2014 Madrid Conference on Applied Mathematics in honor of Alfonso Casal,
Electronic Journal of Differential Equations, Conference 22 (2015), pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MATHEMATICAL ANALYSIS OF A VISCOELASTIC-GRAVITATIONAL LAYERED EARTH MODEL FOR MAGMATIC INTRUSION IN THE DYNAMIC CASE

ALICIA ARJONA, JESÚS ILDEFONSO DÍAZ

Abstract. Volcanic areas present a lower effective viscosity than usually in the Earth's crust. It makes necessary to consider inelastic properties in deformation modelling. As a continuation of work done previously by some of the authors, this work is concerned with the proof that the perturbed equations representing the viscoelastic-gravitational displacements resulting from body forces embedded in a layered Earth model leads to a well-posed problem even for any kind of domains, with the natural boundary and transmission conditions. A homogeneous or stratified viscoelastic half-space has often been used as a simple earth model to calculate the displacements and gravity changes. Here we give a constructive proof of the existence of weak solutions and we show the uniqueness and the continuous dependence with respect to the initial data of weak solutions of the dynamic coupled viscoelastic-gravitational field equations.

1. Introduction

The study of ground deformation in volcano has been an important issue during last decades. There is a wide amount of literature on methodologies for modelling elastic and viscoelastic response when a source is embedded in media. The Mogi model [27] is the simplest analytical solution for a point source of pressure in an elastic half-space to interpret ground deformation. However, pure elastic models do not allow to reproduce gravity changes in some events. Therefore, the computation of gravity changes and deformations is advisable in order to do a correct interpretation. For an example, see the following and references therein [31, 32, 34, 35] and [15, 16, 18, 19, 20, 21]. Analytical and numerical solutions for modelling ground deformations and gravity changes have been devised and used in literature [4, 5, 6, 7, 10, 12, 28, 29, 36]. These models consider different source geometries representing magma such as spherical sources [27], ellipsoidal point sources [5] and sources due to pressurisation of the magma chamber [8]. As we shall see the coupling with the stationary equation for the potential gravity leads to many

2010 *Mathematics Subject Classification.* 35K10, 35L10, 35Q86, 35Q74, 46E35, 86A60.
Key words and phrases. Gravity changes; viscoelastic-gravitational earth model; weak solution; iterative algorithm; continuous dependence; uniqueness of solutions.

Published November 20, 2015.

new difficulties with respect to the pure viscoelastic models (see, e.g. [13] and its references).

In volcanic areas the presence of inhomogeneous materials and high temperature bodies reduce the effective viscosity of the Earth's crust. Therefore, inelastic properties of the Earth's crust must be taken into account [8, 21]. In this way Mogi's model was generalized to viscoelastic rheology in [8]. Rundle [34] presented a stratified viscoelastic half-space taking into account the interaction between the mass of the intrusion and the ambient gravity field and the effect caused by the change of pressure in the magmatic system. A theoretical and computational methods for the calculation of viscoelastic-gravitational displacements resulting from strike-slip faulting was described in [37]. The flow properties of the medium must also be considered. A Maxwell viscoelastic fluid was used by Pollitz [30] instead of Maxwell rheologies [17, 19, 22, 23, 34]. In this last case, the solution of the governing equations can be obtained from elastic solution employing the correspondence principle [24]. A propagator matrix technique is used to obtain the analytical solution of the elastic problem (see its description in [15, 31]).

The objective of this work is to prove that the perturbed equations representing the viscoelastic-gravitational displacements resulting from body forces embedded in a layered Earth model leads to a well-posed problem even for any kind of domains, with the natural boundary and transmission conditions. The existence and uniqueness of weak solutions of the elastic-gravitational problem was demonstrated in [3] and the stabilization to solutions of the associated stationary system was proved in [2]. We give here an additional constructive proof of the existence of weak solutions and we show the uniqueness and the continuous dependence with respect to the initial data of weak solutions of the coupled viscoelastic-gravitational field equations.

