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An elliptic–parabolic equation with a nonlocal term for the transient regime of a plasma in a Stellarator

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Abstract

We prove the existence and the regularity of weak solutions of a nonlocal elliptic–parabolic free-boundary problem involving the notions of relative rearrangement and monotone rearrangement. The problem arises in the study of the dynamics of a magnetically confined fusion plasma in a Stellarator device when the dimensional analysis on the characteristic times suggests to neglect the inertial acceleration in presence of a time dependent magnetic field.

Keywords: Elliptic–parabolic problem; Relative rearrangement; Free-boundary problem; Plasma physics

1. Introduction

We study the existence and regularity of solutions for the following elliptic–parabolic problem: given Ω, an open regular bounded set of $\mathbb{R}^2$, and a positive time

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$T>0$, we seek $u:[0,T]\times\Omega\to\mathbb{R}$ satisfying the *nonlocal problem*

$$(\mathscr{P})\begin{cases}\dfrac{\partial}{\partial t}\beta(u)-\Delta u=aG(u)+J(u) & \text{in }]0,T[\times\Omega,\\ u(t,x)=\gamma & \text{on }]0,T[\times\partial\Omega,\\ \beta(u(0,x))=\beta(u_0(x)) & x\in\Omega,\end{cases}$$

where G, J are defined as

$$G(u)(t,x)=a\left[F_v^2-\lambda\int_{|u(t)>0|}^{|u(t)>u_+(t,x)|}[u_+(t)]'_*(\sigma)(u_+(t))_*(\sigma)b_{*u}(t,\sigma)\,d\sigma\right]_+^{1/2}, \quad (1)$$

$$J(u)(t,x)=\lambda u_+(t,x)[b(x)-b_{*u(t)}(|u(t)>u(t,x)|)] \quad (2)$$

and $\beta(r):=\min(0,r)=-r_-$ for $r\in\mathbb{R}$. The coefficients a,b are given functions in $L^\infty(\Omega)$ such that $a\not\equiv 0$ and $b>0$ a.e. in Ω, while $\lambda>0$, $F_v>0$ and $\gamma<0$ are given constants. Here, u_* denotes the decreasing rearrangement of u, b_{*u} is the relative rearrangement of b with respect to u (see Section 2 below for the definitions) and $|E|$ denotes the Lebesgue measure of a set E.

Problem $(\mathscr{P})$ appears in the mathematical treatment of a bidimensional model describing the quasi-stationary processes that occur in the magnetic confinement of a fusion plasma in a Stellarator device. This model is derived from the 3D MHD system by means of an averaging method. The unknown u is called the *flux function* and its gradient represents the components of the averaged magnetic field confining the plasma. Some indications on the derivation of $(\mathscr{P})$ are given in Appendix A. Here, we just remark that $(\mathscr{P})$ can be viewed as a free-boundary problem, since the interface separating the *elliptic* and *parabolic* domains (i.e., $\{u>0\}$ and $\{u<0\}$ respectively) is a priori unknown. Physically, these two domains correspond with the *plasma region*, i.e. the region in the Stellarator device where the plasma is confined, and the *vacuum region* (i.e., where is present no plasma) respectively. In the first of these regions the plasma can be regarded as being, at each instant of time t, in magnetohydrodynamic equilibrium (this can be justified by dimensional analysis on the characteristic times), while time-dependent diffusion processes take place in the vacuum region (notice that here, the equation in $(\mathscr{P})$ reduces to the linear heat equation).

Two main difficulties appear in the study of existence of solutions of problem $(\mathscr{P})$. The first one comes from the nonlocal terms in $(\mathscr{P})$ since they do not depend on $\beta(u)$ but on u and moreover, they are only known to be continuous under strong regularity hypothesis for u. At the same time, the elliptic–parabolic character of $(\mathscr{P})$ poses big problems, since we have not information on the time derivative of u when $u\geqslant 0$. Hence, we cannot expect to obtain regular solutions for this problem. Let us point out the existence of many papers where elliptic–parabolic problems are treated, most of them appearing in the context of partially saturated flows in porous media (see, e.g., [4,24,25,37] and the references therein). In this sense, we recall that when deducing Darcy's law for flows in porous media by homogenization methods nonlocal terms appear, although they are usually neglected by assuming some hypothesis (see, for instance, Remark 1 in [13]). We shall also mention the important work of Alt and Luckhaus [1] where a general study for elliptic–parabolic systems is carried out. As we already mentioned, the main differences between the model we are interested in and the references above lie in the fact that β is not strictly increasing, and what is even more important, in the nonlocal character of the nonlinearities G and J. Also, concerning the nonlocal character of $(\mathscr{P})$, we mention that nonlocal parabolic problems arise very often in the literature, e.g., in the study of invasion phenomena in Biology. See, for instance, the works of Alvino et al. [2] or Pazy [32] and more recently, Chang and Chipot [10] or Antonsev et al. [3]. Finally, let us point out that the stationary problem associated to $(\mathscr{P})$, describing the stationary equilibria of a plasma confined in a Stellarator device, has been studied in previous papers; in particular the existence of solution for that problem was stated in [17] (see also [16] for the case of a current carrying Stellarator), while the study of the uniqueness of solution was carried out in [14].

The structure of the rest of the paper is as follows: In Section 2 we state our main results concerning the existence of global weak solutions for problem $(\mathscr{P})$; earlier in this section, we shall recall the notions of the relative and monotone rearrangements of a function, as well as some of their properties. Section 3 contains the proofs of the main results of the paper stated in the preceding section. Due to the nonlocal term involved in $(\mathscr{P})$ a special notion of weak solution for this problem will be introduced in Section 2. The relation between this notion of weak solution and the *standard* one (i.e., in the sense of distributions) will be analyzed in Section 4. Finally, the paper ends with a section devoted to the modelling.

2. Notations and statement of the main results

The goal of this section is the statement of the main results of this work, concerning the existence of solution for problem $(\mathscr{P})$. In order to achieve a better understanding of these results and of the difficulties underlying their proofs, we start recalling the notion of the rearrangement functions in $(\mathscr{P})$ as well as some of their properties.

Let Ω be a bounded and connected open measurable set of $\mathbb{R}^2$ (we assume a 2d-setting motivated by the physical modelling but the definitions and results that follows hold for any dimension $N>1$: see, e.g., [29,31]). Given $T>0$ and a measurable function $u:(0,T)\times\Omega\to\mathbb{R}$, the *distribution function of* u (with respect to x), $\mu:(0,T)\times\mathbb{R}\to[0,|\Omega|]$, is given by

$$\mu(t,\theta) = meas\{x\in\Omega\colon\ u(t,x)>\theta\}.$$

It is well-known that, for $t\in(0,T)$ fixed, the function $\mu(t,\cdot)$ is decreasing and right semicontinuous. For a fixed $t\in(0,T)$ and $\theta\in\mathbb{R}$, the Lebesgue measure of the sets $\{x\in\Omega\colon\ u(t,x)>\theta\}$ and $\{x\in\Omega\colon\ u(t,x)=\theta\}$ will be represented by $|u(t)>\theta|$

($= \mu(t,\theta)$) and $|u(t) = \theta|$ respectively. We shall say that $u(t) = u(t,.)$ has a *flat region* at the level θ if $|u(t) = \theta|$ is strictly positive. We recall that given a measurable function v defined in Ω there exists at most a countable family D of flat regions $P_v(\theta_i) := \{v = \theta_i\}$; we denote by $P(v) = \bigcup_{i\in D} P_v(\theta_i)$ the union of all the flat regions of v. We shall use the notation $\Omega_* := (0, |\Omega|)$.

The generalized inverse of μ (with respect to the second variable) is called the *decreasing rearrangement of u* with respect to x and it is defined as the function $u_* : (0,T) \times [0,|\Omega|] \to \bar{\mathbb{R}}$ such that

$$u_*(t,\sigma) = \inf\{\theta\in\mathbb{R}\colon |u(t) > \theta| \leqslant \sigma\} \ (= (u(t))_*(\sigma)).$$

We want to emphasize that for a fixed $t\in(0,T)$, the decreasing rearrangement of $u(t)$ has the same properties as the usual rearrangement of time-independent functions since $u_*(t,\theta) = (u(t))_*(\theta)$—when no confusion is feared, we shall set $u_*(t,\cdot) = u_*(t)$—(see, for instance, [29–31,40]). In particular, $u_*(t)$ is decreasing, $u_*(t)$ and $u(t)$ are equimeasurable (i.e., $|u_*(t) > \theta| = |u(t) > \theta|$), and the mapping $u\in L^p((0,T)\times\Omega) \mapsto u_* \in L^p((0,T)\times\Omega_*)$ is a contraction for $1 \leqslant p \leqslant +\infty$. Also, if $u(t)$ does not have flat regions, then $\mu(t,\cdot)$ and $u_*(t)$ are continuous and $u_*(t,\mu(t,v(x))) = v(x)$ for a.e. $(t,x)\in Q$.

Finally, if $u\in L^1(0,T; W^{1,p}(\Omega))$, $1\leqslant p \leqslant +\infty$, then $u_* \in L^1(0,T; W^{1,p}_{\mathrm{loc}}(\Omega_*))$ (see, for instance, [17,33,35]); in that case, for a fixed $t\in(0,T)$, we shall write $u'_*(t)$ for the derivative $\frac{du_*(t,\sigma)}{d\sigma}$ (remark that this is the notation employed in the definition of $G(u)$ in $(\mathscr{P})$).

Still, we need to recall another notion: the relative rearrangement of a function which, as we show in Appendix A (see (48)), is closely related to the concept of *averaging over a magnetic surface* largely used in the study of magnetically confined plasmas (see, e.g., [20,22]). In order to introduce this concept we consider, for a given $b\in L^p((0,T)\times\Omega)$, $1\leqslant p\leqslant +\infty$, the function w defined on $(0,T)\times(0,|\Omega|)$ by

$$w(t,\sigma) = \int_{\{x: u(t)(x) > u_*(t,\sigma)\}} b(t,x)\,dx + \int_0^{\sigma - |u(t) > u_*(t,\sigma)|} \left(b(t)|_{\{u(t)(x) = u_*(t,\sigma)\}}\right)_*(s)\,ds,$$

where $b(t)|_{\{u(t) = u_*(t,\sigma)\}}$ is the restriction of $b(t)$ to the set $\{x\colon u(t)(x) = u_*(t,\sigma)\}$ $(= P(u_*(t,\sigma)))$. Then, the *relative rearrangement of b with respect to u* is the function $b_{*u}\in L^p((0,T)\times\Omega_*)$ given by

$$b_{*u}(t,\sigma) := \frac{\partial w(t,\sigma)}{\partial\sigma}.$$

This function satisfies $\|b_{*u}\|_{L^p((0,T)\times\Omega_*)} \leqslant \|b\|_{L^p(Q)}$ (see, for instance, [17,30,31]).

The notion of relative rearrangement of a function was first introduced by Mossino and Temam (see [31]) for the study of some stationary differential equations arising in plasma physics, and it was later extended by Mossino and Rakotoson [30] for time-dependent functions. In [31], it is shown that the relative rearrangement b_{*u} can be regarded as the directional differential of the mapping $u \mapsto u_*$ in the direction of b. In particular, if $b \in L^p(\Omega)$ and $u \in L^1(\Omega)$ then

$$\frac{(u+\lambda b)_* - u_*}{\lambda} \rightharpoonup b_{*u} \text{ weakly in } L^p(\Omega_*) \text{ when } \lambda \to 0$$

(if $p = \infty$, the above convergence takes place in the weakly-$*$ topology). Like before, we note that here t plays the role of a parameter. In particular, for $t \in (0, T)$ fixed, the relative rearrangement of u can be seen as

$$b_{*u}(t, \sigma) = (b(t))_{*u(t)}(\sigma) \quad \text{for } \sigma \in \Omega_*,$$

and shares the same properties as the relative rearrangement for time-independent functions (see, for example, [30]). We point up that, in contrast with the usual monotone rearrangement, the relative rearrangement mapping (for a fixed $b \in L^p(Q)$) does not always present good continuity properties with respect to u (i.e., regarded as the functional that maps $v \in L^p(Q) \to b_{*v} \in L^p((0, T) \times \Omega_*)$) (see, e.g., [31]). Further details on the relative rearrangement can be found, for instance, in [17,29,31,33,35].

Having discussed the nonlocal terms in problem $(\mathscr{P})$, let us proceed with the statement of the existence results for this problem. Due to the complexity of the nonlinear terms G and J appearing in the formulation of $(\mathscr{P})$, a natural approach to this problem seems to use a Galerkin method. Here, the main difficulty that we encounter is to guarantee the continuity of these nonlinearities. In particular, a very delicate point is that the continuity of these terms requires strong convergence of the Galerkin sequences in Sobolev spaces. Indeed, the less restrictive result for the continuity of the derivative of the decreasing rearrangement $(u(t))'_*$ with respect to $u(t)$ for a fixed t (i.e., regarded as the functional $v \mapsto v'_*$) states that for any sequence $(v_n)_{n \in \mathbb{N}}$ converging to a function v in $W^{1,p}(\Omega)$-strong for $p > 2$, then

$$(v_n)'_* \to v'_* \text{ strongly in } L^q(\Omega) \quad \text{for any } 1 \leqslant q < q_c := \frac{1}{1 - \frac{1}{2} + \frac{1}{p}}, \tag{3}$$

provided that $v \in W^{2,r}(\Omega)$ for some $r > 1$ (see for instance [17,18,33]). Moreover, in the case of the relative rearrangement it turns out that the only known results show that if $v_n \to v$ in $W^{1,r}(\Omega)$ for some $1 < r \leqslant +\infty$, then

$$b_{*v_n} \to b_{*v} \text{ strongly in } L^p(\Omega_*),$$

$$b_{*v_n}(|v_n > v_n(\cdot)|) \to b_{*v}(|v > v(\cdot)|) \text{ strongly in } L^p(\Omega) \tag{4}$$

when $n \to +\infty$, provided that

$$meas\{x \in \Omega: |\nabla v(x)| = 0\} = meas\{x \in \Omega: |\nabla v_n(x)| = 0\} = 0. \tag{5}$$

(for the proof of these results see, for instance, [17,18,33]). Therefore, the continuity of the nonlinear term J in $(\mathscr{P})$ requires a very restrictive (and noneasy to check) condition given by (5); furthermore, this condition is not always satisfied.

