Failure of the strong maximum principle for linear elliptic with singular convection of non-negative divergence

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Abstract

In this paper we study existence, uniqueness, and integrability of solutions to the Dirichlet problem $-\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(E(x)u) + f$ in a bounded domain of $\mathbb{R}^N$ with $N \geq 3$. We are particularly interested in singular E with $\mathrm{div}E \geq 0$. We start by recalling known existence results when $|E| \in L^N$ that do not rely on the sign of $\mathrm{div}E$. Then, under the assumption that $\mathrm{div}E \geq 0$ distributionally, we extend the existence theory to $|E| \in L^2$. For the uniqueness, we prove a comparison principle in this setting. Lastly, we discuss the particular cases of E singular at one point as $Ax/|x|^2$, or towards the boundary as $\mathrm{div}E \sim \mathrm{dist}(x, \partial\Omega)^{-2-\alpha}$. In these cases the singularity of E leads to u vanishing to a certain order. In particular, this shows that the strong maximum principle fails in the presence of such singular drift terms E.

1 Introduction

It is well known that many relevant applications leads to the presence of a convection term in the correspondent model which, in its simplest formulation, leads to a boundary value problem for linear elliptic second order equation of the following type:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(uE(x)) + f(x) & \Omega \\ u = 0 & \partial\Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an open, bounded set and we assume that $M \in L^\infty(\Omega)^{N\times N}$ is elliptic

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \qquad \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega.$$

According to the regularity of the right hand side datum $f(x)$ it is natural to search the solution in the energy space $W_0^{1,2}(\Omega)$ (case of $f \in H^{-1}(\Omega)$: see, e.g. [18, 15, 1]), or in a larger Sobolev space if f is singular (see [1]); when $f \in L^1(\Omega)$, see, for instance, [8], or when $L^1(\Omega, \delta)$ with $\delta(x) = d(x, \partial\Omega)$, see, e.g., [7, 13].

In the mentioned references it assumed that the convection term is regular (for instance $E \in W^{1,\infty}(\Omega)$) and that it satisfies an additional condition which helps to have a maximum principle:

$$\mathrm{div}\, E \geq 0 \text{ a.e. on } \Omega. \tag{2}$$

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More recently, some effort has been devoted to get an existence and regularity theory under more general conditions on the convection term E were carried out by different authors (see, e.g. [1], [5] and their references). For instance, solutions in the energy space can be considered under the conditions $|E| \in L^N(\Omega)$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$. In [13]) and [12] the authors study the case in which $|E| \in L^N(\Omega)$ and $\operatorname{div} E = 0$ in Ω and $E \cdot n = 0$ on $\partial\Omega$, $f \in L^1(\Omega, \delta)$.

In this paper, we will show that (2) makes $\operatorname{div} E$ behave like a non-negative potential in the Schrödinger case, and we can apply techniques from that setting. See, for example, [12, 13, 14, 16]. We focus on the case where (2) holds in distributional sense.

The paper is structured as follows. First, in Section 2 we review known results for the case $|E| \in L^N$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$ which were published in [1], were show there is a unique weak solution of (1) that can be constructed by approximation. In Section 3 we show that if $|E| \in L^2(\Omega)$, $\operatorname{div} E \geq 0$, and $f \in L^m(\Omega)$ for some $m > 1$ then the same approximation procedure converges to a weak solution of (1), and we give some a priori bounds for this solution. In Section 4 we show that, if we also assume $f \in L^{\frac{2N}{N+2}}(\Omega)$, then this constructed solution is the unique weak solution of (1).

Then we move to discussing interesting examples that follow out this setting. In Section 5 we focus on the case

$$E(x) = A\frac{x}{|x|^2}, \tag{3}$$

which is somehow in the limit of theory since it is not in $L^N(\Omega)$ but it is in $L^r(\Omega)$ for $r \in [1, N)$. In [5] the authors examined the more general class case

$$|E| \leq \frac{|A|}{|x|}. \tag{4}$$

The authors show existence of solutions u, where the summability is reduced as $|A|$ is increased. Their results indicate that the sign of A should play a role, but the application of Hardy's inequality (which they use in a crucial way) is not able to detect this fact. In Theorem 16 we show that if $N > 1$, $f \in L^m(\Omega)$ for suitable m, and $A > 0$ then we can use the sign of $\operatorname{div} E$ to deduce that

$$u_A \to 0 \text{ in } L^1(\Omega) \text{ as } A \to +\infty.$$

By the contrary, when $A < 0$ we cannot improve the result in [5]. Notice that this is similar to the equation $L(u_B) + B\, u_B = f$, where as $B \to \infty$ we have $u_B \to 0$.

Lastly, in Section 6, we discuss the case where E is singular only on the boundary. We present an example showing that if $\operatorname{div} E$ behaves like $\delta^{-2-\gamma}$ for some $\gamma > 0$ and f is bounded, then the solutions are flat on the boundary, i.e.

$$|u(x)| \leq C \operatorname{dist}(x, \partial\Omega)^\alpha \text{ for some } \alpha > 1.$$

This is in contrast to the case $E = 0$, where the solution have non-trivial normal derivative. Again, we use the fact that $\operatorname{div} E$ acts as a potential. However, in the Schrödinger equation is sufficient that $V(x) \geq C\delta^{-2}$ to get flat solutions, whereas for E we need a strictly larger exponent (see Remark 21). Questions of this type are quite relevant in the framework of linear Schrödinger equations associated to singular potential since they can be understood as complements to the Heisenberg Incertitude Principle (see, e.g. [10, 11, 12, 13, 17, 14]).

We conclude with some further comments and open problems in Section 7.

