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FINITE SPEED OF PROPAGATION AND WAITING TIME FOR LOCAL SOLUTIONS OF DEGENERATE EQUATIONS IN VISCOELASTIC MEDIA OR HEAT FLOWS WITH MEMORY

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Abstract. The finite speed of propagation (FSP) was established for certain materials in the 70's by the American school (Gurtin, Dafermos, Nohel, etc.) for the special case of the presence of memory effects. A different approach can be applied by the construction of suitable super and sub-solutions (Crandall, Nohel, Díaz and Gomez, etc.). In this paper we present an alternative method to prove (FSP) which only uses some energy estimates and without any information coming from the characteristics analysis. The waiting time property is proved for the first time in the literature for this class of non-local equations.

Dedicated to Professor David Kinderlehrer on occasion of his 75th birthday.

1. Introduction

The main goal of this paper is to get some qualitative properties, such as finite speed of propagation and waiting time effect, for local in space solutions (i.e. independently of any possible boundary conditions) of some nonlinear evolution equations involving a nonlocal in time term as in the following formulation:

$$\left\{\begin{array}{l} \frac{\partial u}{\partial t} = \frac{\partial (\sigma_1(u_x))}{\partial x} + \frac{\partial}{\partial x}\left(\int_0^t \gamma(x,t,s)\sigma_0(u_x(x,s))ds\right) + \tilde{f}(x,t), \\ u(x,0) = u_0(x). \end{array}\right. \tag{1.1}$$

Formulations as (1.1) arise in many different contexts after making some easy transformations. This, specially the case, when modelling different mechanical phenomena of viscoelastic media. Indeed, if we introduce the displacement vector, as usual in Continuum Mechanics, by

$$\mathbf{u} = \mathbf{x} - \xi, \qquad \mathbf{x}(0) = \xi,$$

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the spatial velocity $\mathbf{v}$ is given by $\mathbf{v} = \dfrac{\partial \mathbf{u}}{\partial t}$ ($\mathbf{v}(\mathbf{x},t)$ and $\mathbf{x}(t;\xi)$ are respectively the Euler and the Lagrangian coordinates). Calculating the acceleration as the *material derivative* of the speed

$$D_t\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\,\mathbf{v}$$

and assuming that the second term on the right-hand side is sufficiently small (because $\mathbf{v}$ or $|\nabla\,\mathbf{v}|$ is small), we get

$$D_t\mathbf{v} = \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

In the case of some one-dimensional motions with constant density (which can be re-scaled as to be the unit) we can assume that

$$\mathbf{u}(\mathbf{x},t) = (u(x,t),0,0),\ \mathbf{x} = (x,0,0),\ \mathbf{f} = (f(x,t),0,0),$$

and that the components of the stress tensor $\mathbf{S}$ have the form

$$S^{11} = \sigma, \qquad S^{ij} = 0 \quad \text{for } i = 1,2,3,\ j = 2,3.$$

With the above simplifications, the momentum balance law takes the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} + f(x,t). \tag{1.2}$$

Now we make the constitutive assumption typical of viscous-elastic media: the stress tensor $\mathbf{S}$ is a function of the deformation gradient of the displacement and the speed. In our case, this constitutive law can be written as

$$\sigma = \sigma(u_x, u_{xt}). \tag{1.3}$$

Then, from (1.2) and (1.3) we get the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma(u_x(x,t), u_{xt}(x,t))}{\partial x} + f(x,t). \tag{1.4}$$

Obviously some initial conditions must be given:

$$u(x,0) = u_0(x),\ \ u_t(x,0) = \varphi(x). \tag{1.5}$$

Equations of the type of (1.4) occur in various problems concerning the motions of viscous-elastic media and was intensively studied in the literature. For instance, in [1] it was proved existence of solutions to the equation

$$u_{tt} = u_{xxt} + \sigma(u_x)_x$$

and in [17] it was investigated the existence, uniqueness and stability of solutions of the equation

$$\rho_0 u_{tt} = \lambda u_{xxt} + \sigma'(u_x)u_{xx}.$$

The mixed initial-boundary value problem for the equations of nonlinear one-dimensional viscoelasticity were considered in [10],[19].