This fact will be reflected in the introduction of a special notion of weak solution for problem $(\mathscr{P})$.

Going back to the question of the convergence of the Galerkin sequence in the strong $W^{1,p}$-topology, we observe that this is usually obtained by combining a priori estimates for u, its time derivative $\frac{\partial u}{\partial t}$ and a compactness result of Lions–Aubin's type (see, e.g., [27]). However, since $\frac{\partial \beta(u)}{\partial t}$ vanishes whenever $u \geqslant 0$, we have not any information on the time derivative of u for $u \geqslant 0$, and hence it is not possible to obtain such estimates for problem $(\mathscr{P})$. Let us point up that this difficulty did not appear in previous works treating elliptic–parabolic equations since, either the continuity of their nonlinear terms did not require such a strong convergence, or they did not depend on the unknown u but on $\beta(u)$, for which it was possible to derive a priori estimate as well as for its time derivative $\frac{\partial \beta(u)}{\partial t}$ (see, for instance, [1,37] and the references therein). Also, we mention that any other time-discretization scheme would be a useful tool to approach the existence of solution for problem $(\mathscr{P})$—see, for instance, the one used by Alt and Luckhaus [1]—, but in either case the difficulties that would appear would be of similar nature.

In order to avoid the lack of information on $\frac{\partial u}{\partial t}$, rather than looking for solutions of $(\mathscr{P})$, we shall consider a family of uniformly parabolic problems $(\mathscr{P}_\alpha)$ approximating $(\mathscr{P})$, obtained by replacing β with β_α, where

$$\beta_\alpha(r) := -r_- + \alpha r_+, \quad \text{for } 0 < \alpha \leqslant 1 \ \left(\alpha = \tfrac{1}{n},\ n \in \mathbb{N}\right).$$

That is $\beta_\alpha(r) = \beta(r) + \alpha r_+$ and thus, $(\mathscr{P}_\alpha)$ approximates $(\mathscr{P})$ as $\alpha \to 0$. In this way, the study of the existence of a global weak solution of $(\mathscr{P}_\alpha)$ for $\alpha > 0$ is the object of our first main result. This is stated in the following terms:

Theorem 2.1. *Let $u_0 \in H^1(\Omega)$ and $\beta(\sigma) = \min(0,\sigma) + \alpha\sigma_+$ with $0 < \alpha \leqslant 1$ and $\beta(u_0) \in L^\infty(\Omega)$. Then there exists at least a couple $(u_\alpha, \hat{b_\alpha})$, $u_\alpha \in L^2(0,T;H^2(\Omega))$ and $\hat{b_\alpha} \in L^\infty(Q)$ satisfying*:

(i) $(u_\alpha - \gamma) \in L^2(0,T;H_0^1(\Omega))$, $\frac{\partial u_\alpha}{\partial t} \in L^2(Q)$,

(ii) $\frac{\partial}{\partial t}\beta_\alpha(u_\alpha) - \Delta u_\alpha = aG(u_\alpha) + \lambda(u_\alpha)_+ |b - \hat{b_\alpha}|$,
$\beta_\alpha(u_\alpha)|_{t=0} = \beta_\varepsilon(u_0)$ *(or equivalently* $u_\alpha|_{t=0} = u_0$),

(iii) $\forall \phi : \mathbb{R} \to \mathbb{R}$ *continuous*, $\forall \theta \in \mathbb{R}$

$$\int_{\{x: u_\alpha(t)(x) > \theta\}} \hat{b_\alpha}\phi(u_\alpha - \gamma)\,dx = \int_{\{x: u_\alpha(t)(x) > \theta\}} b\phi(u_\alpha - \gamma)\,dx$$

for a.e. $t \in]0,T[$ *and* $\operatorname{essinf}_\Omega b \leqslant \hat{b_\alpha} \leqslant \operatorname{esssup}_\Omega b$.

Moreover, if $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ then $u_\alpha \in L^\infty(Q)$.

We observe that the function u_α above does not correspond to the standard notion of weak solution for evolution problems (nor even *mild solution* in the sense of Benilan (see, e.g., [5])), since the term $b_{*u_\alpha}(|u_\alpha(t) > u_\alpha(t,\cdot)|)$ does not appear in the equation satisfied by u_α (see (ii) above), but it is replaced by a $L^\infty(Q)$ function $\hat{b}_\alpha$. The connection between $\hat{b}_\alpha$ and the relative rearrangement b_{*u_α} will be cleared up in Section 4. As we have already mentioned, the presence of this function instead of b_{*u_α} is motivated by the strong hypothesis (see (5)) required for the continuity of J. In fact, since (5) is not always satisfied, throughout this paper we shall only look for solutions of this type. More precisely, we shall use the following definition of weak solution:

Definition 2.1. Let $\tilde{\beta} \in \mathscr{C}(\mathbb{R})$ a nondecreasing function and denote by $(\tilde{\mathscr{P}})$ the problem obtained when β is replaced by $\tilde{\beta}$ in $(\mathscr{P})$. Then we will say that $\tilde{u}$ is a *weak solution* of $(\tilde{\mathscr{P}})$ if $(\tilde{u} - \gamma) \in L^2(0,T; H_0^1(\Omega))$, $\frac{\partial}{\partial t}\beta(\tilde{u}) \in L^2(0,T; H^{-1}(\Omega))$ and:

(1) (*Relative rearrangement condition*) there exists a bounded function $b^{\tilde{u}} \in L^\infty(Q)$, satisfying for *a.e.* $t \in (0,T)$, for all $\theta \in \mathbb{R}$ and for all $\varphi \in C(\mathbb{R})$ with $\varphi(v(t)) \in L^1(\Omega)$ that

$$\int_{\{x:\tilde{u}(t,x)>\theta\}} b^{\tilde{u}} \varphi(\tilde{u}(t,x) - \gamma)\, dx = \int_{\{x:\tilde{u}(t,x)>\theta\}} b\varphi(\tilde{u}(t,x) - \gamma)\, dx$$

and

$$ess \inf_{\Omega} b \leqslant b^{\tilde{u}} \leqslant ess \sup_{\Omega} b,$$

(2) $\frac{\partial}{\partial t}\beta(\tilde{u}) - \Delta\tilde{u} = aG(\tilde{u}) + \lambda(\tilde{u}) + [b - b^{\tilde{u}}]$ in $\mathscr{D}'(\Omega)$ for a.e. t, and $\beta(\tilde{u})|_{t=0} = \beta(u_0)$.

Remark 2.1. Notice that $\beta(\tilde{u}) = -\tilde{u}_- = T^*_{-\gamma}(v) - \gamma$ where $T^*_{-\gamma}$ is the truncation at level $(-\gamma)$ function

$$T^*_{-\gamma}(r) = \begin{cases} r & \text{if } r < -\gamma, \\ -\gamma & \text{if } r \geqslant -\gamma. \end{cases}$$

On the other hand, since $T^*_{-\gamma}(\cdot)$ is a Lipschitz function, we have that $T^*_{-\gamma}(\tilde{u} - \gamma) \in L^2(0,T; H_0^1(\Omega))$. Then, using that condition 1 implies that $\frac{\partial}{\partial t}(T^*_{-\gamma}(\tilde{u} - \gamma)) \in L^2(0,T; H^{-1}(\Omega))$, by well-known interpolation results [39] we conclude that $T^*_{-\gamma}(\tilde{u} - \gamma)$ and $\beta(\tilde{u}) \in C([0,T]; L^2(\Omega))$ and so the restriction $\beta(\tilde{u})|_{t=0}$ is well defined.

A detailed analysis of the relation existing between the notions of weak solution introduced above and the standard one is carry out in Section 4. In particular, we show that when $\tilde{u}(t)$ does not possess flat regions for a.e. $t \in (0,T)$, $b^{\tilde{u}}$ coincides in a weak sense with the relative rearrangement $b_{*\tilde{u}}$.

In our second main result we finally address the question of existence of solution for problem $(\mathscr{P})$. As we said, this will be proved by taking the limit $\alpha\to 0$ in the sequence of solutions $(u_\alpha)_{\alpha>0}$ given in Theorem 2.1. This result can be stated as follows:

Theorem 2.2. *Let* $(u_\alpha)_{\alpha>0}$ *be the sequence given in Theorem* 2.1. *For any* $\phi\in L^2(\Omega)$ *and* $h>0$, *define*

$$\phi_{h\alpha}(t)=\frac{1}{h}\int_t^{t+h}\int_\Omega u_\alpha(\sigma,x)\phi(x)\,dx\,d\sigma \tag{6}$$

and assume that for a.e. $t\in(0,T)$ *and* $\forall\phi\in L^2(\Omega)$:

$$\lim_{h\to 0}\lim_{\alpha\to 0}\phi_{h\alpha}(t)=\lim_{\alpha\to 0}\lim_{h\to 0}\phi_{h\alpha}(t). \tag{7}$$

Then there exists (u,b^u) *being a weak solution to problem* $(\mathscr{P})$. *Furthermore,*

$$u\in L^2(0,T;\mathscr{C}^1(\Omega)) \quad \textit{and} \quad \beta(u)\in\mathscr{C}([0,T];L^2(\Omega)).$$

To end this section, let us make some remarks on the theorem above. First, we observe that assumption (7) provides a sufficient condition for passing to the limit $\alpha\to 0$ in the sequence $(u_\alpha)_{\alpha>0}$. Indeed, (7) together with the compactness result obtaining by Rakotoson and Temam [36] yields to the convergence $u_\alpha\to u$ in the strong topology of $L^2((0,T)\times L^2(\Omega))$. However, we note that this convergence is not enough for taking the limit in the nonlinear terms G and J (see (3) and (5)) and that some further work will be needed. We also point up that assumption (7) replaces the *strong concentration condition*—usually given by estimates of $(\frac{\partial u_\alpha}{\partial t})_{\alpha>0}$ independent of α—required in the compactness result of Aubin–Lions type (see, e.g., [39]); furthermore, hypothesis (7) seems to be sharp in this framework for the solvability of $(\mathscr{P})$ (see [38]).

3. Proof of the main results

As we have said in the preceding section, the proofs of Theorems 2.1 and 2.2 are based in the use of a Galerkin method as well as in passing to the limit $\alpha\to 0$ for the sequence of solutions $(u_\alpha)_{\alpha>0}$. We have devoted the first part of this section to the obtention of some estimates, since the use of the above mentioned tools deeply relies on the obtention of suitable a priori estimates. We point out that these results are stated for weak solutions in the sense of Definition 2.1 of a generic problem $(\tilde{\mathscr{P}})$, obtained when β is replaced in $(\mathscr{P})$ by some $\tilde{\beta}$. Also in this first subsection, we have included a result on the continuity of the nonlinearity G, which will be used all along the section.

Finally, in the second and third parts of this section we carry out the proofs of Theorems 2.1 and 2.2, respectively.

3.1. Some a priori estimates for weak solutions

First of all, let us remark that the nonlinear term G is uniformly bounded since $0 \leqslant G(r) \leqslant F_v$, $\forall r \in \mathbb{R}$ (see definition (1)). Moreover, the following result concerning the continuity of this term holds:

Theorem 3.1. *Let v^ε be a sequence in $W^{1,2+\delta}(\Omega)$, for some $\delta > 0$ and $v \in H^2(\Omega)$ such that v^ε converges to v in $W^{1,2+\delta}(\Omega)$ as $\varepsilon \to 0$. Then, $\lim_{\varepsilon \searrow 0} G(v^\varepsilon + \gamma)(x) = G(v + \gamma)(x)$ for a.e. $x \in \Omega$.*

Proof. Let θ be the characteristic function of the set $P(v_*)$. From the above convergence we get $\frac{dv_*^\varepsilon}{d\sigma} \to \frac{dv_*}{d\sigma}$ in $L^q(\Omega_*)$ as $\varepsilon \searrow 0$ with q given in (3). Thus $\lim_{\varepsilon \searrow 0} \theta \frac{dv_*^\varepsilon}{d\sigma} = 0$ in $L^q(\Omega_*)$. Arguing as in [34], we conclude that

$$(1-\theta) b_{*v^\varepsilon} \rightharpoonup (1-\theta) b_{*v} \text{ weakly-star in } L^\infty(\Omega_*). \tag{8}$$

Now, if we set $I(v^\varepsilon(x))$ the interval given by $[|v^\varepsilon > v_+^\varepsilon(x)|,\ |v^\varepsilon > 0|]$, then for all $\sigma \in \bar{\Omega}_*$, $\sigma \neq |v > v_+(x)|$, $\sigma \neq |v > 0|$, one has

$$\lim_{\varepsilon \searrow 0} (1-\theta)\chi_{I(v^\varepsilon(x))}(\sigma) = (1-\theta)\chi_{I(v(x))}(\sigma). \tag{9}$$

From relation (8) and (9) and the fact $\lim_\varepsilon \left|\theta \frac{dv_*^\varepsilon}{ds}\right|_q = 0$, one finds (for a.e. $x \in \Omega$, using the fact that $v_{+*}^\varepsilon(\sigma) \to v_{+*}$ in $L^r(\Omega_*)$, $r < \infty$) that

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{|v^\varepsilon > 0|}^{|v^\varepsilon > v_+^\varepsilon(x)|} \frac{dv_*^\varepsilon}{d\sigma}(\sigma) b_{*v^\varepsilon}(\sigma) p'(v_*^\varepsilon(\sigma))\, d\sigma \\
&\quad = \int_{|v>0|}^{|v > v_+(x)|} \frac{dv_*}{d\sigma}(\sigma)(1-\theta)^2 b_{*v}(\sigma) p'(v_*(\sigma))\, d\sigma \\
&\quad = \int_{|v>0|}^{|v > v_+(x)|} \frac{d}{d\sigma}(p(v_*(\sigma))) b_{*v}(\sigma)\, d\sigma.
\end{aligned}$$

The above limit implies the result. □

Remark 3.1. Let us emphasize that, although the weak convergence of the sequence $(b_{*v^\varepsilon})_{\varepsilon>0}$ in $G(v^\varepsilon)$ is only known for $(v^\varepsilon)_{\varepsilon>0}$ not having flat regions (see, e.g., [17,31]), theorem above establishes that the continuity of G holds without assuming such condition.

