Homogenization in Chemical Reactive Flows

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Abstract

This paper deals with the homogenization of two nonlinear models for chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid grains (or reactive obstacles). In the first model, the chemical reactions take place on the walls of the grains, while in the second one the fluid penetrates the grains and the reactions take place therein. The effective behavior of these reactive flows is described by a new elliptic boundary-value problem containing an extra zero-order term which captures the effect of the chemical reactions.

Key words: homogenization, reactive flows, variational inequality, monotone graph.

1 Introduction

The general question which will make the object of this paper is the homogenization of chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid grains (or reactive obstacles). We will focus our attention on two nonlinear problems which describe the motion of a reactive fluid having different chemical features. For a nice presentation of the chemical aspects involved in our first model (and also for some mathematical and historical backgrounds) we refer to S.N. Antontsev et al. [1], J. Bear [4], J.I. Díaz [16], [17], [18] and W.S. Norman [24]. For the second model, the interested reader can consult the books by U. Hornung [19] and W.S. Norman [24] and the references therein.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let us introduce a set of periodically distributed reactive obstacles. As a result, we obtain an open set Ω^ε which will be referred to as being the *exterior domain*; ε represents a small parameter related to the characteristic size of the reactive obstacles.

The first nonlinear problem studied in this paper concerns the stationary reactive flow of a fluid confined in Ω^ε, of concentration u^ε, reacting on the boundary of the obstacles. A simplified version of this problem can be written as follows:

$$(1.1)\qquad \begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, ν is the exterior unit normal to Ω^ε, $a > 0$, $f \in L^2(\Omega)$ and S^ε is the boundary of our exterior medium $\Omega \setminus \overline{\Omega^\varepsilon}$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D_f > 0$.

The semilinear boundary condition on S^ε in problem (1.1) describes the chemical reactions which take place locally at the interface between the reactive fluid and the grains. From strictly chemical point of view, this situation represents, equivalently, the effective reaction on the walls of the chemical reactor between the fluid filling Ω^ε and a chemical reactant located in the rigid solid grains.

The function g in (1.1) is assumed to be given. Two model situations will be considered; the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$ and the case of a maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the subdifferential of a convex lower semicontinuous function G. These two general situations are well illustrated by the following important practical examples:

$$a) \quad g(v) = \frac{\alpha v}{1+\beta v}, \quad \alpha, \beta > 0 \quad \text{(Langmuir kinetics)}$$

and

$$b) \quad g(v) = |v|^{p-1}v, \quad 0 < p < 1 \quad \text{(Freundlich kinetics)}.$$

The exponent p is called *the order of the reaction.* In some applications the limit case ($p = 0$) is of great relevance (see Remark 2.8). It is worth remarking that if we assume $f \geq 0$, one can prove (see, e.g. [18]) that $u^\varepsilon \geq 0$ in $\Omega \setminus \overline{\Omega^\varepsilon}$ and $u^\varepsilon > 0$ in Ω^ε, although u^ε is not uniformly positive, except in the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$, as, for instance, in example a).

The existence and uniqueness of a weak solution of (1.1) can be settled by using the classical theory of semilinear monotone problems (see, for instance, [8], [16] and [22]). As a result, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \bigcap H^2(\Omega^\varepsilon)$, where

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}.$$

Moreover, ~~if~~ in the second model situation, which is in fact the most general case we treat here, with Ω^ε we associate the following nonempty convex subset of V^ε:

$$K^\varepsilon = \{v \in V^\varepsilon \mid G(v)|_{S^\varepsilon} \in L^1(S^\varepsilon)\}, \tag{1.2}$$

then u^ε is also known to be characterized as being the unique solution of the following variational problem:

$$\left\{ \begin{array}{l} \text{Find } \; u^\varepsilon \in K^\varepsilon \; \text{ such that} \\ D_f \displaystyle\int_{\Omega^\varepsilon} Du^\varepsilon D(v^\varepsilon - u^\varepsilon)dx - \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx + a\,\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle \geq 0 \quad \forall v^\varepsilon \in K^\varepsilon, \end{array} \right. \tag{1.3}$$

where μ^ε is the linear form on $W_0^{1,1}(\Omega)$ defined by

$$\langle \mu^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} \varphi d\sigma \quad \forall \varphi \in W_0^{1,1}(\Omega).$$

From a geometrical point of view, we shall just consider periodic structures obtained by removing periodically from Ω, with period εY (where Y is a given hyper-rectangle in $\mathbb{R}^n$), an elementary reactive obstacle T which has been appropriated rescaled and which is strictly included in Y, i.e. $\overline{T} \subset Y$.

As usual in homogenization, we shall be interested in obtaining a suitable description of the asymptotic behavior, as ε tends to zero, of the solution u^ε in such domains. We will wonder, for example, whether the solution u^ε converges to a limit u as $\varepsilon \to 0$. And if this limit exists, can it be characterized?

In the second model situation (in absence of any additional regularity on g), the solution u^ε, properly extended to the whole of Ω, converges to the unique solution of the following variational inequality:

$$(1.4)\quad \begin{cases} u \in H_0^1(\Omega) \\ \int\limits_\Omega QDuD(v-u)dx \geq \int\limits_\Omega f(v-u)dx - a\dfrac{|\partial T|}{|Y \setminus T|}\int\limits_\Omega (G(v)-G(u))dx, \quad \forall v \in H_0^1(\Omega). \end{cases}$$

Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$(1.5)\qquad q_{ij} = D_f\left(\delta_{ij} + \frac{1}{|Y \setminus T|}\int\limits_{Y\setminus T}\frac{\partial \chi_j}{\partial y_i}dy\right)$$

in terms of the functions χ_i, $i = 1, ..., n$, solutions of the so-called cell problems

$$(1.6)\qquad \begin{cases} -\Delta\chi_i = 0 \ \text{ in } \ Y \setminus T, \\ \dfrac{\partial(\chi_i + y_i)}{\partial\nu} = 0 \ \text{ on } \ \partial T, \\ \chi_i \ \ Y- \ \text{ periodic.} \end{cases}$$

Remark that if g is smooth, then g is the classical derivative of G.

The chemical situation behind the second nonlinear problem we will treat in this paper is slightly different that the previous one; it also involves a chemical reactor containing reactive grains, but we assume that now there is an internal reaction inside the grains, instead just on their boundaries. In fact, it is therefore a transmission problem with an unknown flux on the boundary of each grain.

To simplify matters, we shall just focus on the case of a function g which is continuous, monotone increasing and such that $g(0) = 0$; examples $a)$ and $b)$ are both covered by this class of functions $g's$ and, of course, both are still our main practical examples.

A simplified setting of this kind of models is as follows:

$$(1.7)\qquad \begin{cases} -D_f\Delta u^\varepsilon = f \quad \text{ in } \Omega^\varepsilon, \\ -D_p\Delta v^\varepsilon + ag(v^\varepsilon) = 0, \quad \text{ in } \Omega \setminus \overline{\Omega^\varepsilon} \\ -D_f\dfrac{\partial u^\varepsilon}{\partial\nu} = D_p\dfrac{\partial v^\varepsilon}{\partial\nu} \quad \text{ on } S^\varepsilon, \\ u^\varepsilon = v^\varepsilon \quad \text{ on } S^\varepsilon, \\ u^\varepsilon = 0 \quad \text{ on } \partial\Omega, \end{cases}$$

where D_p is a second diffusion coefficient characterizing the granular material filling the reactive obstacles. As in the previous case, the classical semilinear theory guarantees the well-posedness of this problem.

