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Numerical Experiments Regarding the Distributed Control of Semilinear Parabolic Problems

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Abstract—This work deals with some numerical experiments regarding the distributed control of semilinear parabolic equations of the type

$$y_t - y_{xx} + f(y) = u\chi_\omega, \qquad \text{in } (0,1)\times(0,T),$$

with Neumann and initial auxiliary conditions, where ω is an open subset of $(0,1)$, f is a C^1 nondecreasing real function, u is the output control and $T>0$ is (arbitrarily) fixed. Given a target state y_T we study the associated *approximate controllability problem* (given $\epsilon>0$, find $u\in L^2(0,T)$, such that $\|y(T;u)-y_T\|_{L^2(0,1)}\le\epsilon$) by passing to the limit (when $k\to\infty$) in the *penalized optimal control problem* (find u_k as the minimum of $J_k(u) = 1/2\ \|u\|^2_{L^2(0,T)} + (k/2)\|y(T;u)-y_T\|^2_{L^2(0,1)}$). In the superlinear case (e.g., $f(y)=|y|^{n-1}y$, $n>1$) the existence of *two obstruction functions* $Y_{\pm\infty}$ shows that the approximate controllability is only possible if $Y_{-\infty}(x,T)\le y_T(x)\le Y_\infty(x,T)$ for a.e. $x\in(0,1)$. We carry out some numerical experiments showing that, for a fixed k, the "minimal cost" $J_k(u)$ (and the norm of the optimal control u_k) for a superlinear function f becomes much larger when this condition is not satisfied. We also compare the values of $J_k(u)$ (and the norm of the optimal control u_k) for a fixed y_T associated with two nonlinearities: one sublinear and the other one superlinear.

Keywords—Controllability, Semilinear parabolic problem, Numerical approximation, Adjoint system, Distributed control, Implicit-scheme, Large solutions.

1. INTRODUCTION

This work deals with some numerical experiments regarding the control of semilinear equations of the type

$$P(u)\begin{cases} y_t - y_{xx} + f(y) = u\chi_\omega, & \text{in } (0,1)\times(0,T),\\ \dfrac{\partial y}{\partial x}(0,t) = \dfrac{\partial y}{\partial x}(1,t) = 0, & \text{for } t\in(0,T),\\ y(x,0) = y_0(x), & \text{in } (0,1),\end{cases}$$

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where f is a C^1 nondecreasing real function, u is the output control, ω is an open subset of $(0,1)$, $T>0$ is (arbitrarily) fixed, and y_0 is a given function (for instance $y_0 \in H_0^1(0,1)$).

Given an arbitrary target state y_T (we can assume, for simplicity, that $y_T \in C^0([0,1])$), the associated *approximate controllability problem* consists of, given an arbitrary $\varepsilon > 0$, find $u \in L^2(\omega \times (0,T))$ such that $\|y(T;u) - y_T\|_{L^2(0,1)} \leq \varepsilon$, where $y(T;u)$ denotes the solution of $P(u)$ at time T.

It is well known (see [1,2]) that the answer is positive if "f is sublinear at infinity" ($|f(s)| \leq M(|s|+1)$ for $|s|$ large and for some $M>0$). In the "superlinear at infinity" case

$$|f(s)| \geq M\left(|s|^n + 1\right), \qquad \text{for } |s| \text{ large and for some } n > 1 \text{ and } M > 0, \tag{1}$$

the answer is negative. This type of negative results can be proved in different ways: via an energy argument (see, e.g., the case of control on the Neumann boundary condition, due to Bamberger, in [3]) or via some pointwise obstruction phenomenon (see [4] for problem $P(u)$ and [2] for other problems).

It is also well known that in the sublinear case, the solution to the controllability problem can be obtained by passing to the limit (as $k \to \infty$) in the *penalized optimal control problem* in which the control u_k is found as the minimum of the functional

$$J_k(u) = \frac{1}{2}\,\|u\|^2_{L^2(\omega\times(0,T))} + \frac{k}{2}\,\|y(T;u) - y_T\|^2_{L^2(0,1)} \tag{2}$$

(see [5] for the linear case and [6] for the semilinear case).

For the superlinear case the approximate controllability was obtained in [7] under the assumption

$$Y_{-\infty}(x,T) \leq y_T(x) \leq Y_{\infty}(x,T), \qquad \text{for a.e. } x \in (0,1), \tag{3}$$

where $Y_{\pm\infty}$ are the "largest solutions". In our case, $Y_{\pm\infty}$ are the solutions of the problem

$$P(\pm\infty)\begin{cases} y_t - y_{xx} + f(y) = 0, & \text{in } ((0,1)\setminus\bar{\omega})\times(0,T),\\ \dfrac{\partial y}{\partial x}(0,t) = \dfrac{\partial y}{\partial x}(1,t) = 0, & \text{for } t \in (0,T),\\ y(\cdot,t) = \pm\infty, & \text{on } \partial\omega \times (0,T),\\ y(x,0) = y_0(x), & \text{in } (0,1), \end{cases}$$

(the existence of such large solutions requires f to be superlinear, i.e., to satisfy (1)). Notice that the special case of $y_T \equiv 0$ is included (see, e.g., [8], for other results on null controllability).