We start the study of the a priori estimates by showing that any weak solution u_α to problem $(\mathscr{P}_\alpha)$ is uniformly bounded (with respect to α) in L^∞-norm. First, we prove that $\beta_\alpha(u_\alpha)(t)$ is bounded in $L^\infty(\Omega)$ for every $t \in [0, T]$. This result is stated in a more general framework.

Theorem 3.2. *Let $\tilde{\beta}\in\mathscr{C}(\mathbb{R})$ be a nondecreasing Lipschitz continuous function. Then, any weak solution of the problem $(\tilde{\mathscr{P}})$ associated to this $\tilde{\beta}$ satisfies*

$$|\tilde{\beta}(\tilde{u}(t))-\tilde{\beta}(\gamma)|_{L^\infty(\Omega)}\leqslant|a|_\infty F_v t+|\tilde{\beta}(u_0)-\tilde{\beta}(\gamma)|_{L^\infty(\Omega)}\quad \forall t\in[0,T].$$

Proof. Let T_k be the truncation operator at level $k\in\mathbb{N}$ given by

$$T_k(\sigma)=\begin{cases}\sigma & \text{if } |\sigma|\leqslant k,\\ k\operatorname{sign}\sigma & \text{if } |\sigma|>k,\end{cases}$$

and let $g_m(\sigma)=|\sigma|^{m-2}\sigma$, for any integer $m\geqslant 2$. Then, if $\tilde{u}$ is a weak solution of $(\tilde{\mathscr{P}})$ then

$$w_{m,k}:=g_m\circ T_k(\tilde{\beta}(\tilde{u})-\tilde{\beta}(\gamma))\in L^\infty(Q)\cap L^2(0,T;H_0^1(\Omega)).$$

Let us multiply the equation satisfied by $v(t):=\tilde{u}(t)-\gamma$ by $w_{m,k}(t)$. Then, by the relative rearrangement condition, one has:

$$\left\langle\frac{\partial}{\partial t}\tilde{\beta}(v+\gamma),w_{m,k}(t)\right\rangle+\int_\Omega\nabla v(t,x)\cdot\nabla w_{m,k}(t,x)\,dx=\int_\Omega aG(v+\gamma)w_{m,k}\,dx, \tag{10}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Define

$$y_{m,k}(t)=\int_\Omega dx\int_0^{\tilde{\beta}(v(t)+\gamma)+\gamma}g_m\circ T_k(\sigma)\,d\sigma.$$

Then, we can use the integration by parts formula (see [1]) obtaining

$$\frac{d}{dt}y_{m,k}(t)=\left\langle\frac{\partial}{\partial t}\tilde{\beta}(v(t)+\gamma),w_{m,k}(t)\right\rangle. \tag{11}$$

Also, $\int_\Omega\nabla v(t,x)\cdot\nabla w_{m,k}(t,x)\,dx\geqslant 0$ since g_m and the truncation operator T_k are nondecreasing. Thus, from the estimate $0\leqslant G(v+\gamma)\leqslant F_v$, (10) and (11), we get via the Hölder inequality, that

$$\frac{d}{dt}y_{mk}(t)\leqslant|a|_\infty F_v\int_\Omega|w_{m,k}(t,x)|\,dx\leqslant|a|_\infty F_v|\Omega|^{\frac{1}{m}}\left(\int_\Omega|w_{m,k}(t,x)|^{\frac{m}{m-1}}\right)^{1-\frac{1}{m}}. \tag{12}$$

A simple calculation allows us to write the above integral as

$$\int_\Omega|w_{m,k}(t,x)|^{\frac{m}{m-1}}\,dx=my_{m,k}(t)-mk^{m-1}\int_\Omega(|\tilde{\beta}(v+\gamma)-\tilde{\beta}(\gamma)|-k)_+\,dx, \tag{13}$$

which, with (12), yields

$$y'_{m,k}(t)\leqslant m^{1-\frac{1}{m}}|\Omega|^{\frac{1}{m}}|a|_\infty F_v y_{m,k}(t)^{1-\frac{1}{m}}. \tag{14}$$

Therefore, integrating

$$y_{m,k}^{\frac{1}{m}}(t) \leqslant m^{-\frac{1}{m}} |\Omega|^{\frac{1}{m}} |a|_{\infty} F_v t + y_{m,k}^{\frac{1}{m}}(0).$$

Finally, using (13) and letting $m \to +\infty$ and $k \to +\infty$ we get the desired result. □

Let us see next that, when $\tilde{\beta} = \beta_\alpha$ for $0 \leqslant \alpha \leqslant 1$, it is possible to obtain an uniformly in time estimate for $(u_\alpha)_+(t)$ in the L^∞-norm, which implies that the whole solution is bounded in the L^∞ topology.

Theorem 3.3. *Assume that $\tilde{\beta}(\sigma) = -\sigma_-$. Then, for any weak solution $\tilde{u}$, one has*

$$|\tilde{u}_+(t)|_\infty \leqslant \frac{1}{4\pi} |a|_\infty F_v |\Omega|, \quad \textit{for all } t \in [0, T].$$

If $\tilde{\beta}(\sigma) = -\sigma_- + \alpha\sigma_+$, $0 < \alpha \leqslant 1$, the above result holds, provided that $\frac{\partial \tilde{u}}{\partial t} \in L^1(Q)$. In particular $\tilde{u} \in L^\infty(Q)$.

Proof. We shall first consider the case $\alpha = 0$. Since $\frac{\partial \beta(\tilde{u})}{\partial t} = -\frac{\partial \tilde{u}_-}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$, one gets that for all $\theta > 0$ and for a.e. $t \in]0, T[$

$$\left\langle \frac{\partial \tilde{u}_-}{\partial t}(t), (\tilde{u}_+(t) - \theta)_+ \right\rangle = 0. \tag{15}$$

Then, multiplying the equation satisfied by $\tilde{u}$ by $(\tilde{u}(t) - \theta)_+$ and using the rearrangement condition we obtain

$$\int_{\{\tilde{u}_+(t) > \theta\}} |\nabla \tilde{u}_+(t)|^2 \, dx = \int_\Omega aG(\tilde{u})(\tilde{u}_+(t) - \theta)_+ \, dx. \tag{16}$$

Differentiating this last relation with respect to θ and using the fact that $0 \leqslant G(\tilde{u}) \leqslant F_v$, we get, after applying the Hölder inequality

$$-\frac{d}{d\theta} \int_{\{\tilde{u}_+(t) > \theta\}} |\nabla \tilde{u}_+(t)|^2 \, dx \leqslant |a|_\infty F_v |\tilde{u}_+(t) > \theta|.$$

Arguing as in [40] (see also the exposition made in [29]) and using the *De Giorgi isoperimetric inequality* we derive that, for a.e. $t \in [0, T]$,

$$|\tilde{u}_+(t)|_\infty \leqslant \frac{1}{4\pi} |a|_\infty F_v |\Omega|. \tag{17}$$

In particular, combining Theorem 3.2 and this result, we get $\tilde{u} \in L^\infty(Q)$. Now, let $0 < \alpha \leqslant 1$. We argue as before (see also, [30]), multiplying by $(\tilde{u}(t) - \gamma)_+$ and differentiating with respect to the parameter θ, we obtain for a.e. $\theta > 0$

$$\alpha \int_{\{\tilde{u}_+(t)>\theta\}} \frac{\partial}{\partial t}\tilde{u}_+(t)\,dx - \frac{d}{d\theta}\int_{\{\tilde{u}_+(t)>\theta\}} |\nabla \tilde{u}_+(t)|^2\,dx = \int_{\{\tilde{u}_+(t)>\theta\}} aG(\tilde{u})\,dx. \quad (18)$$

We use again the bounds for G and Talenti's method mentioned before (see also [15], for an analogous argument) to arrive to

$$-4\pi s \frac{\partial}{\partial s}\tilde{u}_{+*}(t,s) \leqslant |a| F_v s - \alpha \int_0^s \frac{\partial}{\partial t}\tilde{u}_{+*}(t,\sigma)\,d\sigma \quad (19)$$

for all $s \in (0, |\Omega|)$. Let $K(t,s) = \int_0^s \tilde{u}_{+*}(t,\sigma)\,d\sigma$; (19) leads to the following partial differential inequality:

$$\begin{cases} \alpha \dfrac{\partial}{\partial t}K(t,s) - 4\pi s \dfrac{\partial^2}{\partial s^2}K(t,s) & \leqslant |a|_\infty F_v s \\ K(t,0) = 0, & \dfrac{\partial K}{\partial s}(t,|\Omega|) = 0. \end{cases}$$

We consider the function $\hat{K}(s)$ satisfying the ordinary differential equation

$$|a|_\infty F_v s = -4\pi s \frac{d^2\hat{K}}{ds^2}, \quad \hat{K}(0) = 0, \quad \frac{d\hat{K}}{ds}(|\Omega|) = 0;$$

that is, $\hat{K}(s) = -\frac{|a|_\infty F_v}{4\pi}s^2 + \frac{|a|_\infty F_v}{4\pi}|\Omega|$. Then, we can apply a comparison principle between the problems satisfied by K and $\hat{K}$ (see [15]) to get that $\hat{K}(t,s) \leqslant \hat{K}(s)$ for all $s \in [0, |\Omega|]$. In particular we deduce that

$$|\tilde{u}_+(t)|_{L^\infty(\Omega)} \leqslant \left|\frac{d\hat{K}}{ds}(0)\right| = \frac{|a|_\infty F_v |\Omega|}{4\pi}. \qquad \square$$

We finish this section giving an energy estimate for the weak solutions of $(\tilde{\mathscr{P}})$.

Theorem 3.4. *Any weak solution $\tilde{u}$ of $(\tilde{\mathscr{P}})$ satisfies the following estimate*

$$\int_0^t \int_\Omega |\nabla \tilde{u}(\sigma,x)|^2\,dx\,d\sigma + \int_\Omega dx \int_0^{u_0(x)} \tilde{\beta}(\sigma+\gamma)\,d\sigma$$
$$\leqslant \int_\Omega u_0 \tilde{\beta}(u_0+\gamma) + |a|_\infty F_v \int_0^t d\sigma \int_\Omega |\tilde{u}-\gamma|(\sigma,x)\,dx$$

for all $t \in [0,T]$.

Proof. *Let* $v := \tilde{u} - \gamma$. Multiplying the equation by $v(t)$, using the relative rearrangement condition and the *integration by parts formula* (see the proof of Theorem 3.2) one has

$$\frac{d}{dt}\int_\Omega \psi^*(\tilde{\beta}(v(t)+\gamma))\,dx + \int_\Omega |\nabla v(t,x)|^2\,dx = \int_\Omega aG(v(t)+\gamma)v(t)\,dx, \qquad (20)$$

where

$$\int_\Omega \psi^*(\tilde{\beta}(v(t)+\gamma))\,dx = \int_\Omega v(t)\tilde{\beta}(v(t)+\gamma)\,dx - \int_\Omega dx \int_0^{v(t,x)} \tilde{\beta}(\sigma+\gamma)\,d\sigma$$

(notice that ψ^* is just the *Legendre transform* of the term $\int_0^s \tilde{\beta}(\sigma)\,d\sigma$). Integrating (20) with respect to t, dropping some nonnegative term and using the relation $0 \leqslant G(v+\gamma) \leqslant F_v$ we derive the stated result. □

Finally, combining the Sobolev Poincaré inequality and the Schwartz inequality, we easily derive.

Corollary 3.1. *For all $t \in [0,T]$*

$$\int_0^t \int_\Omega |\nabla \tilde{u}(\sigma,x)|^2 + 2\int_\Omega \int_0^{u_0(x)} \tilde{\beta}(\sigma+\gamma)\,d\sigma\,dx \leqslant 2\int_\Omega u_0\tilde{\beta}(u_0)\,dx + \frac{|a|_\infty^2 F_v^2 |\Omega|}{\lambda_1} t,$$

where λ_1 is the first eigenvalue of the homogeneous Dirichlet problem associated to the Laplace operator.

