Pointwise gradient estimates and stabilization for Fisher-KPP type equations with a concentration dependent diffusion

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Abstract

We prove a pointwise gradient estimate for the bounded weak solution of the Cauchy problem associated to the quasilinear Fisher-KPP type equation $u_t = \varphi(u)_{xx} + \psi(u)$ when φ satisfies that $\varphi(0)$=0, and $\psi(u)$ is vanishing only for levels $u = 0$ and $u = 1$. As a first consequence we prove that the bounded weak solution becomes instantaneously a continuous function even if the initial datum is merely a discontinuous bounded function. Moreover the obtained estimates also prove the stabilization of the gradient of bounded weak solutions as $t \to +\infty$ for suitable initial data.

1 Introduction

In this note we shall get some pointwise a priori estimates on the spatial derivatives of the solutions of quasilinear Fisher-KPP type

$$(CP)\begin{cases} u_t = \varphi(u)_{xx} + \psi(u) & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

where

$$\varphi \in C^0([0,+\infty)) \text{ is strictly increasing, } \varphi(0)\text{=}0, \tag{1.1}$$

$$\psi \in C^0([0,+\infty)),\ \psi(0) = \psi(1) = 0,\ \psi(u) > 0 \text{ if } u \in (0,1) \text{ and } \psi(u) < 0 \text{ if } u \in (1,+\infty). \tag{1.2}$$

We shall always assume that

$$0 \leq m_{u_0} \leq u_0(x) \leq M_{u_0} \quad \text{a.e.} \quad x \in \mathbb{R}, \tag{1.3}$$

for some $m_{u_0} \geq 0$ and $M_{u_0} > 0$. Notice that due to assumption (1.2) the behaviour of solutions is quite different according the cases $M_{u_0} \leq 1$ and $m_{u_0} > 1$. We shall also assume some additional regularity on φ and ψ but merely on the the values selected trough the initial condition (1.3). More precisely, we shall also assume that

$$\varphi \in W^{2,1}_{loc}((\overline{m}_{u_0}, \overline{M}_{u_0})),\ \psi \in W^{1,1}_{loc}((\overline{m}_{u_0}, \overline{M}_{u_0})) \text{ and } [(\varphi'\psi)']_{-} \in L^1(\overline{m}_{u_0}, \overline{M}_{u_0}), \qquad (1.4)$$

where $\overline{m}_{u_0} = \min(1, m_{u_0})$, $\overline{M}_{u_0} = \max(1, M_{u_0})$ and, in general, $[h]_{-} := \max(0, -h)$ for any $h \in \mathbb{R}$.

If $\varphi'(0) = 0$ (or/and $\varphi'(1) = 0$) then the diffusion coefficient $D(u) = \varphi'(u)$ vanishes and the equation degenerates at the points $(t, x) \in (0, +\infty) \times \mathbb{R}$ where $u = 0$ (or/and $u = 1$) and a notion of weak solution should be defined. The lack of classical solution also, arise even for a linear diffusion $\varphi(s) = s$, when function $\psi(u)$ is not Hölder continuous.

Definition. *By a bounded weak solution of the Cauchy problem (CP) we mean a nonnegative function u such that $u \in L^\infty((0,T)\times\mathbb{R})$ for any $T > 0$ and satisfies the identity*

$$\int_0^T \int_{\mathbb{R}} [\zeta_t u + \zeta_{xx}\varphi(u) + \zeta\psi(u)]dxdt + \int_{\mathbb{R}} \zeta(0, x)u_0(x)dx = 0, \qquad (1.5)$$

for any $\zeta \in D([0,T)\times\mathbb{R})$.