If we define θ^ε as being:

$$\theta^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & x \in \Omega^\varepsilon, \\ v^\varepsilon(x) & x \in \Omega \setminus \overline{\Omega^\varepsilon}, \end{cases}$$

and we introduce

$$A = \begin{cases} D_f Id & \text{in } Y \setminus T \\ D_p Id & \text{in } T, \end{cases}$$

then our main result of convergence for this model shows that θ^ε converges weakly in $H_0^1(\Omega)$ to the unique solution of the following homogenized problem:

$$\begin{cases} -\sum_{i,j=1}^{n} a_{ij}^0 \frac{\partial^2 u}{\partial x_i \partial x_j} + a\frac{|T|}{|Y \setminus T|} g(u) = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Here, $A^0 = ((a_{ij}^0))$ is the homogenized matrix, whose entries are defined as follows:

$$a_{ij}^0 = \frac{1}{|Y|} \int\limits_Y \left(a_{ij} + a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) dy, \tag{1.9}$$

in terms of the functions χ_j, $j = 1, ..., n$, solutions of the so-called cell problems

$$\begin{cases} -\mathrm{div}(AD(y_j + \chi_j)) = 0 & \text{in } Y, \\ \chi_j - Y \text{ periodic.} \end{cases} \tag{1.10}$$

Notice that the two reactive flows studied in this paper, namely (1.1) and (1.7), lead to completely different effective behaviors. The macroscopic problem (1.4) arises from the homogenization of a boundary-value problem in the exterior of some periodically distributed obstacles and the zero-order term occurring in (1.4) has its origin in this particular structure of the model. The influence of the chemical reactions taking place on the boundaries of the reactive obstacles is reflected in the appearance of this zero-order extra-term. On the other hand, the second model is again a boundary-value problem, but this time in the whole domain Ω, with discontinuous coefficients. Its macroscopic behavior (see (1.8)) also involves a zero-order term, but of a completely different nature; it is originated in the chemical reactions occurring inside the grains.

The approach we used is the so-called energy method introduced by L. TARTAR [25], [26] for studying homogenization problems. It consists of constructing suitable test functions that are used in our variational problems. However, it is worth mentioning that the Γ-convergence of integral functionals involving oscillating obstacles could be a successful alternative. Extensive references on this topic can be found in the monographs of G. DAL MASO [15] and of A. BRAIDES AND A. DEFRANCESCHI [7]. For example, our main result in Chapter 2 (cf. Theorem 2.6) can also be interpreted as a Γ-convergence-type result for the functionals

$$v \longmapsto \frac{1}{2} D_f \int\limits_{\Omega^\varepsilon} DvDvdx + a\langle \mu^\varepsilon, G(v) \rangle - \int_{\Omega^\varepsilon} fvdx + I_{K^\varepsilon}(v)$$

(where I_{K^ε} is the indicator function of the set K^ε, i.e. I_{K^ε} is equal to zero if v belongs to K^ε and $+\infty$ otherwise) to the limit functional

$$v \longmapsto \frac{1}{2} \int\limits_\Omega QDvDvdx + a\frac{|\partial T|}{|Y \setminus T|} \int\limits_\Omega G(v)dx - \int_\Omega fvdx,$$

which is the energy functional associated to (1.3).

Also, let us mention that another possible way to get the limit problem (1.8) could be to use the two-scale convergence technique, coupled with periodic modulation, as in [6].

Regarding our second problem, i.e. chemical reactive flows through periodic array of cells, a related work was completed by HORNUNG ET AL. [21] using nonlinearities which are essentially different from the ones we consider in the present paper. The proof of these authors is also different, since it is mainly based on the technique of two-scale convergence, which, as already mentioned, proves to be a successful alternative for this kind of problems. However, we have decided to use the energy method, coupled with monotonicity methods and results from the theory of semilinear problems, because it offered us the possibility to cover the nonlinear cases of practical importance mentioned above.

The structure of our paper is as follows: first, let us mention that we shall just focus on the case $n \geq 3$, which will be treated explicitly. The case $n = 2$ is much more simpler and we shall omit to treat it. In Section 2 we start by analyzing the first nonlinear problem, namely (1.1). We begin with the case of a monotone smooth function g and we prove the convergence result using the energy method. Next, we treat the case of a maximal monotone graph, by writing our microscopic problem in the form of a variational inequality. The case of a reactive flow penetrating a periodical structure of grains is addressed in Section 3.

Finally, notice that throughout the paper, by C we shall denote a generic fixed strictly positive constant, whose value can change from line to line.

2 Chemical reactions on the walls of a chemical reactor

In this section, we will be concerned with the stationary reactive flow of a fluid confined in the exterior of some periodically distributed obstacles, reacting on the boundaries of the obstacles. We will treat separately the situation in which the nonlinear function g in (1.1) is a monotone smooth function satisfying the condition $g(0) = 0$ and the situation in which g is a maximal monotone graph with $g(0) = 0$.

Let Ω be a smooth bounded connected open subset of $\mathbb{R}^n$ $(n \geq 3)$ and let $Y = [0, l_1[\times ... [0, l_n[$ be the representative cell in $\mathbb{R}^n$. Denote by T an open subset of Y with smooth boundary ∂T such that $\overline{T} \subset Y$. We shall refer to T as being *the elementary obstacle*.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. For each ε and for any integer vector $k \in \mathbb{Z}^n$, set T_k^ε the translated image of εT by the vector $kl = (k_1 l_1, ..., k_n l_n)$:

$$T_k^\varepsilon = \varepsilon(kl + T).$$

The set T_k^ε represents the obstacles in $\mathbb{R}^n$. Also, let us denote by T^ε the set of all the obstacles contained in Ω, i.e.

$$T^\varepsilon = \bigcup \left\{ T_k^\varepsilon \ \mid \ \overline{T_k^\varepsilon} \subset \Omega, \ k \in \mathbb{Z}^n \right\}.$$

Set

$$\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}.$$

Hence, Ω^ε is a periodical domain with periodically distributed obstacles of size of the same order as the period. Remark that the obstacles do not intersect the boundary $\partial\Omega$.

Let

$$S^\varepsilon = \cup \{ \partial T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega, \ k \in \mathbb{Z}^n \}.$$

So

$$\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon.$$

We shall also use the following notations:

$$|\omega| = \text{ the Lebesgue measure of any measurable subset } \omega \text{ of } \mathbb{R}^n,$$

$$\chi_{\omega} = \text{the characteristic function of the set } \omega,$$

$$Y^* = Y \setminus \overline{T},$$

and

(2.1) $$\rho = \frac{|Y^*|}{|Y|}.$$

Moreover, for an arbitrary function $\psi \in L^2(\Omega^\varepsilon)$, we shall denote by $\widetilde{\psi}$ its extension by zero inside the obstacles:

$$\widetilde{\psi} = \begin{cases} \psi & \text{in} \quad \Omega^\varepsilon, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega^\varepsilon} \end{cases}$$

and, also, for any open subset $D \subset \mathbb{R}^n$ and for any function $g \in L^1(D)$, we set

(2.2) $$\mathcal{M}_D(g) = \frac{1}{|D|} \int_D g dx.$$

In the sequel we reserve the symbol # to denote periodicity properties.

2.1 Setting of the problem

As already mentioned, we are interested in studying the behavior of the solution, in such a periodical domain, of the following problem:

(2.3) $$\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, ν is the exterior unit normal to Ω^ε, $a > 0$, $f \in L^2(\Omega)$ and g is assumed to be given. Two model situations will be considered; the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$ and the case of a maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the subdifferential of a convex lower semicontinuous function G. These two general situations are well illustrated by the following important practical examples:

$$a) \quad g(v) = \frac{\alpha v}{1 + \beta v}, \quad \alpha, \beta > 0 \quad \text{(Langmuir kinetics)}$$

and

$$b) \quad g(v) = |v|^{p-1} v, \quad 0 < p < 1 \quad \text{(Freundlich kinetics)}.$$

The exponent p is called *the order of the reaction.* It is worth remarking that if we assume $f \geq 0$, one can prove (see, e.g. [18]) that $u^\varepsilon \geq 0$ in $\Omega \setminus \overline{\Omega^\varepsilon}$ and $u^\varepsilon > 0$ in Ω^ε, although u^ε is not uniformly positive except in the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$, as, for instance, in example a). Moreover, since u represents a concentration, it could be natural to assume that $f \leq 1$, and again one can prove that, in this case, $u \leq 1$. Without loss of generality, in what follows we shall assume that $D_f = 1$.

2.2 First model situation: g smooth

Let g be a continuously differentiable function, monotonously non-decreasing and such that $g(v) = 0$ iff $v = 0$. We shall suppose that there exist a positive constant C and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$\left|\frac{\partial g}{\partial v}\right| \leq C(1 + |v|^q). \tag{2.4}$$

Let us introduce the functional space

$$V^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega \right\},$$

with

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega^\varepsilon)} \, .$$

The weak formulation of problem (2.3) (written for $D_f = 1$) is:

$$\begin{cases} \text{Find } u^\varepsilon \in V^\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx + a\varepsilon \int_{S^\varepsilon} g(u^\varepsilon) \varphi d\sigma = \int_{\Omega^\varepsilon} f \varphi dx \quad \forall \varphi \in V^\varepsilon. \end{cases} \tag{2.5}$$

By classical existence results (see [8]), there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \cap H^2(\Omega^\varepsilon)$ of problem (2.3).