2 Known results when $|E| \in L^N$

For $m \leq N$ we define the Sobolev conjugate exponent

$$m^* = \frac{mN}{N-m}, \qquad m^{**} = (m^*)^* = \frac{mN}{N-2m}$$

We have that $m^{**} \in [1, \infty]$ for $\frac{N}{N+2} \leq m \leq \frac{N}{2}$. Notice that $m^* \geq 2$ if and only if $m \geq \frac{2N}{N+2} = (2^*)'$. Notice that, since $m \geq 1$ we have $m^* \geq m$. In order to compute explicit a priori estimates, we use the Sobolev embedding constant S_p such that, for $1 < p < +\infty$

$$S_p \|u\|_{L^{p^*}(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}. \tag{5}$$

We point out the relevance of the constants, for $N > 2$ of $(2^*)' = \frac{2N}{N+2}$. This constant can be take uniformly for $\mathbb{R}^N$. Since we are going to require the Sobolev embedding for $p = 2$, we assume that $N \geq 3$. In [1] the author proves the following existence theorem with a priori estimates.

Theorem 1 ([1])**.** *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ and $|E| \in L^N(\Omega)$. Then, there exists a unique weak solution u in the sense that*

$$u \in W_0^{1,2}(\Omega) \text{ is such that } \int_\Omega M(x)\nabla u \nabla v = \int_\Omega u\, E(x) \cdot \nabla v + \int_\Omega f(x)\, v(x), \ \forall\, v \in W_0^{1,2}(\Omega).$$

and it satisfies:

1. *Logarithmic estimate*

$$\left(\int_\Omega |\log(1 + |u|)|^2 \right)^{\frac{2}{2^*}} \leq \frac{1}{S_2^2 \alpha^2} \int_\Omega |E|^2 + \frac{2}{S_2^2 \alpha} \int_\Omega |f|,$$

2. *Gradient estimate: there exists $C = C(\alpha, N)$ such that*

$$\int_\Omega |\nabla u|^2 \leq C \left(\|E\|_{L^N}^2 + \|f\|^2_{L^{\frac{2N}{N+2}}} \right). \tag{6}$$

3. *Stampacchia-type summability: For $m \in [\frac{2N}{N+2}, \frac{N}{2})$ there exists a constant $C = C(m, \alpha, N, \|E\|_{L^N})$ such that*

$$\|u\|_{m^{**}} \leq C \|f\|_m. \tag{7}$$

4. *Stampacchia-type boudedness: Let $r > N$ and $m > \frac{N}{2}$. There exists C such that*

$$\|u\|_{L^\infty} \leq C(m, r, \alpha, \|f\|_{L^m}, \|E\|_{L^r}). \tag{8}$$

Remark 2. The natural theory for this problem in energy space is precisely $|E| \in L^N(\Omega)$, since in the weak formulation we need to justify a term of the form $Eu\nabla v$, where $u, v \in W_0^{1,2}(\Omega)$. This means that $u \in L^{2^*}$ whereas $\nabla v \in L^2$. So we always have that $uE \in L^2(\Omega)$.

Since the construction is done of solutions in the proof of Theorem 5 is achieved by approximation, we have that

Corollary 3. *The solutions constructed in Theorem 5 satisfy* (7) *and* (8).

In [1] the main tool to study the linear problem (1) are the auxiliary non-linear Dirichlet problems

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = -\operatorname{div}\left(\dfrac{u_n}{1+\frac{1}{n}u_n}E_n(x)\right) + f_n(x) & \Omega \\ u = 0 & \partial\Omega, \end{cases} \tag{9}$$

where the authors take $f_n = T_n(f)$ a truncation of f through the family

$$T_n(s) = \begin{cases} s & |s| \ \leq k, \\ k\,\operatorname{sign}(s) & |s| > k, \end{cases}$$

and $E_n = \frac{E}{1+\frac{1}{n}E}$. We will take advantage of a similar approximation.

Remark 4. Since the problem is linear, for $t \in \mathbb{R}$ we have that tu is solution of

$$-\operatorname{div}(M(x)\nabla[t\,u]) = -\operatorname{div}([t\,u]E(x)) + t\,f(x),$$

and that E does not change. Thus

$$t^2 \int_\Omega |\nabla u|^2 \leq C\left(\|E\|_{L^N}^2 + \|E\|_{L^2}^2 + t\|f\|_1 + t^2\|f\|^2_{L^{\frac{2N}{N+2}}}\right).$$

Dividing by t^{-2} and taking the limit as $t \to \infty$ gives

$$\int_\Omega |\nabla u|^2 \leq C\|f\|^2_{L^{\frac{2N}{N+2}}}. \tag{10}$$

Notice that in Theorem 12 we will prove this fact for the case $\operatorname{div} E \geq 0$.

3 Existence theory when $|E| \in L^2$ and $\operatorname{div} E \geq 0$

The structural assumption in this section is the following:

$$\begin{cases} E \ \text{belongs to the Lebesgue space} \ (L^2(\Omega))^N, \\ \operatorname{div} E \geq 0 \text{ in } \mathcal{D}'(\Omega), \ \text{ that is } \displaystyle\int_\Omega E\cdot\nabla\phi \leq 0, \ \forall\, 0 \leq \phi \in W_0^{1,2}(\Omega). \end{cases} \tag{11}$$

Theorem 5. *Assume* (11) *and*

$$f \in L^m(\Omega),\ 1 < m < \frac{N}{2}, \tag{12}$$

and let $p = \min\{2, m^*\}$. *Then, there exists a weak solution* u *in the sense that*

$$u \in W_0^{1,p}(\Omega) \textit{ is such that } \int_\Omega M(x)\nabla u\nabla v = \int_\Omega u\,E(x)\cdot\nabla v + \int_\Omega f(x)\,v(x),\ \forall\ v \in W_0^{1,\infty}(\Omega). \tag{13}$$

Furthermore, it satisfies

$$\begin{cases} \|u\|_{W_0^{1,m^*}(\Omega)} \leq C_m\,\|f\|_m, & \textit{if } 1 < m < \frac{2N}{N+2}; \\ \|u\|_{W_0^{1,2}(\Omega)} + \|u\|_{m^{**}} \leq \tilde{C_m}\,\|f\|_m, & \textit{if } \frac{2N}{N+2} \leq m \leq \frac{N}{2}. \end{cases} \tag{14}$$

Remark 6. Due to the gradient estimates, we can extend (13) to all $v \in W_0^{1,q}(\Omega)$ by approximation, where $q = p'$.

