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Uniqueness of the boundary behavior for large solutions to a degenerate elliptic equation involving the ∞–Laplacian

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Abstract. In this note we estimate the maximal growth rate at the boundary of viscosity solutions to

$$-\Delta_\infty u + \lambda |u|^{m-1} u = f \quad \text{in } \Omega \qquad (\lambda > 0,\ m > 3).$$

In fact, we prove that there is a unique explosive rate on the boundary for large solutions. A version of Liouville Theorem is also obtained when $\Omega = \mathbb{R}^{\mathrm{N}}$.

Unicidad del comportamiento en la frontera de las soluciones explosivas de una ecuación elíptica degenerada asociada al ∞–Laplaciano

Resumen. En esta nota estimamos la tasa maxima de crecimiento en la frontera de las soluciones de viscosidad de

$$-\Delta_\infty u + \lambda |u|^{m-1} u = f \quad \text{en } \Omega \qquad (\lambda > 0,\ m > 3).$$

De hecho, mostramos que sólo hay una única tasa de explosión en la frontera para esas soluciones explosivas. También obtenemos una versión del Teorema de Liouville para el caso $\Omega = \mathbb{R}^{\mathrm{N}}$.

1. Introduction.

It is clear that an arbitrary function u can reach the infinity value $u = +\infty$ on a manifold in many ways. This is not the case when u solves certain PDE equations. We prove that if u solves

$$-\Delta_\infty u + \lambda |u|^{m-1} u = f \quad \text{in } \Omega \qquad (\lambda > 0), \tag{1}$$

the condition

$$u = +\infty \quad \text{on } \partial\Omega$$

only is satisfied in a unique way, provided $m > 3$. Here Ω denotes a bounded open set of $\mathbb{R}^{\mathrm{N}}$. We explicit the boundary behavior in Theorem 2 below. Several consequences can be pointed out. We remark that the behavior depends only on the distance to the boundary $\partial\Omega$ and the structure of (1). Moreover, as we indicate later, the condition $m > 3$ is *sharp* (see Remark 3 and Proposition 2).

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By solutions we mean *viscosity solutions* in the sense introduced by M.G.Crandall, H.Ishii and P.L.Lions in [3]. This kind of solutions is the most appropriate notion in a wide class of nonlinear partial differential equation, as it occurs if the leading part contains terms like

$$\Delta_\infty u \doteq \sum_{i,j=1}^{\mathrm{N}} \mathrm{D}_i u \mathrm{D}_{ij} u \mathrm{D}_j u,$$

called the ∞-*Laplacian operator* because, in an suitable sense (see [1]), it is the limit case of the p– Laplacian operator

$$\Delta_p u \doteq \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

There exist several justifying the relevance of this operator. For instance, in [6] we develop an idea due to G. Aronsson (see [1] or [4]) relative to the Calculus of Variations involving L^∞ functionals when the minimization is taken in the set of functions such that

$$u = +\infty \quad \text{on } \partial\Omega$$

leading, in this way, to a *state constraint problem* (see [5] for a similar problem).

So the paper is devoted with solutions with uniform blow up at the boundary so-called *explosive solutions* or *large solutions*. In Section 2 we estimate the maximal behavior at boundary of solutions. As it is shown also in Section 3, this maximality property is, in fact, the unique behavior on the boundary available for large solutions.

We point out that in G. Díaz and R. Letelier [7] the assumption

$$m > p - 1$$

was proved to be as the necessary condition the existence of large solution to the quasilinear problem

$$-\Delta_p u + \lambda |u|^{m-1} u = f \quad \text{in } \Omega.$$

Then, some kind of resemblance between the ∞–Laplacian and the p–Laplacian arises in the case $p = 4$. We study it in the more detailed paper [6] where existence and uniqueness results for large solutions are obtained.