The solution u^ε of problem (2.3) being defined only on Ω^ε, we need to extend it to the whole of Ω to be able to state the convergence result. In order to do that, let us recall the following well-known extension result (see [10]):

Lemma 2.1 *There exists a linear continuous extension operator $P^\varepsilon \in \mathcal{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap$ $\cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ and a positive constant C, independent of ε, such that*

$$\|P^\varepsilon v\|_{L^2(\Omega)} \leq C \, \|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \leq C \, \|\nabla v\|_{L^2(\Omega^\varepsilon)} \, ,$$

for any $v \in V^\varepsilon$. ∎

An immediate consequence of the previous lemma is the following Poincaré's inequality in V^ε :

Lemma 2.2 *There exists a positive constant C, independent of ε, such that*

$$\|v\|_{L^2(\Omega^\varepsilon)} \leq C \, \|\nabla v\|_{L^2(\Omega^\varepsilon)} \, ,$$

for any $v \in V^\varepsilon$. ∎

The main result of this section is the following one:

Theorem 2.3 *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the variational problem (2.5) such that*

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \textit{weakly in } \; H_0^1(\Omega),$$

where u is the unique solution of

$$\text{(2.6)} \qquad \begin{cases} -\sum_{i,j=1}^{n} q_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} + a\dfrac{|\partial T|}{|Y^*|}g(u) = f & \textit{in } \Omega, \\ u = 0 \quad \textit{on } \partial\Omega. \end{cases}$$

Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$\text{(2.7)} \qquad q_{ij} = \delta_{ij} + \frac{1}{|Y^*|}\int_{Y^*} \frac{\partial \chi_j}{\partial y_i} dy$$

in terms of the functions χ_i, $i = 1, ..., n$, solutions of the so-called cell problems

$$\text{(2.8)} \qquad \begin{cases} -\Delta\chi_i = 0 \quad \textit{in} \ \ Y^*, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \quad \textit{on} \ \ \partial T, \\ \chi_i \ \ Y- \ \textit{periodic}. \end{cases}$$

The constant matrix Q is symmetric and positive-definite. ▪

Proof. We divide the proof into four steps.

First step. Let $u^\varepsilon \in V^\varepsilon$ be the solution of the variational problem (2.5) and let $P^\varepsilon u^\varepsilon$ be the extension of u^ε inside the obstacles given by Lemma 2.1. Taking $\varphi = u^\varepsilon$ as a test function in (2.5), using Schwartz and Poincaré's inequalities, we easily get

$$||P^\varepsilon u^\varepsilon||_{H_0^1(\Omega)} \leq C.$$

Consequently, by passing to a subsequence, still denoted by $P^\varepsilon u^\varepsilon$, we can assume that there exists $u \in H_0^1(\Omega)$ such that

$$\text{(2.9)} \qquad P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega).$$

It remains to identify the limit equation satisfied by u.

Second step. In order to get the limit equation satisfied by u we have to pass to the limit in (2.5). For getting the limit of the second term in the left hand side of (2.5), let us introduce, for any $h \in L^{s'}(\partial T)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} h(\frac{x}{\varepsilon})\varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [9] that

$$\text{(2.10)} \qquad \mu_h^\varepsilon \to \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))',$$

where

$$\langle \mu_h, \varphi \rangle = \mu_h \int_\Omega \varphi dx,$$

with

$$\mu_h = \frac{1}{|Y|}\int_{\partial T} h(y) d\sigma.$$

sobra

In the particular case in which $h \in L^\infty(\partial T)$ (or even h is constant) we have

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

In what follows, we shall denote by μ^ε the above introduced measure in the particular case in which $h = 1$. Notice that in this case μ_h becomes $\mu_1 = \dfrac{|\partial T|}{|Y|}$.

Let us prove now that for any $\varphi \in \mathcal{D}(\Omega)$ and for any $v^\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$, we get

$$\varphi g(v^\varepsilon) \rightharpoonup \varphi g(v) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega), \tag{2.11}$$

where

$$\overline{q} = \frac{2n}{q(n-2)+n}.$$

To prove (2.11), let us first note that

$$\sup \|\nabla g(v^\varepsilon)\|_{L^{\overline{q}}(\Omega)} < \infty. \tag{2.12}$$

Indeed, from the growth condition (2.4) imposed to g, we get

$$\int_\Omega \left|\frac{\partial g}{\partial x_i}(v^\varepsilon)\right|^{\overline{q}} dx \le C \int_\Omega \left(1 + |v^\varepsilon|^{q\overline{q}}\right) \left|\frac{\partial v^\varepsilon}{\partial x_i}\right|^{\overline{q}} dx \le$$

$$\le C(1 + (\int_\Omega |v^\varepsilon|^{q\overline{q}\gamma}\, dx)^{1/\gamma})(\int_\Omega |\nabla v^\varepsilon|^{\overline{q}\delta}\, dx)^{1/\delta},$$

where we took γ and δ such that $\overline{q}\delta = 2$, $1/\gamma + 1/\delta = 1$ and $q\overline{q}\gamma = 2n/(n-2)$. Notice that from here we get $\overline{q} = \dfrac{2n}{q(n-2)+n}$. Also, since $0 \le q < n/(n-2)$, we have $\overline{q} > 1$. Now, since

$$\sup \|v^\varepsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} < \infty,$$

we get immediately (2.12). Hence, to get (2.11), it remains only to prove that

$$g(v^\varepsilon) \to g(v) \quad \text{strongly in } L^{\overline{q}}(\Omega). \tag{2.13}$$

But this is just a consequence of the following well-known result (see [15] and [22]):

Theorem 2.4 *Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e.*

a) for every v the function $G(\cdot, v)$ is measurable with respect to $x \in \Omega$.

b) for every (a.e.) $x \in \Omega$, the function $G(x, \cdot)$ is continuous with respect to v.

Moreover, if we assume that there exists a positive constant C such that

$$|G(x,v)| \le C\left(1 + |v|^{r/t}\right),$$

with $r \ge 1$ and $t < \infty$, then the map $v \in L^r(\Omega) \mapsto G(x, v(x)) \in L^t(\Omega)$ is continuous in the strong topologies. ∎

Indeed, since

$$|g(v)| \leq C(1+|v|^{q+1}),$$

applying the above theorem for $G(x,v) = g(v)$, $t = \overline{q}$ and $r = (2n/(n-2)) - r'$, with $r' > 0$ such that $q+1 < r/t$ and using the compact injection $H^1(\Omega) \hookrightarrow L^r(\Omega)$ we easily get (2.13).

Finally, from (2.10) (with $h = 1$) and (2.11) written for $v^\varepsilon = P^\varepsilon u^\varepsilon$, we conclude

$$\langle \mu^\varepsilon, \varphi g(P^\varepsilon u^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|}\int_\Omega \varphi g(u)dx \quad \forall \varphi \in \mathcal{D}(\Omega) \tag{2.14}$$

and this ends this step of the proof.

Third step. Let ξ^ε be the gradient of u^ε in Ω^ε and let us denote by $\widetilde{\xi^\varepsilon}$ its extension with zero to the whole of Ω, i.e.

$$\widetilde{\xi^\varepsilon} = \begin{cases} \xi^\varepsilon & \text{in} \quad \Omega^\varepsilon, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega^\varepsilon}. \end{cases}$$

Obviously, $\widetilde{\xi^\varepsilon}$ is bounded in $(L^2(\Omega))^n$ and hence there exists $\xi \in (L^2(\Omega))^n$ such that

$$\widetilde{\xi^\varepsilon} \rightharpoonup \xi \quad \text{weakly in } (L^2(\Omega))^n. \tag{2.15}$$

Let us see now which is the equation satisfied by ξ. Take $\varphi \in \mathcal{D}(\Omega)$. From (2.5) we get

$$\int_\Omega \widetilde{\xi^\varepsilon}\cdot\nabla\varphi dx + a\varepsilon\int_{S^\varepsilon} g(u^\varepsilon)\varphi d\sigma = \int_\Omega \chi_{\Omega^\varepsilon} f\varphi dx. \tag{2.16}$$

Now, we can pass to the limit, with $\varepsilon \to 0$, in all the terms of (2.16). For the first one, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \widetilde{\xi^\varepsilon}\cdot\nabla\varphi dx = \int_\Omega \xi\cdot\nabla\varphi dx. \tag{2.17}$$

For the second term, using (2.14), we get

$$\lim_{\varepsilon\to 0} a\varepsilon\int_{S^\varepsilon} g(u^\varepsilon)\varphi d\sigma = a\frac{|\partial T|}{|Y|}\int_\Omega g(u)\varphi dx. \tag{2.18}$$

It is not difficult to pass to the limit in the right-hand side of (2.16). Since

$$\chi_{\Omega^\varepsilon} f \rightharpoonup \frac{|Y^*|}{|Y|} f \quad \text{weakly in } L^2(\Omega),$$

we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \chi_{\Omega^\varepsilon} f\varphi dx = \frac{|Y^*|}{|Y|}\int_\Omega f\varphi dx. \tag{2.19}$$

Putting together (2.17)-(2.19), we have

$$\int_\Omega \xi\cdot\nabla\varphi dx + a\frac{|\partial T|}{|Y|}\int_\Omega g(u)\varphi dx = \frac{|Y^*|}{|Y|}\int_\Omega f\varphi dx \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Hence ξ verifies

$$-\operatorname{div}\xi + a\frac{|\partial T|}{|Y|}g(u) = \frac{|Y^*|}{|Y|}f \quad \text{in } \Omega. \tag{2.20}$$

It remains now to identify ξ.