Here, in this paper, we shall assume that the function $\sigma(r,q)$ may include some (x,t) dependence but always under the following growth conditions:

$$\begin{cases} \sigma \equiv \sigma(x,t,s,r,q) = \gamma(x,t,s)\sigma_0(r) + \dfrac{\partial \sigma_1(r)}{\partial r} q, \text{ for any } s \le t,\ r, q \in \mathbb{R}, \\ C_2|r|^p \le \sigma_1(r) r \le C_1 |r|^p, \quad 2 < p < \infty, \text{ for any } r \in \mathbb{R}, \\ |\sigma_0(r)| \le C_3 |r|^{p-1} \text{ for any } r \in \mathbb{R}\text{-} \end{cases} \tag{1.6}$$

The above given function $\gamma(x,t,s)$ (which we assume to be bounded) may contain many diverse information (as, for instance, some memory effects, etc.). Notice that under (1.6) equation (1.2) can be rewritten in the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2(\sigma_1(u_x))}{\partial x \partial t} + \frac{\partial(\gamma(x,t,s)\sigma_0(u_x))}{\partial x} + f(x,t). \tag{1.7}$$

Hence, integrating in t we arrive to the formulation (1.1) with

$$\tilde{f}(x,t) := \varphi(x) - \frac{\partial \sigma_1(u_{0x}(x))}{\partial x} + \int_0^t f(x,s) ds. \tag{1.8}$$

Problem (1.1) also arises in the study of heat flows with memory. In that case a very general starting point is the balance

$$\frac{\partial}{\partial t}\left(u(t,x) + \int_0^t b(t-s) u(s,x) x \right) = d_0 \sigma(u_x)_x + \int_0^t a(t-s)\Psi(u_x(s,x))_x s \tag{1.9}$$

where the functions σ and Ψ are assumed to be increasing real-valued functions such that $\sigma(0) = \Psi(0) = 0$. See, e.g., the expositions made in [29], [28] and [11] and their references.

We point out that in most of the papers in the previous literature it was assumed some similar conditions to (1.6) but for an uniformly elliptic diffusion term, $p = 2$. Our main goal in this paper is the consideration of the degenerate case $p > 2$. In the recent decades, the evolution equations with memory terms for $p > 2$ have been also considered. For instance, in [3], it was considered the Dirichlet problem for the evolution p-Laplace equation with a nonlocal term

$$\begin{cases} u_t - \Delta_p u = \int_0^t g(t-s)\Delta_p u(x,s)\, ds + \Theta(x,t,u) + f(x,t) & \text{in } Q = \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz-continuous boundary, $g(s)$ is a given memory kernel, Θ is a given function and Δ_p denotes the p-Laplace operator $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right), \quad 1 < p < \infty$. Existence and uniqueness of solutions for this problem were proved. Also were established that the disturbances from the data propagate with finite speed and the waiting time effect is possible. Once again, problem (1.10) appears in the mathematical description of the heat propagation in materials with memory where the heat flux may depend on the past history of the process.

The question of the solvability, and the long time behavior of solutions of the abstract nonlinear Volterra equations of the type

$$u_t(t) - Bu(t) + \int_0^t g(t-s) A u(s)\, ds = f(t),$$

and nonlocal equations of similar structure, were studied in many papers of the past literature [5, 9, 13, 20, 23, 26, 31]. For instance, in [26, 27] the solution $u(t)$ takes values in a reflexive Banach space W, u_t is an element of the dual space W', and A and B are the operators given by the subdifferentials of two convex, lower semicontinuous and proper functions. An example of such an equation is furnished by the problem