More recently, some results on the approximate controllability of the projections on finite dimensional subspaces were obtained by Khapalov [9] (see also its references) for the superlinear case (1). Global exact steady-state controllability results have been obtained in [10].

A remarkable fact is that, for the sublinear case, the approximate controllability property holds for any open subset ω (as small as we want), but it may fail when the control domain ω is reduced to a single point (pointwise control). Furthermore, for linear cases it can be proved (see [11]) that the controllability property in the pointwise control is true for what is called *strategic* control points. When looking for these points one find that, for instance, for the linear heat equation, the non-rationals numbers are strategic and, if the problem is symmetric with respect to a rational number, this number is also an strategic point for that problem. Anyway, it is always possible to consider the associate optimal control problems similar to that with cost functional (2) and perform a similar analysis. This analysis has been carried out in [12] for several problems with the control point $x = 1/2$ and the suitable symmetries referred above holding (although for nonlinear cases it is not guaranteed that this point is strategic).

The main goal of this work is to carry out some numerical experiments on the *penalized optimal control problem* for difference target states y_T and different nonlinear terms $f(y)$. We illustrate the fact that, for a fixed k, the "minimal cost" $J_k(u)$ (and the norm of the optimal control u_k) for a superlinear function f becomes much larger when (3) is not satisfied (see numerical test # 1, # 2 and # 3 below). We also compare the values of $J_k(u)$, the norm of the optimal control u_k, and $\|y(T;u) - y_T\|$, for a fixed y_T, associated to two different nonlinearities: one sublinear ($f(y) = \operatorname{arctg}(y)$) and the other one superlinear ($f(y) = y^3$).

2. PROBLEM FORMULATION

Let us consider a given target function $y_T \in L^2(0,1)$. We define the control space as $\mathcal{U} = L^2(\omega \times (0,T))$. The goal is to find a control $u \in \mathcal{U}$ so that $y(T)$ is *close* to y_T at a minimal cost for the control, where $y(x,t)$ is the (unique) solution of $P(u)$). We recall that a *weak* formulation of $P(u)$ is provided by $y \in L^2(0,T;H^1(0,1)) \cap H^1(0,T;(H^1(0,1))') \subset \mathcal{C}([0,T] : L^2(\Omega))$, such that

$$f(y) \in L^1(0,T;L^1(0,1)), \qquad \text{and} \qquad \forall z \in L^2(0,T;H^1(0,1)) \cap L^\infty(Q)$$
$$\int_0^T \langle y_t, z\rangle_{(H^1)'\times H^1}\,dt + \int_0^T\int_0^1 y_x z_x\,dx\,dt + \int_0^T\int_0^1 f(y)z\,dx\,dt = \int_0^T\int_\omega uz\,dx\,dt,$$
$$y(x,0) = y_0(x).$$

The existence and uniqueness of weak solution becomes standard after the work by Brezis-Browder [13]. Moreover, we can prove the boundedness of the solution even for unbounded controls.

PROPOSITION 1. *The weak solution y of problem $\mathcal{P}(u)$ satisfies $y \in L^\infty((0,1) \times (0,T))$.*

PROOF. Due to the monotonicity of function f, it is well-known that

$$|y(t,x)| \le |h(t,x)|, \qquad \text{for any } t \in [0,T] \text{ and a.e. } x \in \Omega, \tag{4}$$

where h is the (unique) solution of the linear equation

$$(\mathrm{LHE})\begin{cases} h_t - h_{xx} = u\chi_\omega, & \text{in } (0,1)\times(0,T),\\ \dfrac{\partial h}{\partial x}(0,t) = \dfrac{\partial h}{\partial x}(1,t) = 0, & \text{for } t \in (0,T),\\ h(x,0) = y_0(x), & \text{in } (0,1).\end{cases}$$

But this equation has a unique solution

$$h \in L^2\left(0,T;H^2(0,1)\right) \cap H^1\left(0,T;L^2(0,1)\right) \subset \mathcal{C}\left([0,T] : H^1(0,1)\right) \subset L^\infty(Q).$$

The proof is based on Theorem 1.1 of Chapter 4 and Theorem 3.1 of Chapter 1 of [14]. This proves the result thanks to (4). ∎

For every $k \in \mathbb{N}$, we define the *cost function* J_k by

$$J_k(v) = \frac{1}{2}\|v\|^2_{\mathcal{U}} + \frac{k}{2}\|y(T) - y_T\|^2_{L^2(0,1)}, \qquad \forall v \in \mathcal{U}.$$

The control problem is then

$$(\mathcal{C}P_k)\begin{cases} \text{find } u_k \in \mathcal{U}, \text{ such that}\\ J_k(u_k) \le J_k(v), \qquad \forall v \in \mathcal{U}.\end{cases}$$

A common way to solve this problem is to solve the problem

$$J_k'(u) = 0,$$

where J_k' denotes the Gateaux differential of J_k.