We say that u_n weak solution of (9) if $u \in W_0^{1,2}(\Omega)$ is such that

$$\int_\Omega M(x)\nabla u_n \nabla v = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} E_n(x)\cdot\nabla v + \int_\Omega f_n(x)\, v(x), \quad \forall\, v \in W_0^{1,2}(\Omega). \tag{15}$$

The existence of a weak solution if $E_n \in L^2(\Omega)^N$ is a consequence of the Schauder theorem. The proof of Theorem 5 is based on the following approximation lemma

Lemma 7. *Assume $E_n = E$, (11), (12), and $f_n = T_n(f)$. Then, for any weak solution u_n of* (15) *we have that*

$$\begin{cases} \|u_n\|_{W_0^{1,m^*}(\Omega)} \le C_m \|f\|_m, & \text{if } 1 < m < \frac{2N}{N+2}; \\ \|u_n\|_{W_0^{1,2}(\Omega)} + \|u_n\|_{m^{**}} \le \tilde{C_m} \|f\|_m, & \text{if } \frac{2N}{N+2} \le m \le \frac{N}{2}. \end{cases} \tag{16}$$

where

$$C_m \text{ does not depend on } E. \tag{17}$$

Proof. Our proof is the same of [4], since we will see that the contribution of new term on E is a negative number. We use $T_k(u_n)|T_k(u_n)|^{2\gamma-2}$ as test function in (15), $\gamma = \frac{m^{**}}{2^*}$; we repeat it is possible since every $T_k(u_n)$ has exponential summability. Note that $2\gamma - 1 > 0$ since $m > 1$. Thus we have

$$\int_\Omega M(x)\nabla u_n \nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} E(x)\cdot\nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) + \int_\Omega f_n(x)\, T_k(u_n)|T_k(u_n)|^{2\gamma-2}.$$

To study the second integral, we define the function

$$H_\gamma(s) = \int_0^s \frac{t\,|t|^{2\gamma-2}}{1+\frac{1}{n}|t|}\,dt.$$

It is easy to check that $H_\gamma(s) \ge 0$ for all $s \in \mathbb{R}$. Thus, using the sign condition on $\operatorname{div} E$ we have that

$$\begin{aligned} &\int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} E(x)\cdot\nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) \\ &\quad = \int_\Omega (2\gamma-1)\frac{T_k(u_n)\,|T_k(u_n)|^{2\gamma-2}}{1+\frac{1}{n}|T_k(u_n)|} E(x)\cdot\nabla T_k(u_n) \\ &\quad = \int_\Omega h_\gamma(T_k(u_n))\, E(x)\cdot\nabla T_k(u_n) = \int_\Omega E(x)\cdot\nabla[H_\gamma(T_k(u_n))] \le 0. \end{aligned}$$

Hence, we have that

$$\int_\Omega M(x)\nabla u_n \nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) \le \int_\Omega f_n(x)\, T_k(u_n)|T_k(u_n)|^{2\gamma-2},$$

which is the starting point of [4] and we get the estimates

$$\begin{cases} \|T_k(u_n)\|_{W_0^{1,m^*}(\Omega)} \le C_m \|f\|_m, & \text{if } 1 < m < \frac{2N}{N+2}; \\ \|T_k(u_n)\|_{W_0^{1,2}(\Omega)} + \|T_k(u_n)\|_{m^{**}} \le \tilde{C_m} \|f\|_, & \text{if } \frac{2N}{N+2} \le m \le \frac{N}{2}. \end{cases}$$

Letting $k \to \infty$ we recover (16). □

With this lemma, we can pass to the limit to prove Theorem 5.

Proof of Theorem 5. Up to subsequences, the sequence $\{T_k(u_n)\}$ constructed above, weakly converges (in $W_0^{1,m^*}(\Omega)$ or in $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$) and it is possible to pass to to some u (note that $u \in L^{m^{**}}(\Omega)$). Recall that $E \in (L^2)^N$. To pass to the limit in

$$\int_\Omega \frac{u_n}{1 + \frac{1}{n}|u_n|} E(x) \cdot \nabla v$$

we need to assume $1 \geq \frac{1}{2} + \frac{1}{m^{**}}$, that is precisely the assumption $\frac{1}{2} + \frac{2}{N} \geq \frac{1}{m}$. Thus we pass also to the limit in (16). □

Remark 8. Note that, once more it is possible to develop an approximate method in order to prove the existence when $E \in L^r$. Indeed, let $E_0 \in L^r$, $r > 1$ and $E_n \in L^2$ converging to E_0 in L^r. Define now u_n in the corresponding way, we can use the statement of (17), so that we can say that estimates (14) still old for this new sequence $\{u_n\}$ and once more we can pass to the limit and we prove the existence if

$$1 \geq \frac{1}{r} + \frac{1}{m} - \frac{2}{N}$$

We can provide further a priori estimates when $\operatorname{div} E \geq 0$

Proposition 9. *The solutions constructed in Theorem 5 satisfy the following additional estimates:*

1. *(L^1 estimate) If* $\operatorname{div} E \in L^1(\Omega)$ *then we have that*

$$\int_\Omega |u| \operatorname{div} E \leq \int_\Omega |f|. \tag{18}$$

2. *(L^q estimate) If* $\operatorname{div} E \geq c_0 > 0$ *and* $m > 1$ *then*

$$\|u\|_{L^m} \leq \frac{q}{q-1} \frac{\|f\|_{L^m}}{c_0}. \tag{19}$$

Remark 10. Notice that (19) blows up as $m \to 1$. In fact, it is known that that the case $m = 1$ does not satisfy such an estimates.