The generality assumed on $\varphi(u)$, not necessarily associated to any power growth law, is an additional difficulty since for the porous media type equation, $\varphi(u) = u^m$, $m \geq 1$, many well-known properties could be used. The existence of a bounded weak solution is not far from several results available in the literature. For nondegenerate case ($\varphi'(s) > 0$ for any $s \geq 0$) and $\psi(u)$ regular enough we refer to the book [LSU]. For a more general framework (including the case of possible discontinuous bounded initial data and the generality assumed here on φ, ψ) see the exposition made in [Di]. In particular, it is shown in this reference that the uniqueness of a bounded weak solution holds under the additional condition on $\psi(u)$ on $[0, \overline{M}_{u_0}]$ where $\overline{M}_{u_0} = \max(1, M_{u_0})$:

$$\left\{ \begin{array}{c} \psi(u) = \psi_L(u) + \psi_d(u) \text{ with } \psi_L(u) \text{ Lipschitz continuous on } [0, \overline{M}_{u_0}] \\ \text{and } \psi_d(u) \text{ continuous and nonincreasing on } [0, \overline{M}_{u_0}]. \end{array} \right. \qquad (1.6)$$

Let us mention, in particular that, as in [OlKalCh], bounded weak solution can be constructed as limit of classical solutions associated to regularized data $\varphi_n(u), \psi_n(u)$ and $u_{0,n}(x)$. Notice that in fact this bounded weak solution satisfies that $u(t, x) \leq \max(1, M_{u_0})$ for any $t \geq 0$ and a.e. $x \in \mathbb{R}$, as it follows from the structural assumption (1.2): indeed, it is well known that the solutions of the ordinary differential equation $U'(t) = \psi(U(t))$, with $U_0 \in [0, M_{u_0}]$ satisfy that $U(t) \in [0, M_{u_0}]$ and that $U(t) \to 1$ as $t \to +\infty$ if $U_0 > 0$, so that it suffices to apply a comparison argument to get that $0 \leq u(t, x) \leq U(t)$ for any $t \geq 0$ and a.e. $x \in \mathbb{R}$. A similar argument shows that $u(t, x) \geq \min(1, m_{u_0})$ for any $t \geq 0$ and a.e. $x \in \mathbb{R}$.

The main goal of this short Note is to get some pointwise gradient estimates proving, among other thinks, that any bounded weak solution becomes instantaneously, for any

$t > 0$, a continuous function. The continuity of bounded weak solutions is a delicate point due, again, to the generality assumed on $\varphi(u)$ and $\psi(u)$. For instance, the general scope used in [DB] does not applies to our framework (its condition $[A_1]$ requires $\varphi'(0) = 0$). Much closer to our goals are the regularity and gradient estimates obtained in [Ar], [Kal] and [Ke] (see also [He-Va], [Sou] and the general exposition made in [Va]) but again the additional conditions assumed in those papers on $\varphi(u)$ and $\psi(u)$ do not allow their application under the mere assumptions (1.1) and (1.2). In any case, the continuity of bounded weak solutions will be obtained in this paper trough some pointwise gradient estimates which have their own relevance. In particular, the explicit time dependence in the estimates (not investigated in the above mentioned references) will allow us to get some applications to the study of the stabilization of solutions as $t \to +\infty$ for suitable initial data.

We shall derive here some gradient estimates on function $\varphi(u)$ (sometimes called Bernstein estimates) by using the method proposed in the paper Benilan and Díaz [BeDi] (see also the pioneering exposition [Be] where similar estimates were first proved for (CP) but without source term, i.e. for $\psi(u) \equiv 0$).

Theorem. *Assume (1.1), (1.2), (1.3), (1.6) and (1.4). Then,*

a) the bounded weak solution of (CP) *satisfies that* $u \in C^0((0,+\infty)\times\mathbb{R}))$, *for any* $T > 0$ *we have* $\varphi(u)_x \in L^1(0,T : L^\infty(\mathbb{R})) \cap L^\infty(T,+\infty : L^\infty(\mathbb{R}))$ *and for any* $t > 0$ *we have the gradient estimate*

$$\begin{gathered} \|\varphi(u(t,.))_x\|^2_{L^\infty(\mathbb{R})} \leq \frac{\overline{M}_{u_0}\varphi(\overline{M}_{u_0})}{t} + \\ +2(\int_{\overline{m}_{u_0}}^{\|u(t,.)\|_{L^\infty(\mathbb{R})}} \left(\int_w^{\overline{M}_{u_0}} \left[(\varphi'\psi)'\right]_- (s)ds\right) dw + \overline{m}_{u_0} \int_{\overline{m}_{u_0}}^{\overline{M}_{u_0}} \left[(\varphi'\psi)'\right]_- (s)ds \\ + \max\nolimits_{\tau\in[0,t]} \left[\varphi'\psi\right]_+ (\|u(\tau,.)\|_{L^\infty(\mathbb{R})})). \end{gathered} \tag{1.7}$$

b) if $M_{u_0} \leq 1$ *and* $m_{u_0} > 0$, *or if* $m_{u_0} \geq 1$, *we have*

$$\lim_{t\to+\infty} \|\varphi(u(t,.))_x\|_{L^\infty(\mathbb{R})} = 0. \tag{1.8}$$

