0,\ a.e.\ x \in \Omega.$$

Analoguously, $u^0 \leq 0,\ f \leq 0,\ (u^0,f) \not\equiv (0,0),\ \Longrightarrow u(t,x) < 0\ \forall t > 0,\ a.e.\ x \in \Omega.$ We study here the case where u^0 and f are not globally and simultanuously positive or negative. We prove first

Theorem 2.1. *Assume that the following hypothesis are satisfied*

$$f \in L^2_{loc}((0;\infty); L^p(\Omega)),\ \ p > N, \tag{2.1}$$

$$u^0 \in L^2(\Omega), \tag{2.2}$$

$$\exists C \geq 0,\ f_\infty \in L^p(\Omega),\ p > N,\ f_\infty \leq 0,\ \textit{such that}$$

$$f(t,x) \geq f_\infty(x) - Ce^{(a-\lambda_2)t}\varphi_1(x)\ a.e.\ (x,t) \in Q, \tag{2.3}$$

$$u_1^0 > (u_{f_\infty})_1 + \frac{C}{\lambda_2 - \lambda_1}, \tag{2.4}$$

with u_{f_∞} solution of the elliptic problem $E(f_\infty, a)$, and $\varphi_1 > 0$ principal eigenfunction, solution of $E(0, \lambda_1)$ with $\| \varphi_1 \|_{L^2} = 1$. Then, we have:

i) if $f = 0$, there exists $t^* \geq 0$ and $K > 0$ such that

$$u(t,x) \geq K e^{(a-\lambda_2)t} \varphi_1(x), \ \forall t > t^*, \ a.e.\, x \in \Omega,$$

ii) if $f \neq 0$, there exists $\delta > 0$, $t^* \geq 0$ and $K > 0$ such that, for all $a \in]\lambda_1, \lambda_1 + \delta[$ with $\lambda_1 + \delta < \lambda_2$,

$$u(t,x) \geq K e^{(a-\lambda_2)t} \varphi_1(x), \ \forall t > t^*, \ a.e.\, x \in \Omega.$$

In particular, in both cases, $\forall t > t^*$,

$$u(t,x) > 0 \ a.e.\, x \in \Omega, \ and \ \frac{\partial u}{\partial n}(t,x) < 0 \text{ on } \partial\Omega.$$

It is interesting to examine a special, but very representative, example:

Example 1. *Consider $\mathcal{P}_a(f, u^0)$ where $f \equiv f_1\varphi_1 + f_2\varphi_2$ and $u^0 = u_1^0\varphi_1 + u_2^0\varphi_2$, with f_1 and f_2 constants. The solution u to $\mathcal{P}(f, u^0)$ is then:*

$$u(t,.) = u_1(t)\varphi_1 + u_2(t)\varphi_2$$

or, more precisely:

$$\begin{aligned} u(t,x) &= [e^{(a-\lambda_1)t}(u_1^0 - (\frac{f_1}{\lambda_1 - a})) + (\frac{f_1}{\lambda_1 - a})]\varphi_1(x) \\ &+ [e^{(a-\lambda_2)t}(u_2^0 - (\frac{f_2}{\lambda_2 - a})) + (\frac{f_2}{\lambda_2 - a})]\varphi_2(x). \end{aligned}$$

Then, for $a < \lambda_1$, for any u^0, f and $t \to \infty$, we get that

$$u(t,x) \to u_\infty = (\frac{f_1}{\lambda_1 - a})\varphi_1 + (\frac{f_2}{\lambda_2 - a})\varphi_2,$$

with u_∞ solution of the associated elliptic problem $E(f, a)$. By the classical Maximum Principle for elliptic problems, $f \geq 0$ implies $u_\infty \geq 0$.

If $\lambda_1 < a < \lambda_2$ and $u_1^0 - (\frac{f_2}{\lambda_2 - a}) \neq 0$ then $u(t,x) \to \pm\infty$ according to the sign of $u_1^0 - (\frac{f_2}{\lambda_2 - a})$.

Finally if $\lambda_1 < a < \lambda_2$, and if $u_1^0 - \frac{f_1}{\lambda_1 - a} = 0$, $u(t,x) \to u_\infty$, and hence, if $f \geq 0$ (resp. $f \leq 0$) with $\lambda_1 < a < \lambda_1 + \delta$, and δ defined in [4], u is negative (resp. positive) for large t.