Now, it is easy to prove (see, e.g., [11,15]) that

$$J_k'(v) = v + p|_\omega,$$

i.e.,

$$(J_k'(v), w) = \int_0^T \int_\omega (v + p)w \, dx \, dt, \qquad \forall\, w \in \mathcal{U},$$

where p is the solution of the adjoint system

$$\begin{aligned} -p_t - p_{xx} + f'(y)p &= 0, && \text{in } Q, \\ \frac{\partial p}{\partial x}(0,t) = \frac{\partial p}{\partial x}(1,t) &= 0, && \text{for } t \in (0,T), \\ p(T) &= k(y(T;v) - y_T), && \text{in } (0,1), \end{aligned}$$

and $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{U}$ defined by $(u,v) = \int_0^T \int_\omega uv \, dx \, dt$. Notice that Proposition 1 guarantees that $f'(y) \in L^\infty((0,1) \times (0,T))$.

3. TIME DISCRETIZATION

We consider the time discretization step Δt, defined by $\Delta t = T/N$, where N is a positive integer. Then, if $t^n = n\Delta t$, we have $0 < t^1 < t^2 < \cdots < t^N = T$. We approximate then problem $(\mathcal{C}P)$ by the following finite-dimensional minimization problem:

$$(\mathcal{C}P_k)^{\Delta t} \begin{cases} \text{find } u^{\Delta t} = \{u^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \text{ such that} \\ J_k^{\Delta t}(u) \le J_k^{\Delta t}(v), \ \forall\, v = \{v^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \end{cases}$$

with the *time discrete control space* $\mathcal{U}^{\Delta t} = L^2(\omega) \times \mathbb{R}^N$ and

$$J_k^{\Delta t}(v) = \frac{\Delta t}{2} \sum_{n=1}^N \|v^n\|^2_{L^2(\omega)} + \frac{k}{2}\left((1-\theta)\left\|y^{N-1} - y_T\right\|^2_{L^2(0,1)} + \theta \left\|y^N - y_T\right\|^2_{L^2(0,1)}\right),$$

where $\theta \in (0,1]$ and $\{y^n\}_{n=1}^N$ is defined from the solution of the following *second-order accurate time discretization* scheme of problem $(P(u))$:

$$y^0 = y_0,$$

$$\frac{y^1 - y^0}{\Delta t} - \frac{\partial^2}{\partial x^2}\left(\frac{2}{3}y^1 + \frac{1}{3}y^0\right) + f\left(y^1\right) = \frac{2}{3}v^1\chi_\omega, \qquad \text{in } (0,1),$$

$$\frac{\partial y^1}{\partial x}(0) = \frac{\partial y^1}{\partial x}(1) = 0,$$

and for $n \ge 2$,

$$\frac{(3/2)y^n - 2y^{n-1} + (1/2)y^{n-2}}{\Delta t} - \frac{\partial^2}{\partial x^2}y^n + f(y^n) = v^n\chi_\omega, \qquad \text{in } (0,1),$$

$$\frac{\partial y^n}{\partial x}(0) = \frac{\partial y^n}{\partial x}(1) = 0.$$

REMARK. We have used an *implicit* scheme. We could also have used a *semi-implicit* scheme, treating *implicitly* the diffusion term and *explicitly* the reaction term (as done in [11,15,16] for the case of the diffusion and advection terms of the Burgers equation), but this choice may imply the necessity of choosing a very small time step Δt, in particular for reaction-dominated problem as the one we are treating.

4. FULL DISCRETIZATION

We consider the space discretization step h, defined by $h = 1/I$, where I is a positive integer. Then, if $x_i = (i-1)h$, we have $0 = x_1 < x_2 < \cdots < x_I < x_{I+1} = 1$. We approximate $H^1(0,1)$ by

$$V_h = \left\{ z \in C^0[0,1] : z|_{(x_i,x_{i+1})} \in P_1,\ i = 1,\dots,I \right\},$$

where P_1 is the space of the polynomials of degree least or equal than one and $\mathcal{U}$ by $\mathcal{U}_h^{\Delta t} = (\mathcal{U}_h)^N$, where

$$\mathcal{U}_h = \left\{ z : z \in C^0(\bar{\omega}) : z|_{(x_i,x_{i+1})} \in P_1,\ \forall\, i = 1,\dots,I, \text{ such that } (x_i, x_{i+1}) \subset \omega \right\}.$$

We define a_h by

$$a_h(y,z) = \int_0^1 y_x z_x \,\mathrm{d}x.$$

We approximate then problem (CP_k) by the following finite-dimensional minimization problem:

$$(CP_k)_h^{\Delta t} \begin{cases} \text{Find } u_h^{\Delta t} = \{u^n\}_{n=1}^N \in \mathcal{U}_h^{\Delta t}, \text{ such that} \\ J_{k,h}^{\Delta t}(u_h^{\Delta t}) \le J_{k,h}^{\Delta t}(v),\ \forall\, v = \{v^n\}_{n=1}^N \in \mathcal{U}_h^{\Delta t}; \end{cases}$$

with

$$J_{k,h}^{\Delta t}(v) = \frac{\Delta t}{2} \sum_{n=1}^{N} \|v^n\|_{L^2(\omega)}^2 + \frac{k}{2}\left((1-\theta) \left\| y_h^{N-1} - y_T \right\|_{L^2(0,1)}^2 + \theta \left\| y_h^N - y_T \right\|_{L^2(0,1)}^2 \right),$$

where $\theta \in (0,1]$ and $\{y_h^n\}_{n=1}^N$ is defined from the solution of the following full discretization of problem $(P(u))$:

$$y_h^0 \in V_h,$$
$$(y_h^0, z) = (y_0, z), \qquad \forall\, z \in V_h;$$

$$y_h^1 \in V_h,$$
$$\left(\frac{y_h^1 - y_h^0}{\Delta t}, z \right) + a_h\left(\frac{2}{3} y_h^1 + \frac{1}{3} y_h^0, z \right) + \left(f\left(y_h^1\right), z \right) = \frac{2}{3} \int_\omega v^1 z \,\mathrm{d}x, \qquad \forall\, z \in V_h;$$

and for $n \ge 2$,

$$y_h^n \in V_h,$$
$$\left(\frac{(3/2) y_h^n - 2 y_h^{n-1} + (1/2) y_h^{n-2}}{\Delta t}, z \right) + a_h\left(y_h^n, z\right) + \left(f\left(y_h^n\right), z \right) = \int_\omega v^n z \,\mathrm{d}x, \qquad \forall\, z \in V_h.$$

In the above algorithm $(\cdot,\cdot)$ denotes the scalar product in $L^2(0,1)$, that is,

$$(f,g) = \int_0^1 f(x) g(x) \,\mathrm{d}x \qquad \forall\, f, g \in L^2(0,1).$$

As for the continuous case, to solve problem $(CP)_h^{\Delta t}$, we look for the solution $u_h^{\Delta t}$ of

$$\frac{\partial J_h^{\Delta t}}{\partial v}\left(u_h^{\Delta t}\right) = 0.$$

Computing $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$ is more complicated than in the continuous case but, following the same approach, we can show that

$$\left\langle \frac{\partial J_{k,h}^{\Delta t}}{\partial v}(v), w \right\rangle = \Delta t \sum_{n=1}^{N} \int_\omega (v^n + p^n)\, w^n \, \mathrm{d}x,$$

where $\{p_h^n\}_{n=1}^{N+2}$ is the solution of

$$p_h^{N+2} \in V_h,$$
$$(p_h^{N+2}, z) = -8l\,(1-\theta) \int_0^1 (y_h^{N-1} - y_T) z \, \mathrm{d}x - 2l\theta \int_0^1 (y_h^N - y_T)\, z \, \mathrm{d}x, \qquad \forall\, z \in V_h;$$

$$p_h^{N+1} \in V_h,$$
$$(p_h^{N+1}, z) = -2l\,(1-\theta) \int_0^1 (y_h^{N-1} - y_T) z \, \mathrm{d}x, \qquad \forall\, z \in V_h;$$

and for $n = N, \cdots, 1$,

$$p_h^n \in V_h,$$
$$\left(\frac{(3/2)p_h^n - 2p_h^{n+1} + (1/2)p_h^{n+2}}{\Delta t}, z \right) + a_h(p_h^n, z) + (f'(y_h^n)\, p_h^n, z) = 0, \qquad \forall\, z \in V_h.$$

Now, once we know how to compute $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$, we use a *quasi-Newton method* á la BFGS (see, e.g., [17] for BFGS algorithms and their implementations) to compute the solution of the fully discrete control problem $(\mathcal{C}P)_h^{\Delta t}$.

5. NUMERICAL EXPERIMENTS

In all the tests considered we have taken $\omega = (0.4, 0.5)$, $T = 1$, $I = 100$, $N = 500$, $k = 10^{12}$, and $y_0 = 0$ (notice that this implies $y(x,t;0) \equiv 0$). We use, for our algorithm, $\theta = 3/2$. Further, if v_p $(p = 1, 2, \dots)$ is the sequence of controls we get from the BFGS algorithm, we use the following stopping criteria: we stop iterating after step p if either

$$\left\| \frac{\partial J_h^{\Delta t}}{\partial v}(u_p) \right\|_\infty \le 10^{-5}$$

or

$$\frac{J_h^{\Delta t}(u_{p-1}) - J_h^{\Delta t}(u_p)}{\max\left\{ \left|J_h^{\Delta t}(u_{p-1})\right|, \left|J_h^{\Delta t}(u_p)\right|, 1 \right\}} \le 2 \cdot 10^{-9}.$$

We have considered three different tests, depending on the target function.

5.1. Test 1: $y_T \equiv 5$

In Figure 1 (respectively, Figure 2) we have shown the super-solution $Y_\infty(T)(\cdots)$, the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (respectively, $f(y) = \operatorname{arctg}(y)$). The corresponding control functions have been represented in Figures 3 and 4.

In Figure 5 (respectively, Figure 6) we have shown the graphic of $\|y(t) - y_T\|_{L^2(0,1)}$, $t \in [0,1]$, when $f(y) = y^3$ (respectively, $f(y) = \operatorname{arctg}(y)$).

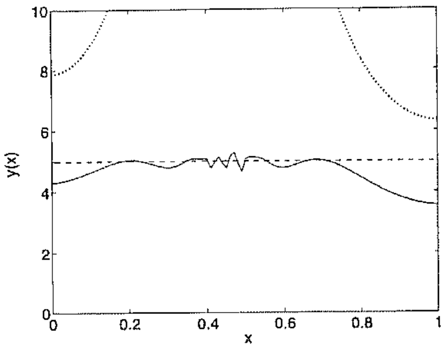

Figure 1. The target function (- -), the large solution (· ·) and controlled (–) states at time T, for $f(y) = y^3$.