2. The problem

We consider here an Earth model composed of several viscoelastic-gravitational layers. We also consider the contribution of source terms which represent magmatic intrusion, corresponding to body forces acting on the medium. This is due to both volumetric change of wall of the chamber and sudden emplacement of a mass into the medium as a result of a new material injection into a magmatic chamber. The coupled model for deformation and variation of gravity is given by the following system of partial differential equations:

$$\begin{gathered}
\rho\mathbf{u}(\mathbf{t},\mathbf{x})_{tt} - \gamma\Delta\mathbf{u}(\mathbf{t},\mathbf{x})_t - \Delta\mathbf{u}(\mathbf{t},\mathbf{x}) - \frac{1}{1-2\nu}\nabla(\operatorname{div}\mathbf{u}(\mathbf{t},\mathbf{x})) \\
- \frac{\rho g}{\mu}\nabla(\mathbf{u}(\mathbf{t},\mathbf{x})\cdot\mathbf{e}_z) + \frac{\rho g}{\mu}\mathbf{e}_z\operatorname{div}\mathbf{u}(\mathbf{t},\mathbf{x}) + \frac{\rho}{\mu}\nabla\phi(t,\mathbf{x}) \\
= \mathbf{f}_u(t,\mathbf{x}), \\
-\Delta\phi(t,\mathbf{x}) - 4\pi\rho G\operatorname{div}\mathbf{u}(\mathbf{t},\mathbf{x}) = f_\phi(t,\mathbf{x}) \quad \text{in } (0,T)\times\Omega,
\end{gathered} \tag{2.1}$$

where $\mathbf{u}$ denotes the displacement, ϕ gravitational perturbed potential, ν the Poisson's ratio, ρ the unperturbed density of the medium, g the externally imposed gravitational acceleration, μ is the rigidity, $\gamma\Delta\mathbf{u}_t$ is a term introduced by the viscoelasticity of each layer, G universal gravitational constant, $\mathbf{e}_z$ is the unit vector pointing in the positive z-direction (down into the medium) and $\mathbf{f}_u$ and f_ϕ the body forces. Let us consider spatial domain Ω as union of p layers "overlay", that we will denote Ω_i, $i = 1, \dots, p$. Each layer is given through a common horizontal open set, $\omega \subset \mathbb{R}^2$, and so

$$\Omega_1 := \omega \times (d_1, d_1 + d_2), \quad \Omega_2 := \omega \times (d_1 + d_2, d_1 + d_2 + d_3), \dots; \tag{2.2}$$

that is,

$$\Omega_i := \omega \times \Big(\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j \Big) \subset \mathbb{R}^3, \tag{2.3}$$

where $i = 1, \dots, p-1$, and

$$\Omega_p := \omega \times (H, H + d_r), \tag{2.4}$$

where $H := \sum_{j=1}^{i-1} d_j$ and d_r can be equal to $+\infty$.

Let $\mathbf{u}^i : [0,T] \times \Omega_i \to \mathbb{R}^3$ be the displacement vector in each layer where T is an arbitrary time, $\mathbf{u}^i = (u_x^i, u_y^i, u_z^i)$ which depends on $\mathbf{x} =(x, y, z)$ and $t \in [0, T]$. The system (2.1) has been reached on each layer.

To the set of partial differential equations we will add the following boundary conditions (see Figure 2). Regarding to displacement field we prescribe on the side boundary, $\partial_l \Omega_i$, for $i = 1, \dots, p$, that:

$$\mathbf{u}^i(t, \mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in \partial_l \Omega_i, t \in (0, T) \tag{2.5}$$

on the upper boundary of the first layer $\partial_+ \Omega_1$,

$$\frac{\partial \mathbf{u}^1(t, \mathbf{x})}{\partial z} = \mathbf{0} \quad \mathbf{x} \in \partial_+ \Omega_1, t \in (0, T), \tag{2.6}$$

and that on the bottom boundary $\partial_- \Omega_p$,

$$\mathbf{u}^p(t, \mathbf{x}) = \mathbf{0} \quad \mathbf{x} \in \partial_- \Omega_p, t \in (0, T). \tag{2.7}$$