3.2. Existence of solution to problem $(\mathscr{P}_\alpha)$ for $\alpha>0$: Proof of Theorem 2.1

For the proof of this Theorem we shall proceed in the following way: first, for a fixed $\alpha>0$, we shall consider a more regular family of problems $(\mathscr{P}_\varepsilon)$ approximating $(\mathscr{P}_\alpha)$, which we shall solve by means of a Galerkin method; later, we shall look for estimates of the sequence $(u_\varepsilon)_{\varepsilon>0}$ uniform in ε. The solutions to $(\mathscr{P}_\alpha)$ will be found as the limit of this sequence when $\varepsilon \to 0$.

Thus, let $\alpha>0$ be fixed. For $0<\varepsilon<1$ (countable), we consider the family of functions $\beta_\varepsilon \in C^\infty(\mathbb{R})$ satisfying

(i) $\alpha \leqslant \beta'_\varepsilon \leqslant 1+\alpha$, $\beta_\varepsilon(0)=0$, $\beta_\varepsilon \to \beta \in H^1_{loc}(\mathbb{R})$.
(ii) $|\beta_\varepsilon(\sigma)-\beta(\sigma)| \leqslant 2\varepsilon$, for any $\sigma \in \mathbb{R}$.

We have

Theorem 3.5. *Assume $u_0 \in H^1(\Omega)\cap L^\infty(\Omega)$. Then there exist $(w^\varepsilon, \hat{b^\varepsilon}) \in L^\infty(Q)^2$ satisfying the following problem $(\mathscr{P}_\varepsilon)$:*

(i) $w^\varepsilon \in L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega))$, $\frac{\partial w^\varepsilon}{\partial t} \in L^2(Q)$,
(ii) $\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon+\gamma) - \Delta w^\varepsilon = aG(w^\varepsilon+\gamma) + \lambda(w^\varepsilon+\gamma)_+[b-\hat{b^\varepsilon}]$,
$\beta_\varepsilon(w^\varepsilon)|_{t=0} = \beta_\varepsilon(u_0-\gamma)$ *(or equivalently* $w^\varepsilon|_{t=0} = u_0-\gamma$*)*,

(iii) $\forall \phi : \mathbb{R} \to \mathbb{R}$ *continuous*, $\forall \theta \in \mathbb{R}$

$$\int_{\{x:w^{\varepsilon}(t)(x)>0\}} \hat{b^{\varepsilon}} \phi(w^{\varepsilon})\, dx = \int_{\{x:w^{\varepsilon}(t)(x)>0\}} b\phi(w^{\varepsilon})\, dx$$

for a.e. $t \in]0, T[$ *and* $\operatorname{essinf}_{\Omega} b \leqslant \hat{b^{\varepsilon}} \leqslant \operatorname{esssup}_{\Omega} b$.

Remark 3.2. Notice that the above result states that $u_{\varepsilon} := w^{\varepsilon} + \gamma$ is a weak solution in the sense of Definition 2.1 of problem $(\tilde{\mathscr{P}})$ when $\tilde{\beta} = \beta_{\varepsilon}$ and that we are to prove that the sequence u_{ε} converges in some sense to u_{α}, where u_{α} is the solution to problem $(\mathscr{P}_{\alpha})$ given in Theorem 2.1.

Remark 3.3. For the sake of simplicity in the exposition, throughout this section we shall always use the notations $w^{\varepsilon} := u_{\varepsilon} - \gamma$ and $w_{\alpha} := u_{\alpha} - \gamma$ since these functions belongs to $L^1(0, T; H_0^1(\Omega))$.

The proof will be divided into several steps.

3.2.1. The Galerkin method I: Existence of solution for a family of finite dimensional problems $(\mathscr{P}_{\varepsilon,m})$

Let $(\lambda_k, \varphi_k)_{k \geqslant 1}$ be the eigenvalues and eigenfunctions associated to the Laplacian operator $-\Delta$ on Ω with zero Dirichlet boundary conditions, i.e.

$$-\Delta\varphi_k = \lambda_k \varphi_k, \quad \varphi_k \in H_0^1(\Omega).$$

We denote by V_m the vector space spanned by $\{\varphi_1, \ldots, \varphi_m\}$. For all $v \in V_m$, $v = \sum_{i=1}^{m} v^i \varphi_i$. We consider the following approximate problem: find

$$w_m \in L^1(0, T; V_m), \quad w_m(t) = \sum_{i=1}^{m} w_m^i(t) \varphi_i,$$

satisfying

$$\int_{\Omega} \left(\frac{\partial}{\partial t} \beta_{\varepsilon}(w_m(t) + \gamma) \right) \varphi_k\, dx + \int_{\Omega} \nabla w_m(t) \cdot \nabla \varphi_k\, dx$$
$$= \int_{\Omega} a(x) G(w_m + \gamma) \varphi_k\, dx + \int_{\Omega} J(w_m + \gamma) \varphi_k\, dx \qquad (\mathscr{P}_{\varepsilon,m})$$

for $k = 1, \ldots, m$ and the initial condition $w_m(0) = P_m(u_0 - \gamma)$, where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m.

Theorem 3.6. *There exists* w_m *solution of problem* $(\mathscr{P}_{\varepsilon,m})$. *Furthermore, if* $a \not\equiv 0$ *then there exists* k_0 *such that* $w_m \not\equiv 0$ *for* $m \geqslant k_0$.

Proof. The above problem can be written as a nonlinear ordinary differential system for the functions $w_m^1(t), \dots, w_m^m(t)$, $w_m(t) = \sum_{i=1}^m w_m^i(t)\varphi_i$. Indeed, $w_m^i(t)$ with $i = 1, \dots, m$, verify

$$\sum_{i=1}^m a_{ik}(w_m(t))w_m^{i\prime}(t) + \sum_{i=1}^m b_{ik} w_m^i(t) = \hat{\mathscr{I}}_k(w_m(t)), \quad k = 1, \dots, m,$$

$$w_m^i(0) = \text{the } i\text{th component of } P_m(u_0 - \gamma), \tag{21}$$

where for $i,\ k = 1, \dots, m$ we have set

$$\begin{aligned} a_{ik}(w_m(t)) &:= \int_\Omega \beta_\varepsilon'(w_m + \gamma)\varphi_i\varphi_k\, dx, \\ b_{ik} &:= \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_k\, dx, \\ \hat{\mathscr{I}}_k(w_m(t)) &:= \int_\Omega a(x)G(w_m(t) + \gamma)\varphi_k\, dx + \int_\Omega J(w_m(t) + \gamma)\varphi_k\, dx. \end{aligned}$$

To prove the existence of a solution of the above initial value problem we need the following result:

Lemma 3.1. *The function $\hat{\mathscr{I}}_k : V_m \to \mathbb{R}$ is continuous, $k = 1, \dots, m$.*

Proof. Let $v \in V_m$. Then

$$\hat{\mathscr{I}}_k(v) = \int_\Omega a(x)G(v + \gamma)\varphi_k\, dx + \int_\Omega J(v + \gamma)\varphi_k\, dx,$$

with G and J given by (1) and (2). Indeed, we observe that if $v \in V_m\backslash\{0\}$ then v has not flat regions since it is an analytical function. Therefore, the map $v \in V_m\backslash\{0\} \mapsto b_{*v}(|v > v(\cdot)|) \in L^p(\Omega)$ is strongly continuous for any finite p (see, for instance, [17,18,33]). Moreover, as $J(\gamma) = 0$ (we recall that $\gamma < 0$) and $|b_{*v}|_{L^\infty} \leqslant |b|_\infty$, we deduce that the map $v \in V_m \mapsto J(v + \gamma) \in L^p(\Omega)$ is strongly continuous.

Next, we proceed to show that the map $v \in V_m \mapsto \int_{\Omega_*} \chi_{I(v,x)}(\sigma)(v + \gamma)'_{+*}(\sigma)(v + \gamma)_{+*}(\sigma)b_{*v}(\sigma)\, d\sigma$ is continuous in $L^p(\Omega)$ for some $p \geqslant 1$, where $\chi_{I(v,x)}$ denotes the characteristic function of the interval $I(v,x) := [|v + \gamma > (v + \gamma)_+(x)|,\ |v + \gamma > 0|]$, $x \in \bar{\Omega}$. Arguing as before, $v \in V_m\backslash\{0\} \mapsto b_{*v} \in L^p(\Omega_*)$ and $v \in V_m\backslash\{0\} \mapsto (v + \gamma)'_{+*}(v + \gamma)_{+*} \in L^q(\Omega_*)$ are strongly continuous for any finite p and $q \in [1, 2)$ (see (3)), and since $\gamma < 0$, we obtain that $v \in V_m \mapsto (v + \gamma)'_{+*}(v + \gamma)_{+*}b_{*v} \in L^q(\Omega_*)$ is also continuous. So, let $(v_j)_{j \geqslant 1}$ be a sequence of V_m converging to v, if $v \neq 0$ we have that $\chi_{I(v_j,x)}$ converges to $\chi_{I(v,x)}$ in $L^r(\Omega_*)$ for every r finite and every $x \in \bar{\Omega}$. Hence, using again that $\gamma<0$ we deduce that for every $x\in\bar{\Omega}$:

$$\lim_{j\to+\infty}\int_{\Omega_*}\chi_{I(v_j,x)}(\sigma)[p(v_{j*}+\gamma)]'(\sigma)b_{*v_j}(\sigma)\,d\sigma$$
$$=\int_{\Omega_*}\chi_{I(v,x)}(\sigma)[p(v_*+\gamma)]'(\sigma)b_{*v}(\sigma)\,d\sigma$$

Noting that $(v+\gamma)_*=v_*+\gamma$, $b_{*(v+\gamma)}=b_{*v}$ (see, for instance, [17]) we find

$$G(v_j+\gamma)(x)\underset{j\to\infty}{\longrightarrow}G(v+\gamma)(x),\quad \text{a.e. } x\in\Omega.$$

Now, as $0\leqslant G(v_j+\gamma)\leqslant F_v$, the Lebesgue dominate convergence yields the continuity of the map $v\in V_m\to\int_\Omega a(x)G(v+\gamma)\varphi_k\,dx$. $\square$

We still need some a priori estimate for the sequence $(w_m)_{m\in\mathbb{N}}$:

Lemma 3.2. *If* w_m *is a solution of* $(\mathscr{P}_{\varepsilon,m})$ *then*

(i) $\forall\phi:\mathbb{R}\to\mathbb{R}$ *Borelian with* $\phi(w_m(t))\in L^1(\Omega)$ *we have*

$$\int_\Omega J(w_m(t)+\gamma)\phi(w_m(t))\,dx=0$$

(ii) w_m *remains in a bounded set of* $L^2(0,T;H_0^1(\Omega))$ *as* $m\to+\infty$ *and satisfies the following estimates, for all* $t\in[0,T]$

$$\int_0^t\int_\Omega|\nabla w_m(\sigma,x)|^2\,dx\,d\sigma+2\int_\Omega dx\int_0^{w_m(0)}\beta_\varepsilon(\sigma+\gamma)\,d\sigma$$
$$\leqslant 2\int_\Omega w_m(0)\beta_\varepsilon(w_m(0)+\gamma)\,dx+\frac{|a|_\infty^2F_v^2|\Omega|t}{2\lambda_1}.$$

Proof. Since $w_m\not\equiv 0$ (we recall that $a\not\equiv 0$), then for all t, *meas*$\{x\in\Omega\colon |\nabla w_m(t,x)|=0\}=0$. Thus, the properties of the relative rearrangement (see Lemma 4.1 in Section 4) yield

$$\int_\Omega b_{*w_m}(|w_m(t)>w_m(t,x)|)\phi(w_m(t,x))=\int_\Omega b\phi(w_m(t,x))\,dx$$

from where (i) follows. For the proof of (ii), we take $w_m(t)$ as test function in the equation $(\mathscr{P}_{\varepsilon,m})$ and using the above property:

$$\int_\Omega\frac{\partial}{\partial t}\beta_\varepsilon(w_m(t)+\gamma)w_m(t)\,dx+\int_\Omega|\nabla w_m(t,x)|^2\,dx=\int_\Omega aG(w_m+\gamma)w_m(t).$$

Now, the proof follows exactly the same idea as for the proof of Corollary 3.1. $\square$

End of the proof of Theorem 3.6. Since $\{\varphi_1, \dots, \varphi_m\}$ is free and β'_ε verifies $\beta'_\varepsilon \in C^1(\mathbb{R})$, $0 < \alpha \leqslant \beta'_\varepsilon < 2$, the matrix of coefficients $a_{ik}(w_m(t))$ is invertible. So, by Cauchy–Peano's theorem, the nonlinear differential system (21) has a maximal solution defined on some interval $[0, T_m]$. The above a priori estimates on w_m show that in fact $T_m = T$, $\forall m \geqslant 1$. Finally, if $a \not\equiv 0$, then $\int_\Omega a(x)\varphi_{k_0}(x)\,dx \neq 0$ for some k_0; therefore, if we assume that $w_m \equiv 0$ for some $m \geqslant k_0$, as $G(0+\gamma) = F_v$, we would arrive to $\int_\Omega a(x)\varphi_{k_0}(x)\,dx = 0$ which is a contradiction. $\square$

3.2.2. The Galerkin method II: Additional a priori estimates for $(\mathscr{P}_{\varepsilon,m})$

In order to pass to the limit $m \to \infty$ and to obtain a solution to $(\mathscr{P}_\varepsilon)$ we need some information on the time derivative of w_m

Lemma 3.3. *The sequence* $\frac{\partial w_m}{\partial t}$ *remains in a bounded set of* $L^2(Q)$ *as* $m \to \infty$. *More precisely*

$$\int_0^T \left|\frac{\partial w_m}{\partial t}\right|_2^2 dt + ess \sup_{t \in [0,T]} \frac{1}{\alpha} |\nabla w_m(t)|_2^2$$
$$\leqslant \frac{1}{\alpha}|\nabla u_0|_2^2 + \frac{2}{\alpha^2}\left(||a||_\infty^2 F_v^2 |\Omega| T + \lambda^2 \left(\operatorname*{osc}_\Omega b\right)^2 ||w_m||_{L^2(Q)}^2 \right),$$

where $\operatorname{osc}_\Omega b$ *denotes the oscillation of* b *in* Ω.