We point out that conclusion b) shows the stabilization of the gradient of $\varphi(u(t,.))$ and gives an idea about how fast the bounded weak solution becomes spatially uniform, i.e. $\varphi(u(t,.))$ stabilizes toward 1 as $t \to +\infty$. Indeed, if we assume that $M_{u_0} \leq 1$ and $m_{u_0} > 0$ (respectively $m_{u_0} \geq 1$) then, by comparing $u(t,x)$ with the solution of the ordinary differential equation $U'(t) = \psi(U(t))$, $U(0) = U_0$, with $U_0 = m_{u_0}$ (respectively $U_0 = M_{u_0}$) we deduce that $\|u(t,.) - 1\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to +\infty$. Nevertheless, the spatial variability could be very complex if, for instance, $u_0(x)$ is an highly oscillating function (as it is the case, e.g. of $u_0(x) = \frac{1}{2} + (\frac{1}{2} - U_0)\sin n\pi x$, with n very large and $U_0 \in (0,1)$). Estimate (1.8) shows that $u(t,.)$ becomes each time spatially more uniform as $t \to +\infty$. A deeper result, which, in fact, was the origin of our special motivation in this problem, will be proved in a subsequent paper [DiKa] to this one, in which it will be shown (in the line of [KaR]) that under some additional conditions on the data, φ, ψ and u_0, the bounded weak solution converges, in a suitable way, as $t \to \infty$, to the travelling wave of the equation linking, from $-\infty$ to $+\infty$ the two existing constant stationary solutions ($\overline{u}_\infty(x) \equiv 1$ and $\underline{u}_\infty(x) \equiv 0$). This is a very rich subject, specially after the pioneering results of the celebrated paper by Kolmogorov, Petrovsky and Piscunov [KPP], of so great relevance in genetic theory (see Fisher [Fi]). We send the reader to [KaR] and its references, for many other results on the asymptotic behaviour, as $t \to +\infty$, of bounded weak solutions of this type of problems.

2 Bernstein estimates and the main result

Like in [BeDi], it is useful to start by considering regular solutions of the more general equation

$$u_t = \sigma(u, u_x)_x + \beta(u, u_x) \tag{2.9}$$

($\beta(u, u_x) \equiv 0$ in [BeDi]). The main idea of the method presented in [BeDi] is to find an auxiliary nonlinear parabolic operator $\mathcal{L}$ in such a way that a pointwise gradient estimate of the type

$$\sigma(u, u_x) \leq k(t, x, u) \tag{2.10}$$

holds as a consequence of the comparison principle applied to a function of the form $p = \sigma(u, u_x) \ / \ k(.,.,u)$, in the kernel of operator $\mathcal{L}$, and the constant function $p \equiv 1$, i.e.

$$\mathcal{L}(p) = 0 \leq \mathcal{L}(1). \tag{2.11}$$

In this way, the possible choices of the function $k(t, x, u)$ are determined by the condition $\mathcal{L}(1) \geq 0$. Another characteristic of the method is that it can be used *locally in time and space* and, in particular, it illustrates some regularizing effects (for instance some gradient bounds take place for any $t > 0$, independently of whether it already holds at $t = 0$ or not). It was in this way how it was obtained some pointwise gradient estimates for the unperturbed equation ($\beta \equiv 0$) when $\varphi(u)$ is merely supposed to be a maximal monotone graph (see Corollary 6 of [BeDi]).