Fourth step. In order to identify ξ, we shall make use of the solutions of the cell-problems (2.8). For any fixed $i = 1, ..., n$, let us define

$$\Phi_{i\varepsilon}(x) = \varepsilon \left(\chi_i(\frac{x}{\varepsilon}) + y_i \right) \quad \forall x \in \Omega^\varepsilon, \tag{2.21}$$

where

$$y = \frac{x}{\varepsilon}.$$

By periodicity

$$P^\varepsilon \Phi_{i\varepsilon} \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega). \tag{2.22}$$

obstacles

Let η_i^ε be the gradient of $\Phi_{i\varepsilon}$ in Ω^ε. Denote by $\widetilde{\eta_i^\varepsilon}$ the extension by zero of η_i^ε inside the ~~holes~~. From (2.21), for the j-component of $\widetilde{\eta_i^\varepsilon}$ we get

$$\left(\widetilde{\eta_i^\varepsilon}\right)_j = \left(\widetilde{\frac{\partial \Phi_{i\varepsilon}}{\partial x_j}}\right) = \left(\widetilde{\frac{\partial \chi_i}{\partial y_j}}(y)\right) + \delta_{ij}\chi_{Y*}$$

and hence

$$\left(\widetilde{\eta_i^\varepsilon}\right)_j \rightharpoonup \frac{1}{|Y|}\left(\int_{Y^*} \frac{\partial \chi_i}{\partial y_j} dy + |Y^*|\,\delta_{ij} \right) = \frac{|Y^*|}{|Y|} q_{ij} \quad \text{weakly in } L^2(\Omega). \tag{2.23}$$

On the other hand, it is not difficult to see that η_i^ε satisfies

$$\begin{cases} -\text{div}\, \eta_i^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \eta_i^\varepsilon \cdot \nu = 0 & \text{on } S^\varepsilon. \end{cases} \tag{2.24}$$

Now, let $\varphi \in \mathcal{D}(\Omega)$. Multiplying the first equation in (2.24) by φu^ε and integrating by parts over Ω^ε we get

$$\int_{\Omega^\varepsilon} \eta_i^\varepsilon \cdot \nabla\varphi u^\varepsilon dx + \int_{\Omega^\varepsilon} \eta_i^\varepsilon \cdot \nabla u^\varepsilon \varphi dx = 0.$$

So

$$\int_{\Omega} \widetilde{\eta_i^\varepsilon} \cdot \nabla\varphi P^\varepsilon u^\varepsilon dx + \int_{\Omega^\varepsilon} \eta_i^\varepsilon \cdot \nabla u^\varepsilon \varphi dx = 0. \tag{2.25}$$

On the other hand, taking $\varphi\Phi_{i\varepsilon}$ as a test function in (2.5) we obtain

$$\int_{\Omega^\varepsilon} (\nabla u^\varepsilon \cdot \nabla\varphi)\Phi_{i\varepsilon} dx + \int_{\Omega^\varepsilon} (\nabla u^\varepsilon \cdot \nabla\Phi_{i\varepsilon})\varphi dx + a\varepsilon \int_{S^\varepsilon} g(u^\varepsilon)\varphi\Phi_{i\varepsilon} d\sigma = \int_{\Omega^\varepsilon} f\varphi\Phi_{i\varepsilon} dx$$

which, using the definitions of $\widetilde{\xi^\varepsilon}$ and $\widetilde{\eta_i^\varepsilon}$, gives

$$\int_{\Omega} \widetilde{\xi^\varepsilon} \cdot \nabla\varphi P^\varepsilon \Phi_{i\varepsilon} dx + \int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \eta_i^\varepsilon \varphi dx + a\varepsilon \int_{S^\varepsilon} g(u^\varepsilon)\varphi\Phi_{i\varepsilon} d\sigma = \int_{\Omega} f\chi_{\Omega^\varepsilon}\varphi P^\varepsilon \Phi_{i\varepsilon} dx.$$

Now, using (2.25), we get

$$\int_{\Omega} \widetilde{\xi^\varepsilon} \cdot \nabla\varphi P^\varepsilon \Phi_{i\varepsilon} dx - \int_{\Omega} \widetilde{\eta_i^\varepsilon} \cdot \nabla\varphi P^\varepsilon \Phi_{i\varepsilon} dx + a\varepsilon \int_{S^\varepsilon} g(u^\varepsilon)\varphi\Phi_{i\varepsilon} d\sigma = \int_{\Omega} f\chi_{\Omega^\varepsilon}\varphi P^\varepsilon \Phi_{i\varepsilon} dx. \tag{2.26}$$

hacer

Let us pass to the limit in (2.26). Firstly, using (2.15) and (2.22), we have

$$\lim_{\varepsilon\to 0}\int_{\Omega}\widetilde{\xi^{\varepsilon}}\cdot\nabla\varphi P^{\varepsilon}\Phi_{i\varepsilon}dx=\int_{\Omega}\xi\cdot\nabla\varphi x_i dx. \tag{2.27}$$

On the other hand, (2.9) and (2.23) imply that

$$\lim_{\varepsilon\to 0}\int_{\Omega}\widetilde{\eta_i^{\varepsilon}}\cdot\nabla\varphi P^{\varepsilon}u^{\varepsilon}dx=\frac{|Y^*|}{|Y|}\int_{\Omega}q_i\cdot\nabla\varphi u dx, \tag{2.28}$$

where q_i is the vector having the j-component equal to q_{ij}.

Because the boundary of T is smooth, of class C^2, $P^{\varepsilon}\Phi_{i\varepsilon}\in W^{1,\infty}(\Omega)$ and $P^{\varepsilon}\Phi_{i\varepsilon}\to x_i$ strongly in $L^{\infty}(\Omega)$. Then, since $g(P^{\varepsilon}u^{\varepsilon})P^{\varepsilon}\Phi_{i\varepsilon}\to g(u)x_i$ strongly in $L^{\overline{q}}(\Omega)$ and $g(P^{\varepsilon}u^{\varepsilon})P^{\varepsilon}\Phi_{i\varepsilon}$ is bounded in $W^{1,\overline{q}}(\Omega)$, we have $g(P^{\varepsilon}u^{\varepsilon})P^{\varepsilon}\Phi_{i\varepsilon}\rightharpoonup g(u)x_i$ weakly in $W^{1,\overline{q}}(\Omega)$. So

$$\lim_{\varepsilon\to 0}a\varepsilon\int_{S^{\varepsilon}}g(u^{\varepsilon})\varphi\Phi_{i\varepsilon}d\sigma=a\frac{|\partial T|}{|Y|}\int_{\Omega}g(u)\varphi x_i dx. \tag{2.29}$$

Finally, for the limit of the right-hand side of (2.26), since $\chi_{\Omega^{\varepsilon}}f\rightharpoonup\frac{|Y^*|}{|Y|}f$ weakly in $L^2(\Omega)$, using again (2.22) we have

$$\lim_{\varepsilon\to 0}\int_{\Omega}f\chi_{\Omega^{\varepsilon}}\varphi P^{\varepsilon}\Phi_{i\varepsilon}dx=\frac{|Y^*|}{|Y|}\int_{\Omega}f\varphi x_i dx. \tag{2.30}$$