Proof. Multiplying $(\mathscr{P}_{\varepsilon,m})$ by $\frac{dw_m^j(t)}{dt}$ and adding these equations for $j = 1, \dots, m$, we get

$$\int_\Omega \beta'_\varepsilon(w_m(t)+\gamma)|w'_m(t)|^2\,dx + \tfrac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w_m(t)|^2\,dx$$
$$= \int_\Omega a(x)G(w_m+\gamma)w'_m(t)\,dx + \int_\Omega J(w_m+\gamma)w'_m\,dx,$$

where $w'_m(t) = \frac{\partial w_m}{\partial t}$. The first assumption on β_ε and the estimates $0 \leqslant G \leqslant F_v$, $|J(w_m(t)+\gamma)| \leqslant \lambda(w_m(t)+\gamma)_+ \operatorname{osc}_\Omega b$ yield

$$\alpha|w'_m(t)|_2^2 + \frac{1}{2}\frac{d}{dt}|\nabla w_m(t)|_2^2$$
$$\leqslant |a|_\infty F_v |w'_m(t)|_1 + \lambda \operatorname*{osc}_\Omega b \int_\Omega (w_m+\gamma)_+ w'_m(t)\,dx$$
$$\leqslant |a|_\infty F_v |\Omega|^{1/2} |w'_m(t)|_2 + \lambda \operatorname*{osc}_\Omega b |w_m(t)|_2 |w'_m(t)|_2,$$

where we have used the Holder inequality and the fact that $\gamma<0$. Applying the Young inequality,

$$\alpha|w_m'(t)|_2^2+\frac{1}{2}\frac{d}{dt}|\nabla w_m(t)|_2^2\leqslant\delta C_0|w_m'(t)|_2^2+\frac{1}{4\delta}\left(|a|_\infty F_v|\Omega|^{1/2}+\lambda\ \underset{\Omega}{\operatorname{osc}}\ b|w_m(t)|_2^2\right).$$

From the choice of δ and integrating in $]0,t[$, $t\leqslant T$, we have

$$\int_0^t|w_m'(s)|_2^2\,ds+\frac{1}{\alpha}|\nabla w_m(t)|_2^2$$
$$\leqslant\frac{1}{\alpha}|\nabla w_m(0)|_2^2+\frac{1}{\alpha^2}C_0\left(|a|_\infty F_v|\Omega|_T^{1/2}+\lambda\ \underset{\Omega}{\operatorname{osc}}\ b\int_0^t|w_m(\sigma)|_2^2\,d\sigma\right),$$

which leads to the estimate stated in the lemma. □

Corollary 3.2. *The sequence* $(w_m)_{m\geqslant1}$ *remains bounded in* $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))\cap\mathscr{C}([0,T];\ H_0^1(\Omega))$.

Proof. The bound in $H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$ follows from the estimates obtained in the two precedent lemmas. In order to show that w_m remains in a bounded set of $L^2(0,T;H^2(\Omega))$, we consider the orthogonal projection of $L^2(\Omega)$ onto V_m. The equation satisfied by w_m is equivalent to

$$\begin{cases}P_m\left(\dfrac{\partial}{\partial t}\beta_\varepsilon(w_m(t)+\gamma)\right)-\Delta w_m=P_m(aG(w_m(t)+\gamma)+J(w_m(t)+\gamma)),\\ w_m(t)\in V_m,\ \text{ for a.e. } t\in(0,T).\end{cases}\tag{22}$$

Lemma 3.2 and the estimate $0\leqslant G\leqslant F_v$ ensure that $aG(w_m(t)+\gamma)+J(w_m(t)+\gamma)$ lies in a bounded set of $L^2(Q)$. Also, since $0<\beta_\varepsilon'\leqslant2$, Lemma 3.3 implies that $\frac{\partial}{\partial t}\beta_\varepsilon(w_m(t)+\gamma)$ is bounded in $L^2(Q)$. From Eq. (22), we infer that Δw_m remains in a bounded set of $L^2(Q)$, and thus w_m is bounded in $L^2(0,T;H^2(\Omega))$. Finally, by using standard results (see, e.g., [28, Chapter I]):

$$Y:=H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\hookrightarrow\mathscr{C}([0,T];H_0^1(\Omega)),$$

we obtain the remaining estimates. □

3.2.3. The limit $m\to\infty$: Existence of solution for $(\mathscr{P}_\varepsilon)$

End of the proof of Theorem 3.5. The estimates above show that there exist a subsequence of $(w_m)_{m\geqslant1}$, which we also denote by $(w_m)_{m\geqslant1}$, and $w^\varepsilon\in Y$ such that $w_m\rightharpoonup w^\varepsilon$ weakly in Y, and so, by compactness results (see, for instance, [27]) we get

$$w_m\to w^\varepsilon \text{ strongly in } L^2(0,T;\ W_0^{1,p}(\Omega)),\ \text{ with } p\in[2,+\infty).\tag{23}$$

From the uniform bound $\operatorname{essinf}_\Omega b \leqslant b_{*(w_m+\gamma)}(|w_m(t) > w_m(t;\cdot)|) \leqslant \operatorname{esssup}_\Omega b$, we obtain the existence of $\hat{b^\varepsilon} \in L^\infty(Q)$ such that

$$b_{*(w_m+\gamma)}(|w_m(t) > w_m(t,\cdot)|) \rightharpoonup \hat{b^\varepsilon} \text{ weakly-star in } L^\infty(Q).$$

Analogously, as $|G(w_m(t,x)+\gamma)| \leqslant F_v$ a.e. in Q, there exists $G^\varepsilon_\infty \in L^\infty(Q)$ such that $G(w_m+\gamma) \rightharpoonup G^\varepsilon_\infty$ in $L^\infty(Q)$ weakly-star. Thus, w^ε is a solution of the following limit problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon+\gamma) - \Delta w^\varepsilon = aG^\varepsilon_\infty + \lambda(w^\varepsilon+\gamma)_+[b-\hat{b^\varepsilon}], \\ w(0) = u_0 - \gamma, \\ w \in Y \cap L^2(0,T;H^2(\Omega)). \end{cases} \tag{24}$$

Therefore, to finish the proof it remains to identify the term G^ε_∞ and to check that the limit function $\hat{b^\varepsilon}$ satisfies the rearrangement condition. This will be done in the following two lemmas:

Lemma 3.4. $G^\varepsilon_\infty = G(w^\varepsilon+\gamma)$.

Proof. Indeed, from (23) we can deduce that there exists a subsequence of w_m, which we will denote also by w_m, such that $w_m(t) \to w^\varepsilon(t)$ strongly in $W^{1,p}(\Omega)$ for $p \in [2,+\infty)$, a.e. t, and $w^\varepsilon(t) \in H^2(\Omega)$; thus, we may appeal to Lemma 3.1 to conclude. □

Lemma 3.5. *For any* $\phi \in C(\mathbb{R})$ *and* $\forall \theta \in \mathbb{R}$

$$\int_{\{x:w^\varepsilon(t,x)>\theta\}} \widehat{b^\varepsilon}\phi(w^\varepsilon)\,dx = \int_{\{x:w^\varepsilon(t,x)>\theta\}} b\phi(w^\varepsilon)\,dx \quad \textit{for a.e. } t \in]0,T[,$$

and

$$\operatorname*{essinf}_\Omega b \leqslant \widehat{b^\varepsilon}(t,x) \leqslant \operatorname*{esssup}_\Omega b \ \ \textit{a.e. in } Q.$$

Proof. For fixed t, it suffices to prove the equality for θ such that $|w^\varepsilon(t) = \theta| = 0$. Let $b^\varepsilon_m(t,x) = b_{*w_m}(|w_m(t) > w_m(t,x)|)$. By the properties of the relative rearrangement (see Lemma 4.1 in Appendix A), we know that

$$\int_{\{x:w_m(t,x)>\theta\}} b^\varepsilon_m(t,x)\phi(w_m(t,x))\,dx = \int_{\{x:w_m(t,x)>\theta\}} b\phi(w_m(t,x))\,dx. \tag{25}$$

Thus, we deduce the result from (25), since $\phi(w_m(t)) \to \phi(w^\varepsilon(t)) \in L^2(\Omega)$ and $\lim \chi_{\{x:w_m(t)>\theta\}}(x) = \chi_{\{x:w^\varepsilon(t)>\theta\}}(x)$ for a.e. $x \in \Omega$ a.e. $t \in]0,T[$. □

3.2.4. Proof of Theorem 2.1: The limit $\varepsilon \to 0$

First, we observe that, since w^ε is a weak solution of $(\mathscr{P}_\varepsilon)$, it satisfies the estimates obtained in the preceding section (see Theorems 3.2, 3.3 and 3.4) and these are uniform in ε and in α (note that these estimates were not obtained for w^ε but for $u_\varepsilon := w^\varepsilon + \gamma$). Therefore, there exists a positive constant c, independent of α and ε, such that

$$\int_0^T \int_\Omega |\nabla w^\varepsilon(\sigma,x)|^2 \, d\sigma \, dx \leqslant c \quad \text{and} \quad |\beta_\varepsilon(w^\varepsilon+\gamma)|_{L^\infty(Q)} \leqslant c.$$

Thus, since $0 \leqslant G(w^\varepsilon+\gamma) \leqslant F_v$, we may assume the existence of w_α and G_α such that $w^\varepsilon \rightharpoonup w_\alpha$ weakly in $L^2(0,T;H_0^1(\Omega))$ and $G(w^\varepsilon+\gamma) \rightharpoonup G_\alpha$ weakly-star in $L^\infty(Q)$ as $\varepsilon \to 0$, for a subsequence of $(w^\varepsilon)_{\varepsilon>0}$ that we have denoted again $(w^\varepsilon)_\varepsilon$. On the other hand, the following estimate holds:

$$\int_0^T \left|\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon(t)+\gamma)\right|_2^2 dt \leqslant |\nabla u_0|_2^2 + 2\left(|a|_\infty^2 F_v^2 |\Omega| T + M\lambda^2 \operatorname*{osc}_\Omega^2 b\right) \tag{26}$$

for a positive constant M. Indeed, multiplying in $(\mathscr{P}_\varepsilon)$ by $\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon+\gamma)$ and applying the Hölder inequality, we get

$$\begin{aligned}&\int_\Omega \left|\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon(t)+\gamma)\right|^2 dx + \int_\Omega \nabla w^\varepsilon \cdot \nabla\left(\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon+\gamma)\right) dx \\ &\quad \leqslant \left(|a|_\infty F_v |\Omega|^{1/2} + \lambda \operatorname*{osc}_\Omega b |(w^\varepsilon(t)+\gamma)|_2\right)\left|\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon(t)+\gamma)\right|_2.\end{aligned}$$

Then, the identity

$$\int_\Omega \nabla w^\varepsilon \cdot \nabla\left(\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon+\gamma)\right) dx = \frac{1}{2}\int_\Omega \beta_\varepsilon'(w^\varepsilon(t)+\gamma)\frac{\partial}{\partial t}|\nabla w^\varepsilon(t,x)|^2 \, dx,$$

and the assumption $\alpha \leqslant \beta_\varepsilon' \leqslant 2$, yield

$$\begin{aligned}&\left|\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon(t)+\gamma)\right|_2^2 + \frac{\alpha}{2}\frac{d}{dt}|\nabla w^\varepsilon(t)|_2^2 \\ &\quad \leqslant \left(|a|_\infty F_v |\Omega|^{1/2} + \lambda \operatorname*{osc}_\Omega b |(w^\varepsilon(t)+\gamma)|_2\right)\left|\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon(t)+\gamma)\right|_2.\end{aligned}$$

Estimates (26) follows from the above, after applying Poincare's and Young's inequalities. Moreover, since $\alpha \leqslant \beta_\varepsilon' \leqslant 2$, we obtain from (26) that

$$\left|\frac{\partial w^\varepsilon}{\partial t}\right|_{L^2(Q)}^2 \leqslant \frac{1}{\alpha}|\nabla u_0|_2^2 + \frac{2}{\alpha^2}\left(|a|_\infty^2 F_v^2 |\Omega| T + M\lambda^2 \operatorname*{osc}_\Omega^2 b\right).$$

