

*Original article*

## Global stability for convection when the viscosity has a maximum

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**Abstract.** Until now, an unconditional nonlinear energy stability analysis for thermal convection according to Navier–Stokes theory had not been developed for the case in which the viscosity depends on the temperature in a quadratic manner such that the viscosity has a maximum. We analyse here a model of non-Newtonian fluid behaviour that allows us to develop an unconditional analysis directly when the quadratic viscosity relation is allowed. By unconditional, we mean that the nonlinear stability so obtained holds for arbitrarily large perturbations of the initial data. The nonlinear stability boundaries derived herein are sharp when compared with the linear instability thresholds.

**Key words:** Thermal convection, nonlinear stability, energy method

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### 1 Introduction

In everyday life the viscosity of a fluid is usually strongly dependent on temperature [1–3]. A large change in the viscosity with temperature may have a pronounced effect on Bénard convection, which is the cellular motion that ensues when a layer of fluid is heated from below. In particular, the increase or decrease of the viscosity with temperature has been shown experimentally to give rise, respectively, to decreasing or increasing fluid motion in the centre of the Bénard cell. This important work was due to Tippelskirch [4]. While the theory of linearised instability of the problem of thermal convection with a temperature-dependent viscosity is well known, the corresponding *nonlinear stability theory* is incomplete. By using a generalised energy analysis, Richardson [1] established nonlinear stability bounds that were very sharp when compared with those of linearised instability theory when the viscosity was a linearly decreasing function of temperature. Richardson's work employed a non-trivial Lyapunov function and was an intricate analysis. It was extended by Capone and Gentile [2,3] to allow for a more general viscosity–temperature relationship of exponential form but essentially one that had a bounded derivative. These are important articles and were the first to derive nonlinear energy stability bounds for Bénard convection with a temperature-dependent viscosity. However, they suffer from two drawbacks. The first is that the analysis holds for two stress-free surfaces bounding the fluid layer. The second is that the stability thresholds derived are *conditional*, in the sense that the initial data is restricted and tends to zero upon approaching the critical Rayleigh number. In this paper we propose a way to overcome the above obstacles when the viscosity is quadratic in temperature, by employing a non-Newtonian model of fluid behaviour.

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For many fluids the viscosity–temperature relationship is essentially a decreasing exponential one. In this case, for temperature differences that are not too large, one can approximate the viscosity by a function that is linear in the temperature field. However, not all fluids may be adequately modelled by a linear viscosity–temperature relationship. For example, liquid sulphur and bismuth are fluids for which the viscosity achieves a maximum as a function of temperature [5]. For such fluids a quadratic relation of the form

$$\nu(T) = \nu_0 [1 - \gamma(T - T_0)^2] \quad (1.1)$$

is necessary to reflect the maximum with temperature. In (1.1),  $\nu_0$ ,  $T_0$ , and  $\gamma$  are constants. Richardson [1] also developed a nonlinear, conditional energy method to handle the viscosity given by (1.1). The goal of this work is to incorporate (1.1) into an unconditional (for all initial data) nonlinear stability analysis of the Bénard problem for a non-Newtonian fluid.

In the context of thermal convection in a porous medium, nonlinear energy stability analyses have been developed by Richardson [1] and by Qin and Chadam [6]. Their work derived *conditional* (initial-data-dependent) stability thresholds. Payne and Straughan [7] employed  $L^p$  energy functionals to develop an *unconditional* nonlinear stability theory for the porous Bénard problem with a temperature-dependent viscosity provided that one of the Forchheimer theories is used. Straughan [8] has also developed an unconditional analysis with a viscosity that is linear in temperature by using Ladyzhenskaya’s models of fluid behaviour. The analysis developed here is necessarily different from that of Straughan [8], due to the different viscosity (1.1), which is quadratic rather than linear in temperature. In fact, the higher order temperature dependence in (1.1) requires the introduction of an  $L^4$  norm in the energy rather than lower order ones. Also, the non-Newtonian continuum theory employed in this paper is different from the Ladyzhenskaya theories used by Straughan [8].

Convection theory with a temperature- or concentration-dependent viscosity is a highly active area and there has been much recent mathematical work (see articles by Capone and Gentile [2,3], Diaz and Galiano [9, 10], Flavin and Rionero [11–13], Galiano [14], Payne et al. [15], Payne and Straughan [7], Qin and Chadam [6], Richardson [1], and Straughan [8]). These articles demonstrate the need for a well-established theory of convection that takes into account the variation of fluid properties such as viscosity or conductivity with temperature. In particular, Diaz and Galiano [9, 10] and Galiano [14] have established the existence of solutions to the fluid equations with temperature-varying viscosity and thermal conductivity. The present paper is devoted to studying a well-defined nonlinear stability problem for a non-Newtonian fluid for the case in which the viscosity is a quadratic function of the temperature. To the best of our knowledge, this is the first *unconditional* nonlinear energy stability analysis for Bénard convection in a *fluid* for which the viscosity is a quadratic function of the temperature. The nonlinear stability thresholds we derive are very sharp when compared with those of linearised instability theory.