Coming back to the a priori estimate, let Q be an open regular set of $(0, +\infty) \times \mathbb{R}$ and let assume that u is a classical solution of the equation. We assume also that $u \in C^{2,3}(Q)$ such that

$$0 \leq \overline{m}_{u_0} = \min(1, m_{u_0}) \leq u(t, x) \leq \overline{M}_{u_0} = \max(1, M_{u_0}) \ \text{ on } Q, \tag{2.12}$$

and, at this step, that $\sigma, \beta \in C^2([0, \overline{M}_{u_0}] \times \mathbb{R})$. As indicated at the Introduction, the property $0 \leq \overline{m}_{u_0} \leq u(t, x) \leq \overline{M}_{u_0}$ comes from the structural assumption (1.2) by comparing u with $U(t)$ solution of the ordinary differential equation $U'(t) = \psi(U(t))$ once we assume appropriate boundary conditions on ∂Q, nevertheless here property (2.12) is assumed as an a priori condition.

Our main goal is to obtain pointwise estimates as (2.10) for suitable auxiliary functions k. As mentioned before, we start by assuming the regularity conditions: $k \in$

$C^2(Q \times [0, \overline{M}_{u_0}])$ and $k \neq 0$. The principal ingredient of the method is to find a (possibly nonlinear) parabolic operator, $\mathcal{L}$, vanishing when applied to the function

$$p(t,x) := \frac{\sigma(u(t,x), u_x(t,x))}{k(t,x,u)} \qquad (t,x) \in Q \tag{2.13}$$

Using (2.9) we find

$$u_t = (kp)_x + \beta(u,\ u_x) = (k_x + k_u + k_u u_x)p + kp_x + \beta(u,\ u_x) \tag{2.14}$$

and differentiating (2.14) with respect to x,

$$\begin{aligned} u_{tx} &= (k_{xx} + 2k_{xu}u_x + k_{uu}u_x^2 + k_u u_{xx})p + 2(k_x + k_u u_x)p_x + kp_{xx} + \beta_u u_x + \beta_{u_x} u_{xx} \\ &= \sigma_u u_x + \sigma_{u_x} u_{xx} + \beta_u u_x + \beta_{u_x} u_{xx}. \end{aligned} \tag{2.15}$$

Differentiating (2.13) with respect to t we have

$$p_t k + p(k_t + k_u u_t) = \sigma_u u_t + \sigma_{u_x} u_{tx}.$$

Multiplying (2.14) by σ_u and (2.15) by σ_{u_x} and adding the resulting expressions we find that

$$\widetilde{\mathcal{L}}(p, u_x) = 0$$

where

$$\begin{aligned} \widetilde{\mathcal{L}}(p, u_x) = p_t k + pk_t &+ (pk_u - \sigma_u)\{(k_x + k_u u_x)p + kp_x + \beta\} \\ &- \sigma_{u_x}[(k_{xx} + 2k_{xu}u_x + k_{uu}u_x^2)p + 2(k_x + k_u u_x)p_x + kp_{xx} + \beta_u u_x] \\ &- (k_u p + \beta_{u_x})[p_x k + p(k_x + k_u u_{x)} - \sigma_u u_x]. \end{aligned}$$

In order to obtain an expression independent of u_x we assume

$$\sigma_{u_x} > 0. \tag{2.16}$$

Thus we can define $S(u, \cdot) := \sigma(u, \cdot)^{-1}$ and so $u_x = S(u, kp)$. Finally, in order to find an operator $\mathcal{L}$ such that $\mathcal{L}(p) = 0$ we define $\mathcal{L}$ by

$$\mathcal{L}(p) := \widetilde{\mathcal{L}}(p, S(u, kp))$$

(the dependence on u of the right hand side is understood as a (t,x)-dependent coefficient). The exact form of $\mathcal{L}(p)$ is irrelevant for our purposes. Our main aim is to compare p with 1 and so the relevant expression is $\mathcal{L}(1)$, i.e.

$$\mathcal{L}(1) = k_t + (k_u - \sigma_u(\widetilde{S}(k))(k_x + \beta(\widetilde{S}(k))) - \sigma_{u_x}(\widetilde{S}(k))\{k_{xx} + 2k_{xu}\widetilde{S}(k) \tag{2.17}$$

$$+ k_{uu}\widetilde{S}(k)^2 + \beta_u(\widetilde{S}(k))\widetilde{S}(k)\} - \beta_{u_x}(\widetilde{S}(k))\{k_x + k_u\widetilde{S}(k) - \sigma_u(\widetilde{S}(k))\widetilde{S}(k)\}$$

where we used the notation $\sigma_u(\widetilde{S}(k)) = \sigma_u(u, \widetilde{S}(k))$ and $\widetilde{S}(k) := S(u,k)$.