In particular, (26) implies that $\left(\frac{\partial}{\partial t}\beta_\varepsilon(w^\varepsilon(t)+\gamma)\right)_{\varepsilon>0}$ remains in a bounded subset of $L^2(Q)$. Hence, from the equation in $(\mathscr{P}_\varepsilon)$ and using the above estimates we deduce that $\Delta w^\varepsilon \in L^2(Q)$ uniformly bounded in ε, which, by elliptic regularity, implies that $(w^\varepsilon)_{\varepsilon>0}$ is in a bounded subset of $L^2(0,T;H^2(\Omega))$. Hence, since $w^\varepsilon \rightharpoonup w_\alpha$ in $L^2(Q)$, we obtain that in fact $w^\varepsilon \rightharpoonup w_\alpha$ weakly in $L^2(0,T;H^2(\Omega))$ as $\varepsilon \to 0$. Furthermore, by the Rellich–Kondrachov compact embedding and using a standard compactness result (see, for instance, [27]), we deduce that $w^\varepsilon \to w_\alpha$ strongly in $L^2(0,T;W_0^{1,p}(\Omega))$ for $p \geqslant 2$. This last strong convergence allows us to pass to the limit $\varepsilon \to 0$ in the term $G(w^\varepsilon+\gamma)$. Indeed, since $w^\varepsilon(t) \to w_\alpha(t)$ strongly in $W_0^{1,p}(\Omega)$, $p \geqslant 2$, for a.e. $t \in (0,T)$, by Theorem 3.1 we get

$$G(w^\varepsilon+\gamma)(t,x) \to G(w_\alpha+\gamma)(t,x) \quad \text{a.e. } (t,x) \in Q, \text{ as } \varepsilon \to 0.$$

Finally, the uniform bound $\operatorname{essinf}_\Omega b \leqslant \hat{b^\varepsilon} \leqslant \operatorname{esssup}_\Omega b$ implies the existence of $\hat{b}_\alpha \in L^\infty(Q)$ such that $\hat{b^\varepsilon} \rightharpoonup \hat{b}_\alpha$ weakly-star in $L^\infty(Q)$ as $\varepsilon \to 0$. Arguing as in Lemma 3.5 we obtain that the couple $(u_\alpha, \hat{b}_\alpha)$ with $u_\alpha := w_\alpha + \gamma$ satisfies the rearrangement condition. □

3.3. Existence of solution for problem $(\mathscr{P})$ $(\alpha = 0)$: Proof of Theorem 2.2

In Theorem 2.1 we have proved the existence of a couple $(u_\alpha, \hat{b}_\alpha)$ being a weak solution to problem $(\mathscr{P}_\alpha)$ for $\alpha > 0$; i.e., if $w_\alpha := u_\alpha + \gamma$, then $(w_\alpha, \hat{b}_\alpha)$ verifies

$$(\mathscr{P}_\alpha)\begin{cases} -\dfrac{\partial}{\partial t}(w_\alpha+\gamma)_- + \alpha\dfrac{\partial}{\partial t}(w_\alpha+\gamma)_+ - \Delta w_\alpha \\ \quad = aG(w_\alpha+\gamma) + \lambda(w_\alpha+\gamma)_+[b-\hat{b}_\alpha] \quad \text{in } (0,T)\times\Omega, \\ w_\alpha = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ w_\alpha(0,x) = u_0 - \gamma \quad \text{in } \Omega. \end{cases}$$

Therefore, collecting the estimates proved in Section 3.1, we can conclude that $(w_\alpha)_{\alpha>0}$ remains in a bounded subset of $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$. Thus, we can extract a subsequence of (w_α), again denoted by (w_α), such that

$$w_\alpha \rightharpoonup w \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \text{ and weakly-star in } L^\infty(Q). \tag{27}$$

Also, from the uniform bound of the terms G and $\hat{b}_\alpha$, we deduce the existence of $h \in L^\infty(Q)$ such that

$$aG(w_\alpha+\gamma) + \lambda(w_\alpha+\gamma)_+[b-\hat{b}_\alpha] \overset{*}{\rightharpoonup} h \text{ weakly-star in } L^\infty(Q). \tag{28}$$

Moreover, the following estimate holds:

Lemma 3.6. *The sequence* $(w_\alpha+\gamma)_-$ *remains in a bounded set of* $L^\infty(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ *as* α *goes to zero. Furthermore,*

$$\int_0^t \left|\frac{\partial}{\partial t}(w_\alpha(\sigma)+\gamma)_-\right|^2_{L^2(\Omega)} d\sigma + \int_\Omega |\nabla(w_\alpha(t)+\gamma)_-|^2\,dx$$

$$\leqslant |\nabla(w_0+\gamma)_-|^2_{L^2(\Omega)} + T|\Omega||a|^2_\infty F_v^2.$$

Proof. We multiply by $\frac{\partial}{\partial t}(w_\alpha+\gamma)_-$ the equation satisfied by w_α and we use the *integration by parts* formula:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(w_\alpha(t)+\gamma)_-|^2\,dx = \int_\Omega \frac{\partial}{\partial t}(w_\alpha(t)+\gamma)_- \Delta w_\alpha(t)\,dx,$$

that can be justified, thanks to the above regularity, by using a smooth approximation (see, e.g., [42]). We find

$$\left|\frac{\partial}{\partial t}(w_\alpha(t)+\gamma)_-\right|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(w_\alpha(t)+\gamma)_-|^2\,dx \leqslant M\left|\frac{\partial}{\partial t}(w_\alpha(t)+\gamma)_-\right|_{L^1(\Omega)}.$$

Applying the Young inequality and integrating, we arrive to

$$\int_0^t \left|\frac{\partial}{\partial t}(w_\alpha(\sigma)+\gamma)_-\right|^2_{L^2(\Omega)} d\sigma + \int_\Omega |\nabla(w_\alpha(t)+\gamma)_-|^2\,dx$$

$$\leqslant |\nabla(w_0+\gamma)_-|^2_{L^2(\Omega)} + T|\Omega|M^2$$

from where the desired result follows. □

Hence, the above lemma and using a standard compactness result (see, e.g., [27] or [42]) we deduce the existence of $z\in L^2(Q)$ and of a subsequence of $((w_\alpha+\gamma)_-)$, denoted again by $((w_\alpha+\gamma)_-)$, such that $-(w_\alpha+\gamma)_-\to z$ in $L^2(Q)$ as α goes to zero. In fact:

Lemma 3.7. *The following identity is verified*:

$$z = -(w+\gamma)_- = \beta(w+\gamma).$$

Proof. The $\mathbb{R}^2$-graph β generates a maximal monotone operator A on $L^2(0,T;L^2(\Omega))$ (see [8, Chapter II]), defined as

$$Av = -(v+\gamma)_- \quad \forall v\in L^2(0,T;L^2(\Omega)).$$

From the weak convergence of $(w_\alpha+\gamma)_{\alpha>0}$ in $L^2(0,T;H_0^1(\Omega))$ and the strong convergence of $(\beta(w_\alpha+\gamma))_{\alpha>0}$ in $L^2(0,T;L^2(\Omega))$ we know that

$$(w_\alpha+\gamma)\rightharpoonup w+\gamma \text{ weakly in } L^2(Q),$$

$$\beta(w_\alpha+\gamma)\to z \text{ strongly in } L^2(Q).$$

Thus, using the properties of the maximal monotone operators (see [8]) we arrive to $((w_\alpha+\gamma),z)\in A$ and so $z=\beta(w_\alpha+\gamma)$ which ends the proof of the lemma. □

Therefore, using the lemma above and the convergences (27) and (28), it is clear that w satisfies the following limit problem $(\mathscr{P}_0)$

$$\begin{cases} -\dfrac{\partial}{\partial t}(w+\gamma)_- - \Delta w = h, \\ w\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q), \\ (w+\gamma)_- = u_{0-}. \end{cases}$$

Furthermore, from Lemma 3.6 and Agmon–Douglis–Nirenberg's elliptic regularity, we deduce from the equation above that

$$w\in L^2(0,T;H^2(\Omega)). \tag{29}$$

We are to prove that $u:=w+\gamma$ is a weak solution to $(\mathscr{P})$. To this end, it remains to identify the function h. For this purpose we need (see Theorem 3.1) the strong convergence of $w_\alpha(t)$ in $W^{1,2+\delta}(\Omega)$ with $\delta>0$ and a.e. $t\in(0,T)$. As we announced in Section 2, we shall obtain this convergence by using a compactness result due to Rakotoson and Temam [36], for what we shall need the following proposition:

Proposition 3.1. *Given $\phi\in L^2(\Omega)$ and $h>0$, let $\phi_{h\alpha}$ be defined by (6). Then, $w_\alpha(t)\rightharpoonup w(t)$ in $L^2(\Omega)$ for a.e. $t\in(0,T)$ if and only if (7) holds.*

Proof. Assume (7) then, from the boundedness of $(w_\alpha)_{\alpha>0}$ given in Theorem 2.1 it follows the existence of a subsequence, that we still denote by $(w_\alpha)_{\alpha>0}$, such that

$$w_\alpha\rightharpoonup w \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

In particular,

$$\int_0^T\int_\Omega w_\alpha(t,x)\psi(t)\phi(x)\,dx\,dt\to\int_0^T\int_\Omega w(t,x)\psi(t)\phi(x)\,dx\,dt$$

$\forall\psi\in L^2(0,T)$ and $\forall\phi\in L^2(\Omega)$, when $\alpha\to 0$. Let us fix $t_0\in(0,T), h>0$ small enough and define the sequence $(\psi_h)_{h>0}\subset L^2(0,T)$ as

$$\psi_h(t)=\tfrac{1}{h}\chi_{[t_0,t_0+h]}(t),$$

where $\chi_{[t_0,t_0+h]}$ denotes the characteristic function of $[t_0, t_0+h]$. Then,

$$\int_0^T \int_\Omega w_\alpha(t,x)\psi_h(t)\phi(x)\,dx\,dt \to \int_0^T \int_\Omega w(t,x)\psi_h(t)\phi(x)\,dx\,dt$$

when $\alpha \to 0$ and by (7) this convergence is uniformly in h. Then, passing to the limit $h \to 0$ in the above expression and taking into account that for any integrable function the complementary of its Lebesgue points is a set of zero measure (see, e.g., [8, p. 140]) we get

$$\int_\Omega w_\alpha(t_0,x)\phi(x)\,dx \to \int_\Omega w(t_0,x)\phi(x)\,dx \quad \text{when } \alpha \to 0, \text{ a.e. } t_0 \in (0,T),$$

and thus $w_\alpha(t) \rightharpoonup w(t)$ in $L^2(\Omega)$-weak a.e. $t \in (0,T)$. Assume now that the $L^2(\Omega)$-weak convergence of $(w_\alpha(t))_{\alpha>0}$ to $w(t)$ holds for a.e. $t \in (0,T)$. We always have

$$\begin{aligned}\lim_{h\to 0}\lim_{\alpha\to 0} \phi_{h\alpha}(t) &= \lim_{h\to 0}\left[\int_0^T \int_\Omega w(t,x)\psi_h(t)\phi(x)\,dx\,dt\right]\\ &= \int_\Omega w(t,x)\phi(x)\,dx \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Moreover, by the Lebesgue theorem,

$$\begin{aligned}\lim_{\alpha\to 0}\lim_{h\to 0} \int_0^T \int_\Omega w_\alpha(t,x)\psi_h(t)\phi(x)\,dx\,dt &= \lim_{\alpha\to 0}\int_\Omega w_\alpha(t,x)\phi(x)\,dx\\ &= \int_\Omega w(t,x)\phi(x)\,dx \quad \text{a.e. } t\in(0,T),\end{aligned}$$

and so (7) holds. □

End of the proof of Theorem 2.2. From assumption (7) it follows that $w_\alpha(t) \rightharpoonup w(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$. We can then appeal to the compactness result given in [36] and we deduce that $w_\alpha \to w$ in $L^2(Q)$ and for a.e. $t \in]0,T[$. Then, Lemma 3.6 and the bound of w_α in $L^\infty(Q)$ yield:

$$\lim_{\alpha\searrow 0}\int_Q w_\alpha \frac{\partial}{\partial t}(w_\alpha+\gamma)_-\,dx\,dt = \int_Q w\frac{\partial}{\partial t}(w+\gamma)_-\,dx\,dt \tag{30}$$

and

$$\lim_{\alpha\searrow 0}\alpha\int_Q w_\alpha \frac{\partial}{\partial t}(w_\alpha+\gamma)_+\,dx\,dt = 0.$$

Since the second member in $(\mathscr{P}_\alpha)$ converges weakly to h in $L^2(Q)$, then

$$\lim_{\alpha\searrow 0}\int_Q w_\alpha(aG(w_\alpha+\gamma)+\lambda(w_\alpha+\gamma)_+[b-\hat{b}_\alpha])\,dx\,dt = \int_Q wh\,dx\,dt.$$

Multiplying by w_α the first equation of $(\mathscr{P}_\alpha)$ one deduces from $(\mathscr{P}_0)$:

$$\lim_{\alpha\searrow 0}\int_Q |\nabla w_\alpha|^2\,dx\,dt = \int_Q w\frac{\partial}{\partial t}(w+\gamma)_-\,dx\,dt + \int_Q hw\,dx\,dt = \int_Q |\nabla w|^2\,dx\,dt. \quad (31)$$

Thus, the weak convergence of w_α to w in $L^2(0,T;H_0^1(\Omega))$ and (31) implies that

$$w_\alpha(t)\underset{\alpha\searrow 0}{\longrightarrow} w(t) \text{ in } H_0^1(\Omega) \text{ for a.e. } t\in]0,T[.$$