We prove a priori estimates under the assumption of sign for bounded (or even smooth) E, which we now know will hold for approximations.

Remark 11. It could seem that approximations of E might loose the sign of the divergence. Nevertheless, when $E \in L^1(\Omega)$ with $\operatorname{div} E \geq 0$ we can construct and approximating sequence such that $E_n \in W^{1,\infty}(\Omega)$ and $\operatorname{div} E_n \geq 0$. For example, take $E_n = \rho_n * E$, where ρ_n are some positive mollifiers. Then $\operatorname{div} E_n = \rho_n \star \operatorname{div} E \geq 0$.

Proof of Proposition 9. Assume first that $E \in (L^N)^N$, and $f \in L^m$ for $m \geq \frac{2N}{N+2}$. Then, we can deal with the unique solution $u \in W_0^{1,2}(\Omega)$ that exists by Theorem 1. Due to the construction by approximation in Theorem 5 the estimates pass to the limit in the construction. Take $h \in W^{1,\infty}(\mathbb{R})$ such that $h(0) = 0$. We take $v = h(u)$ as a test function we can write

$$\alpha \int_\Omega h'(u)|\nabla u|^2 \leq \int_\Omega M(x)\nabla u \cdot \nabla h(u) = \int_\Omega uE \cdot \nabla h(u) + \int_\Omega f h(u).$$

We can write

$$u\nabla h(u) = uh'(u)\nabla u = \nabla F(u)$$

where $F(s) = \int_0^s \tau\, h'(\tau) d\tau$. Hence

$$\alpha \int_\Omega h'(u)|\nabla u|^2 \le \int_\Omega E\cdot \nabla F(u) + \int_\Omega f h(u).$$

We can integrate by parts again to deduce

$$\alpha \int_\Omega h'(u)|\nabla u|^2 + \int_\Omega F(u) \operatorname{div} E \le \int_\Omega f h(u). \tag{20}$$

Now we prove both items

- *Item 1.* Let us consider $h_\varepsilon(s) = T_\varepsilon(s)/\varepsilon$. Then $h'_\varepsilon \ge 0$ and $|h_\varepsilon| \le 1$ and, hence, in (20)

  $$\int_\Omega F_\varepsilon(u) \operatorname{div} E \le \int_\Omega |f|.$$

  It is clear that $F_\varepsilon(s) \to |s|$ a.e. as $\varepsilon \to 0$. Then,

  $$\int_\Omega |u| \operatorname{div} E \le \int_\Omega |f|.$$

- *Item 2.* Let us take, for $m > 1$, $h(s) = |s|^{m-1}$ then

  $$F(s) = (m-1)\int_0^s |\tau|^{m-2} \operatorname{sign}(\tau)\tau d\tau = \frac{m-1}{m} s^m.$$

  Hence, going back to (20)

  $$c_0 \frac{m-1}{m}\|u\|_{L^m}^m \le \frac{m-1}{m}\int_\Omega |u|^m \operatorname{div} E \le \int_\Omega f|u|^{m-1} \le \|f\|_{L^m}\|u\|_{L^m}^{m-1}.$$

  Hence, we simplify

  $$\|u\|_{L^m} \le \frac{m}{m-1}\frac{\|f\|_{L^m}}{c_0}.$$

□

4 Comparison principle and well-posedness

To show uniqueness of solutions we prove a weak maximum principle.

Theorem 12. *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ and (11). Then, if $u \in W_0^{1,2}(\Omega)$ is a solution of (13) then*

$$\|\nabla u^+\|_2 \le \frac{1}{\alpha S_2}\|f^+\|_{\frac{2N}{N+2}}.$$

Hence, there is, at most, one solution of (13). Furthermore, if $f \ge 0$ then $u \ge 0$.

We first prove the following lemma

Lemma 13. *Let $m > 1$, $E \in L^{r'}(\Omega)$ with $0 \le \operatorname{div} E \in \mathcal{D}'(\Omega)$. Then, we have that*

$$-\int_\Omega E\nabla v \ge 0 \qquad \forall\, 0 \le v \in W_0^{1,r}(\Omega). \tag{21}$$

Proof. By definition of having a sign in distributional sense, for $0 \le \varphi \in \mathcal{C}_c^\infty(\Omega)$, we have that

$$-\int_\Omega E\nabla\varphi = \langle \operatorname{div} E, \varphi\rangle \ge 0.$$

For $0 \le v \in W_0^{1,r}(\Omega)$, we can find a sequence sequence $0 \le \varphi_n \in \mathcal{C}_c^\infty(\Omega)$, and $\varphi_n \to v$ in $W_0^{1,r}(\Omega)$. In particular, $\nabla\varphi \to \nabla v$ in $L^r(\Omega)$. We can pass to the limit in the estimate. □

Proof of Theorem 12. Let u be a solution. Take $v = u^+$ as a test function. Then, applying the previous lemma

$$\alpha \int_\Omega |\nabla u^+|^2 \le \int_\Omega E\nabla \frac{u_+^2}{2} + \int_\Omega f u^+ \le \|f\|_{(2^*)'} \|u^+\|_{2^*} \le \frac{1}{S}\|f\|_{(2^*)'}\|\nabla u^+\|_2.$$