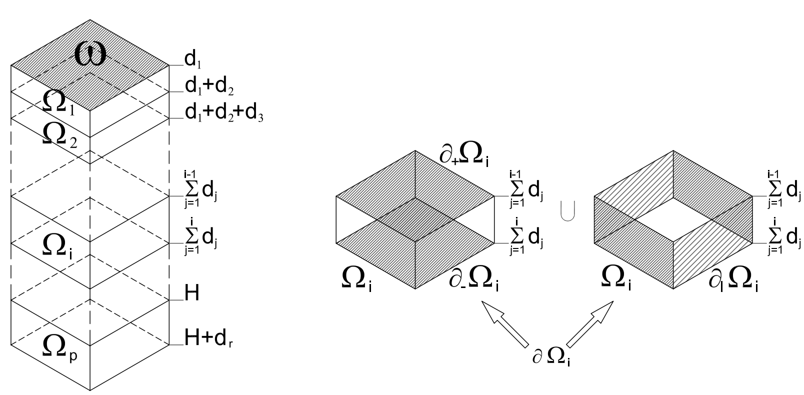

FIGURE 1. Domain of the problem.

In general, we can assure only that the first derivatives of $\mathbf{u}$ are continuous on the boundaries of the layers, that is, on the boundary between layers. We will require "transmission conditions" between both upper and bottom boundaries of the layers excepting on the first and the last layers. Therefore, the next conditions on $\partial_-\Omega_i = \partial_+\Omega_{i+1}$, with $i = 1, \dots, p-1$ are as follows:

$$\begin{gathered} \mathbf{u}^i(t,\mathbf{x}) = \mathbf{u}^{i+1}(t,\mathbf{x}) \quad \mathbf{x} \in \partial_-\Omega_i,\ t \in (0,T), \\ \frac{\partial \mathbf{u}^i(t,\mathbf{x})}{\partial z} = \frac{\partial \mathbf{u}^{i+1}(t,\mathbf{x})}{\partial z} \quad \mathbf{x} \in \partial_-\Omega_i,\ t \in (0,T). \end{gathered} \tag{2.8}$$

In relation to the gravitational perturbed potential we will assume that on side boundary $\partial_l\Omega_i$ for $i = 1, \dots, p$, it holds:

$$\phi(t,\mathbf{x}) = 0 \quad \mathbf{x} \in \partial_l\Omega_i,\ t \in (0,T), \tag{2.9}$$

on the upper boundary of the first layer $\partial_+\Omega_1$;

$$\phi^1(t,\mathbf{x}) = \phi_0(t,\mathbf{x}) \quad \mathbf{x} \in \partial_+\Omega_1, t \in (0,T), \tag{2.10}$$

and on the bottom boundary, $\partial_-\Omega_p$;

$$\phi^p(t,\mathbf{x}) = 0 \quad \mathbf{x} \in \partial_-\Omega_p, t \in (0,T). \tag{2.11}$$

Like before, we will require transmission conditions between upper and bottom boundary of the next layers excepting on the first and the last layers. So, we must have, on $\partial_-\Omega_i = \partial_+\Omega_{i+1}$ with $i = 1, \dots, p-1$, the following conditions:

$$\begin{gathered} \phi^i(t,\mathbf{x}) = \phi^{i+1}(t,\mathbf{x}) \quad \mathbf{x} \in \partial_-\Omega_i, t \in (0,T), \\ \frac{\partial \phi^i(t,\mathbf{x})}{\partial z} = \frac{\partial \phi^{i+1}(t,\mathbf{x})}{\partial z} \quad \mathbf{x} \in \partial_-\Omega_i, t \in (0,T). \end{gathered} \tag{2.12}$$

Finally we prescribe initial conditions for the displacements and the velocities:

$$\begin{gathered} \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{u}_t(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega. \end{gathered} \tag{2.13}$$

Since the multilayered structure of the domain introduce some possible abrupt changes on the second derivatives of solutions the existence of classical solutions of the problem looks artificial and we must introduce a suitable weak formulation notion of the problem which mathematical analysis is the main object of this paper.

Remark 2.1. Here, and in what follows, $H^1(\Omega)$ denotes the Sobolev space

$$H^1(\Omega) = \{\psi \in L^2(\Omega) : \frac{\partial \psi}{\partial x_i} \in L^2(\Omega), i = 1,2,3\},$$

where L^2 is the space of all square integrable functions. Both spaces have structure of Hilbert space (see, e.g. [9], for more details).