In fact, as w_α remains in a bounded set of $L^2(0,T;H^2(\Omega))$, from the above convergence we deduce, by the Gagliardo–Nirenberg interpolation, that

$$w_\alpha(t)\underset{\alpha\searrow 0}{\longrightarrow} w(t) \text{ in } W^{1,p}(\Omega) \text{ for a.e. } t\in]0,T[\text{ and } 2\leqslant p\leqslant 4.$$

Since $w(t)\in H^2(\Omega)$, we may appeal Lemma 3.1 to derive that

$$\lim_{\alpha\searrow 0} G(w_\alpha(t)+\gamma)(x) = G(w(t)+\gamma)(x)$$

for a.e. $(t,x)\in Q$. From the boundedness of $\hat{b}_\alpha$, there exists $\hat{b}_0$ such that $\hat{b}_\alpha \rightharpoonup \hat{b}_0$ weakly-star in $L^\infty(Q)$, $\operatorname{essinf}_\Omega b\leqslant \hat{b}_0\leqslant \operatorname{esssup}_\Omega b$ and

$$\int_{\{x:w(t)(x)>\theta\}} b\phi(w(t))\,dx = \lim_{\alpha\searrow 0}\int_{\{x:w_\alpha(t)(x)>\theta\}} \hat{b}_\alpha\phi(w_\alpha(t))\,dx$$
$$= \int_{\{x:w(t)(x)>\theta\}} \hat{b}_0\phi(w(t))\,dx$$

for a.e. $t\in]0,T[$, for any $\phi\in C(\mathbb{R})$ and for all $\theta\in\mathbb{R}$. Then, passing to the limit in $\mathscr{D}'(Q)$

$$\lim_{\alpha\searrow 0}(aG(w_\alpha+\gamma) + \lambda(w_\alpha+\gamma)_+[b-\hat{b}_\alpha]) = aG(w+\gamma) + (w+\gamma)_+[b-\hat{b}_0]$$

which implies that: $h = aG(w+\gamma) - (w+\gamma)_+[b-\hat{b}_0]$. Using the equation in problem $(\mathscr{P}_0)$ we deduce that $(u,\hat{b}_0)$, $u=w+\gamma$, is a weak solution of $(\mathscr{P})$. By arguing as in Lemma 3.3, we deduce that $\beta(u)\in\mathscr{C}([0,T];L^2(\Omega))$ and $u\in L^2(0,T;H^2(\Omega))$, we can then use Sobolev embedding to conclude that $u\in L^2(0,T;\mathscr{C}^1(\Omega))$. □

Remark 3.4. We conjecture that condition (7) holds under stronger regularity on the initial datum (and so on the approximating solutions w_α).

4. On the weak solution for problem $(\mathscr{P})$

In this section we analyze the relation between the notion of the weak solutions that we are considering here (Definition 2.1) and the *standard* notion of weak solutions (satisfying $(\mathscr{P})$ in the sense of distributions $\mathscr{D}'(Q)$). We start with a lemma, already used in the preceding section, that establishes the connection between the relative rearrangement $b_{*\tilde{u}}$ and function $b^{\tilde{u}}$ appearing in Definition 2.1:

Lemma 4.1. *Let v be in $L^1(\Omega)$ without any flat zones, i.e.* $\text{meas}(P(v)) = 0$, *and* $b \in L^1(\Omega)$. *For all* $\phi \in C(\mathbb{R})$, *we have*

$$\int_\Omega b_{*v}(|v > v(x)|)\phi(v)(x)\,dx = \int_\Omega b\phi(v)\,dx.$$

In particular, if $\tilde{u} := (v + \gamma)$ *is a weak solution to* $(\tilde{\mathscr{P}})$ *then,*

$$\int_\Omega b^{\tilde{u}}\phi(\tilde{u})(t,x)\,dx = \int_\Omega b_{*\tilde{u}}(|\tilde{u}(t) > \tilde{u}(t,x)|)\phi(\tilde{u})(t,x)\,dx \quad a.e.\ t \in (0,T).$$

Proof. Since v does not have flat regions it follows that $v_*(\mu(v)) = v$ and so

$$\int_\Omega b(x)\phi(v(x))\,dx = \int_\Omega b(x)\phi(v_*(\mu(v(x))))\,dx.$$

The *mean value operator property* proved in [31] (see also [29]) allows us to write the above integral as an integral over Ω_* by using the relative rearrangement in the following way:

$$\int_\Omega b(x)\phi(v_*(\mu(v(x))))\,dx = \int_{\Omega*} b_{*v}(\sigma)\phi(v_*(\sigma))\,d\sigma.$$

Since the function ϕ is continuous, it is a Borel function and the identity $\phi(v_*) = [\phi(v)]_{*v}$ holds (see, e.g., [31]). Thus, using once more that $meas(P(v)) = 0$ and the *mean value operator property* we get

$$\int_{\Omega*} b_{*v}(\sigma)\phi(v_*(\sigma))\,d\sigma = \int_{\Omega*} b_{*v}(\sigma)[\phi(v)]_{*v}(\sigma)\,d\sigma = \int_\Omega b_{*v}(|v > v(x)|)\phi(v(x))\,dx$$

which end the proof. □

Thus, the relation between the notion of the weak solutions and the *standard* notion of weak solutions is given by

Theorem 4.1. *Let* $\tilde{u} \in L^2(0,T;H^1(\Omega))$ *such that* $\tilde{u}(t,\cdot)$ *has not flat regions for a.e.* $t \in]0,T[$:

(a) *If* $\tilde{u}$ *is a weak solution of* $(\tilde{\mathscr{P}})$ (*in the sense of Definition* 2.1) *and there exists a Borel map* $g^{\tilde{u}} : \mathbb{R} \to \mathbb{R}$ *such that* $g^{\tilde{u}} \circ \tilde{u} = b^{\tilde{u}}$ *then* $\tilde{u}$ *satisfies* $(\tilde{\mathscr{P}})$ *in the sense of distributions* $\mathscr{D}'(Q)$.
(b) *Conversely, if* $\tilde{u}$ *satisfies* $(\tilde{\mathscr{P}})$ *in the sense of distributions* $\mathscr{D}'(Q)$, *then it is a weak solution in the sense of Definition* 2.1.

Proof. For the first part of the theorem it suffices to show that $b^{\tilde{u}}(t,x) = b_{*\tilde{u}}(|\tilde{u}(t) > \tilde{u}(t,x)|)$ for a.e. $(t,x) \in Q$. First, let us observe that since $\tilde{u}$ satisfies the rearrangement condition (see (i) in Definition 2.1) we deduce, by an approximating argument, that for any Borel function φ on $\mathbb{R}$ with $\varphi(v(t)) \in L^1(\Omega)$

$$\int_\Omega b^v \varphi(\tilde{u}(t,x))\,dx = \int_\Omega b\varphi(\tilde{u}(t,x))\,dx, \tag{32}$$

and, since $\tilde{u}(t,\cdot)$ does not have any flat regions a.e. $t \in (0,T)$, from Lemma 4.1 we have:

$$\int_\Omega b\varphi(\tilde{u}(t,x))\,dx = \int_\Omega b_{*\tilde{u}}(|\tilde{u}(t) > \tilde{u}(t,x)|)\varphi(\tilde{u}(t,x))\,dx. \tag{33}$$

We consider now the function

$$\varphi(\sigma) = b_{*\tilde{u}}(|\tilde{u}(t) > \sigma|) - g^{\tilde{u}}(\sigma), \quad \sigma \in \mathbb{R} \tag{34}$$

which is a Borel function in $\mathbb{R}$ for almost every t. Thus, from (32), (33) and (34) it follows

$$\int_\Omega (b_{*\tilde{u}}(|\tilde{u}(t) > \tilde{u}(t,x)|) - g^{\tilde{u}} \circ \tilde{u}(t,x))^2\,dx = 0.$$

In order to prove the second result in the statement, let $\tilde{u}$ be a *usual* weak solution of $(\mathscr{P})$. That is, $\tilde{u} \in L^2(0,T;H^1(\Omega))$ such that $\beta(\tilde{u}) \in H^1(0,T;H^{-1}(\Omega))$ satisfying $(\tilde{\mathscr{P}})$ in the sense of $\mathscr{D}'(Q)$. We set $b^{\tilde{u}}(t,x) := b_{*\tilde{u}}(|\tilde{u}(t) > \tilde{u}(t,x)|)$, for $(t,x) \in Q$. It remains to check that this choice of $b^{\tilde{u}}$ verifies the relative rearrangement condition. Indeed, as $\tilde{u}(t,\cdot)$ has not flat region, we can argue as in Lemma 4.1 obtaining that, for a.e. t and for all $\theta \in \mathbb{R}$,

$$\begin{aligned}\int_{\{x:\ \tilde{u}(t,x) > \theta\}} b^{\tilde{u}} \varphi(\tilde{u}(t,x))\,dx &= \int_{\{\sigma:\ \tilde{u}_*(t,\sigma) > \theta\}} b_{*\tilde{u}} \varphi(\tilde{u}_*(t,\sigma))\,d\sigma \\ &= \int_{\{x:\ \tilde{u}(t,x) > \theta\}} b\varphi(\tilde{u}(t,x))\,dx.\end{aligned}$$

The estimate $ess \inf_\Omega b \leqslant b^{\tilde{u}} \leqslant ess \sup_\Omega b$ follows from the bound of the relative rearrangement (see, e.g., the definition of the relative rearrangement in Section 2 or [30,31]) and from the fact that $b \in L^\infty(\Omega)$. □

Next, we provide a sufficient condition for the weak solution u of problem $(\mathscr{P})$ not having flat regions:

Theorem 4.2. *Let u be a weak solution of $(\mathscr{P})$ and assume that $a(x)$ does not have any flat regions. Then, if*

$$2\lambda \|b\|_\infty < \left(\frac{1-v}{v}\right) \inf_\Omega a^2, \tag{35}$$

where $v := \frac{2\lambda\|b\|_\infty S|\Omega|}{F_v^2} < 1$ *and* $S := |u_+|_\infty$, *it follows that* $u(t)$ *has not flat regions for a.e.* $t \in]0, T[$.

Proof. Suppose that there exists $I \subset]0, T[$, $|I| > 0$, and a constant $c \in \mathbb{R}$ such that

$$meas\{x \in \Omega\colon\ u(t,x) = c\} > 0, \quad \text{for a.e. } t \in I.$$

Then, if $c \leqslant 0$, by Stampacchia's theorem we can deduce that $\Delta u(t) = 0$ a.e. on $S_c := \{x \in \Omega\colon\ u(t,x) = c\}$. Let us denote by $v := u_- \in H^1(0, T; L^2(\Omega))$, then it follows (see, e.g., [30, p. 63]) that there exists a constant K_c such that $\frac{\partial u_-(t)}{\partial t} = K_c$ a.e. on S_c. Thus, necessarily

$$K_c = a(x)F_v \quad \text{for a.e. } x \in S_c, \text{ a.e. } t \in I,$$

and so a has a flat region of positive measure ($\{x\colon\ a = 0\}$ if $K_c = 0$, and $\left\{x\colon\ a = \frac{F_v}{K_c}\right\}$ otherwise), whose measure is at least $meas(S_c)$. Hence, the assumption on a is violates. Let us assume now that $c > 0$. In this case, necessarily

$$0 = aG(u) + \lambda u_+(b - b^u) \text{ a.e. on } S_c, \text{ and a.e. } t \in I.$$

But then, using the estimates that we have on $G(u)$, u_+ and b^u, we arrive to

$$(2\lambda\|b\|_\infty \|u_+\|_\infty)^2 \geqslant \inf_\Omega a^2[F_v^2 - 2\lambda\|b\|_\infty \|u_+\|_\infty^2],$$

which is in contradiction with (35). □

Furthermore, the following corollary holds:

Corollary 4.1. *Let us assume that* $meas\{x \in \Omega\colon\ |\nabla w^\varepsilon(t,x)| = 0\} = 0$, $meas\{x \in \Omega\colon\ |\nabla w_\alpha(t,x)| = 0\} = 0$ *and* $meas\{x \in \Omega\colon\ |\nabla w(t,x)| = 0\} = 0$ *for a.e.* t, *then* w^ε, w_α *and* w *satisfy the equations of the respective problems* $(\mathscr{P}_\varepsilon)$, $(\mathscr{P}_\alpha)$ *and* $(\mathscr{P})$ *in the sense of distributions* $\mathscr{D}'(Q)$.