$$u_t(x,t) - u_{xx}(x,t) = \int_0^t g(t-s)(\sigma(u_x(x,s)))_x \, ds + f(x,t)$$

with a sufficiently smooth function σ - see [9]. For the semilinear equation (1.10) with $p = 2$ and $\Theta \neq 0$ the same questions were addressed by many authors, see, e.g., [6, 7, 8] and references therein. Nonexistence of global solutions (a finite time blow-up) for semilinear equations was studied in [22, 24, 25]. Similar results are also known for nonlocal parabolic equations and boundary conditions of other types, see, e.g., [14, 16, 21]. Doubly nonlinear nonlocal parabolic equations

$$\partial_t \beta(u) - \operatorname{div} \sigma(\nabla u) = \int_0^t g(t-s) \operatorname{div} \sigma(\nabla u(s)) \, ds + f(x,t,u)$$

were studied in [30] in an abstract setting. In [15] was investigated the existence of weak solutions of a class of quasilinear hyperbolic integro-differential equations describing viscoelastic materials.

In contrast to most of the previous studies on the derivation of the finite speed of perturbation for viscoelastic media we shall not use any characteristic argument but purely some suitable energy arguments in the spirit of the monograph [4]. Our arguments are of a different nature to some other energy methods which need some information obtained trough the characteristics (see, e.g. [33] and [32]). As far as we know, the waiting time property was never before obtained in the literature for the class of nonlocal problems of the type (1.1) (see Remark 1 below).

2. Finite speed of propagation and the waiting time effect

Given, $\Omega = (-L, L)$, we consider (local in space) weak solutions to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial(\sigma_1(u_x))}{\partial x} + \frac{\partial}{\partial x}\left(\int_0^t \gamma(x,t,s)\sigma_0(u_x(x,s))ds\right) + \tilde{f}(x,t), \tag{2.1}$$

in the class of functions

$$u \in W \equiv C\left([0,T]; L^2_{loc}(\Omega)\right) \cap L^p\left(0,T; W^{1,p}_{loc}(\Omega)\right)$$

and satisfying the initial condition

$$u(x,0) = u_0(x) \quad x \in \Omega. \tag{2.2}$$

Our main assumption on the initial condition $u_0 \in W^{1,p}_{loc}(\Omega)$ is that it represents a finite propagation in the sense that

$$u_0(x) \equiv 0, \text{ when } |x| \leq \rho_0 < L. \tag{2.3}$$

As mentioned before, we assume that $\forall\, r \in \mathbb{R}$

$$C_2|r|^p \leq \sigma_1(r)r \leq C_1|r|^p, \quad 2 < p < \infty, \tag{2.4}$$

$$|\sigma_0(r)| \leq C_3|r|^{p-1}. \tag{2.5}$$

Thus, any local weak solution satisfies, for any ball $B_\rho \subset \Omega$, and for some $M > 0$,

$$\sup_{0\le t\le T} \int_{B_\rho} u^2 dx + \int_0^T \int_{B_\rho} |u_x|^p dx d\tau \le M. \tag{2.6}$$

Results on the existence of solutions in this class of functions (once we specify the boundary conditions) are well known in the literature (see for example [2, 4]). Our main result is the following:

Theorem 1. *Assume* (2.4), (2.5) *and also (2.3) and*

$$\tilde f(x,t) \equiv 0,\ |x| \le \rho_0 \ \textit{for a.e.}\ t \in (0,T).$$

Let $u(x,t)$ be a local weak solution of problem (2.1), (2.2) *satisfying* (2.6). *Then $u(x,t)$ possesses the finite speed of propagation property (FSP) in the following sense:*
there exist $t^* \in (0,T]$ and a function $\rho(t)$, with $0 < \rho(t) < \rho_0$, $\rho(0) = \rho_0$, such that

$$u(x,t) = 0 \quad \textit{for}\ |x| < \rho(t),\ 0 \le t \le t^*.$$

The function $\rho(t)$ satisfies

$$\rho^{1+\alpha}(t) = \rho_0^{1+\alpha} - Ct^\kappa \tag{2.7}$$

for some positive constants α, κ depending only on p, and $C \equiv C(p,T,M)$. Moreover, if