Hence we get

$$\int_{\Omega}\xi\cdot\nabla\varphi x_i dx-\frac{|Y^*|}{|Y|}\int_{\Omega}q_i\cdot\nabla\varphi u dx+a\frac{|\partial T|}{|Y|}\int_{\Omega}g(u)\varphi x_i dx=\frac{|Y^*|}{|Y|}\int_{\Omega}f\varphi x_i dx. \tag{2.31}$$

Using Green's formula and equation (2.20), we have

$$-\int_{\Omega}\xi\cdot\nabla x_i\varphi dx+\frac{|Y^*|}{|Y|}\int_{\Omega}q_i\cdot\nabla u\varphi dx=0\quad\text{in }\Omega.$$

The above equality holds true for any $\varphi\in\mathcal{D}(\Omega)$. This implies that

$$-\xi\cdot\nabla x_i+\frac{|Y^*|}{|Y|}q_i\cdot\nabla u=0\quad\text{in }\Omega. \tag{2.32}$$

Writing (2.30) by components, derivating with respect to x_i, summing after i and using (2.19), we conclude that

$$\frac{|Y^*|}{|Y|}\sum_{i,j=1}^{n}q_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}=\operatorname{div}\xi=-\frac{|Y^*|}{|Y|}f+a\frac{|\partial T|}{|Y|}g(u),$$

which means that u satisfies

$$-\sum_{i,j=1}^{n}q_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+a\frac{|\partial T|}{|Y^*|}g(u)=f\quad\text{in }\Omega.$$

Since $u\in H_0^1(\Omega)$ (i.e. $u=0$ on $\partial\Omega$) and u is uniquely determined, the whole sequence $P^{\varepsilon}u^{\varepsilon}$ converges to u and Theorem 2.3 is proved. ∎

Remark 2.5 *As already mentioned, it is worth remarking that if we assume $f \geq 0$, the function g in example a) is indeed a particular example of our first model situation. Moreover, the growth condition (2.4) for g holds with $q = 0$, hence we get $\overline{q} = 2$ and convergence (2.11) holds in $H_0^1(\Omega)$. Since g is Lipschitz continuous, one can prove (see* J.I. DÍAZ *[18]) that the solution of the homogenized problem is also strictly positive on Ω. This will not be the case when g is not necessarily regular.*

2.3 Second model situation: the case of a monotone graph

In this subsection we shall treat the case in which the function g appearing in (1.1) is a single-valued maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, satisfying the condition $g(0) = 0$. Also, if we denote by $D(g)$ the domain of g, i.e. $D(g) = \{\xi \in \mathbb{R} \mid g(\xi) \neq \varnothing\}$, then we suppose that $D(g) = \mathbb{R}$. Moreover, we assume that g is continuous and there exist $C \geq 0$ and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$|g(v)| \leq C(1 + |v|^q). \tag{2.33}$$

Notice that the second important practical example $b)$ mentioned in the Introduction is a particular example of such a single-valued maximal monotone graph.

We know that in this case there exists a lower semicontinuous convex function G from $\mathbb{R}$ to $]-\infty, +\infty]$, G proper, i.e. $G \not\equiv +\infty$ such that g is the subdifferential of G, $g = \partial G$ (G is an indefinite "integral" of g). Let $G(v) = \int\limits_0^v g(s)ds$.

Define the convex set

$$K^\varepsilon = \left\{ v \in V^\varepsilon \mid G(v)|_{S^\varepsilon} \in L^1(S^\varepsilon) \right\}. \tag{2.34}$$

For a given function $f \in L^2(\Omega)$ the weak solution of the problem (2.3) is also the unique solution of the following variational inequality:

$$\begin{cases} \text{Find } \ u^\varepsilon \in K^\varepsilon \ \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} Du^\varepsilon D(v^\varepsilon - u^\varepsilon)dx - \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx + a\,\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle \geq 0 \quad \forall v^\varepsilon \in K^\varepsilon. \end{cases} \tag{2.35}$$

First, let us notice that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \cap H^2(\Omega^\varepsilon)$ of the above variational inequality (see [8]). Also, notice that it is well-known that the solution u^ε of the variational inequality (2.35) is also the unique solution of the minimization problem:

$$\begin{cases} u^\varepsilon \in K^\varepsilon, \\ J^\varepsilon(u^\varepsilon) = \inf\limits_{v \in K^\varepsilon} J^\varepsilon(v), \end{cases}$$

where

$$J^\varepsilon(v) = \frac{1}{2}\int_{\Omega^\varepsilon} |Dv|^2\, dx + a\,\langle \mu^\varepsilon, G(v)\rangle - \int_{\Omega^\varepsilon} fv dx.$$

Introduce the following functional defined on $H_0^1(\Omega)$:

$$J^0(v) = \frac{1}{2}\int_\Omega QDvDvdx + a\frac{|\partial T|}{|Y^*|}\int_\Omega G(v)dx - \int_\Omega fvdx.$$

The main result of this subsection is the following one:

Theorem 2.6 *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the variational inequality (2.35) such that*

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \textit{weakly in } \; H_0^1(\Omega),$$

where u is the unique solution of the minimization problem

$$\left\{ \begin{array}{l} \textit{Find } u \in H_0^1(\Omega) \textit{ such that} \\ J^0(u) = \inf\limits_{v \in H_0^1(\Omega)} J^0(v). \end{array} \right. \tag{2.36}$$

Moreover, $G(u) \in L^1(\Omega)$. Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries were defined by (2.7)-(2.8).

Notice that u also satisfies

$$\left\{ \begin{array}{l} -\sum\limits_{i,j=1}^{n} q_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + a \dfrac{|\partial T|}{|Y^*|} g(u) = f \quad \text{ in } \Omega, \\ u = 0 \quad \text{ on } \partial\Omega. \end{array} \right.$$

Proof of Theorem 2.6 Let u^ε be the solution of the variational inequality (2.35). We shall use the same extension $P^\varepsilon u^\varepsilon$ as in the previous case (given by Lemma 2.1). It is not difficult to see that $P^\varepsilon u^\varepsilon$ is bounded in $H_0^1(\Omega)$. So by extracting a subsequence, one has

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } \; H_0^1(\Omega). \tag{2.37}$$

Let $\varphi \in \mathcal{D}(\Omega)$. By classical regularity results $\chi_i \in L^\infty$. Using the boundedness of χ_i and φ, there exists $M \geq 0$ such that

$$\left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^\infty} \left\| \chi_i \right\|_{L^\infty} < M.$$

Let

$$v^\varepsilon = \varphi + \sum_i \varepsilon \frac{\partial \varphi}{\partial x_i}(x) \chi_i(\frac{x}{\varepsilon}). \tag{2.38}$$

Then $v^\varepsilon \in K^\varepsilon$ which will allow us to take it as a test function in (2.35). Moreover, $v^\varepsilon \to \varphi$ strongly in $L^2(\Omega)$.

Let us compute Dv^ε:

$$Dv^\varepsilon = D\varphi + \sum_i \frac{\partial \varphi}{\partial x_i}(x) D\chi_i(\frac{x}{\varepsilon}) + \varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x) \chi_i(\frac{x}{\varepsilon}).$$

So

$$Dv^\varepsilon = \sum_i \frac{\partial \varphi}{\partial x_i}(x)(\mathbf{e}_i + D\chi_i(\frac{x}{\varepsilon})) + \varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x) \chi_i(\frac{x}{\varepsilon}),$$

where $\mathbf{e}_i$, $1 \leq i \leq n$, are the elements of the canonical basis in $\mathbb{R}^n$.