Proof. Following Lemma 4, as the sequence w_m given in Theorem 3.6 satisfies that $meas\{x \in \Omega\colon\ |\nabla w_m(t,x)| = 0\} = 0$ for a.e. $t \in]0, T[$, then the condition $meas\{x \in \Omega\colon\ |\nabla w^\varepsilon(t,x)| = 0\} = 0$ implies that

$$b_{*w_m(t)}(|w_m(t) > w_m(t,\cdot)|) \to b_{*w^\varepsilon(t)}(|w^\varepsilon(t) > w^\varepsilon(t,\cdot)|) \quad \text{in } L^p(\Omega),$$

as $m \to +\infty$, for $1 < p < +\infty$. Thus, we obtain that

$$\hat{b_\varepsilon} = b_{*w^\varepsilon(t)}(|w^\varepsilon(t) > w^\varepsilon(t,x)|).$$

Repeating the same argument for w_α, we get $\hat{b_\alpha} = b_{*w_\alpha(t)}(|w_\alpha(t) > w_\alpha(t,x)|)$ and $\hat{b_0} = b_{*w(t)}(|w(t) > w(t,x)|)$, and so the conclusion is reached. $\square$

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Appendix A. Derivation of $(\mathscr{P})$

In this section we shall briefly describe the derivation of problem $(\mathscr{P})$, through the modeling of the *quasi-stationary processes* occurring in a Stellarator machine. Let us start saying that the Stellarators are a class of toroidal devices for the confinement of fusion plasmas by means of magnetic fields (for a detailed description of this machine and of the processes taking place in it the reader is referred to [11,19,26]). The nuclear fusion plasma used in these reactors is assumed to be an ideal fluid and so the ideal incompressible MHD system is used to derive our model, since it provides a single-fluid description of the macroscopic plasma behavior. The equations of MHD are given by

$$\frac{D\mathbf{v}}{Dt} = \mathbf{J} \times \mathbf{B} - \nabla p, \tag{A.1}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\mu}\mathbf{J}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{A.2}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0. \tag{A.3}$$

In these equations the electromagnetic variables are the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$ and the current density $\mathbf{J}$. We denote by $\mathbf{v}$ the fluid velocity, p the fluid pressure and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ the convective derivative. The parameter μ represents the electric conductivity. In Stellarators machines the plasma is assumed to occupy an unknown region of the toroidal cavity (the *plasma region*), surrounded by another region which is vacuous (the *vacuum region*). In the plasma region we shall assume $\mu = 0$, i.e., the plasma is a perfect conductor and so Eqs. (A.1)–(A.3) become the system of ideal MHD. In the vacuum region we shall take $\mu = 1$.

Quasi-stationary processes in a Stellarator device appears when we work in a slow reference time-scale: the resistive diffusion time-scale. In this characteristic times scale's, plasma would evolve through a series of states each of which would be very near to an equilibrium, i.e., at each instant t, plasma can be regarded as being in

MHD equilibrium (see, e.g., [21]). Following [6, Chapter IV] (see also [21]) a dimensional analysis is done in order to neglect some quantities in the MHD system (A.1)–(A.3) and to retain only the principal ones. In particular, it follows that the term $\frac{D\mathbf{v}}{Dt}(t)$ in (A.1) is negligible compared to $\nabla p(t)$ in the plasma region at time t. Thus,

$$\nabla p(t) = \mathbf{J}(t) \times \mathbf{B}(t) \tag{A.4}$$

is satisfied at each instant t in this region and so, at each instant t, plasma is in MHD equilibrium. We point out that very little is known about the existence of solution for the equilibria equation (A.4) in a 3D nonsymmetric torus; in particular, we mention the work of Bruno and Laurence [9] for configurations close to axisymmetry (see also, [12] for the case of a axisymmetric Stellarator). From (A.4) it follows that, for any instant t,

$$\mathbf{B}(t) \cdot \nabla p(t) = 0 \quad \text{and} \quad \mathbf{J}(t) \cdot \nabla p(t) = 0. \tag{A.5}$$

Then, the pressure is constant on each magnetic surface (i.e., surface generated by the magnetic field lines) and they have to be (see, for instance, [19]) nested toroids, but not necessarily symmetric or of circular cross-section. Thus, in order to get a better description of it is useful to introduce a set of special toroidal coordinates *Boozer vacuum coordinates system* ([7]) (ρ, θ, ϕ), where $\rho = \rho(x, y, z)$ is an arbitrary function which is constant on each nested toroid and $\theta = \theta(x, y, z)$ is the poloidal coordinate which is constant on any toroidal circuit but changes by 2π over a poloidal circuit (here by a toroidal circuit we mean any closed loop that encircles the axis of the torus once, and by a poloidal circuit a closed loop that encircles the minor axis once) and $\phi = \phi(x, y, z)$ is the toroidal coordinate. For a vacuum configuration (i.e. without any plasma) the magnetic field $\mathbf{B}_v$ may be written (see [11]) in covariant form as

$$\mathbf{B}_v = F_v \nabla \phi, \tag{A.6}$$

where F_v is a constant (which customary is taken as positive).

The Stellarators-type configurations are very complicated due to the fully three-dimensional nature of the device (see, for instance, [11,23] or [26]). To simplify the model to a two-dimensional problem different averaging methods were used for the study of stationary models, such as the methods developed by Greene and Johnson [21], and by Hender and Carreras [23]. Following the last reference we may decompose the magnetic field in terms of its toroidally averaged and rapidly varying parts. For a general function f this decomposition takes the form

$$f = \langle f \rangle + \tilde{f}, \quad \text{with } \langle f \rangle := \frac{1}{2\pi} \int_0^{2\pi} f \, d\phi.$$

Using a suitable assumption on the Stellarator geometry (the Stellarator expansion hypothesis) Hender and Carreras [23] show that the second equation in (A.3) leads to the equation

$$\frac{\partial}{\partial\rho}\left(\rho\left\langle\frac{B^{\rho}(t)}{D}\right\rangle\right)+\frac{\partial}{\partial\theta}\left(\left\langle\frac{B^{\theta}(t)}{D}\right\rangle\right)=0,$$

where B^i are the contravariant components of the vacuum magnetic field, $i=\rho,\theta,\phi$, and D is the Jacobian of the Boozer coordinate system. Thus, it follows the existence of a *potential* function, the *averaged poloidal flux function* $\psi=\psi(t,\rho,\theta)$, defined at each instant t by

$$\left\langle\frac{B^{\rho}(t)}{D}\right\rangle=\frac{1}{\rho}\frac{\partial\psi(t)}{\partial\theta}\quad\text{and}\quad\left\langle\frac{B^{\theta}(t)}{D}\right\rangle=-\frac{\partial\psi(t)}{\partial\rho}. \tag{A.7}$$

They also show that, when MHD equilibrium (A.4) exists, then $\langle B_\phi\rangle$ is a function ψ alone and the same for $\langle p\rangle$ (recall (A.5)). Thus, in our case, as Eq. (A.4) is satisfied in the plasma region, by introducing the usual notation $F(\psi):=\langle B_\phi\rangle$ and $p(\psi):=\langle p\rangle$, and following Hender and Carreras [23], we obtain that ψ satisfies the following Grad–Shafranov type equation in the plasma region $\Omega_p(t)$, at each $t\geqslant 0$:

$$-\Delta\psi=aF(\psi)+F(\psi)F'(\psi)+bp'(\psi). \tag{A.8}$$

That is, ψ satisfies, at each instant, the above equilibrium equation in Ω_p, where a and b are bounded functions that depend on geometrical aspects of the device and such that $b>0$.

In fact, in [23] the above equation is obtained for a different second order elliptic operator $\mathscr{L}$, but here, for the sake of simplicity we have replaced it by the Laplacian one. A study of the case of $\mathscr{L}$ would follow the same arguments that we have developed in this paper, with the addition of some technical details (see [17]).

As we have already pointed out, Eq. (A.8) only holds on the (averaged) region occupied by the plasma. In order to analyze the processes that takes place in the vacuum region at time t, $\Omega_p(t)$, we follow once more [6, Chapter IV], obtaining, after using (A.2), (A.3) and relation (A.6), that the equation satisfied by ψ in $\Omega_v:=\bigcup_{t\in[0,T]}\{t\}\times\Omega_v(t)$ is

$$-\Delta\psi=aF_v-\frac{\partial\psi}{\partial t},$$

Once obtained the equations satisfied by ψ in the plasma and vacuum regions, we can give a global formulation as a free boundary problem by using that in the vacuum region $\nabla p=0$. Indeed, the free-boundary, that separates the plasma and vacuum regions, is a magnetic surface, and thus, as $p=p(\psi)$, we can identify (after normalizing) the free boundary as the level line $\{\psi(t)=0\}$, the plasma region as $\Omega_p(t)=\{\psi(t)>0\}$ (and thus $\{p>0\}$) and the vacuum region by $\Omega_v(t)=\{\psi(t)<0\}$ (and $\{p=0\}$). It is well-known that it is not possible to obtain the pressure from the

MHD system and some constitutive law must be assumed. Here, for simplicity, we shall assume the quadratic law (see, e.g., Temam [41])

$$p(\psi) = \frac{\lambda}{2}[\psi_+]^2, \quad \psi_+ = \max\{\psi, 0\} \tag{A.9}$$

which is compatible with the above normalization. Finally, to give an unified formulation for the present model, we extend the *unknown* function $F(\psi)$ for negative values of ψ by using (A.6) and so we must find $\psi(t, \mathrm{x})$ and $F : \mathbb{R} \to \mathbb{R}_+$ such that $F(s) = F_v$ for any $s \leqslant 0$, satisfying

$$-\frac{\partial \psi_-}{\partial t} - \Delta\psi = aF(\psi) + F(\psi)F'(\psi) + \lambda b\psi_+. \tag{A.10}$$

The above equation is satisfied in a bidimensional open set Ω obtained after averaging the physical three-dimensional domain. The boundary condition that must verify ψ results as a consequence of assuming that the wall of the device is perfectly conducting and it is expressed as (see [11])

$$\psi(t) = \gamma \quad \text{on }]0, T[\times\partial\Omega, \tag{A.11}$$

where γ is a negative constant.

In order to complete the formulation of the problem under consideration we must add the *Stellarator condition* imposing a zero net current within each flux magnetic surface. This condition, which comes from the design of the external conductors in Stellarators, conforms one of the characteristic of this type of reactors for nuclear fusion. According to the averaging method by Hender and Carreras [23] this condition can be expressed (see [11]) for a.e. $t \in]0, T[$ as

$$\int_{\{\psi(t) \geqslant \tau\}} [F(\psi)F'(\psi) + \lambda b\psi_+]\rho \, d\rho \, d\theta = 0 \quad \text{for any } \tau \in [\inf \psi, \sup \psi]. \tag{A.12}$$

Summarizing we arrive to the mathematical formulation of the model describing the evolution of equilibria in a plasma confined in a Stellarator, as the following *inverse problem* $(\mathscr{P}_I)$: Let Ω be a bounded open regular set of $\mathbb{R}^2$ and $T>0$, find $u : [0, T] \times \Omega \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$, such that $F(s) = F_v$ for any $s \leqslant 0$ and (u, F) satisfying (A.10)–(A.12) and the boundary condition $u(0, x)_- = (u_0(x))_-$ for a given $u_0 : \Omega \to \mathbb{R}$.

In order to determine the unknown nonlinearity F we can reformulate problem $(\mathscr{P}_I)$ by means of the notion of relative rearrangement, as it was done in [11,17] for the stationary case. Here, we shall just give some indications on how to arrive to problem $(\mathscr{P})$ and we shall omit the Lemmas and proofs needed for this derivation, since they are slights modifications of the results appearing in [17]. In this sense, following [11] (see also [17]) we can differentiate (formally) in the Stellarator condition (A.12) with respect to θ and using the notation in (A.9) (notice that we have also use the notation $u = \psi$), we find that

$$F(\theta)F'(\theta) = -p'(\theta)\frac{\int_{u(t)^{-1}(\theta)} \frac{b(x)}{|\nabla u(t,x)|} dH_1(x)}{\int_{u(t)^{-1}(\theta)} \frac{dH_1(x)}{|\nabla u(t,x)|}} \quad \text{a.e. } t \in (0,T), \tag{A.13}$$

where H_1 is the one-dimensional Hausdorf measure. We remark that functions of similar nature to (A.13) appears very often in the mathematical treatment of the study of equilibria in magnetically confined plasmas and they are related with the concept of *averaging over magnetic surfaces* (see [6,21–23,26]). From the mathematical point of view, (A.13) can be expressed in terms of the relative rearrangement b_{*u}. In particular, in (A.4) it was proved that when u is regular enough (i.e., $|\nabla u(t)|^{-1} \in L^1(\Omega)$) we can set $\theta = u_*(t,\sigma)$, $\sigma \in \Omega_*$, and thus (A.13) becomes

$$F(u_*(t,\sigma))F'(u_*(t,\sigma)) = -p'(u_*(t,\sigma))b_{*u}(t,\sigma) \quad \text{a.e. } (t,\sigma) \in (0,T) \times \Omega_*,$$

from where we obtain $(\mathscr{P})$. More precisely, we have the following result (whose proof follows the same arguments of the one of Theorem 6 in [17]):

Theorem 5.1. *Let* $u : [0,T] \times \Omega \to \mathbb{R}$ *a measurable function such that, for a.e.* $t \in (0,T)$, $u(t) \in W^{1,+\infty}(\Omega)$ *and* $\text{meas}\{x \in \Omega: |\nabla u(t,x)| = 0\} = 0$ *and set* $\hat{m} = \text{essinf}\, u$ *and* $M = \text{esssup}\, u$. *Assume that* $\hat{m}(t) = \inf_\Omega u(t) \leqslant 0$ *a.e.* $t \in (0,T)$. *If* (u,F) *is a solution of* $(\mathscr{P}_I)$ *such that* $F \in W^{1,\infty}(\hat{m},M)$ *and* $F(\theta) = F_v$ *for* $\theta \leqslant 0$, *then* u *is a solution of* $(\mathscr{P})$ *and*

$$F(\theta) = \left[F_v^2 - 2\int_0^{\theta_+} p'(s) b_{*u(t)}(|u(t) > s|)\, ds\right]_+^{1/2},$$

for a.e. t *and for all* $\theta \in (\hat{m}(t), M(t))$. *Conversely, if* u *is a solution of* $(\mathscr{P})$, *then the pair* (u,G) *is a solution of* $(\mathscr{P}_I)$ *and* $G \in W^{1,\infty}(\hat{m}(t), M(t))$ *for a.e.* $t \in (0,T)$.

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