We recover the estimate. □

Theorem 14. *Let $f \in L^m(\Omega)$ and $E \in L^r(\Omega)$ such that $0 \le \operatorname{div} E \in \mathcal{D}'(\Omega)$ and*

$$\begin{cases} \dfrac{1}{\min\{2^*, m^{**}\}} + \dfrac{1}{r} \le 1 & \text{if } \frac{N}{N+2} \le m \le \frac{N}{2} \\ \dfrac{1}{2^*} + \dfrac{1}{r} \le 1 & \text{otherwise} \end{cases} \tag{22}$$

Then, taking $q = \min\{2, m^*\}$ there exists a solution of

$$u \in W_0^{1,q}(\Omega) \quad \text{such that} \quad \int_\Omega M(x)\nabla u \nabla v = \int_\Omega uE\nabla v + \int_\Omega fv, \quad \forall v \in W_0^{q',\infty}(\Omega). \tag{23}$$

Furthermore, if $m \ge \frac{2N}{N+2}$ and $r \ge N$ it is the unique solution of (13).

Proof. Let $f_k = T_k(f)$ where T_k is the cut-off function. By considering

$$E_k = \rho_k \star E$$

as point out in Remark 11 the sign condition is preserved. Since the mollifiers are non-negative and integrate to 1, we even have $\min E_k \ge \min E$.
By Proposition 9 there exists a unique weak u_k solution of (13). Since the $\cdot^*$ operation is monotone, then $q^* = \min\{2^*, m^{**}\}$. The sequence u_k is uniformly bounded in $W_0^{1,q}(\Omega)$. Therefore, by the Sobolev embedding theorem, it is uniformly bounded on $L^{q^*}(\Omega)$.
Up to a subsequence, there exists $u \in W_0^{1,q}(\Omega)$ such that

$$\begin{aligned} \nabla u_k &\rightharpoonup \nabla u && \text{in } L^q(\Omega) \\ u_k &\rightharpoonup u && \text{in } L^{q^*}(\Omega). \end{aligned}$$

Since $M \in L^\infty(\Omega)^{N\times N}$, $E_k \to E \in L^r(\Omega)^N$ strongly and (22) we have that

$$\begin{aligned} M(x)\nabla u_k &\rightharpoonup M(x)\nabla u && \text{in } L^q(\Omega) \\ u_k E_k &\rightharpoonup uE && \text{in } L^1(\Omega). \end{aligned}$$

Therefore, we can pass to the limit in the weak formulation for $v \in W_0^{1,\infty}(\Omega)$. If $m \ge \frac{2N}{N+2}$ and $r \ge N$, then $uE \in L^2(\Omega)$ and it is a solution of (13) by approximation. □

5 Convection with singularity at one point

With the approach developed in this paper we are able to study the special situation

$$E = A\frac{x}{|x|^2} \qquad \text{where } A > 0 \tag{24}$$

which is somehow in the limit of theory since it is not in $L^N(\Omega)$ but it is in $L^r(\Omega)$ for $r \in [1, N)$. In [5] the authors examined the framework of drifts such that

$$|E| \le \frac{|A|}{|x|}, \tag{25}$$

The authors show existence of solutions u, where the summability is reduced as $|A|$ is increased. They prove

Theorem 15 ([5])**.** *Let $f \in L^m(\Omega)$ and $|E| \le |A|/|x|$. Then*

1. *If $|A| < \frac{\alpha(N-2m)}{m}$ and $m \in [\frac{2N}{N+2}, \frac{N}{2})$ then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.*
2. *If $|A| < \frac{\alpha(N-2m)}{m}$ and $m \in (1, \frac{2N}{N+2})$ then $u \in W_0^{1,m^*}(\Omega)$.*
3. *If $|A| < \alpha(N-2)$ and $m = 1$ then $\nabla u \in (M^{\frac{N}{N-1}}(\Omega))^N$ and $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.*

Above, $M^{\frac{N}{N-1}}$ denotes the Marcinkiewicz space (see [5] for the definition and some properties). The argument in [5] is based on Hardy's inequality

$$\left(\frac{N-2}{N}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \le \int_{\mathbb{R}^N} |\nabla u|^2. \tag{26}$$

We are able to extend this result to distinguish depending on the sign of A. Our result is the following

Theorem 16. *Let $f \in L^m(\Omega)$ for some $m > 1$ and* (24)*. Then, there exists a solution of* (23)*, and it satisfies the estimates in Proposition 9. Furthermore, $u_A \to 0$ as $A \to \infty$ in the sense that*

$$\int_\Omega \frac{|u_A(x)|}{|x|^2} \le \frac{1}{A(N-2)} \int_\Omega |f|.$$

We point out that, if $m > \frac{2N}{N+2}$, we have furthermore $uE \in L^2(\Omega)$.