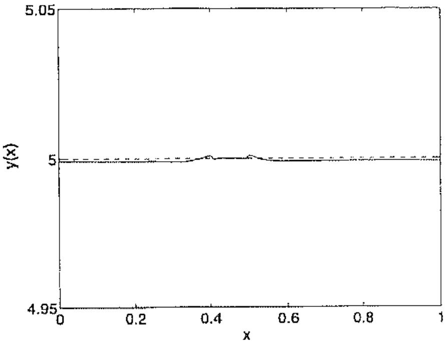

Figure 2. The target function (- -) and the controlled (–) state at time T, for $f(y) = \operatorname{arctg}(y)$.

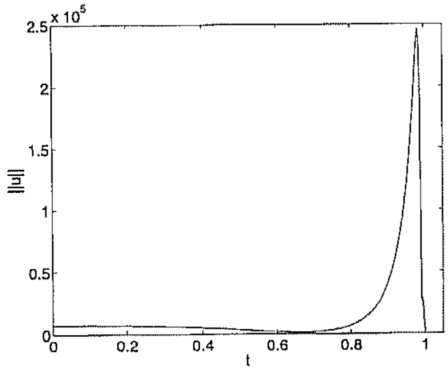

Figure 3. $\|u(t)\|$, for $f(y) = y^3$.

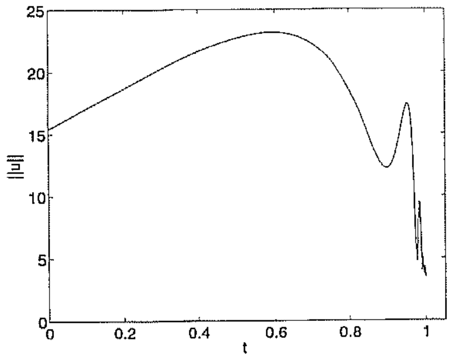

Figure 4. $\|u(t)\|$, for $f(y) = \operatorname{arctg}(y)$.

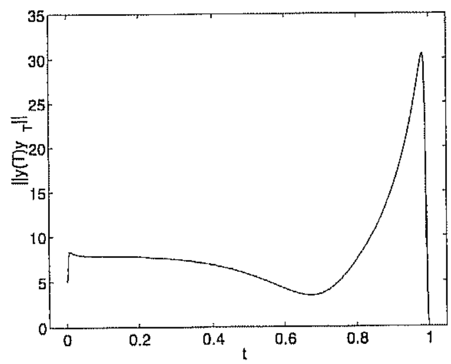

Figure 5. $\|y(t) - y_T\|$, for $f(y) = y^3$.

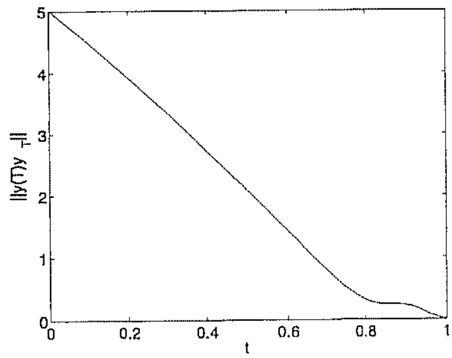

Figure 6. $\|y(t) - y_T\|$, for $f(y) = \operatorname{arctg}(y)$.

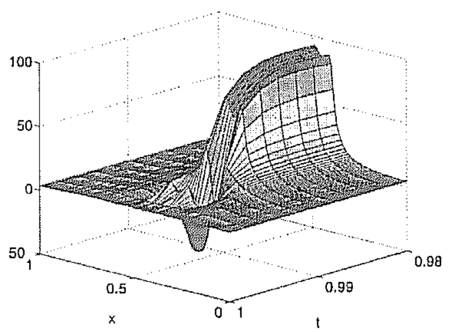

Figure 7. Graphic of $y(x,t)$ ($t \in [0.98, 1]$), for $f(y) = y^3$.

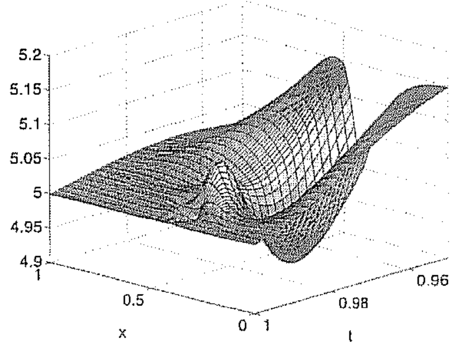

Figure 8. Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = \operatorname{arctg}(y)$.

Table 1. Computational results.

	$f(y) = y^3$	$f(y) = \text{arctg}\,(y)$
$\|y(0;T) - y_T\|$ ($= \|y_T\|$)	5	5
$\|y(u;T) - y_T\|$	0.5613	$8.69 \cdot 10^{-4}$
$\|u\|$	$4.2476 \cdot 10^4$	19.4185
$J(0)$	$1.25 \cdot 10^{13}$	$1.25 \cdot 10^{13}$
$J(u)$	$1.5933 \cdot 10^{11}$	$3.7797 \cdot 10^5$

In Figure 7 (respectively, Figure 8) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.98, 1]$ and $f(y) = y^3$ (respectively, when $t \in [0.95, 1]$ and $f(y) = \text{arctg}\,(y)$).