The real strength of a nonlinear energy stability analysis is when it can be employed to yield sharp stability thresholds that are valid for *all* initial data, or at least for a large set of initial data. This has been stressed by Straughan (p. 157 of [16]) and several recent articles have explicitly addressed this question in a variety of contexts (see articles by Budu [17], Flavin and Rionero [11–13], Lombardo et al. [18], Payne and Straughan [7], and Straughan [19,8]). Flavin and Rionero [11–13], in particular, employed a “natural” transformation to cope with the situation in which the thermal conductivity is a nonlinear function of temperature, at least for a certain class of nonlinearities. We point out that energy methods in general for various problems in partial differential equations are addressed in the books of Antontsev et al. [20], Doering and Gibbon [21], Flavin and Rionero [22], and Straughan [23].

## 2 The non-Newtonian fluid model

We introduce a constitutive equation in which the viscosity is composed of two parts, one involving the usual viscosity term as a function of temperature and the other being a nonlinear function of the symmetric part of the velocity gradient. Such a model is a generalisation to the temperature -dependent viscosity case of a well-known equation in non-Newtonian fluid mechanics (cf. p. 228 of [20]). Our constitutive theory also resembles the generalised second grade fluid model of Man and Sun [24] and Massoudi and Phuoc [25], although they were interested in other questions such as glacier flow.

Thus, our basic equations of momentum, continuity, and energy balance are

$$\begin{aligned} \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k} 2\nu_0 \left\{ [1 - \gamma(T - T_0)^2] D_{ik} \right\} + 2\nu_1 \frac{\partial}{\partial x_k} (|\mathbf{D}|^{2\mu} D_{ik}) + g\alpha k_i T, \\ \frac{\partial v_i}{\partial x_i} &= 0, \quad \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \kappa \Delta T. \end{aligned} \quad (2.1)$$

Here  $v_i$ ,  $T$ , and  $p$  are the velocity, temperature, and pressure in the fluid,  $g, \alpha, \rho$ , and  $\kappa$  are gravity, thermal expansion coefficient, density (constant), and thermal diffusivity,  $\mathbf{k}=(0,0,1)$ , and  $D_{ik} = (v_{i,k} + v_{k,i})/2$ . Standard indicial notation is used throughout with a repeated index summing from 1 to 3. The coefficients  $\nu_1$  and  $\mu$  are positive constants and throughout this work we select  $\mu = 1$ . This choice is consistent with other non-Newtonian fluid models, e.g., the equations for a fluid of third grade [17,26].

To study the Bénard problem, we suppose that the fluid occupies the layer  $z \in (0, d)$ ,  $(x, y) \in \mathbb{R}^2$ . Gravity is in the negative  $z$ -direction and the planes  $z = 0, d$  are, respectively, held at fixed temperatures  $T_L$  and  $T_U$ , with  $T_L > T_U$ .

The equations in (2.1) admit the steady solution

$$\bar{T} = T_L - \beta z, \quad \bar{v}_i \equiv 0, \quad (2.2)$$

where

$$\beta = \frac{T_L - T_U}{d}. \quad (2.3)$$

The steady pressure  $\bar{p}(z)$  is found from the momentum equation. It is the nonlinear stability of this solution that is the object of this paper.

We introduce perturbations  $(u_i, \theta, \pi)$  by  $v_i = \bar{v}_i + u_i$ ,  $T = \bar{T} + \theta$ , and  $p = \bar{p} + \pi$  to system (2.1), and rescale the equations for the perturbation quantities to make them non-dimensional. The scalings that we employ are those of Richardson (p. 58 of [1]), who refers everything to an average viscosity,  $\nu_m = d^{-1} \int_0^d \nu(\bar{T}) dz$ , and takes  $T_0 = T_L$ . Thus, we write  $t = t^* d^2 / \nu_m$ ,  $P = U \nu_m \rho / d$ ,  $Pr = \nu_m / \kappa$ ,  $u_i = u_i^* U$ ,  $T^\# = U \sqrt{\beta \nu_m / \kappa g \alpha}$ , and  $U = \nu_m / d$ , and then introduce the Rayleigh number  $Ra = R^2 = \alpha g \beta d^4 / \kappa \nu_m$ , the non-dimensional viscosity variation  $\Gamma = \gamma \beta^2 d^2$ , and the non-dimensional non-Newtonian coefficient  $\omega = 2\nu_1 \nu_m^{2\mu-1} / d^{4\mu}$ . The non-dimensional perturbation equations then become (where all stars are omitted although it is understood that the equations are non-dimensional)