The following result is a trivial consequence of the maximum principle:

Lemma 1. *Assume (2.16),*

$$k(t,x,u) > 0 \qquad on\ Q \times (0, \overline{M_{u_0}}), \tag{2.18}$$

$$\mathcal{L}(1) \geq 0 \quad in\ Q, \tag{2.19}$$

$$\sigma(u,u_x)(t,x) \leq k(t,x,u(t,x)), \quad (t,x) \in \partial_p Q, \tag{2.20}$$

where $\partial_p Q$ denotes the parabolic boundary of Q. Then

$$\sigma(u,u_x)(t,x) \ \leq k(t,x,u(t,x)) \ \ for\ any \ \ (t,x) \in \overline{Q}. \tag{2.21}$$

If we particularize the study to the special case of

$$\sigma(u,u_x) = \varphi'(u)u_x \ \text{ and } \beta(u,\ u_x) = \psi(u)$$

then

$$\widetilde{S}(k) = \frac{k}{\varphi'(u)}$$

and choosing k under the form

$$k(t,x,u) = \frac{\theta(u)}{\sqrt{t}}$$

(like in (24) of [BeDi])) we obtain that

$$\mathcal{L}(1) = -\frac{k(u)}{t\varphi'(u)}\{\frac{\varphi'(u)}{2} + \theta''(u)\theta(u) + t(\Phi(u)\frac{\theta'(u)}{\theta(u)} - \Phi'(u))\} \tag{2.22}$$

with

$$\Phi(u) = \psi(u)\varphi'(u).$$

Some suitable choice of function $\theta(u)$ allows to get the following result

Lemma 2. *For any given $T > 0$ and for any $(t,x) \in Q$* and *$t \in (0,T]$ we have the pointwise estimate*

$$\begin{gathered}\varphi(u(t,x))_x^2 \leq \\ \tfrac{1}{t}(u(t,x)\varphi(\overline{M}_{u_0}) - \int_{\overline{m}_{u_0}}^{u(t,x)} \varphi(s)ds \\ +2T(\int_{\overline{m}_{u_0}}^{u(t,x)} \left(\int_w^{\overline{M}_{u_0}} [(\varphi'\psi)']_-(s)ds\right) dw + \overline{m}_{u_0} \int_{\overline{m}_{u_0}}^{\overline{M}_{u_0}} [(\varphi'\psi)']_-(s)ds \\ + \max_{(\tau,y)\in Q} [\varphi'\psi]_+ (u(\tau,y))),\end{gathered} \tag{2.23}$$

where $\overline{m}_{u_0} = \min(1, m_{u_0})$ and *$\overline{M}_{u_0} = \max(1, M_{u_0})$.*

Proof. As in (53) of [BeDi] we take

$$\theta(u) = \sqrt{\alpha u - j(u)}$$

with $\alpha > 0$ and $j(u)$ a convex function to be determined. Using (2.22) the inequality $\mathcal{L}(1) \geq 0$ is implied once we get

$$\varphi'(u) - j''(u) - \frac{(\alpha - j'(u))^2}{2\theta(u)^2} + 2t(\Phi(u)\frac{(\alpha - j'(u))}{2\theta(u)^2} - \Phi'(u)) \leq 0.$$

So, in particular, it is enough to have

$$j''(u) \geq \varphi'(u) + 2T\left[\Phi'\right]_-(u), \tag{2.24}$$

(recall that and $[h]_- := \max(0, -h)$) and

$$\alpha \geq j'(u) + 2T\left[\Phi\right]_+(u). \tag{2.25}$$

By integrating the identity associated to (2.24) we see that this condition is implied if

$$j'(u) = \varphi(u) + 2T\int_{\overline{m}_{u_0}}^{u}\left[\Phi'\right]_-(s)ds,$$

and thus

$$j'(\overline{M}_{u_0}) = \varphi(\overline{M}_{u_0}) + 2T\int_{\overline{m}_{u_0}}^{\overline{M}_{u_0}}\left[\Phi'\right]_-(s)ds,$$