3. Weak formulation

Following some ideas already introduced in the previous work by the authors concerning the stationary problem [3] we define the energy space $V = V_u \times V_\phi$ as cross product of the energy spaces for the displacement and for the perturbed gravitational potential, V_u and V_ϕ respectively, where

$$\begin{aligned} V_\phi := \Big\{ &(\phi^1,\dots,\phi^p) \in \prod_{i=1}^p H^1(\Omega_i) \colon \phi^i = 0 \text{ on } \partial_l\Omega_i \forall\ i = 1\dots p, \phi^1 = 0 \text{ on } \partial_+\Omega_1 \\ &\text{and } \phi^p = 0 \text{ on } \partial_-\Omega_p, \phi^i = \phi^{i+1},\ \frac{\partial \phi^i(\mathbf{x})}{\partial z} = \frac{\partial \phi^{i+1}(\mathbf{x})}{\partial z} \\ &\text{on } \partial_-\Omega_1 \cup \partial_+\Omega_2 \cup \partial_-\Omega_2 \cup \dots \cup \partial_+\Omega_p,\ \text{for } i = 1\dots p-1 \Big\} \end{aligned}$$

and

$$V_u := \Big\{(\mathbf{u}^1,\dots,\mathbf{u}^p) \in \prod_{i=1}^p H^1(\Omega_i)^3 : \mathbf{u}^i = 0 \text{ on } \partial_l\Omega_i \text{ for } i=1\dots p, \mathbf{u}^1 = 0 \text{ on } \partial_+\Omega_1$$
$$\text{and } \mathbf{u}^p = 0 \text{ on } \partial_-\Omega_p, \mathbf{u}^i = \mathbf{u}^{i+1} \text{ and } \frac{\partial \mathbf{u}^i}{\partial z} = \frac{\partial \mathbf{u}^{i+1}}{\partial z}$$
$$\text{on } \partial_-\Omega_1 \cup \partial_+\Omega_2 \cup \partial_-\Omega_2 \cup \dots \cup \partial_+\Omega_p, \text{ for } i = 1\dots p-1\Big\}.$$

Regarding boundary data, ϕ_0 shall be extended to the interior of the domain Ω_1 such that

$$\begin{gathered} \widehat{\phi}_0 \in L^q(0,T : H^1(\Omega_1)), \quad \widehat{\phi}_0(t,\mathbf{x}) = \phi_0(t,\mathbf{x}) \text{ in } (0,T)\times\partial_+\Omega_1, \\ \widehat{\phi}_0(\mathbf{x}) = 0 \text{ in } (0,T)\times(\partial_-\Omega_1\cup\partial_l\Omega_1), \text{ for some } 2\le q\le+\infty. \end{gathered} \tag{3.1}$$

We will suppose, at least, that

$$\phi_0 \in L^q(0,T : \prod_{i=1}^p H^1(\Omega_i)) \text{ and satisfies (3.1),} \tag{3.2}$$

and under the following regularity on the data:

$$\mathbf{f}_u \in L^2(0,T : \prod_{i=1}^p H^{-1}(\Omega_i)^3), \tag{3.3}$$
$$f_\phi \in L^q(0,T : \prod_{i=1}^p H^{-1}(\Omega_i)), \tag{3.4}$$

for some $2 \le q \le +\infty$ and

$$\mathbf{u}_0, \mathbf{v}_0 \in V_u.$$

To introduce the weak solution definition we will follow similar arguments already introduced in the previous work by the authors for the stationary case. We start by assuming that $(\mathbf{u},\phi)$ is a classical solution of system (2.1). Let $(\mathbf{w},\theta) \in C^2([0,T] : V_u \times V_\phi)$ be test functions. We multiply the first equation of (2.1) by $\mathbf{w}^i(t,\mathbf{x})$, and the second equation by $\theta^i(t,\mathbf{x})$. Integrating by parts and applying Green's formula, we arrive, in a natural way, to the definition of the weak solution of the problem.