$$\begin{aligned} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{\partial \pi}{\partial x_i} + R\theta k_i + (f d_{ij})_{,j} + h(z\theta d_{ij})_{,j} - \zeta(d_{ij}\theta^2)_{,j} + \omega(|\mathbf{d}|^{2\mu} d_{ij})_{,j}, \\ \frac{\partial u_i}{\partial x_i} &= 0, \quad Pr \left( \frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} \right) = R w + \Delta \theta, \end{aligned} \quad (2.4)$$

where  $w = u_3$  and

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad f(z) = \frac{6}{(3-\Gamma)}(1-\Gamma z^2), \quad h = \frac{12\Gamma Pr}{R(3-\Gamma)}, \quad \zeta = \frac{6\Gamma Pr^2}{R^2(3-\Gamma)}. \quad (2.5)$$

The coefficient  $\Gamma$  is such that  $0 < \Gamma < 1$ , which is consistent with the viscosity being positive.

The equations in (2.4) hold on the domain  $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$ . The boundary conditions on the perturbations become

$$u_i = 0, \quad z = 0, 1, \quad \theta = 0, \quad z = 0, 1, \quad (2.6)$$

and  $u_i, \theta, \pi$  satisfy a plane-tiling periodic shape in the  $(x, y)$  plane (cf. p. 51 of [23]). The cell that arises due to this plane-tiling form is denoted by  $V$ .

### 3 Unconditional nonlinear stability

We develop two energy equations by multiplying (2.4)<sub>1</sub> by  $u_i$  and (2.4)<sub>3</sub> by  $\theta$ , and integrating over  $V$ . This yields, with  $\mu = 1$ ,

$$\frac{d}{dt} \frac{1}{2} \|\mathbf{u}\|^2 = R(\theta, w) - \int_V f |\mathbf{d}|^2 dx - h \langle z\theta |\mathbf{d}|^2 \rangle - \omega \int_V |\mathbf{d}|^4 dx + \zeta \langle \theta^2 |\mathbf{d}|^2 \rangle, \quad (3.1)$$

$$\frac{d}{dt} \frac{1}{2} Pr \|\theta\|^2 = R(\theta, w) - \|\nabla\theta\|^2. \quad (3.2)$$

Here,  $\|\cdot\|$  and  $(\cdot, \cdot)$  denote the norm and inner product on  $L^2(V)$ ,  $w = u_3$ , and  $\langle \cdot \rangle$  denotes integration over  $V$ .

The first step is to use the arithmetic–geometric mean inequality on the third and last terms on the right in (3.1) to derive, for numbers  $\alpha, \beta > 0$  (to be chosen),

$$\frac{d}{dt} \frac{1}{2} \|\mathbf{u}\|^2 \leq R(\theta, w) - \langle f|\mathbf{d}|^2 \rangle - \langle |\mathbf{d}|^4 \rangle \left( \omega - \frac{\alpha h}{2} - \frac{\zeta\beta}{2} \right) + \frac{h}{2\alpha} \langle z^2\theta^2 \rangle + \frac{\zeta}{2\beta} \langle \theta^4 \rangle. \quad (3.3)$$

Now, let  $\lambda > 0$  be a coupling parameter that we may select optimally later. Then, from (3.2) and (3.3) we form

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\lambda Pr}{2} \|\theta\|^2 \right) &\leq R(1 + \lambda)(\theta, w) - \langle f|\mathbf{d}|^2 \rangle - \lambda \|\nabla\theta\|^2 + \frac{h}{2\alpha} \langle z^2\theta^2 \rangle \\ &\quad + \frac{\zeta}{2\beta} \langle \theta^4 \rangle - \langle |\mathbf{d}|^4 \rangle \left( \omega - \frac{\alpha h}{2} - \frac{\zeta\beta}{2} \right). \end{aligned} \quad (3.4)$$