Using v^ε as a test function in (2.35), we can write

$$\int_{\Omega^\varepsilon} Du^\varepsilon Dv^\varepsilon dx \geq \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon) dx + \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx - a \left\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon) \right\rangle.$$

In fact, we have

$$\int_{\Omega} DP^\varepsilon u^\varepsilon (\widetilde{Dv^\varepsilon}) dx \geq \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon) dx + \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx - a \left\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon) \right\rangle. \tag{2.39}$$

Denote

$$\rho Q\mathbf{e}_j = \frac{1}{|Y^*|}\int_{Y^*}(D\chi_j + \mathbf{e}_j)dy, \tag{2.40}$$

where $\rho = |Y^*| \,/\, |Y|$. Neglecting the term $\varepsilon \sum_i D\frac{\partial\varphi}{\partial x_i}(x)\chi_i(\frac{x}{\varepsilon})$ which actually tends strongly to zero, we can pass immediately to the limit in the left-hand side of (2.39). Hence

$$\int_\Omega DP^\varepsilon u^\varepsilon \widetilde{Dv^\varepsilon}dx \to \int_\Omega \rho Q Du D\varphi dx. \tag{2.41}$$

It is not difficult to pass to the limit in the first term of the right-hand side of (2.39). Indeed, since $v^\varepsilon \to \varphi$ strongly in $L^2(\Omega)$, we get

$$\int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx = \int_\Omega f\chi_{\Omega^\varepsilon}(v^\varepsilon - P^\varepsilon u^\varepsilon)dx \to \int_\Omega f\rho(\varphi - u)dx. \tag{2.42}$$

For the third term of the right-hand side of (2.39), assuming the growth condition (2.33) for the single-valued maximal monotone graph g and reasoning exactly like in the previous subsection, we get

$$G(P^\varepsilon u^\varepsilon) \rightharpoonup G(u) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega)$$

and then

$$\langle \mu^\varepsilon, G(P^\varepsilon u^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|}\int_\Omega G(u)dx.$$

In a similar manner, we obtain

$$\langle \mu^\varepsilon, G(v^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|}\int_\Omega G(\varphi)dx$$

and hence we get

$$a\,\langle \mu^\varepsilon, G(v^\varepsilon) - G(P^\varepsilon u^\varepsilon)\rangle \to a\frac{|\partial T|}{|Y|}\int_\Omega (G(\varphi) - G(u))dx. \tag{2.43}$$

So, it remains to pass to the limit only in the second term of the right-hand side of (2.39). For doing this, we can write down the subdifferential inequality

$$\int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int_{\Omega^\varepsilon} Dw^\varepsilon Dw^\varepsilon dx + 2\int_{\Omega^\varepsilon} Dw^\varepsilon(Du^\varepsilon - Dw^\varepsilon)dx, \tag{2.44}$$

for any $w^\varepsilon \in H_0^1(\Omega)$. Reasoning as before and choosing

$$w^\varepsilon = \overline{\varphi} + \sum_i \varepsilon\frac{\partial\overline{\varphi}}{\partial x_i}(x)\chi_i(\frac{x}{\varepsilon}),$$

where $\overline{\varphi}$ enjoys similar properties as the corresponding φ, the right-hand side of the inequality (2.44) passes to the limit and one has

$$\liminf_{\varepsilon\to 0}\int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int_\Omega \rho Q D\overline{\varphi}D\overline{\varphi}dx + 2\int_\Omega \rho Q D\overline{\varphi}(Du - D\overline{\varphi})dx,$$

for any $\overline{\varphi} \in \mathcal{D}(\Omega)$. But since $u \in H_0^1(\Omega)$, taking $\overline{\varphi} \to u$ strongly in $H_0^1(\Omega)$, we conclude

$$\liminf_{\varepsilon \to 0} \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int_\Omega \rho Q Du Du dx. \tag{2.45}$$

Putting together (2.41)-(2.43) and (2.45), we get

$$\int_\Omega \rho Q Du D\varphi dx \geq \int_\Omega f\rho(\varphi - u)dx + \int_\Omega \rho Q Du Du dx - a\frac{|\partial T|}{|Y|}\int_\Omega (G(\varphi) - G(u))dx,$$

for any $\varphi \in \mathcal{D}(\Omega)$ and hence by density for any $v \in H_0^1(\Omega)$.

So, finally, we obtain

$$\int_\Omega QDuD(v-u)dx \geq \int_\Omega f(v-u)dx - a\frac{|\partial T|}{|Y^*|}\int_\Omega (G(\varphi) - G(u))dx,$$

which gives exactly the limit problem (2.36). This completes the proof of Theorem 2.6. ∎

Remark 2.7 *The choice of the test function (2.38) gives, in fact, a first-corrector term for the weak convergence of $P^\varepsilon u^\varepsilon$ to u.* ∎

Remark 2.8 *We can treat in a similar manner the case of a multi-valued maximal monotone graph, which includes various semilinear classical boundary-value problems, such as Dirichlet or Neumann problems, Robin boundary conditions, Signorini's unilateral conditions, climatization problems (see, for instance, [8], [9], [13] and [14]). We could also include here the case of the so-called zeroth-order reactions, in which, formally, g is given by the discontinuous function $g(v) = 0$, if $v \leq 0$ and $g(v) = 1$ if $v > 0$ (see, for instance, [3]). The correct mathematical treatment needs the problem to be reformulated by using the maximal monotone graph of $\mathbb{R}^2$ associated to the Heaviside function $\beta(v) = \{0\}$ if $v < 0$, $\beta(0) = [0,1]$ and $\beta(v) = 1$ if $v > 0$. The existence and uniqueness of a solution can be found, for instance, in* H. Brézis *[8] and* J.I. Díaz *[16]. The solution is obtained by passing to the limit in a sequence of problems associated to a monotone sequence of Lipschitz functions approximating β and the results of this section remain true. Notice that now the homogenized problem becomes*

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} q_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + a\frac{|\partial T|}{|Y^*|}\beta(u) \ni f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

A curious fact is that this type of problems arises in very different contexts (see, for instance, [27]).

Remark 2.9 *Under the assumptions of this section, g does not need to be Lipschitz continuous (as, for instance, in the second example or in the multivalued example of the previous remark) and so the solution of the homogenized problem may give rise to a "dead zone" (where $u(x) = 0$) when a suitable balance between the "size" of some norm of f and the "size" of the greatest ball included in Ω holds (see* J.I. Díaz *[18]).*

Remark 2.10 *The case of a spherically symmetric isolated particle under singular reaction kinetics was considered by* J.M. Vega and A. Liñán *[28].*

3 Chemical reactive flow through grains

As already mentioned in Introduction, the chemical situation behind the second nonlinear problem we will treat here involves a chemical reactor with the grains constituted by solid catalyst particles. We assume that now the chemical reactions take place inside the grains, instead just on their boundaries. In fact, the problem corresponds to a transmission problem between the solutions of two separated equations. A simplified version of this kind of models can be formulated as follows:

$$
\begin{cases}
-D_f \Delta u^\varepsilon = f \quad \text{in } \Omega^\varepsilon, \\
-D_p \Delta v^\varepsilon + a g(v^\varepsilon) = 0, \quad \text{in } \Pi^\varepsilon \\
-D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = D_p \dfrac{\partial v^\varepsilon}{\partial \nu} \quad \text{on } S^\varepsilon, \\
u^\varepsilon = v^\varepsilon \quad \text{on } S^\varepsilon, \\
u^\varepsilon = 0 \quad \text{on } \partial\Omega.
\end{cases} \tag{3.1}
$$

Here, $\Pi^\varepsilon = \Omega \setminus \overline{\Omega^\varepsilon}$, ν is the exterior unit normal to Ω^ε, a, D_f, $D_p > 0$, $f \in L^2(\Omega)$ and g is a continuous function, monotonously non-decreasing and such that $g(v) = 0$ iff $v = 0$. Moreover, we shall suppose that there exist a positive constant C and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$|g(v)| \leq C(1 + |v|^{q+1}).$$

Notice that examples $a)$ and $b)$ are both covered by this class of functions $g's$ and, of course, both are still our main practical examples.

Let us consider again the functional space

$$V^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon) \;\mid v = 0 \text{ on } \partial\Omega \right\}$$

and introduce the space

$$H^\varepsilon = \left\{ w^\varepsilon = (u^\varepsilon, v^\varepsilon) \;\Big|\; u^\varepsilon \in V^\varepsilon, v^\varepsilon \in H^1(\Pi^\varepsilon),\; u^\varepsilon = v^\varepsilon \text{ on } S^\varepsilon \right\},$$

with the norm

$$\|w^\varepsilon\|^2_{H^\varepsilon} = \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \|\nabla v^\varepsilon\|^2_{L^2(\Pi^\varepsilon)} \cdot$$

The variational formulation of problem (3.1) is the following one:

$$
\begin{cases}
\text{Find } w^\varepsilon \in H^\varepsilon \text{ such that} \\
D_f \displaystyle\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx + D_p \int_{\Pi^\varepsilon} \nabla v^\varepsilon \cdot \nabla \psi dx + a \int_{\Pi^\varepsilon} g(v^\varepsilon) \psi dx = \int_{\Omega^\varepsilon} f \varphi dx \quad \forall (\varphi, \psi) \in K^\varepsilon.
\end{cases} \tag{3.2}
$$

Under the above structural hypotheses and the conditions fulfilled by H^ε, it is well-known by classical existence and uniqueness results (see [8] and [22]) that (3.2) is a well-posed problem.