Proof. Since $N \ge 3$ we know that $|E| \in L^2(\Omega)$ and that

$$\operatorname{div} E = r^{1-N}\frac{\partial}{\partial r}(r^{N-1}Ar^{-1}) = \frac{A(N-2)}{|x|^2} \tag{27}$$

is non-negative and it is in $L^1(\Omega)$. Then, we have satisfies the existence theory of Theorem 5. Due to Proposition 9 and (27) the estimate follows. □

6 Convection with singularity on the boundary

The aim of this section is to understand the case where E is regular inside Ω but blows up towards $\partial\Omega$. For the sake of simplicity we present and example, which as mentioned in Section 7 can be generalised, but the computations become quite technical. Let us consider φ_1 the first eigenfunction of $-\Delta$ with Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We normalise it so that $\|\nabla\varphi_1\|_{L^\infty} = 1$. It is known that there exists $C > 0$ such that

$$0 < C \operatorname{dist}(x, \partial\Omega) \leq \varphi_1(x) \leq C^{-1} \operatorname{dist}(x, \partial\Omega), \qquad \forall x \in \Omega.$$

and near $\partial\Omega$ we have that

$$|\nabla\varphi(x)| \geq C > 0.$$

We focus our efforts on the particular case

$$E = -\varphi_1^{-1-\gamma}\nabla\varphi_1, \qquad \text{for some } \gamma > 0, \tag{28}$$

and $f \in L_c^\infty(\Omega)$, the space of functions bounded with compact support in Ω. The aim of this section is to prove

Theorem 17. *Let E be given by (28), $M = I$ and $f \in L_c^\infty(\Omega)$. Then, there exists a unique $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $uE \in L^\infty(\Omega)$ and u is a weak solution in the sense that (13). Furthermore, u is flat on the boundary in the sense that*

$$\text{for all } \alpha \geq 1 \text{ we have that} \qquad |u(x)| \leq C_\alpha \operatorname{dist}(x, \partial\Omega)^\alpha, \qquad \text{for a.e. } x \in \partial\Omega. \tag{29}$$

It is immediate to compute that

$$\begin{aligned} \operatorname{div} E &= (1+\gamma)\varphi_1^{-2-\gamma}|\nabla\varphi_1|^2 - \varphi_1^{-1-\gamma}\Delta\varphi_1 \\ &= (1+\gamma)\varphi_1^{-2-\gamma}|\nabla\varphi_1|^2 + \lambda_1\varphi_1^{-\gamma} \end{aligned}$$

Hence $\operatorname{div} E(x) \geq c \operatorname{dist}(x, \partial\Omega)^{-2-\gamma}$ near the boundary. Notice that E and $\operatorname{div} E$ are not in $L^1(\Omega)$. We start the proof with a lemma

Lemma 18. *In the assumptions of Theorem 14, assume furthermore that $\operatorname{div} E \in L^1_{loc}(\Omega)$. Then $u \operatorname{div} E \in L^1(\Omega)$, still satisfying estimate (18).*

Proof. We consider the approximating sequence for Theorem 14. For the approximation we know that

$$\int_\Omega |u_n| \operatorname{div} E_n \leq \int_\Omega |f|.$$

Let us fix $K \Subset \Omega$. We have that

$$\int_K |u_n| \operatorname{div} E_n \leq \int_\Omega |f|.$$

Since we know that $\operatorname{div} E_n \to \operatorname{div} E$ in $L^1(K)$, we have that, up to a further subsequence, the sequence converges a.e. in K. Hence, applying Fatou's lemma

$$\int_K |u| \operatorname{div} E \leq \int_\Omega |f|.$$

Since this estimate in uniform in K, we can take $K_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq h\}$ and deduce, as $h \to 0$, that (18) holds. □

The solution found in Theorem 17 is unique in a certain class. We provide a uniqueness result extending Theorem 12, which can itself be generalised to a larger framework

Lemma 19. *Assume that* $u \in H_0^1(\Omega)$, $E \in L^\infty_{loc}(\Omega)$, $u|E| \in L^2(\Omega)$, $\operatorname{div} E \geq 0$ *distributionally, and* $f \in L^{\frac{2N}{N+2}}(\Omega)$. *Then*

$$\|\nabla u^+\|_2 \leq \frac{1}{\alpha S_2} \|f^+\|_{\frac{2N}{N+2}}$$

In particular, there is at most one weak solution of the problem.

Proof. We want to repeat the argument in Theorem 12, i.e. taking $v = u_+$ in the weak formulation and using that

$$-\int_\Omega uE \cdot \nabla u_+ \geq 0.$$

We prove this formula by approximation. Take $\eta \in C_c^\infty(\Omega)$. There exits $K \Subset \Omega$ and $\phi_m \in C_0^\infty(K)$ such that $\phi_m \to u_+\eta$ in $H_0^1(\Omega)$. We have that

$$-\int_\Omega \phi_m E \cdot \nabla \phi_m = \left\langle \operatorname{div} E, \frac{\phi_m^2}{2} \right\rangle \geq 0.$$

Since $E \in L^\infty(K)$ we pass to the limit to deduce

$$-\int_\Omega (u_+\eta) \cdot E\nabla(u_+\eta) \geq 0.$$

Now we expand

$$\int_\Omega (u_+\eta)E \cdot \nabla(u_+\eta) = \int_\Omega u_+^2 \eta E \cdot \nabla \eta + \int_\Omega u_+\eta^2 E \cdot \nabla u_+$$

Now we take $\eta_m \nearrow 1$. In particular $\eta_m(x) = \eta_0(m\varphi_1(x))$ where η_0 is non-decreasing, $\eta_0(s) = 0$ if $s \leq 1$ and $\eta_0(s) = 1$ if $s > 2$. Clearly $\|\nabla \eta_m\|_{L^\infty} \leq Cm$. Since $u_+ \in H_0^1(\Omega)$ then $u_+(x)/\varphi_1(x) \in L^2(\Omega)$ by Hardy's inequality. And we compute

$$\left| \int_\Omega u_+^2 \eta_m \cdot E\nabla \eta_m \right| \leq \int_{\varphi_1(x) \leq \frac{1}{m}} \frac{u_+}{\varphi_1} \frac{\varphi_1}{m} |uE| Cm \leq C \int_{\varphi_1(x) \leq \frac{1}{m}} \frac{u_+}{\varphi_1} |uE| \to 0$$

since $\frac{u_+}{\varphi_1}|uE| \in L^1(\Omega)$ and the size of the domain tends to zero. We conclude, by Dominated Convergence that

$$0 \geq \int_\Omega (u_+\eta_m) \cdot E\nabla(u_+\eta_m) \to \int_\Omega u_+ E \cdot \nabla u_+ = \int_\Omega uE \cdot u_+. \qquad \square$$

We are finally ready to prove the result.