Definition 3.1. We assume the above regularity on the functions $\mathbf{f}_u$, f_ϕ, ϕ_0, $\mathbf{u}_0$ and $\mathbf{v}_0$. We say that $\{\mathbf{u},\phi\}$ is a weak solution of the problem (2.1) with the above mentioned boundary conditions if $(\mathbf{u},\phi-\phi_0) \in L^2(0,T : V)$, $\mathbf{u}_{tt} \in L^2(0,T : V_u')$ and for any test function $(\mathbf{w},\theta) \in L^2(0,T : V)$, $\mathbf{v} \in H^1(0,T : V_u')$ the following equalities hold:

$$\begin{aligned} &\int_0^T \sum_{i=1}^p \Big[\langle \rho^i \mathbf{u}^i_{tt}, \mathbf{w}^i\rangle + \int_{\Omega_i} \Big\{ \frac{1}{1-2\nu^i} \operatorname{div}\mathbf{u}^i \operatorname{div}\mathbf{w}^i \\ &- \frac{\rho^i g}{\mu^i} \nabla(\mathbf{u}^i\cdot\mathbf{e}_z)\cdot\mathbf{w}^i + \frac{\rho^i g}{\mu^i}\mathbf{e}_z \operatorname{div}\mathbf{u}^i\mathbf{w}^i + \nabla\mathbf{u}^i : \nabla\mathbf{w}^i + \gamma^i\nabla\mathbf{u}^i_t : \nabla\mathbf{w}^i\Big\} d\mathbf{x}\Big] dt \\ &= \int_0^T \sum_{i=1}^p \Big[\frac{\rho^i}{\mu^i}\int_{\Omega_i} -\nabla\phi^i\cdot\mathbf{w}^i\, d\mathbf{x} + \langle \mathbf{f}^i_u(t,\cdot), \mathbf{w}^i(t,\cdot)\rangle_{V_u'\times V_u}\Big] dt, \end{aligned} \tag{3.5}$$

and a.e. $t \in (0,T)$,

$$\begin{aligned} &\sum_{i=1}^{p} \int_{\Omega_i} \nabla\phi^i(t,\cdot)\cdot\nabla\theta^i(t,\cdot)d\mathbf{x} \\ &= \sum_{i=1}^{p} \Big[4\pi\rho^i G\int_{\Omega_i} \operatorname{div}\mathbf{u}^i(t,\cdot)\theta^i(t,\cdot)d\mathbf{x} + \langle f^i_\phi(t,\cdot),\theta^i(t,\cdot)\rangle_{V'_\phi\times V_\phi}\Big]. \end{aligned} \tag{3.6}$$

We shall prove that problem (2.1) is well-posed in the Hadamard sense.

Theorem 3.2. *(i) Assumed the regularity on the data $\mathbf{f}_u$, f_ϕ, ϕ_0, $\mathbf{u}_0$ and $\mathbf{v}_0$ then there exists a unique weak solution $\{\mathbf{u},\phi\}$ of the problem (2.1). Moreover $\mathbf{u}_t \in L^\infty(0,T:L^2(\Omega))$, $\mathbf{u} \in L^\infty(0,T:V_u)$, $\phi\in L^q(0,T:V_\phi)$ and there exists a positive constant C (depending on T, Ω_i and the constants ρ^i, μ^i, ν^i,γ^i and G) such that the following continuous dependence estimate holds*

$$\begin{aligned} &\sup_{t\in[0,T]}\sum_{i=1}^{p}\Big[\int_{\Omega_i}|\mathbf{u}^i_t|^2d\mathbf{x}\Big] + \int_0^T\int_{\Omega_i}|\nabla\mathbf{u}^i_t|^2\,d\mathbf{x}\,dt + \sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\mathbf{u}^i|^2d\mathbf{x} \\ &+\sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}(\operatorname{div}\mathbf{u}^i)^2\,d\mathbf{x} + \int_0^T\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\phi^i(t,\mathbf{x})|^2\,d\mathbf{x} \\ &\le C\Big[\int_0^T\sum_{i=1}^{p}\|\mathbf{f}^i_u(t,\cdot)\|^2_{H^{-1}}dt + \int_0^T\sum_{i=1}^{p}\|\mathbf{f}^i_\phi(t,\cdot)\|^2_{H^{-1}}dt \\ &\quad+\int_{\partial_+\Omega_1}|\phi_0(\mathbf{s})\mathbf{v}_0(\mathbf{s})\cdot\mathbf{n}|ds + \sum_{i=1}^{p}\int_{\Omega_i}|\mathbf{v}^i_0(\mathbf{x})|^2d\mathbf{x} \\ &\quad+\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\mathbf{v}^i_0(\mathbf{x})|^2d\mathbf{x} + \sum_{i=1}^{p}\int_{\Omega_i}|\nabla\mathbf{u}^i_0(\mathbf{x})|^2d\mathbf{x} \\ &\quad+\sum_{i=1}^{p}\int_{\Omega_i}\operatorname{div}\mathbf{u}^i_0(\mathbf{x})^2d\mathbf{x} + \int_0^T\int_{\partial_+\Omega_1}\big|\phi^0(t,\mathbf{s})\frac{\partial}{\partial\mathbf{n}}\phi^0(t,\mathbf{s})\big|ds\Big]. \end{aligned} \tag{3.7}$$