2. Interior solutions.

Due to the strong nonlinear structure of the equation (1), $\mathcal{C}^2$ or $\mathcal{W}^{2,p}$ solutions are not available, in general. The non divergence form of the operator $\Delta_\infty u$ enables us to consider the theory of viscosity solutions. We send to [3] (see also [6]) for a detailed explanation of how a function $u \in \mathcal{USC}(\Omega)$ (*upper semi-continuous in* Ω) solves, in the *viscosity sense*,

$$-\Delta_\infty u + \lambda |u|^{m-1} u \le f \quad \text{in } \Omega, \tag{2}$$

for $f \in \mathcal{USC}(\Omega)$. This discontinuous notion is close to the *Weak Maximum Principle*. More precisely, if u is a solution of (2) and $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ verifies

$$-\Delta_\infty v(x) + \lambda |v(x)|^{m-1} v(x) \ge g(x), \quad x \in \Omega \quad (g \in \mathcal{LSC}(\Omega)),$$

then inequality

$$u(x) \le v(x) + \sup_{\partial\Omega}(u - v)_+ + \left(\frac{2^{m-1}}{\lambda} \sup_{\Omega}(f - g)_+\right), \quad x \in \overline{\Omega}, \tag{3}$$

holds, provided that Ω is bounded. Analogously, in [3] one introduces how a function $u \in \mathcal{LSC}(\Omega)$ (*lower semi-continuous in* Ω) solves, in the *viscosity sense*,

$$-\Delta_\infty u + \lambda |u|^{m-1} u \geq f \quad \text{in } \Omega, \tag{4}$$

for $f \in \mathcal{LSC}(\Omega)$. Then one verifies the relative applications to the *Weak Maximum Principle*. In particular, if u is a solution of (4) and $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ verifies

$$-\Delta_\infty v(x) + \lambda |v(x)|^{m-1} v(x) \leq g(x), \quad x \in \Omega \quad (g \in \mathcal{USC}(\Omega)),$$

then we have

$$v(x) \leq u(x) + \sup_{\partial\Omega}(v-u)_+ + \left(\frac{2^{m-1}}{\lambda} \sup_{\Omega}(g-f)_+\right), \quad x \in \overline{\Omega}, \tag{5}$$

provided that Ω is bounded. The notion of solution of

$$-\Delta_\infty u + \lambda |u|^{m-1} u = f \quad \text{in } \Omega.$$

involves (2) and (4) simultaneously. Certainly, the viscosity solutions are consistent with classical solutions.

Our first contribution deals with a classical interior property.

Proposition 1 (Universal interior bounds)

Let us assume $m > 3$. Then there exist a positive constant M depending only on m and λ such that

$$u(x) \leq \mathrm{M}\left(\mathrm{R} - |x - x_0|\right)^{-\frac{4}{m-3}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0) \subset \mathbb{R}^{\mathrm{N}} \tag{6}$$

for any solution u of

$$-\Delta_\infty u + \lambda |u|^{m-1} u \leq 0 \quad \textit{in } \mathbf{B}_{\mathrm{R}}(x_0). \quad \blacksquare$$

Remark 1 This estimate is near the Harnack inequality. Several authors have studied that property for the ∞–Laplacian equation without any perturbation term (see [9], [2] or [1], for example). ■

Remark 2 The above result allows the application of the Perron Method in order to obtain the existence results of [6]. ■

As immediate consequence of (6) follows by letting $\mathrm{R} \to \infty$

Corollary 1 (Liouville Theorem)

Let u be any solution of

$$-\Delta_\infty u + \lambda |u|^{m-1} u \leq 0 \quad \textit{in } \mathbb{R}^{\mathrm{N}}, \quad \lambda > 0,\ m > 3.$$

Then

$$u(x) \leq 0, \quad x \in \mathbb{R}^{\mathrm{N}}. \tag{7}$$

Remark 3 In the above result the assumption $m > 3$ is *sharp*. Indeed, the positive función $u(x) = e^x$ satisfies

$$-\Delta_\infty u + u^3 = 0 \quad \text{in } \mathbb{R}. \quad \square$$

In our study near the boundary we will *tubular* neighborhoods defined by

$$\mathcal{O}_\vartheta^\varsigma = \{x \in \Omega : \varsigma < \operatorname{dist}(x, \partial\Omega) < \vartheta\}, \quad 0 \leq \varsigma < \vartheta.$$