In Table 1, we give some further results about our solutions. The norms considered in all the tables of the present article refer to the L^2–norm of the discrete entries. One of the entries of the table shows the number of discrete parabolic equations the BFGS algorithm has needed to solve (one-half of this number corresponds to the nonlinear state system and the other half corresponds to the linear adjoint system). Further, $y(v;T)$ represents the solution at time T, associated with the control v ($y(0;T)$ represents the solution without control, at time T).

5.2. Test 2: $y_T \equiv 50$

In Figure 9 (respectively, Figure 10) we illustrate the super-solution $Y_\infty(T)$ ($\cdots$), the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (respectively, $f(y) = \text{arctg}\,(y)$). The corresponding control functions have been represented on Figures 11 and 12.

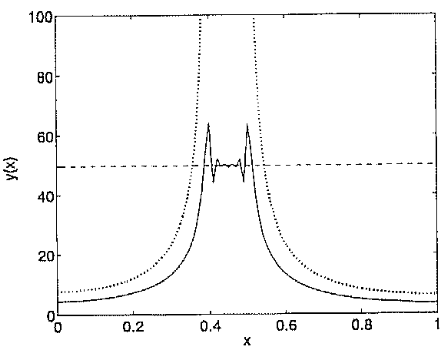

Figure 9. The target function (- -), the large solution (..) and controlled (–) states at time T, for $f(y) = y^3$.

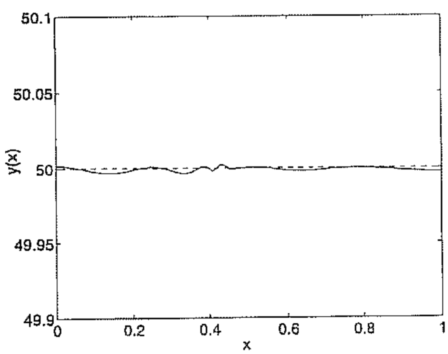

Figure 10. The target function (- -) and the controlled (–) state at time T, for $f(y) = \text{arctg}\,(y)$.

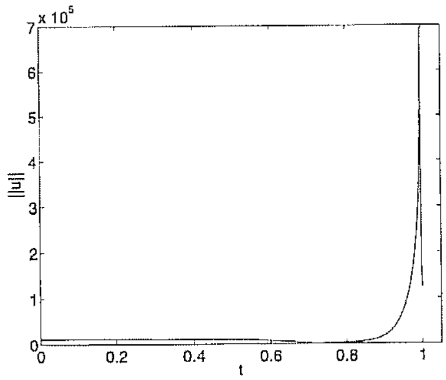

Figure 11. The computed optimal control for $f(y) = y^3$.

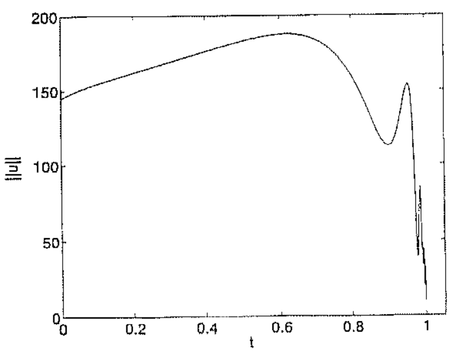

Figure 12. The computed optimal control for $f(y) = \text{arctg}\,(y)$.

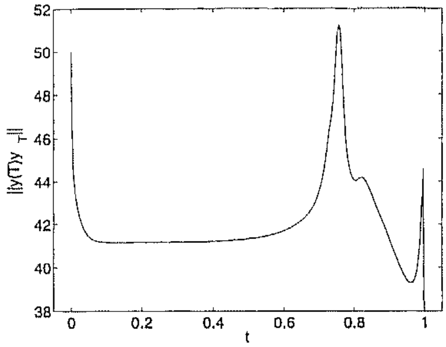

Figure 13. $\|y(t) - y_T\|$, for $f(y) = y^3$.

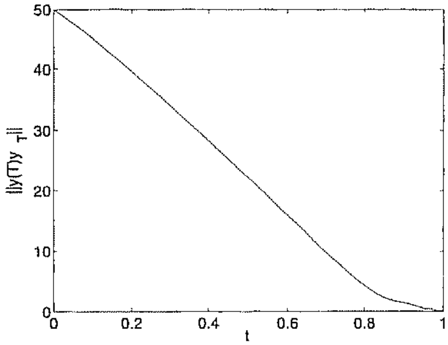

Figure 14. $\|y(t) - y_T\|$, for $f(y) = \operatorname{arctg}(y)$.

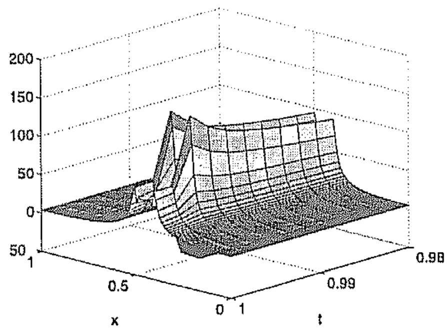

Figure 15. Graphic of $y(x,t)$ ($t \in [0.98, 1]$), for $f(y) = y^3$.