Remark 1. *In the case $f = 0$ (hence $f_\infty = 0$, $C = 0$) condition (2.4) reduces to $u_1^0 > 0$. Generally, if $f \neq 0$, since $u_{f_\infty} > 0$ and $C \geq 0$, we derive from (2.4) that necessarily, $u_1^0 > 0$. Notice that this avoids a contradiction with the parabolic strong maximum principle for $f(t,x)$ satisfying (2.3) and $f(t,x) \leq 0$ a.e. $(x,t) \in Q$.* ■

Corollary 2.2. *Let w be a solution of the linear heat eqution*

$$\mathcal{P}_0(F, u^0) \begin{cases} \partial_t w - \Delta w = F(t,x), & \text{in } Q, \\ w(t,x) = 0, & \text{on }]0;\infty[\times\partial\Omega, \\ w(0,x) = u^0(x), & x \in \Omega. \end{cases}$$

Assume there exists $a \in]\lambda_1, \lambda_2[$ such that u^0 and $f := e^{at}F$ satisfies the assumptions of Theorem 2.1. Then, if $a \in]\lambda_1, \lambda_1 + \delta[$ with δ defined by ([4]), we have:

i) if $f = 0$, there exists $t^* \geq 0$ and $K > 0$ such that

$$w(t,x) \geq K e^{-\lambda_2 t} \varphi_1(x), \ \forall t > t^*, \ a.e.\, x \in \Omega;$$

ii) if $f \neq 0$, there exists $t^* \geq 0$ and $K > 0$ such that

$$w(t,x) \geq K e^{-\lambda_2 t}\varphi_1(x),\ \forall t > t^*,\ a.e.\ x \in \Omega.$$

In particular, in both cases $\forall t > t^*$,

$$w(t,x) > 0 \ a.e.\ x \in \Omega \ \text{and} \ \frac{\partial w}{\partial n}(t,x) < 0 \ \text{on} \ \partial\Omega.$$

Proof of Corollary 1.2. It is sufficient to notice that $\mathcal{P}_0(F,u^0)$ is equivalent to $\mathcal{P}_a(f,u^0)$, for a given parameter a (and the rest of assumptions of Theorem 2.1), by the change of unknown $u(t,x) := e^{at}w(t,x)$ with $F = fe^{-at}$. Moreover, we note that u and w as well as f and F have the same sign . ▩

In order to prove part i) of Theorem 2.1. it is useful to establish a previous result

Proposition 2.3. *(Díaz and Morel [5]). Let v be the solution of $\mathcal{P}_a(0,(u^0)^{\perp})$. Then*

$$|v(t,x)| \leq C \left\|(u^0)^{\perp}\right\|_{L^2(\Omega)} e^{(a-\lambda_2)t}\varphi_1(x) \quad \forall t > 0,\ \text{a.e.}\ x \in \Omega.$$

Proof of Proposition 2.3. We first remark that if v satisfies $\mathcal{P}_0(0,u^0)$, then $v_1 = e^{-\lambda_1 t}u_1^0$. Let us prove that if $u_1^0 = 0$, then $|v(t,x)| \leq C \left\|u^0\right\|_{L^2(\Omega)} e^{-\lambda_2 t}\varphi_1(x)$ $\forall t > 0$, $a.e.\ x \in \Omega$. Indeed, multiplying by v, we get that

$$\frac{d}{dt}\int_\Omega v^2 + 2\lambda_2 \int_\Omega v^2 \leq 0.$$

By the regularizing effect we get that $|v(t,x)| \leq C \left\|u^0\right\|_{L^2(\Omega)} e^{-\lambda_2 t}$ for any $t \geq \tau > 0$, with C only dependent of Ω and τ. But assuming $u^0 \in H^2(\Omega) \cap H_0^1(\Omega)$