(ii) If in addition we assume that

$$\begin{gathered} \widehat{\phi}_0\in H^1(0,T:H^1(\Omega_1)),\quad \phi_0\in H^1(0,T:\prod_{i=1}^{p}H^1(\Omega_i)), \\ f_\phi\in H^1(0,T:\prod_{i=1}^{p}H^{-1}(\Omega_i)), \end{gathered} \tag{3.8}$$

then $\phi\in H^1(0,T:V_\phi)$ and we have the additional continuous dependence estimate

$$\begin{aligned} &\sup_{t\in[0,T]}\sum_{i=1}^{p}\Big[\int_{\Omega_i}|\mathbf{u}^i_t|^2d\mathbf{x}\Big] + \int_0^T\int_{\Omega_i}|\nabla\mathbf{u}^i_t|^2\,d\mathbf{x}\,dt + \sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\mathbf{u}^i|^2d\mathbf{x} \\ &+\sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}(\operatorname{div}\mathbf{u}^i)^2\,d\mathbf{x} + \sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\phi^i(t,\mathbf{x})|^2\,d\mathbf{x} \\ &\le C\Big[\int_0^T\sum_{i=1}^{p}\|\mathbf{f}^i_u(t,\cdot)\|^2_{H^{-1}}dt + \int_0^T\sum_{i=1}^{p}\|\mathbf{f}^i_\phi(t,\cdot)\|^2_{H^{-1}}dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \sum_{i=1}^p \|\frac{\partial}{\partial t}(\mathbf{f}_\phi^i)(t,\cdot)\|_{H^{-1}}^2 dt + \int_{\partial_+\Omega_1} |\phi_0(\mathbf{s})\mathbf{v}_0(\mathbf{s})\cdot\mathbf{n}|ds + \sum_{i=1}^p \int_{\Omega_i} |\mathbf{v}_0^i(\mathbf{x})|^2 d\mathbf{x} \\
&+\sum_{i=1}^p \int_{\Omega_i} |\nabla \mathbf{v}_0^i(\mathbf{x})|^2 d\mathbf{x} + \sum_{i=1}^p \int_{\Omega_i} |\nabla \mathbf{u}_0^i(\mathbf{x})|^2 d\mathbf{x} + \sum_{i=1}^p \int_{\Omega_i} \operatorname{div} \mathbf{u}_0^i(\mathbf{x})^2 d\mathbf{x} \\
&+\int_0^T \int_{\partial_+\Omega_1} |\phi^0(t,\mathbf{s})\frac{\partial}{\partial \mathbf{n}}\phi^0(t,\mathbf{s})|ds + \int_0^T \int_{\partial_+\Omega_1} \big|\phi^0(t,\mathbf{s})\frac{\partial^2}{\partial t\partial \mathbf{n}}\phi^0(t,\mathbf{s})\big|ds\Big],
\end{aligned}
$$

for a suitable positive constant C (depending on T, Ω_i and the constants ρ^i, μ^i, ν^i, γ^i and G).

The existence of weak solutions will be proved by means of an iterative method (which can be very useful to justify the convergence of some numerical algorithms) without requiring any additional time regularity to function ϕ. This allows a great generality on the data. In the second part we shall prove a stronger regularity on the weak solution by a "cancellation method" related to the time differentiability of function ϕ. We show that this holds under some slightly stronger regularity on the data.

4. Proof of Theorem 3.2 part (i)

We shall prove the existence of a weak solution by splitting it in several steps. We first consider two different uncoupled problems: the first one when displacements are known and the second one in which the gradient of the gravitational potential is given.