Proof of Theorem 17. The uniqueness claim is proven in Lemma 19. We now prove the existence and bounds by approximation. We can assume, without loss of generality, that $f \geq 0$. We construct approximations of E given by

$$E_\ell = -(\varphi_1 + \tfrac{1}{\ell})^{-1-\gamma} \nabla \varphi_1.$$

Clearly $E_\ell \in L^\infty(\Omega)$. These are well inside the theory of Theorem 5. Hence, there exists a weak solution $u_\ell \in H_0^1(\Omega)$. We compute

$$\operatorname{div} E_\ell = (1+\gamma)(\varphi_1 + \tfrac{1}{\ell})^{-2-\gamma} |\nabla \varphi_1|^2 + \lambda_1 (\varphi_1 + \tfrac{1}{\ell})^{-1-\gamma} \varphi_1.$$

This is non-negative. Hence, due to Theorem 12 we have that

$$\|\nabla u_\ell\|_{L^2} \le C\|f\|_{L^\infty}.$$

Splitting the behaviour near the boundary and away from the boundary, it is easy to see that $\operatorname{div} E_\ell \ge c_0 > 0$ uniformly. Therefore, due to Proposition 9 we have that

$$\|u_\ell\|_{L^\infty} \le \frac{\|f\|_{L^\infty}}{c_0}. \tag{30}$$

Now we must construct barrier functions. Select a single $\alpha > 0$ and the barrier

$$U = \tfrac{1}{\alpha}(\varphi_1 + \tfrac{1}{\ell})^\alpha.$$

We drop the dependence on ℓ and α to make the presentation below more readable. Plugging it into the equation we get

$$\begin{aligned}
-\Delta U + \operatorname{div}(U E_\ell) &= -\Delta U + \nabla U \cdot \nabla E_\ell + U \operatorname{div} E_\ell \\
&= -(\alpha-1)(\varphi_1 + \tfrac{1}{\ell})^{\alpha-2}|\nabla\varphi_1|^2 + \lambda_1(\varphi_1 + \tfrac{1}{\ell})^{\alpha-1}\varphi_1 \\
&\quad - (\varphi_1 + \tfrac{1}{\ell})^{\alpha-2-\gamma}|\nabla\varphi_1|^2 \\
&\quad + (1+\gamma)(\varphi_1 + \tfrac{1}{\ell})^{\alpha-2-\gamma}|\nabla\varphi_1|^2 + \lambda_1(\varphi_1 + \tfrac{1}{\ell})^{\alpha-1-\gamma}\varphi_1 \\
&\ge \Big(\gamma(\varphi_1 + \tfrac{1}{\ell})^{-\gamma} - (\alpha-1)\Big)(\varphi_1 + \tfrac{1}{\ell})^{\alpha-2}|\nabla\varphi_1|^2.
\end{aligned}$$

This is non-negative if $\varphi_1(x) + \frac{1}{m} \le (\frac{\alpha-1}{\gamma})^{-\frac{1}{\gamma}}$. There exists $\eta_\alpha > 0$ small enough such that

$$f(x) = 0 \qquad \text{and} \qquad \varphi_1(x) \le \tfrac{1}{2}(\tfrac{\alpha-1}{\gamma})^{-\frac{1}{\gamma}}, \qquad \forall x \text{ such that } \operatorname{dist}(x,\partial\Omega) \le \eta_\alpha.$$

We will use the neighbourhood of the boundary $A_\alpha = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \eta_\alpha\}$. Also, we consider the candidate super-solution

$$\overline{u}(x) = U(x)\left(\frac{\alpha}{\min_{\operatorname{dist}(x,\partial\Omega)=\eta_\alpha}\varphi_1(x)^\alpha} + \frac{\alpha}{c_0}\frac{\|f\|_{L^\infty}}{\min_{\operatorname{dist}(x,\partial\Omega)\ge\eta_\alpha}\varphi_1(x)^\alpha}\right).$$

We denote the constant on the right-hand side as C_α. Using the first term of C_α, $\overline{u} \ge u$ when $\operatorname{dist}(x,\partial\Omega) = \eta_\alpha$. Also, $\overline{u} = \frac{1}{m^\alpha} \ge 0 = u$ on $\partial\Omega$. Let us call

$$\overline{f} = -\Delta\overline{u} + \operatorname{div}(\overline{u}E_\ell).$$

By the previous computations, if $\ell \ge 2(\frac{\alpha-1}{\gamma})^{\frac{1}{\gamma}}$, we have $\overline{f} \ge 0 = f$ in A_α, and clearly $\overline{f} \in L^\infty(A_\alpha)$. Hence, due to Theorem 12 we have that

$$0 \le u_\ell(x) \le \overline{u}(x), \qquad x \in A_\alpha.$$

Also, due to (30) and the second part of C_α, we have that

$$0 \le u_\ell(x) \le \overline{u}(x), \qquad x \in \Omega \setminus A_\alpha.$$

Eventually, we deduce that for any $\alpha > 1$ we that

$$0 \le u_\ell(x) \le \tfrac{C_\alpha}{\alpha}(\varphi_1 + \tfrac{1}{\ell})^\alpha, \qquad \forall x \in \Omega \text{ and } \ell \ge 2(\tfrac{\alpha-1}{\gamma})^{\frac{1}{\gamma}}.$$

In particular, picking $\alpha = \gamma + 1$ we deduce that

$$|u_\ell E_\ell| \le \tfrac{C_{\gamma+1}}{\gamma+1} \|\nabla \varphi_1\|_{L^\infty} = \tfrac{C_{\gamma+1}}{\gamma+1}.$$

We deduce that, up to a subsequence,

$$u_\ell \to u \text{ a.e. and strongly in } L^2 \qquad \text{and} \qquad u_\ell \rightharpoonup u \text{ weakly in } H_0^1(\Omega).$$

This implies that $u_\ell E_\ell \to uE$ a.e. And hence uE is bounded. Passing to the limit in the weak formulation by the Dominated Convergence Theorem, the result is proven. □

Remark 20. Notice that the construction of the super-solution above can be done in any dimension $N \ge 1$. However, most of the results in the rest of the paper are only available for $N \ge 3$.