$$\int_\Omega v(t)\varphi_1 = \int_\Omega \frac{dv}{dt}(t)\varphi_1 = 0$$

and so $\int_\Omega \Delta v(t)\varphi_1 = 0$ for $a.e.\ t \in (0,+\infty)$. Moreover, since $\frac{dv}{dt}(t) = \Delta v(t) = 0$ on $(0,+\infty) \times \partial\Omega$, we get that $\Delta v(t)$ satisfies $\mathcal{P}_0(0,\Delta u^0)$. Then, as before, $|\Delta v(t,x)| \leq C \left\|\Delta u^0\right\|_{L^2(\Omega)} e^{-\lambda_2 t} \leq C' \left\|u^0\right\|_{L^2(\Omega)} e^{-\lambda_2 t}$ for any $t \geq \tau > 0$. Finally, if we define $\psi(x)$ as the (unique) solution of $(E(1,0))$, then, by well-known results $|\psi(x)| \leq C\varphi_1(x)$ on Ω, we derive the result by applying the comparison principle to v and $C' \left\|u^0\right\|_{L^2(\Omega)} e^{-\lambda_2 t}\psi(x)$. The conclusion for the general case $u^0 \in L^2(\Omega)$ is obtained by a process of approximation and passing to the limit. ▩

Remark 2. *An analoguous result to Proposition 2.3 stands in Gmira and Veron ([8]).*

In the following, we decompose u, solution of $\mathcal{P}_a(f,u^0)$, as: $u = u_1\varphi_1 + u^{\perp}$. Our results, for $\mathcal{P}_a(f,u^0)$, suggest that there is a balance between the components of the data on the linear subspace spanned by φ_1. Given $u^0 \in L^1_{\varphi_1}(\Omega)$ and $f \in L^1_{loc}((0;\infty);L^1_{\varphi_1}(\Omega))$, we introduce, for $t \geq 0$, the "balance function",

$$B_a(t;f,u^0) := u_1^0 + \int_0^t e^{(\lambda_1 - a)s}f_1(s)ds,$$

where $f_1(t) := \int_\Omega f(t,\cdot)\varphi_1 dx$ and $u_1^0 := \int_\Omega u^0\varphi_1 dx$. We have:

Finally we derive from Proposition 2.3 that

$$\widetilde{u}(t,x) \geq ((u_1^0 - (u_{f_\infty})_1 - \frac{C}{\lambda_2 - \lambda_1})e^{(a-\lambda_1)t} -$$
$$C \left\| (u^0 - u_{f_\infty} - \frac{C\varphi_1(x)}{\lambda_2 - \lambda_1})^\perp \right\|_{L^2(\Omega)} e^{(a-\lambda_2)t} + \frac{C}{\lambda_2 - \lambda_1} e^{(a-\lambda_2)t})\varphi_1(x).$$

By using condition (2.4) we derive that after a finite time, t^*, the coefficient in front of $\varphi_1(x)$ in the above inequality is positive, which gives the result.■

3. A quasilinear problem. In this section, we consider the quasi-linear problem

$$\mathcal{QP}_a(f,u^0) \begin{cases} \partial_t u - \Delta\phi(u) = g(u) + f(t,x), & \text{in } Q, \\ \phi(u) = 0 & \text{on }]0,\infty[\times\partial\Omega, \\ u(0,x) = u^0(x), & x \in \Omega, \end{cases}$$

where $\phi \in C^1(\mathbb{R}), \phi$ is (strictly) increasing with $\phi(0) = 0$ and g is Lipschitz continuous. We have

Theorem 3.1. *Assume that the following hypothesis are satisfied*

$$f \in L^2_{loc}((0;\infty); L^p(\Omega)), \;\; p > N, \; u^0 \in L^2(\Omega), \; \phi(u^0) \in H^2(\Omega), \tag{3.1}$$

$$\phi'(r) \leq M \quad \forall r \in \mathbb{R}, \tag{3.2}$$

$$\exists a > 0, K \geq 0 \, : \, g(r) \geq a\phi(r) - K, \; \forall r \in \mathbb{R}, \tag{3.3}$$

$$\exists C \geq 0, \; f_\infty \in L^p(\Omega), \; p > N, \; f_\infty \leq 0:$$
$$f(t,x) \geq f_\infty(x) - Ce^{M(a-\lambda_2)t}\varphi_1(x) \;\; a.e. \; (x,t) \in Q, \tag{3.4}$$

$$\phi(u^0)_1 > (u_{f_\infty - K})_1 + \frac{C}{\lambda_2 - \lambda_1}, \tag{3.5}$$

$$\Delta\phi(u^0) + a\phi(u^0) \leq -f_\infty(x) + K + \frac{C(a-\lambda_1)\varphi_1(x)}{\lambda_2 - \lambda_1}, \text{ a.e. } x \in \Omega, \tag{3.6}$$