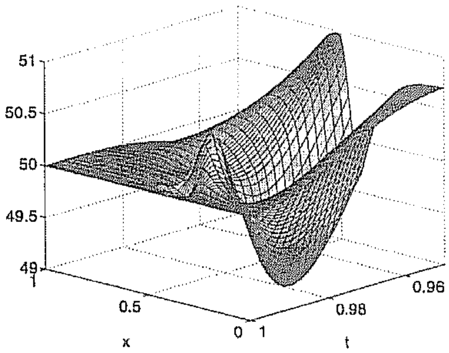

Figure 16. Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = \operatorname{arctg}(y)$.

Table 2. Computational results.

	$f(y) = y^3$	$f(y) = \operatorname{arctg}(y)$
$\|y(0;T) - y_T\|$ ($= \|y_T\|$)	50	50
$\|y(u;T) - y_T\|$	38.4450	$1.615 \cdot 10^{-3}$
$\|u\|$	$5.6426 \cdot 10^4$	$1.6364 \cdot 10^2$
$J(0)$	$1.25 \cdot 10^{15}$	$1.25 \cdot 10^{15}$
$J(u)$	$7.3901 \cdot 10^{14}$	$1.3312 \cdot 10^6$

In Figure 13 (respectively, Figure 14) we illustrate the graphic of $\|y(t) - y_T\|_{L^2(0,1)}$, $t \in [0,1]$, when $f(y) = y^3$ (respectively, $f(y) = \operatorname{arctg}(y)$).

In Figure 15 (respectively, 16) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.98, 1]$ and $f(y) = y^3$ (respectively, when $t \in [0.95, 1]$ and $f(y) = \operatorname{arctg}(y)$).

In Table 2 we give some further results about our solutions.

5.3. Test 3

$$y_T(x) = \begin{cases} 0, & \text{if } x \in (0, 0.5), \\ 8x - 4, & \text{if } x \in (0.5, 0.75), \\ -8x + 8, & \text{if } x \in (0.75, 1). \end{cases}$$

In Figure 17 (respectively, Figure 18) we have shown the super-solutions $Y_\infty(T)$ ($\cdots$), the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (respectively, $f(y) = \operatorname{arctg}(y)$). The corresponding control functions have been represented on Figures 19 and 20.

In Figure 21 (respectively, Figure 22) we have shown the graphic of $\|y(t) - y_T\|_{L^2(0,1)}$, $t \in [0,1]$, when $f(y) = y^3$ (respectively, $f(y) = \operatorname{arctg}(y)$).

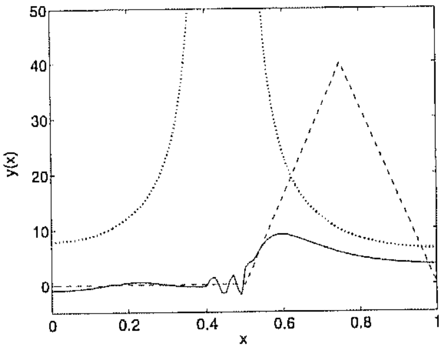

Figure 17. The target function (- -), the large solutions (··) and controlled (–) states at time T, for $f(y) = y^3$.

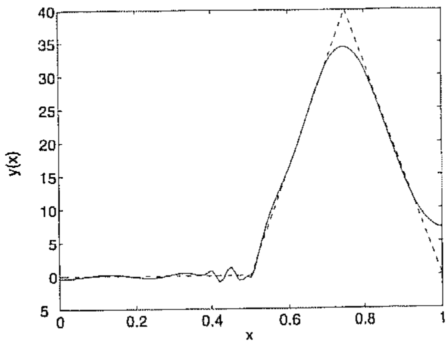

Figure 18. The target function (- -) and the controlled (–) state at time T, for $f(y) = \operatorname{arctg}(y)$.

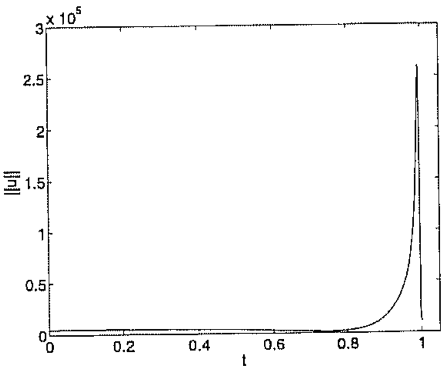

Figure 19. The computed optimal control for $f(y) = y^3$.

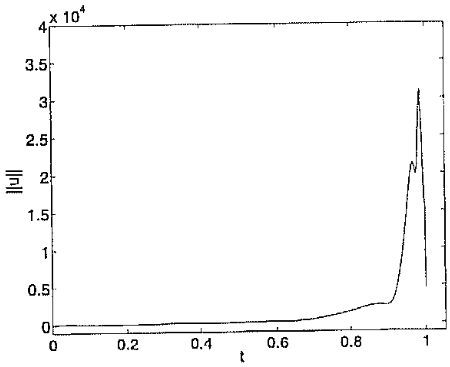

Figure 20. The computed optimal control for $f(y) = \operatorname{arctg}(y)$.