It would appear that we need extra dissipation on the right of (3.4) to deal with the  $\langle \theta^4 \rangle$  term. Hence, we now derive an ‘‘energy’’ identity for  $\langle \theta^4 \rangle$ ,

$$\frac{d}{dt} \frac{a Pr}{4} \langle \theta^4 \rangle = a R(\theta^3, w) - \frac{3a}{4} \|\nabla\theta\|^2, \quad (3.5)$$

where  $a > 0$  is another coupling parameter to be selected. We next use Young’s inequality on the first term on the right of (3.5) and Poincaré’s inequality on the last term to find, for a constant  $\epsilon > 0$ ,

$$\frac{d}{dt} \frac{a Pr}{4} \langle \theta^4 \rangle \leq \langle \theta^4 \rangle \left( \frac{3a R \epsilon^{4/3}}{4} - \frac{3\pi^2 a}{4} \right) + \frac{a R}{4\epsilon^4} \langle w^4 \rangle. \quad (3.6)$$

Next, we estimate  $\langle w^4 \rangle$  by using Hölder’s inequality and the Sobolev inequality to be

$$\langle w^4 \rangle \leq \langle |\mathbf{u}|^4 \rangle \leq m^{1/3} \langle |\mathbf{u}|^6 \rangle^{2/3} \leq m^{1/3} c_S^4 \|\nabla\mathbf{u}\|^4 \leq 4m^{1/3} c_S^4 \langle |\mathbf{d}|^2 \rangle^2 \leq 4m^{4/3} c_S^4 \int_V |\mathbf{d}|^4 dV, \quad (3.7)$$

where  $m$  is the volume of  $\Omega$  and  $c_S$  is the constant in the Sobolev inequality  $\|u\|_{L^6} \leq c_S \|\nabla u\|_{L^2}$ , namely,  $c_S = 2^{1/3}/3^{2/3}\pi^{2/3}$ . We now let  $c_1 = 4m^{4/3}c_S^4$ . By using (3.7), we may derive from (3.6) and (3.4)

$$\frac{dE}{dt} \leq I - D - \|\theta\|_4^4 \left( \frac{3\pi^2 a}{4} - \frac{3a R \epsilon^{4/3}}{4} - \frac{\zeta}{2\beta} \right) - \langle |\mathbf{d}|^4 \rangle \left( \omega - \frac{\alpha h}{2} - \frac{\zeta\beta}{2} - \frac{a c_1 R}{4\epsilon^4} \right), \quad (3.8)$$

where the functions  $E$ ,  $D$ , and  $I$  are defined by

$$E(t) = \frac{1}{2} \|\mathbf{u}\|^2 + \frac{1}{2} \lambda Pr \|\theta\|^2 + \frac{a Pr}{4} \|\theta\|_4^4, \quad (3.9)$$

$$I(t) = R(1 + \lambda)(\theta, w) + \frac{h}{2\alpha} \langle z^2\theta^2 \rangle, \quad (3.10)$$

$$D(t) = \langle f|\mathbf{d}|^2 \rangle + \lambda \|\nabla\theta\|^2. \quad (3.11)$$

The coefficient  $\alpha$  is selected as  $\alpha = 16RPr\sqrt{c_1}/9\pi^4$ , and we then choose the coefficients  $a$ ,  $\beta$ , and  $\epsilon$  optimally. To do this, we put  $a = a' + k\delta$ , where  $\delta$  is crucial to obtain an exponential decay rate. We then pick

$$\frac{3a'}{4} (\pi^2 - R\epsilon^{4/3}) - \frac{\zeta}{2\beta} = 0, \quad (3.12)$$

and then minimize the right hand side of

$$\omega > \frac{\alpha h}{2} + \frac{k\delta c_1 R}{4\epsilon^4} + f(\beta, \epsilon),$$

where  $f(\beta, \epsilon) = \zeta\beta/2 + a'c_1R/4\epsilon^4$ . We solve (3.12) for  $a'$  and substitute in  $f$  and minimize  $f$  with respect to  $\beta$  to find  $\beta = \beta(\epsilon)$ . Next, we use this value in  $f$  and minimise the result with respect to  $\epsilon$ . This yields the values

$$\epsilon = \left( \frac{3\pi^2}{4R} \right)^{3/4}, \quad \beta = \frac{16\sqrt{c_1}R^2}{9\pi^4}, \quad a' = \frac{3\zeta\pi^2}{2\sqrt{c_1}R^2}.$$

**Table 1.** Critical Rayleigh numbers of linear and energy theory

$\Gamma$	$Ra_L$	$Ra_E$	$a_L^2$	$a_E^2$	$\lambda_{crit}$
0	1707.76	1707.76	9.7115	9.7115	1.000000
0.5	1621.16	1616.97	9.6098	9.6047	1.005191
0.9	1354.90	1345.40	9.2394	9.2287	1.014130
0.99	1130.65	1119.32	8.7921	8.7804	1.020256
0.999	1040.56	1028.91	8.5528	8.5413	1.022656