with $u_{(f_\infty - K)}$ solution of the elliptic problem $E(f_\infty - K, a)$. Then:

i) if $f = 0$, there exists $t^* \geq 0$ and $K > 0$ such that

$$u(t,x) \geq Ke^{M(a-\lambda_2)t}\varphi_1(x), \; \forall t > t^*, \; a.e. \; x \in \Omega,$$

ii) if $f \neq 0$ there exists $\delta > 0$, $t^* \geq 0$ and $K^* > 0$ such that, for all $a \in]\lambda_1, \lambda_1 + \delta[$ with $\lambda_1 + \delta < \lambda_2$, we have

$$u(t,x) \geq K^* e^{M(a-\lambda_2)t}\varphi_1(x), \; \forall t > t^*, \; a.e. \; x \in \Omega.$$

In particuliar, $\forall t > t^*$,

$$u(t,x) > 0 \; a.e. \; x \in \Omega \; \text{ and } \; \frac{\partial u}{\partial n}(t,x) < 0 \;\text{ on } \partial\Omega.$$

Proof We show that the function $\underline{u} := \phi^{-1}(w + u_{(f_\infty - K)} + u_R)$ is a subsolution when w satisfies

$$\begin{cases} \frac{1}{M}\partial_t w - \Delta w = aw & \text{in } Q, \\ w = 0 & \text{on }]0,\infty[\times\partial\Omega, \\ w(0,x) = w^0(x) := \phi(u^0) - u_{(f_\infty - K)} - \frac{C\varphi_1(x)}{\lambda_2 - \lambda_1}, & x \in \Omega, \end{cases}$$

with $u_R(t,x) = \frac{C}{\lambda_2 - \lambda_1}e^{M(a-\lambda_2)t}\varphi_1(x)$. Indeed it follows from the Maximum Principle and from Condition (3.6) that (Sattinger [9]) $\partial_t w \leq 0$. Then, using ~~Assumptions~~ a (3.2) and (3.3) we obtain

Proposition 2.4. *Assume (2.2) and* $f \in L^2_{loc}((0;\infty) : L^2(\Omega))$. *Then*

$$u(t,x) = e^{(a-\lambda_1)t} B_a(t; f, u^0)\varphi_1(x) + u^{\perp}(t,x)$$

with $u^{\perp}$ *solution of* $\mathcal{P}_a(f^{\perp}, (u^0)^{\perp})$.

Proof of Proposition 2.4. From the hypotheses, we can write

$$f(t,.) = \sum_{k=1}^{\infty} f_k(t)\varphi_k; \text{ and } u^0 = \sum_{k=1}^{\infty} u_k^0 \varphi_k.$$

By Fourier-Galerkin method, we seek, analoguously, u as

$$u(t,.) = \sum_{k=1}^{\infty} u_k(t)\varphi_k.$$

So u_k satisfies

$$\begin{cases} u_k' + \lambda_k u_k = a u_k + f_k, & \forall k \in \mathbb{N}^*, \\ u_k(0) = u_k^0. \end{cases}$$

Hence $u_k(t) = C_k(t)e^{(a-\lambda_k)t}$ with $C_k(t) = u_k^0 + \int_0^t e^{(\lambda_k - a)s} f_k(s)ds$. Since $C_1(t) = B_a(t; f, u^0)$, using that $u^{\perp}$, solution of $\mathcal{P}_a(f^{\perp}, (u^0)^{\perp})$, is a weak solution orthogonal to φ_1, we derive the desired result.■

Remark 3. *If conditions (2.3) and (2.4) are satisfied, then we have*

$$B_a(t; f, u^0) > (u_{f_\infty})_1 + \frac{C}{\lambda_2 - \lambda_1} e^{-(\lambda_2 - \lambda_1)t} > 0, \forall t \geq 0. \tag{2.5}$$

Motivated by Proposition 2.3, we guess that the mere assumption $B_a(t; f, u^0) > 0$ $\forall t \geq t_1 \geq 0$ *is sufficient for having the conclusion of Theorem 2.1, but it is not our purpose here. We also note that by use of the same change of function as in Corollary 2.2, we have:* $B_a(t; f, u^0) = B_0(t; e^{-at} f, u^0)$.