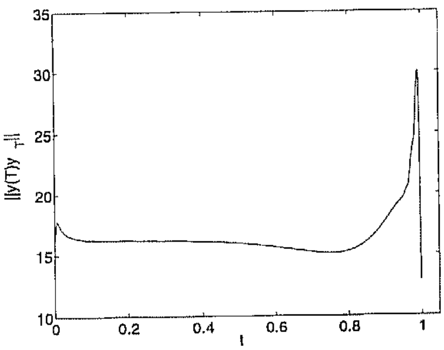

Figure 21. $\|y(t) - y_T\|$, for $f(y) = y^3$.

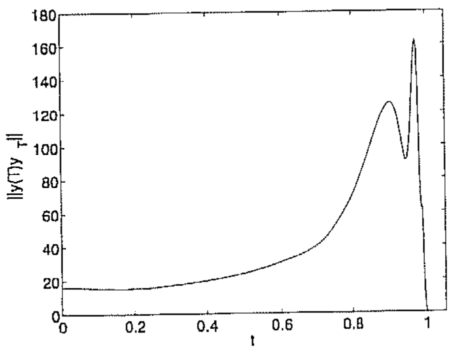

Figure 22. $\|y(t) - y_T\|$, for $f(y) = \operatorname{arctg}(y)$.

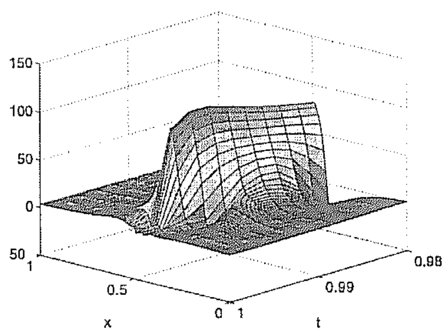

Figure 23. Graphic of $y(x, t)$ ($t \in [0.95, 1]$), for $f(y) = y^3$.

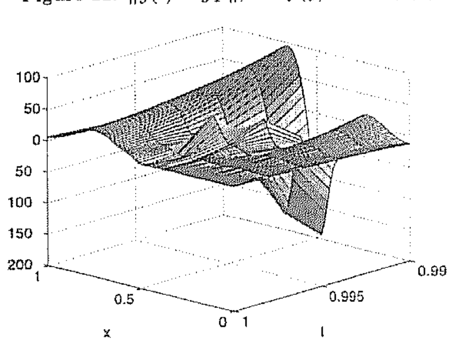

Figure 24. Graphic of $y(x, t)$ ($t \in [0.99, 1]$), for $f(y) = \operatorname{arctg}(y)$.

Table 3. Computational results.

	$f(y) = y^3$	$f(y) = \operatorname{arctg}(y)$
$\|y(0;T) - y_T\|$ $(= \|y_T\|)$	16.3299	16.3299
$\|y(u;T) - y_T\|$	12.920120	1.418373
$\|u\|$	$2.9238 \cdot 10^4$	$5.1621 \cdot 10^3$
$J(0)$	$1.3333 \cdot 10^{14}$	$1.3333 \cdot 10^{14}$
$J(u)$	$8.3466 \cdot 10^{13}$	$1.0059 \cdot 10^{12}$

In Figure 23 (respectively, Figure 24) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.98, 1]$ and $f(y) = y^3$ (respectively, when $t \in [0.99, 1]$ and $f(y) = \operatorname{arctg}(y)$).

In Table 3 we give some further results about our solutions.

6. CONCLUSIONS AND CONJECTURES

Our numerical results give some quantitative information on a result theoretically showed in [2]: when we consider a superlinear at infinity nonlinearity (e.g., $f(y) = y^3$) and the target function y_T does not satisfy (3), then the approximate controllability property fails.

We also (numerically) show the obstruction phenomenon does not appear when f is sublinear at infinity (e.g., $f(y) = \operatorname{arctg}(y)$) and get suitable controls. This is consistent with the theoretical approximate controllability results obtained in [1] (see also [2]).

For the superlinear case, our experiments confirm that, as theoretically proved in [7], when the target function satisfies (3), the controllability property holds. The above mentioned proof in [7] is not constructive and follows a different scheme to the successive *penalized optimal control problems* used in this paper.

A remarkable fact is that, in superlinear cases (and occasionally also in sublinear cases), the solution y oscillates very fast for times $t \in (T - \delta, T)$, getting away from the target state y_T and finally approaching y_T at time T. This is an unstable phenomenon typical of optimal control problems of controllability type, in contrast with the nonoscillating behavior of the solution of stabilization type problems (see, e.g., [18]).

Finally, we point out that the optimal controls obtained in our experiments follow the typical pattern of remaining *close* to zero until the last part of the time interval.

The above numerical experiments lead us to formulate the following conjectures.

A. A theoretical proof of the approximate controllability property for problems with superlinear at infinity nonlinearities and target states satisfying (3) can be also obtained in a constructive way, by means of the *penalized optimal control problems* (CP_k) used in this paper.

B. Fixed a target function y_T satisfying (3), the *cost* (in terms of the norm of the controls) to approximate this function is, in general, much bigger for superlinear cases than for sublinear cases. However, this result can be false if y_T is *small* enough. For instance, when $f(y) = |y|^{p-1}y$, the cost to approximate y_T is much bigger when $p > 1$, except for target functions satisfying $|y_T(x)| \leq 1$. This conjecture is exactly the opposite of the results obtained in [19] for the case of initial value control problems with nonlinearities of the type $f(y) = -y^3$.

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