The following technical result is a very useful tool

Lemma 1 ([8])
Let $\Omega \subset \mathbb{R}^N$ be an open set with bounded boundary of $\mathcal{C}^k$-class. Then there exists a positive constant δ_Ω, depending only on Ω, such that $\mathrm{dist}(\cdot, \partial\Omega) \in \mathcal{C}^k(\overline{\mathcal{O}}^0_{\delta_\Omega})$. *Moreover,* $\mathrm{d}_{\partial\Omega}(\cdot) = \mathrm{dist}(\cdot, \partial\Omega)$ *verifies*

$$|\nabla \mathrm{d}_{\partial\Omega}(x)| = 1, \quad x \in \overline{\mathcal{O}}^0_{\delta_\Omega}. \quad \blacksquare$$

We recall that the function $\mathrm{d}_{\partial\Omega}(\cdot)$ is Lipschitz continuous in the whole space $\mathbb{R}^N$. In fact, it is the unique (viscosity) solution of

$$\begin{cases} |\nabla u| = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The *maximal* behavior available at the boundary is collected now

Theorem 1 (Maximal behavior at the boundary)
Let $\partial\Omega \in \mathcal{C}^2$ and let u be any solution of

$$-\Delta_\infty u + \lambda |u|^{m-1} u \leq f \quad \textit{in } \Omega \qquad (\lambda > 0,\ m > 3) \tag{8}$$

for $f \in \mathcal{USC}(\Omega)$. Let us denote $\hat{q} = \dfrac{4m}{m-3}$.

Then, if

$$\limsup_{\mathrm{d}_{\partial\Omega}(x) \to 0} f(x) \left(\mathrm{d}_{\partial\Omega}(x)\right)^{\hat{q}} \leq f_{\hat{q}} \in [0, \infty[$$

one has

$$\limsup_{\mathrm{d}_{\partial\Omega}(x) \to 0} u(x) \left(\mathrm{d}_{\partial\Omega}(x)\right)^{\frac{\hat{q}}{m}} \leq u_\infty(\hat{q}),$$

where $u_\infty(\hat{q})$ is the positive root of

$$\mathrm{P}_1(\mu) = \lambda \mu^m - \frac{(\hat{q}+m)\hat{q}^3}{m^4} \mu^3 - f_{\hat{q}}.$$

On the other hand, if

$$\limsup_{\mathrm{d}_{\partial\Omega}(x) \to 0} f(x) \left(\mathrm{d}_{\partial\Omega}(x)\right)^{q} \leq f_q \in]0, \infty[, \quad q > \hat{q}$$

the following inequality holds

$$\limsup_{\mathrm{d}_{\partial\Omega}(x) \to 0} u(x) \left(\mathrm{d}_{\partial\Omega}(x)\right)^{\frac{q}{m}} \leq \left(\frac{f_q}{\lambda}\right)^{\frac{1}{m}}. \quad \blacksquare$$

In the proof of the above result (see [6]) we use the representation

$$-\Delta_\infty \left(\mathrm{d}_{\partial\Omega}(x)\right)^{-\alpha} = -\alpha^3(\alpha+1) \left(\mathrm{d}_{\partial\Omega}(x)\right)^{-(3\alpha+4)} + \alpha^3 \left(\mathrm{d}_{\partial\Omega}(x)\right)^{-(3\alpha+3)} \Delta_\infty \mathrm{d}_{\partial\Omega}(x)$$

for $x \in \mathcal{O}^0_{\mathrm{R}}(\partial\Omega)$, $\mathrm{R} < \delta_\Omega$, where the term $\Delta_\infty \mathrm{d}_{\partial\Omega}$ involves the geometry of Ω (see Remark 5 below for some details). In the particular case $\Omega = \mathbf{B}_{\mathrm{R}}(0)$ one has $\delta_{\mathbf{B}_{\mathrm{R}}(0)} = \mathrm{R}$ (see Lemma 1). Moreover, straightforward computations show that the relative distance function

$$\mathrm{d}_{\partial\Omega}(x) = \mathrm{R} - |x|$$

is ∞-harmonic, $i.e.$

$$\Delta_\infty \mathrm{d}_{\partial\Omega}(x) = 0 \quad x \in \mathbf{B}_{\mathrm{R}}(0) \setminus \{0\}.$$

Remark 4 As it is indicated later, the estimates of Theorem 1 are the *best maximal estimate on the behavior at the boundary* of solutions of (1). □

3. Large solutions.

Here we focus the attention on the behavior near the boundary of the large solutions, $i.e.$ satisfying the property

$$\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u(x) = +\infty.$$

The next result is the main contribution in this note. In fact, it is a major key in order to obtain uniqueness of these singular solutions (see [6]).