and

$$j(u) = \int_{\overline{m}_{u_0}}^{u}\varphi(s)ds + 2T\int_{\overline{m}_{u_0}}^{u}\left(\int_{\overline{m}_{u_0}}^{w}\left[\Phi'\right]_-(s)ds\right)dw,$$

where we have taken $j(\overline{m}_{u_0}) = 0$. Finally we can choose

$$\alpha = j'(\overline{M}_{u_0}) + 2T\max_{\tau\in[0,T]}\left[\varphi'\psi\right]_+(\|u(\tau,.)\|_{L^\infty(\mathbb{R})}) =$$
$$= \varphi(M_{u_0}) + 2T\int_{\overline{m}_{u_0}}^{M_{u_0}}\left[\Phi'\right]_-(s)ds + 2T\max_{(\tau,y)\in Q}\left[\varphi'\psi\right]_+(u(\tau,y))).$$

The result comes from the fact that

$$u\int_{\overline{m}_{u_0}}^{\overline{M}_{u_0}}\left[\Phi'\right]_-(s)ds - \int_{\overline{m}_{u_0}}^{u}\left(\int_{\overline{m}_{u_0}}^{w}\left[\Phi'\right]_-(s)ds\right)dw$$
$$= \int_{\overline{m}_{u_0}}^{u}\left(\int_{w}^{\overline{M}_{u_0}}\left[\Phi'\right]_-(s)ds\right)dw + \overline{m}_{u_0}\int_{\overline{m}_{u_0}}^{\overline{M}_{u_0}}\left[\Phi'\right]_-(s)ds.$$

Proof of the Theorem. We regularize functions $\varphi_n(u), \psi_n(u)$ and $u_{0,n}(x)$ and consider the problem (CP) by replacing $\varphi(u)$, $\psi(u)$ and $u_0(x)$ by those regular functions. The above gradient estimates show the existence of a modulus of continuity of u_n (the regular solution of the approximate problem) in x which are independent of the regularized process. Moreover, by the results of Kruzhkov [Kr] we also get a modulus of continuity in both variables x and t which is independent of n. Since we know that u_n converges to the bounded weak solution (see, e.g., the exposition made in [Di]), we can pass to the limit in n and the gradient estimates and the continuity (with the same modulus of continuity) hold for the (unique) bounded weak solution $u(t,x)$ of (CP) corresponding to $\varphi(u)$, $\psi(u)$ and $u_0(x)$.
Estimate (1.7) is obtained from Lemma 2 by making $t = T$ and recalling that T is arbitrary. Finally, to prove part b) it is enough to consider the case of $m_{u_0} \geq 1$ (the case of $M_{u_0} \leq 1$ and $m_{u_0} > 0$ is analogous). By comparing $u(t,x)$ with the solution of the ordinary differential equation $U'(t) = \psi(U(t))$, $U(0) = M_{u_0}$ and using that $U(t) \searrow 1$ we deduce that $1 \leq u(t,x) \leq U(t)$ for any $t > 0$ and a.e. $x \in \mathbb{R}$. Thus, given $\epsilon > 0$ there exists $t_0 > 0$ large enough such that $\max_{s\in[1,U(t)]} \left[\varphi'\psi\right]_+ (s) \leq \epsilon$,

$$\left(\int_1^{\|u(t,.)\|_{L^\infty(\mathbb{R})}} \left(\int_w^{U(t_0+\tau)} \left[(\varphi'\psi)'\right]_- (s) ds \right) dw + \int_1^{U(t_0+\tau)} \left[(\varphi'\psi)'\right]_- (s) ds \right) \leq \epsilon$$

and that $\|u(t,.) - 1\|_{L^\infty(\mathbb{R})} \leq \epsilon$ for any $t \geq t_0$. Starting now on the interval $(t_0 + \tau, +\infty)$, with $\tau > 0$ arbitrary, we get an estimate similar to (1.7) but replacing $\overline{m}_{u_0}$by 1 and $\overline{M}_{u_0}$ by $U(t_0 + \tau)$. Thus, for any $t \in (t_0 + \tau, +\infty)$ we get

$$\|\varphi(u(t,.))_x\|_{L^\infty(\mathbb{R})} \leq \frac{\sqrt{(1+\epsilon)\varphi(1+\epsilon)}}{\sqrt{\tau}} + \sqrt{4\epsilon}. \tag{2.26}$$

Taking $\tau = \tau(\epsilon)$ large enough we arrive to the conclusion.