Let us note that we can write the above microscopic model in the following equivalent weak form:

$$
\begin{cases}
D_f \displaystyle\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi dx + D_p \int_{\Pi^\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi dx + a \int_{\Pi^\varepsilon} g(v^\varepsilon) \varphi dx = \\
\qquad = \displaystyle\int_{\Omega^\varepsilon} f \varphi dx \quad \forall \varphi \in H^1(\Omega),\; \varphi = 0 \quad \text{on } \partial\Omega, \\
D_p \displaystyle\int_{\Pi^\varepsilon} \nabla v^\varepsilon \cdot \nabla \psi dx + a \int_{\Pi^\varepsilon} g(v^\varepsilon) \psi dx = 0 \quad \forall \psi \in H^1_0(\Pi^\varepsilon), \\
\qquad u^\varepsilon = v^\varepsilon \quad \text{on } S^\varepsilon.
\end{cases} \tag{3.3}
$$

Also, note that if we let

$$\theta^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & x \in \Omega^\varepsilon, \\ v^\varepsilon(x) & x \in \Pi^\varepsilon, \end{cases}$$

then (3.3) is a weak form of

$$\begin{cases} -D\Delta\theta^\varepsilon = F & \text{in } \Omega, \\ \theta^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$D = \chi_{\Omega^\varepsilon} D_f + (1 - \chi_{\Omega^\varepsilon}) D_p$$

and

$$F = \chi_{\Omega^\varepsilon} f - (1 - \chi_{\Omega^\varepsilon}) ag.$$

By classical existence results there is a unique solution $\theta^\varepsilon \in H_0^1(\Omega)$ and, by restriction, we obtain u^ε and v^ε as required.

Let us introduce the matrix

$$A = \begin{cases} D_f Id & \text{in } Y \backslash T \\ D_p Id & \text{in } T. \end{cases}$$

The main result of this section is the following one:

Theorem 3.1 *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the variational problem (3.2) such that*

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \textit{weakly in } H_0^1(\Omega),$$

where u is the unique solution of

$$(3.4) \qquad \begin{cases} -\sum_{i,j=1}^{n} a_{ij}^0 \dfrac{\partial^2 u}{\partial x_i \partial x_j} + a\dfrac{|T|}{|Y*|} g(u) = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Here, $A^0 = ((a_{ij}^0))$ is the homogenized matrix, whose entries are defined as follows:

$$(3.5) \qquad a_{ij}^0 = \frac{1}{|Y|} \int_Y \left(a_{ij} + a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) dy,$$

in terms of the functions χ_i, $i = 1, ..., n$, solutions of the so-called cell problems

$$(3.6) \qquad \begin{cases} -div(AD(y_j + \chi_j)) = 0 & \textit{in } Y, \\ \chi_j - Y \textit{ periodic.} \end{cases}$$

The constant matrix A^0 is symmetric and positive-definite. ∎

3.1 A priori estimates

Apart from the results given by Lemma 2.1 and Lemma 2.2, we recall the following well-known result (see, for instance, [20] and [23]):

Lemma 3.2 *There exists a positive constant C, independent of ε, such that*

$$\|v\|^2_{L^2(S^\varepsilon)} \leq C(\varepsilon^{-1} \|v\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon \|\nabla v\|^2_{L^2(\Omega^\varepsilon)}), \tag{3.7}$$

for any $v \in V^\varepsilon$. ∎

Also, in the same spirit of Lemma 6.1 in [11], we can prove immediately:

Lemma 3.3 *There exists a positive constant C, independent of ε, such that*

$$\|v\|^2_{L^2(\Pi^\varepsilon)} \leq C(\varepsilon \|v\|^2_{L^2(S^\varepsilon)} + \varepsilon^2 \|\nabla v\|^2_{L^2(\Pi^\varepsilon)}), \tag{3.8}$$

for every $v \in H^1(\Pi^\varepsilon)$. ∎

In order to describe the effective behavior of u^ε and v^ε, as $\varepsilon \to 0$, we need to prove some a priori estimates for them.

Proposition 3.4 *Let u^ε and v^ε be the solutions of the problem (3.1). There exists a positive constant C, independent of ε, such that*

$$\|P^\varepsilon u^\varepsilon\|_{H^1_0(\Omega)} \leq C, \tag{3.9}$$

$$\left\|\widetilde{v^\varepsilon}\right\|_{L^2(\Omega)} \leq C, \tag{3.10}$$

$$\|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)\times L^2(\Pi^\varepsilon)} \leq C, \tag{3.11}$$

$$\|P^\varepsilon u^\varepsilon - v^\varepsilon\|_{L^2(\Pi^\varepsilon)} \leq C\varepsilon. \tag{3.12}$$ ∎

Proof. Let us take $(u^\varepsilon, v^\varepsilon)$ as a test function in (3.2). Using the properties of f and g, Hölder and Poincaré's inequalities, the first three estimates come immediately. In order to get the fourth one, we shall make use of Lemma 3.3:

$$\|P^\varepsilon u^\varepsilon - v^\varepsilon\|^2_{L^2(\Pi^\varepsilon)} \leq C\left(\varepsilon \|u^\varepsilon - v^\varepsilon\|^2_{L^2(S^\varepsilon)} + \varepsilon^2 \|\nabla(P^\varepsilon u^\varepsilon - v^\varepsilon)\|^2_{L^2(\Pi^\varepsilon)}\right) \leq$$

$$\leq C\varepsilon^2 \left(\|\nabla P^\varepsilon u^\varepsilon\|_{L^2(\Omega)} + \|\nabla v^\varepsilon\|_{L^2(\Pi^\varepsilon)}\right)^2 \leq C\varepsilon^2 \left(\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} + \|\nabla v^\varepsilon\|_{L^2(\Pi^\varepsilon)}\right)^2 \leq C\varepsilon^2,$$

which concludes the proof. ∎

Corollary 3.5 *If u^ε and v^ε are the solutions of the problem (3.1), then, passing to a subsequence, still denoted by ε, there exist $u \in H^1_0(\Omega)$ and $v \in L^2(\Omega)$ such that*

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \textit{weakly in } H^1_0(\Omega), \tag{3.13}$$

$$\widetilde{v^\varepsilon} \rightharpoonup v \quad \textit{weakly in } L^2(\Omega) \tag{3.14}$$

and

$$v = \frac{|T|}{|Y|} u. \tag{3.15}$$ ∎

Proof. The convergence results (3.13)-(3.14) are direct consequences of the estimates (3.9)-(3.10). To prove (3.15), let $\varphi \in L^2(\Omega)$. We have

$$\int_\Omega \widetilde{v^\varepsilon}\varphi dx = \int_{\Pi^\varepsilon} v^\varepsilon \varphi dx = \int_{\Pi^\varepsilon} (v^\varepsilon - P^\varepsilon u^\varepsilon)\varphi dx + \int_{\Pi^\varepsilon} P^\varepsilon u^\varepsilon \varphi dx \quad \forall \varphi \in L^2(\Omega).$$

From Proposition 3.4 we get

$$|\int_{\Pi^\varepsilon} (v^\varepsilon - P^\varepsilon u^\varepsilon)\varphi dx| \leq \|v^\varepsilon - P^\varepsilon u^\varepsilon\|_{L^2(\Pi^\varepsilon)} \|\varphi\|_{L^2(\Omega)} \to 0.$$

Hence, using (3.13) and the fact that $\chi_{\Pi^\varepsilon} \rightharpoonup |T| / |Y|$ weakly in $L^2(\Omega)$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega \widetilde{v^\varepsilon}\varphi dx = \lim_{\varepsilon\to 0} \int_\Omega \chi_{\Pi^\varepsilon} P^\varepsilon u^\varepsilon \varphi dx = \frac{|T|}{|Y|} \int_\Omega u\varphi dx,$$

which gives exactly (3.15). ▪

Finally, let us note that there exists a positive constant C, independent of ε, such that

$$\int_\Omega |\theta^\varepsilon|^2 \, dx \leq C$$

and

$$\int_\Omega |\nabla \theta^\varepsilon|^2 \, dx \leq C.$$

Hence, there exists $\theta \in H_0^1(\Omega)$ such that

$$\theta^\varepsilon \rightharpoonup \theta \qquad \text{weakly in } H_0^1(\Omega)$$

and it is not difficult to see that $\theta = u$. This proves, in fact, the following

Corollary 3.6 *Let θ^ε be defined by*

$$\theta^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & x \in \Omega^\varepsilon, \\ v^\varepsilon(x) & x \in \Pi^\varepsilon. \end{cases}$$

Then, there exists $\theta \in H_0^1(\Omega)$ such that

$$\theta^\varepsilon \rightharpoonup \theta \qquad \textit{weakly in } H_0^1(\Omega),$$

where θ is the unique solution of

$$\begin{cases} -\sum_{i,j=1}^n a_{ij}^0 \dfrac{\partial^2 \theta}{\partial x_i \partial x_j} + a\dfrac{|T|}{|Y^*|} g(\theta) = f & \textit{in } \Omega, \\ \theta = 0 \quad \textit{on } \partial\Omega, \end{cases}$$

and A^0 is given by (3.5)-(3.6), i.e. $\theta = u$, due to the well-posedness of problem (3.4).