Proof of Theorem 2.1. Assume that $f(t,x) \geq \widetilde{f}(t,x) := f_\infty(x) - Ce^{(a-\lambda_2)t}\varphi_1(x)$ *a.e.* $(x,t) \in Q$. From the Maximum Principle, we deduce that $u \geq \widetilde{u}$ with $\widetilde{u}$ solution of $\mathcal{P}_a(\widetilde{f}, u^0)$. We decompose $\widetilde{u}$ as

$$\widetilde{u} = w + u_{f_\infty} + u_R$$

where

w is solution of $\mathcal{P}_a(0, u^0 - u_{f_\infty} - \frac{C\varphi_1(x)}{\lambda_2 - \lambda_1})$,
u_{f_∞} is solution of the elliptic problem $E(f_\infty, a)$,
u_R is solution of $P_a(-Ce^{(a-\lambda_2)t}\varphi_1(x), \frac{C\varphi_1(x)}{\lambda_2-\lambda_1})$.

It is easy to verify that

$$u_R(t,x) = \frac{C}{\lambda_2 - \lambda_1} e^{(a-\lambda_2)t}\varphi_1(x).$$

Moreover, the (elliptic) Antimaximum Principle implies that

$$u_{f_\infty} > 0 \text{ (or } u_{f_\infty} = 0 \text{ if } f_\infty(x) = 0).$$

Also, Proposition 2.4 with $f = 0$ implies

$$w(t,x) = (u_1^0 - (u_{f_\infty})_1 - \frac{C}{\lambda_2 - \lambda_1})e^{(a-\lambda_1)t}\varphi_1(x) + v(t,x)$$

where v is the solution of $\mathcal{P}_a(0, (u^0 - u_{f_\infty} - \frac{C\varphi_1(x)}{\lambda_2-\lambda_1})^{\perp})$.

$$
\begin{gathered}
\partial_t \underline{u} - \Delta\phi(\underline{u}) - g(\underline{u}) - f(t,x) \leq \\
\{\phi'(\phi^{-1}(w + u_{(f_\infty - K)} + u_R))\}^{-1}(w_t + (u_R)_t) - \Delta(w + u_{(f_\infty - K)} + u_R) \\
-a(w + u_{(f_\infty - K)} + u_R) - f_\infty(x) - K - Ce^{M(a-\lambda_2)t}\varphi_1(x) \leq 0.
\end{gathered}
$$

Applying the Comparison Principle and reasoning as in the linear case we derive the result. ∎

Remark 4. *A similar result is established in Bertsch-Peletier ([3]) for the case i) with $g = f = 0$ and without Condition (3.6) . We can find here $t^* > 0$. If $g = f = 0$, the phenomenon of finite extinction time (see e.g. Antontsev, Díaz and Shmarev [2]) shows that Condition (3.2) is necessary*

Remark 5. *A slight change in the proof shows that if we replace (3.2) by $\phi'(r) \geq m > 0$, $\forall r \in \mathbb{R}$, the result still holds by replacing Condition (3.6) by*

$$
\Delta\phi(u^0) + a\phi(u^0) \geq -f_\infty(x) + K + \frac{C(a-\lambda_1)\varphi_1(x)}{\lambda_2 - \lambda_1}, \text{ a.e. } x \in \Omega. \tag{3.7}
$$

Remark 6. *In the special case of $g(u) = au_+ - bu_-$ (where $u_+ := max\{u, 0\}$), the conclusion of Theorem 2.1 shows that, with this class of initial data, the solution of the parabolic problem tends to the positive solution of the stationary problem sudied in Fleckinger, Gossez, Takáč and de Thélin ([7]).*

Remark 7. *Concerning the equation $\partial_t u - \Delta_p u = au^{p-2}u + f(t,x)$ the decomposition in sum of the solution does not hold any more. Nevertheless the Antimaximum Principle is still valid for the elliptic equation ([7]) and hence if $f_\infty < 0$ then $u_{f_\infty} > 0$. A direct study of the equation with the conditions of [6] and with hypotheses on u^0 shows that $u(t,.) \longmapsto u_\infty$ in $L^\infty(\Omega)$ and $u(t,x)$ is positive after a finite time. For the case $a = 0$ and $f = 0$ we have, for some initial data and when $p > 2$, that necessarily $t^* > 0$ (see e.g., [2]). We get extinction in finite time for $1 < p < 2$ (see e.g. [2]).*

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