Remark. The conclusions of the main theorem remain valid under more general assumptions on $\varphi(u)$ and $\psi(u)$ (for instance the strict monotonicity of $\varphi(u)$ can be replaced by the mere condition of φ continuous and nondecreasing) but then the notion of bounded weak solution must be replaced by the more restrictive concept of "bounded entropy solution" (see the exposition made in [Di] and its many references). We also mention that conclusion b) of the main theorem applies to the case in which the solution generates a free boundary (given as the boundary of the level set $\{x \in \mathbb{R},\ u(t,x) = 1\}$, for $t \geq 0$) as it is the case, for instance, of $\varphi(u) = u$ and $\psi(u) = \lambda u\sqrt{|1-u|}sign(1-u)$ for any $\lambda > 0$ (notice that the regularity conditions (1.4) are fulfilled even if $\psi(u)$ is not Lipschitz continuous at $u = 1$).

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References

[Ar] D. G. Aronson. Regularity properties of flows through porous media, *SIAM J. Appl. Math.* **17** (1969), 461-467.

[Be] Ph. Benilan. *Evolution Equations and Accretive Operators*, Lecture Notes, Univ. Kentucky, manuscript, 1981.

[BeDi] Ph. Benilan and J.I. Díaz, Pointwise gradient estimates of solutions of onedimensional nonlinear parabolic problems, *J. Evolution Equations*, **3** (2004) 557-602.

[Di] J.I. Díaz, On some onedimensional reaction-diffusion-convection equations. To appear.

[DiKa] J.I. Díaz and S. Kamin, Convergence to Travelling Waves for Quasilinear Fisher-KPP Type Equations. To appear.

[DB] E. DiBenedetto, Continuity of Weak Solutions to a General Porous Medium Equation, *Indiana Univ. Math. J.* **32** No. 1 (1983), 83–118.

[Fi] R. A. Fisher, The wave of advance of advantageous genes, *Annals of Eugenics* **7** (1937), 355-369.

[He-Va] M.A. Herrero, J.L. Vázquez, The one-dimensional nonlinear heat equation with absorption. Regularity

of solutions and interfaces, *SIAM J. Math. Anal.* **18** (1987) 149-167.

[Kal] A.S. Kalashnikov, The propagation of disturbances in problems of non-linear heat conduction with absorption, *USSR Comput. Math. and Math. Phys.* **14** (1974), 70-85.

[KaR] S. Kamin and P. Rosenau, Convergence to the Travelling Wave Solution for a Non-linear Reaction-Diffusion Equation, *Rendiconti Mat. Acc. Lincei Cl. Sci. Fis. Mat. Natur.* **15** (2004), 271-280.

[Ke] R. Kersner, Degenerate parabolic equations with general nonlinearities, *Nonlinear Anal.* **4** (1984), 1043-1062.

[Kr] S.N. Kruzhkov, Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications, *Math. Zam.* **6,** 1 (1969) 97-98. English tr. in *Math. Notes*, V **6**, (1969) 517-523.

[LSU] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type.* Transl. Math. Monographs, Vol. 23, Amer. Math. Soc., Providence, RI. 1968.

[KPP] A. Kolmogorov, I. Petrovsky and N. Piscunov, Etude de l'equation de la diffusion avec croissance de la quantité de matiere et son application a un probleme biologique, *Bulletin Univ. Moscow, Ser. Internationale, Math., Mec.* **1** (1937), 1-25. English translation in P. Pelce, (ed.), *Dynamics of Curved Fronts*, Academic Press, Boston, 1988, 105-130.

[OlKalCh] O.A. Oleinik, A.S. Kalashnikov and Y.-L. Chzhou, The Cauchy problem and boundary problems for equations of the type of nonstationary filtration, *Izv. Akad. Nauk. SSSR Ser. Mat.* **22** (1958), 667-704 (Russian).

[Sou] Ph. Souplet, An optimal Liouville theorem for radial entire solutions of the porous medium equation

with source, *J. Differential Equations* **246** (2009), 3980-4005.

[Va] J.L. Vázquez, *The Porous Medium Equation. Mathematical Theory*, Oxford Univ. Press, 2007.