Remark 21. For Schrödinger-type equations $-\Delta u + Vu = f$, it is known that if the potential V is greater than $\operatorname{dist}(x, \partial\Omega)^{-2}$ and f is compactly supported, then u is flat the boundary, in the sense that $|u| \le C \operatorname{dist}(x, \partial\Omega)^{1+\varepsilon}$. This means that $\partial_n u = 0$ on $\partial\Omega$. This means that it satisfies Dirichlet and Neumann homogeneous boundary conditions. And it can be extended by 0 outside Ω with higher regularity than H^1. In contrast, the exponent γ in the above result can not be taken as $\gamma = 0$ in order to get flat solutions. Indeed, the convection term $E \cdot \nabla \varphi_1$, in the above computations, is more singular than the term $\varphi_1 \operatorname{div} E$. A very explicit example can be done in one dimension: if we consider $E = -Cx^{-1}$ then this drift does not generate flat solutions since if we take $U(x) = x^\alpha$ then

$$-U'' + (EU)' = (-\alpha x^{\alpha-1} - Cx^{\alpha-1})' = -(\alpha + C)(\alpha - 1)x^{\alpha-2},$$

and this is a supersolution only if $\alpha \le 1$.

Corollary 22. *In the hypothesis of Theorem 17 replace $f \in L_c^\infty(\Omega)$ by*

$$|f(x)| \le C \operatorname{dist}(x, \partial\Omega)^\omega \qquad \textit{for some } \omega \in \mathbb{R}.$$

Then

$$|u(x)| \le \operatorname{dist}(x, \partial\Omega)^\alpha \qquad \textit{for all } \alpha \in (1, \gamma + 2 - \omega)\,.$$

Proof. We maintain the notation of the proof of Theorem 17. We have already shown that, on a neighbourhood of the boundary,

$$-\Delta U + \operatorname{div}(UE_m) \ge \frac{\gamma}{2}(\varphi_1 + \tfrac{1}{m})^{\alpha-2-\gamma} |\nabla \varphi_1|^2 \ge c_1 (\varphi_1 + \tfrac{1}{m})^{\alpha-2-\gamma} \ge c_2 |f|.$$

For α in the range $(1, \gamma + 2 - \omega)$, we can take as a supersolutions for the approximating sequence

$$\overline{u}(x) = U(x) \left(\frac{1}{c_2} + \frac{\alpha}{\min_{\operatorname{dist}(x,\partial\Omega)=\eta_\alpha} \varphi_1(x)^\alpha} + \frac{\alpha}{c_0} \frac{\|f\|_{L^\infty}}{\min_{\operatorname{dist}(x,\partial\Omega)\ge\eta_\alpha} \varphi_1(x)^\alpha} \right). \qquad \square$$

And the rest of the proof remains as in Theorem 17.

7 Further remarks, extensions, and open problems

1. We point that the proofs of our estimates hold in many non-linear settings.

2. Theorem 17 admits many generalisations. For instance, one can consider the case $|E| \leq c_0 \operatorname{dist}(x, \partial\Omega)^{-\gamma-1}$ with $\operatorname{div} E \geq c_1 \operatorname{dist}(x, \partial\Omega)^{-\gamma-2}$ up to suitable conditions on the constants. Also, the techniques in this paper could be extended to the situation where $\operatorname{dist}(x, \partial\Omega)$ is replaced by $\operatorname{dist}(x, \Gamma)$ with a suitable part $\Gamma \subset \partial\Omega$. The case Γ an interior manifold can also be studied.

3. Including a non-negative potential. The same analysis can be performed on the equation

$$-\operatorname{div}(M(x)\nabla u) + a(x)u = -\operatorname{div}(uE(x)) + f(x)$$

   when $a \geq 0$. Furthermore, one will then obtain

$$\int_\Omega |u|(a + \operatorname{div} E) \leq \int_\Omega |f|.$$

   Hence, one can reduce the hypothesis to $a + \operatorname{div} E \geq 0$ in the whole analysis.

4. The study of $a \equiv 1$ is useful in the study of the evolution problem

$$u_t - \operatorname{div}(M(x)\nabla u) + \operatorname{div}(uE(x)) = 0.$$

   For the study of this problem one can write $u_t + Au = 0$ where

$$Au = -\operatorname{div}(M(x)\nabla u) + \operatorname{div}(uE(x)).$$

   In order to obtain solutions in semigroup form in L^p (where $1 \leq p \leq +\infty$), following the theory of accretive operators, it is sufficient that,

$$\|u\|_{L^p} \leq \|u + \lambda Au\|_{L^p}.$$

   Letting $f = u + \lambda Au$, this is precisely what we have proven above, where $M = \lambda I$ and $a \equiv 1$. See also [6].

5. We point out that when $|E| \leq |A|/|x|$, we have that, if $m > \frac{2N}{N+2}$ then $u|E| \in L^2(\Omega)$. It seems possible to extend the uniqueness result (19) to this setting.

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