3.2 Proof of Theorem 3.1

Set

$$\xi^\varepsilon = (\xi_1^\varepsilon, \xi_2^\varepsilon) = (D_f \nabla u^\varepsilon, D_p \nabla v^\varepsilon).$$

From (3.11) it follows that there exists a positive constant C such that

$$\|\xi_1^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C$$

and

$$\|\xi_2^\varepsilon\|_{L^2(\Pi^\varepsilon)} \leq C.$$

If we denote by $\sim$ the zero extension to the whole of Ω of functions defined on Ω^ε or Π^ε, we see that $\widetilde{\xi_1^\varepsilon}$ and $\widetilde{\xi_2^\varepsilon}$ are bounded in $(L^2(\Omega))^n$ and hence there exist $\xi_1, \xi_2 \in (L^2(\Omega))^n$ such that

$$\widetilde{\xi_i^\varepsilon} \rightharpoonup \xi_i \quad \text{weakly in } (L^2(\Omega))^n, \quad i = 1, 2. \tag{3.16}$$

Let us see now which is the equation satisfied by ξ_1 and ξ_2. Let $\phi \in \mathcal{D}(\Omega)$. Taking $(\phi_{|\Omega^\varepsilon}, \phi_{|\Pi^\varepsilon})$ as a test function in (3.2) we get

$$\int_\Omega \widetilde{\xi_1^\varepsilon} \cdot \nabla\phi dx + \int_\Omega \widetilde{\xi_2^\varepsilon} \cdot \nabla\phi dx + a \int_{S^\varepsilon} g(v^\varepsilon)\phi d\sigma = \int_\Omega \chi_{\Omega^\varepsilon} f\phi dx. \tag{3.17}$$

Now, we can pass to the limit, with $\varepsilon \to 0$, in all the terms of (3.17). For the first two, we have

$$\lim_{\varepsilon\to 0} \int_\Omega \widetilde{\xi_1^\varepsilon} \cdot \nabla\phi dx = \int_\Omega \xi_1 \cdot \nabla\phi dx \tag{3.18}$$

and

$$\lim_{\varepsilon\to 0} \int_\Omega \widetilde{\xi_2^\varepsilon} \cdot \nabla\phi dx = \int_\Omega \xi_2 \cdot \nabla\phi dx. \tag{3.19}$$

In order to pass to the limit in the third term, let us notice that, exactly like in Section 2.2, using Theorem 2.4, we can easily prove that for any $\phi \in \mathcal{D}(\Omega)$ and for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$\phi g(z^\varepsilon) \rightharpoonup \phi g(z) \quad \text{strongly in } L^{\overline{q}}(\Omega).$$

In particular, we have

$$\phi g(\theta^\varepsilon) \rightharpoonup \phi g(\theta) \quad \text{strongly in } L^{\overline{q}}(\Omega). \tag{3.20}$$

Now, let us write $a \int\limits_{\Pi^\varepsilon} g(v^\varepsilon)\phi d\sigma$ in the following form:

$$a\int_{\Pi^\varepsilon} g(v^\varepsilon)\phi d\sigma = a\int_{\Pi^\varepsilon} g(\theta^\varepsilon)\phi d\sigma = a\int_\Omega g(\theta^\varepsilon)\phi d\sigma - a\int_{\Omega^\varepsilon} g(\theta^\varepsilon)\phi d\sigma. \tag{3.21}$$

Obviously

$$\lim_{\varepsilon\to 0} a\int_\Omega g(\theta^\varepsilon)\phi d\sigma = a\int_\Omega g(\theta)\phi dx = a\int_\Omega g(u)\phi dx. \tag{3.22}$$

On the other hand, we know that $\chi_{\Omega^\varepsilon} \rightharpoonup \dfrac{|Y^*|}{|Y|}$ weakly in any $L^\sigma(\Omega)$ with $\sigma \geq 1$. In particular, defining q^* such that

$$\frac{1}{\overline{q}} + \frac{1}{q^*} = 1,$$

we see that $q^* \geq 1$ and, consequently,

$$\chi_{\Omega^\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|} \quad \text{weakly in } L^{q^*}(\Omega). \tag{3.23}$$

Hence, from (3.20)-(3.23), we obtain

$$\lim_{\varepsilon \to 0} a \int_{\Pi^\varepsilon} g(v^\varepsilon)\phi d\sigma = a\frac{|T|}{|Y|} \int_{\Omega} g(u)\phi dx. \tag{3.24}$$

It is not difficult to pass to the limit in the right-hand side of (3.17). Since

$$\chi_{\Omega^\varepsilon} f \rightharpoonup \frac{|Y^*|}{|Y|} f \quad \text{weakly in } L^2(\Omega),$$

we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega} \chi_{\Omega^\varepsilon} f\phi dx = \frac{|Y^*|}{|Y|} \int_{\Omega} f\phi dx. \tag{3.25}$$

Putting together (3.18), (3.19), (3.24) and (3.25), we have

$$\int_{\Omega} \xi_1 \cdot \nabla\phi dx + \int_{\Omega} \xi_2 \cdot \nabla\phi dx + a\frac{|T|}{|Y|} \int_{\Omega} g(u)\phi dx = \frac{|Y^*|}{|Y|} \int_{\Omega} f\phi dx \quad \forall\phi \in \mathcal{D}(\Omega).$$

Hence

$$-\operatorname{div}(\xi_1 + \xi_2) + a\frac{|T|}{|Y|} g(u) = \frac{|Y^*|}{|Y|} f \quad \text{in } \Omega. \tag{3.26}$$

It remains now to identify $\xi_1 + \xi_2$. Introducing the auxiliary periodic problem (3.6) and following the same classical procedure like in the last step of the proof of Theorem 2.3, one easily gets

$$\xi_1 + \xi_2 = A^0 \nabla u. \tag{3.27}$$

Since $u \in H_0^1(\Omega)$ (i.e. $u = 0$ on $\partial\Omega$) and u is uniquely determined, the whole sequence $P^\varepsilon u^\varepsilon$ converges to u and Theorem 3.1 is proved. ∎

Remark 3.7 *In (3.1) we took the ratio of our diffusion coefficients to be of order one just for a better comparison between the two situations we intended to deal with: the case in which the chemical reactions take place on the boundary of the grains and the case in which the chemical reactions occur inside them. However, a much more interesting problem would arise if we consider different orders for the diffusion in the "obstacles" and in the "pores". More precisely, if one takes the ratio of the diffusion coefficients to be of order ε^2, then the limit model will be the so-called* double-porosity model. *This scaling preserves the physics of the flow inside the grains, as $\varepsilon \to 0$. The less permeable part of our medium (the grains) contributes in the limit as a nonlinear memory term. In fact, the effective limit model includes two equations, one in T and another one in Ω, the last one containing an extra-term which reflects the remaining influence of the grains (see, for instance, [2], [5], [6], [12], [20]).*

As in Section 2, g does not need to be Lipschitz continuous (as it is the case, for instance, of the second example or the multivalued example of Remark 2.8) and so, again, the solution of the homogenized problem may give rise to a "dead zone" (where $u(x) = 0$) (see J.I. Díaz [18]).

As a matter of fact, some "dead zone" may be formed, this time, at the level of the microscopic problems, since the equation satisfied by function v^ε leads to such type of behaviors when g is not Lipschitz continuous and a suitable balance between the data and the spatial domain is satisfied (see J.I. DÍAZ [18]). It is quite surprising that the macroscopic balance on the data and domain necessary for the formation of "macroscopic dead zone" may take place by passing to the limit in the microscopic system independently if the microscopic condition for the formation of the "microscopic dead zone" holds or not.

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