These values in turn lead to the restriction on  $\omega$

$$\omega > \frac{\alpha h}{2} + \frac{k\delta c_1 R}{4\epsilon^4} + \frac{16\zeta\sqrt{c_1}R^2}{9\pi^4}.$$

The number  $\delta$  is arbitrarily small, and in the limit in which it vanishes, the restriction  $\omega$  must satisfy is

$$\omega > \frac{64\sqrt{c_1}Pr^2}{3\pi^4(3-\Gamma)} \Gamma. \tag{3.13}$$

From inequality (3.8) we now pick  $k = 4/3\pi^2$  and then derive

$$\frac{dE}{dt} \leq -D\left(1 - \frac{1}{R_E}\right) - \delta\|\theta\|_4^4, \tag{3.14}$$

where

$$\frac{1}{R_E} = \max_H \frac{I}{D}, \tag{3.15}$$

$H$  being the space of admissible solutions.

We require  $R_E > 1$  and then use the bound  $f \geq 6(1-\Gamma)/(3-\Gamma) \equiv f_0 > 0$  so that

$$D \geq \min\left\{8f_0\pi^2, \frac{2\pi^2}{Pr}\right\} \left(\frac{1}{2}\|u\|^2 + \frac{\lambda Pr}{2}\|\theta\|^2\right).$$

Then, from (3.14) we may show that

$$\frac{dE}{dt} \leq -KE, \tag{3.16}$$

where

$$K = \left(\frac{R_E - 1}{R_E}\right) \min\left\{\frac{48\pi^2(1-\Gamma)}{(3-\Gamma)}, \frac{2\pi^2}{Pr}, \frac{4\delta}{aPr}\right\}.$$

Inequality (3.16) leads to exponential decay and hence global nonlinear stability (for all initial data). The only requirement that we make is that the non-Newtonian coefficient satisfies (3.13). Since, in practice,  $\Gamma$  is relatively small, this is a weak requirement.

The nonlinear stability critical Rayleigh number is calculated from (3.15) and we put  $R_E = 1$  to obtain the sharpest value. This leads to the following Euler–Lagrange equations for the determination of the nonlinear critical Rayleigh number  $R^2$ :

$$R(1+\lambda)\theta k_i + 2(fd_{ij})_{,j} = -\pi_{,i}, \quad u_{i,i} = 0, \quad R(1+\lambda)w + \frac{27\pi^4\Gamma}{4R^2(3-\Gamma)\sqrt{c_1}} z^2\theta + 2\lambda\Delta\theta = 0. \tag{3.17}$$

Since the linearised equations that we obtain from (2.4) are symmetric, the relevant equations for the linear instability boundary are

$$R\theta k_i + (fd_{ij})_{,j} = -\pi_{,i}, \quad u_{i,i} = 0, \quad \Delta\theta + R w = 0. \tag{3.18}$$

#### 4 Numerical results and conclusions

When the  $\Gamma$  term in (3.17)<sub>3</sub> is absent, (3.17) and (3.18) are identical if we set  $\lambda = 1$ . The flexibility of  $\lambda$  allows us to optimize the nonlinear stability threshold and for  $\Gamma$  small the critical Rayleigh numbers obtained from (3.17) and (3.18) are very close.

Table 1 shows a comparison of the linear instability critical Rayleigh number  $Ra_L$  against the nonlinear stability one,  $Ra_E$ , for  $\Gamma$  varying from 0 to 0.999, which is very close to the limit of the validity of the viscosity relation. It is seen that even for  $\Gamma = 0.999$ , there is only a 1.13% difference.

The  $Ra_E$  thresholds represent the unconditional nonlinear stability boundary provided that condition (3.13) is met. In terms of the original variables, condition (3.13) is

$$\nu_1 > 0.005821 \frac{\nu_0 d^4 \gamma (T_L - T_U)^2}{\kappa^2}. \quad (4.1)$$

This is an experimentally verifiable condition for a general non-Newtonian fluid. In Table 1 there is instability for  $Ra > Ra_L$  and unconditional nonlinear stability for  $Ra < Ra_E$ , provided that condition (4.1) is satisfied. This shows that the linearised theory is likely to be highly satisfactory in predicting instability. Subcritical instabilities can still occur if  $Ra$  lies in the very small band  $(Ra_E, Ra_L)$  or if condition (4.1) is not met. Experiments with a particular non-Newtonian fluid can be constructed to verify the theory.

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