$$\int_{B_\rho} u_0^2 dx + \int_0^T \int_{B_\rho} |\widetilde f(x,\tau)|^2 dx d\tau \le C\,(\rho-\rho_0)_+^{1/(1-\nu)} \tag{2.8}$$

for any $\rho \in (\rho_0, L)$, with

$$\nu = \nu(p) = \frac{2p}{3p-2},$$

then $u(x,t)$ possesses the waiting time property (WTP): there exists $t^* > 0$ such that

$$u(x,t) = 0 \qquad \textit{for any}\ |x| \le \rho_0\ \textit{and any}\ t \in [0,t^*].$$

Remark 1. We point out that the growth estimate given by (2.7) is quite unusual in the literature for this class of integro-differential equation. The main reason is that most of the authors assume $p = 2$ and then the application of the characteristics method leads to estimates on the interface involving expressions of the type $\sigma_1'(u_{0x}(x_\pm))$, for the points $x_\pm$ defining the boundary of the support of u_0. Notice that if we assume conditions (1.6) then we get $\sigma_1'(u_{0x}(x_\pm)) = 0$ at least for initial data which are flat enough near $x_\pm$ (so that $u_{0x}(x_\pm) = 0$) which explains (but it does not proves it!!) the possibility to get the waiting time property.

Proof. We shall prove Theorem 1 by using an energy method similar to the ones presented in the monographs [2, 4].

First of all we introduce the set B_ρ and points S_ρ by

$$B_\rho = \{x, x_0 \in \Omega :\ |x - x_0| < \rho\} \subset \Omega,\ S_\rho = \partial B_\rho.$$

We define the energy functions

$$b(\rho,\tau) = \int_{B_\rho} |u(\cdot,\tau)|^2 dx, \quad \overline{b}(\rho,t) = \ sup_{0\le\tau\le t} b(\rho,\tau),$$

$$E(\rho,t)=\int_0^t\int_{B_\rho}\sigma_1(u_x(x,\tau))u_x(x,\tau)dxd\tau,$$
$$\overline{E}(\rho,t)=\ sup_{0\le\tau\le t}E(\rho,\tau).$$

Since
$$C_1|r|^p\le\sigma_1(r)r\le C_2|r|^p,$$
we get
$$C_1\int_0^t\int_{B_\rho}|u_x|^p\le E(\rho,t)\le C_2\int_0^t\int_{B_\rho}|u_x|^p,$$
and according to (1.6)

$$b(\rho,\tau)+E(\rho,t)\le CM. \tag{2.9}$$

Notice the following important properties on the energy functions:
$$\mathrm{sup}_{0\le\tau\le t}\frac{\partial E(\rho,\tau)}{\partial\rho}=\frac{\partial\overline{E}(\rho,t)}{\partial\rho}=$$
$$=\int_0^t\int_{S_\rho}\sigma_1(u_x(x,\tau))u_x(x,\tau)dSd\tau>0,$$
$$E_t=\frac{\partial E(\rho,t)}{\partial t}=\int_{B_\rho}\sigma_1(u_x(x,\tau))u_x(x,\tau)dx>0.$$
Since we are in the one dimensional case, we have
$$\overline{E}_\rho=\frac{\partial\overline{E}(\rho,t)}{\partial\rho}=$$
$$=\int_0^t\left(\sigma_1(u_x)u_x(-\rho,\tau)+\sigma_1(u_x)u_x(\rho,\tau)\right)d\tau>0.$$
Multiplying equation (2.1) by u,integrating over the cylinder $B_\rho\times(0,t)$ and applying the formula of integration by parts, we arrive to the energy relation
$$\frac{1}{2}\int_{B_\rho}u^2(\cdot,\tau)dx\bigg|_{\tau=0}^{\tau=t}+\int_0^t\int_{B_\rho}\sigma_1(u_x)u_xdxd\tau=I,$$
or, in the notation of the energy functions,