Theorem 2 (Uniqueness of the explosive rate)
Let Ω be a bounded open set of $\mathbb{R}^{\mathrm{N}},\ \mathrm{N}\geq 1$, with $\partial\Omega\in\mathcal{C}^2$. Then all large solution of

$$-\Delta_\infty u+\lambda|u|^{m-1}u=f \quad \text{in } \Omega \qquad (\lambda>0,\ m>3)$$

has a unique explosive rate. More precisely, if $\widehat{q}=\dfrac{4m}{m-3}$, the assumption

$$\lim_{\mathrm{d}_{\partial\Omega}(x)\to 0} f(x)\left(\mathrm{d}_{\partial\Omega}(x)\right)^{\widehat{q}}=f_{\widehat{q}}\in[0,\infty[$$

implies

$$\lim_{\mathrm{d}_{\partial\Omega}(x)\to 0} u(x)\left(\mathrm{d}_{\partial\Omega}(x)\right)^{\frac{\widehat{q}}{m}}=u_\infty(\widehat{q}),$$

where $u_\infty(\widehat{q})$ is the positive root of

$$\mathrm{P}_1(\mu)=\lambda\mu^m-\frac{(\widehat{q}+m)\widehat{q}^3}{m^4}\mu^3-f_{\widehat{q}}.$$

On the other hand,

$$\lim_{\mathrm{d}_{\partial\Omega}(x)\to 0} f(x)\left(\mathrm{d}_{\partial\Omega}(x)\right)^{q}=f_q\in]0,\infty[,\quad q>\widehat{q}$$

leads to

$$\lim_{\mathrm{d}_{\partial\Omega}(x)\to 0} u(x)\left(\mathrm{d}_{\partial\Omega}(x)\right)^{\frac{q}{m}}=\left(\frac{f_q}{\lambda}\right)^{\frac{1}{m}}. \quad \blacksquare$$

Remark 5 The main idea of the proofs of Theorems 1 and 2 lies on the construction of some suitable smooth sub and supersolutions given, respectively, by

$$\begin{cases} \Phi_{\varepsilon,\delta}(x)=(c-\varepsilon)\left(\mathrm{d}_{\partial\Omega}(x)+\delta\right)^{-\alpha}-M,\ x\in\mathcal{O}^0_\vartheta \\ \Psi_{\varepsilon,\delta}(x)=(c+\varepsilon)\left(\mathrm{d}_{\partial\Omega}(x)-\delta\right)^{-\alpha}+M,\ x\in\mathcal{O}^\delta_\vartheta \end{cases} \qquad 0<\delta<\vartheta<\delta_\Omega,\quad \vartheta\ll 1,$$

where c,α and M are positive constants to be chosen. The parameter c is the positive root of the relative polynomials

$$\mathrm{P}_1(\mu)=\lambda\mu^m-\frac{(\hat{q}+m)\hat{q}^3}{m^4}\mu^3-f_{\hat{q}}.$$

or

$$\mathrm{P}_2(\mu)=\lambda\mu^m-f_q$$

and it leads to the *blow up rate*. On the other hand, the *blow up order* α is chosen by means of some adequate balances between the constants q and m. The arguments conclude by passing to the limits $\delta\to\ 0,\ \mathrm{d}_{\partial\Omega}(x)\to 0$ and $\varepsilon\to 0$. $\blacksquare$

Remark 6 Lower terms in the expansion of the behavior will be obtained in a future paper. ■

As it was partially quoted in Remark 3, condition $m > 3$ is *sharp*. Concerning the non–existence of large solutions we have

Proposition 2

The equation

$$-\Delta_\infty u + h(u) \geq 0,$$

possed in a bounded domain $\Omega \subset \mathbb{R}^N$*, has no explosive positive solutions, provided that* h *is a nonnegative continuous function verifying*

$$\sup_{r>0} \frac{h(r)}{r^3} < +\infty. \quad ■ \tag{9}$$

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