$$b(\rho,\tau)|_{\tau=0}^{\tau=t}+E(\rho,t)=\sum_{i=1}^{4}I_i\equiv I, \tag{2.10}$$

where
$$I_1=\int_0^t u(\xi,\tau)\sigma_1(u_x)|_{\xi=-\rho}^{\xi=\rho}\,d\tau,$$
$$I_2=\int_0^t u(\xi,\tau)\int_0^\tau\gamma(x,t,s)\sigma_0(u_x(\xi,s))ds|_{\xi=-\rho}^{\xi=\rho}\,d\tau,$$
$$I_3=-\int_0^t\int_{B_\rho}u_x(x,\tau)\int_0^\tau\gamma(x,t,s)\sigma_0(u_x(x,s))dsdxd\tau,$$

$$I_4 = \int_0^t \int_{B_\rho} u\tilde{f}dxd\tau.$$

First we shall prove the FSP property. Without lost of generality we can assume that $f \equiv 0$. In this case we use that $b(\rho, 0) = 0$ for $\rho \leq \rho_0$ since $u_0(x) = 0$, if $x \in [-\rho_0, \rho_0]$, and that

$$I_4 = \int_0^t \int_{B_\rho} u\tilde{f}dxd\tau = 0.$$

The energy relation (2.10) takes now the form

$$b(\rho, t) + E(\rho, t) = \sum_{i=1}^{3} I_i \equiv I. \tag{2.11}$$

Next we use the multiplicative estimate (for any fixed t)

$$\begin{aligned} [u]^p &:= (|u(-\rho, t) + u(\rho, t)|)^p \\ &\leq C \left(E_t^{\frac{1}{p}} + \rho^{-\delta} b^{\frac{1}{2}} \right)^{p\theta} b^{\frac{p(1-\theta)}{2}}, \end{aligned}$$

where δ and θ are some given positive parameters. We evaluate the terms I_i, $i = 1, 2, 3$ in the following way:

$$\begin{aligned} |I_1| &\leq C \left(\int_0^t [u]^p \right)^{\frac{1}{p}} \left(\int_0^t (|u_x(-\rho, \cdot)|^p + u_x(\rho, \cdot)|^p) \right)^{\frac{p-1}{p}} \\ &\leq C \left(\int_0^t \left(E_t^{\frac{1}{p}} + \rho^{-\delta} b^{\frac{1}{2}} \right)^{p\theta} b^{\frac{p(1-\theta)}{2}} \right)^{\frac{1}{p}} \left(\overline{E}_\rho \right)^{\frac{p-1}{p}} \\ &\leq C \overline{b}^{\frac{(1-\theta)}{2}} \left(\int_0^t \left(E_t + \rho^{-\delta p} b^{\frac{p}{2}} \right)^{\theta} \right)^{\frac{1}{p}} \left(\overline{E}_\rho \right)^{\frac{p-1}{p}} \\ &\leq C \max \left(1, \overline{b}^{\frac{p-2}{2}} \right) \overline{b}^{\frac{(1-\theta)}{2}} t^{1-\theta} \rho^{-\delta\theta} \left(\overline{E} + \overline{b} \right)^{\frac{\theta}{p}} \left(\overline{E}_\rho \right)^{\frac{p-1}{p}} \\ &\leq C t^{1-\theta} \rho^{-\delta\theta} \left(\overline{E} + \overline{b} \right)^{\frac{\theta}{p} + \frac{(1-\theta)}{2}} \left(\overline{E}_\rho \right)^{\frac{p-1}{p}}, \\ |I_2| &\leq C t^{1-\theta} \rho^{-\delta\theta} \left(\overline{E} + \overline{b} \right)^{\frac{\theta}{p} + \frac{(1-\theta)}{2}} \left(\overline{E}_\rho \right)^{\frac{p-1}{p}}, \\ |I_3| &\leq C t^{\kappa} \overline{E}(\rho, t). \end{aligned}$$

Substituting last estimates to (2.11), taking the maximum with respect to t and applying the Young inequality, we arrive to the ordinary differential inequality, with respect to ρ,

$$\overline{E}(\rho, t) \leq \overline{b}(\rho, t) + \overline{E}(\rho, t) \leq C \rho^{-\frac{\alpha}{\nu}} t^{\frac{\chi}{\nu}} \left(\overline{E}_\rho(\rho, t) \right)^{\frac{1}{\nu}},$$

or equivalently

$$\overline{E}^{\nu}(\rho, t) \leq C \rho^{-\alpha} t^{\chi} \overline{E}_\rho(\rho, t). \tag{2.12}$$

Here the time t is considered as a fixed parameter. Integrating the last inequality with respect to ρ, over (ρ, ρ_0), we obtain

$$\overline{E}^{1-\nu}(\rho, t) \leq \overline{E}^{1-\nu}(\rho_0, t) - C \frac{1-\nu}{1+\alpha} \left(\rho_0^{1+\alpha} - \rho^{1+\alpha} \right) t^{-\chi}.$$

Then defining $\rho(t)$ by the formula

$$\rho^{1+\alpha}(t) = \rho_0^{1+\alpha} - \overline{E}^{1-\nu}(\rho_0, t)\frac{1+\alpha}{C(1-\nu)}t^{\chi},$$

and assuming that

$$\overline{E}^{1-\nu}(\rho_0, t) \le CM,$$

we arrive to the desired expression

$$\rho^{1+\alpha}(t) = \rho_0^{1+\alpha} - Ct^{\kappa},$$

and the first assertion of the theorem is proved.

To prove the waiting time property we use the energy relation for $\rho > \rho_0$ and evaluate the additional terms in the following way:

$$I_4 = \int_0^t \int_{B_\rho} u\tilde{f}dxd\tau$$

$$= \int_0^t \int_{B_\rho} \left(u_t(x,0) - \frac{\partial\sigma_1(u_{0x}(x))}{\partial x} + \int_0^\tau f(x,s)ds \right) u(x,\tau)dxd\tau,$$

and

$$I_5 = \int_{B_\rho} u_0^2 dx,$$

which implies

$$|I_4| + |I_5| \le \delta\overline{b}(\rho,t) + C(\delta)t^{\eta} \int_0^t \int_{B_\rho} \left|\tilde{f}\right| dxd\tau.$$

Finally under conditions (1.6) we arrive to the ordinary non-homogeneous differential inequality

$$\overline{E}^{\nu}(\rho,t) \le Ct^{\chi}\left(\rho^{-\alpha}\overline{E}_\rho(\rho,t) + (\rho-\rho_0)_+^{\frac{\nu}{1-\nu}}\right). \tag{2.13}$$

As in ([4]) we can prove that for a sufficiently small $t^* > 0$ and $0 < t \le t^*$ all solutions of the above inequality must satisfy

$$\overline{E}(\rho,t) \le C\left(\rho-\rho_0\right)_+^{\frac{1}{1-\nu}}, \tag{2.14}$$

and the result holds. □

Remark 2. The localization properties can also be studied in a more general class of data in which the function $\sigma = \sigma(r,q)$ is not subject to conditions (1.6). The study is performed in terms of the function

$$w(x,t) = u_t(x,t), \qquad u(x,t) = \int_0^t w(x,\tau)d\tau + u_0(x),$$

which satisfies the equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\sigma\left(\int_0^t w_x(x,\tau)d\tau + u_{0x}(x),\, w_x(x,t)\right)\right) + f(x,t).$$

The energy methods of [4] still apply and we can get similar properties to the (FSP) and (WTP) for the function $w(x,t)$, but we shall not develop it here.

Remark 3. Some other qualitative properties, such as the finite extinction time, for many other nonlocal problems can be obtained trough energy methods (see, e.g., [4] and [12] and its references).

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