4.1. **Uncoupled problem for the potential.** ($\mathbf{u}$ is assumed to be known)

We assume that $\mathbf{u}$ is known, with

$$
\mathbf{u}^i \in H^1(0,T : H^1(\Omega_i)). \tag{4.1}
$$

Let us consider the following problem, which we denotes as $P_1[\phi_0^1, \mathbf{u}^i, \mathbf{f}_\phi^i]$, over the energy space $L^2(0,T : V_\phi)$:

$$
\begin{gathered}
-\Delta\phi^i = 4\pi\rho^i G \operatorname{div} \mathbf{u}^i(t,\mathbf{x}) + f_\phi^i(t,\mathbf{x}) \quad \text{in } (0,T)\times\Omega_i, \\
\phi^i = 0 \quad \text{on } (0,T)\times\partial_l\Omega_i \ \forall i = 1,\dots,p, \\
\phi^i = \phi^{i+1}, \quad \frac{\partial\phi^i}{\partial z} = \frac{\partial\phi^{i+1}}{\partial z} \quad \text{on } (0,T)\times\partial_-\Omega_i = (0,T)\times\partial_+\Omega_{i+1}, \\
\forall i = 1,\dots,p-1, \\
\phi^1 = \phi_0^1 \quad \text{on } (0,T)\times\partial_+\Omega_1, \\
\phi^p = 0 \quad \text{on } (0,T)\times\partial_-\Omega_p.
\end{gathered} \tag{4.2}
$$

Definition 4.1. Assumed the above regularity, a function ϕ is a weak solution of the problem $P_1[\phi_0^1, \mathbf{u}^i, \mathbf{f}_\phi^i]$ if $\phi^* := \phi - \phi_0 \in V_\phi$ and for any test function $\theta \in V_\phi$, and a.e. $t \in (0,T)$ we have

$$
\sum_{i=1}^p \int_{\Omega_i} \nabla\phi^{*i}\cdot\nabla\theta^i \, d\mathbf{x} = \sum_{i=1}^p 4\pi\rho^i G \int_{\Omega_i} (\operatorname{div} \mathbf{u}^i)\,\theta^i \, d\mathbf{x} + \langle f_\phi, \theta\rangle_{V_\phi'\times V_\phi}. \tag{4.3}
$$

The following result was shown in the previous paper by the authors.

Theorem 4.2 ([3]). *Assuming the above regularity on the data $\mathbf{u}^i$, f_ϕ and ϕ_0, there exists a weak solution, ϕ, of the problem (4.2). Moreover, if we denote the Poincaré's constant on Ω_i by $C(\Omega_i)$ we have the estimate*

$$\begin{aligned}\sum_{i=1}^p \int_{\Omega_i} |\nabla\phi^i(t,\mathbf{x})|^2\,d\mathbf{x} &\le \sum_{i=1}^p C(\Omega_i)4\pi\rho^i G\int_{\Omega_i}|\nabla\mathbf{u}^i(t,\mathbf{x})|^2 d\mathbf{x}\\ &\quad + \sum_{i=1}^p \|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2 + 2\int_{\partial_+\Omega_1}\phi^0(t,\mathbf{s})\frac{\partial}{\partial\mathbf{n}}\phi^0(t,\mathbf{s})ds.\end{aligned}$$

Remark 4.3. Note that in fact, since we assume that $\mathbf{u}^i \in H^1(0,T:H^1(\Omega_i))$ it is easy to see that $f_\phi \in H^1(0,T:\prod_{i=1}^p H^{-1}(\Omega_i))$ implies a more regularity with respect to the variable $t: \phi^i \in H^1(0,T:H^1(\Omega_i))$.

4.2. **Uncoupled problem for the displacements.** (ϕ is assumed to be known)
Now we assume given ϕ is given and

$$\phi^i \in L^2(0,T:H^1(\Omega_i)). \tag{4.4}$$

Let us consider the following problem $P_2[\phi^i,\mathbf{f}_u^i]$ in $L^2(0,T:V_u)$:

$$\begin{gathered}\begin{aligned}&\rho^i\mathbf{u}_{tt}^i - \gamma^i\Delta\mathbf{u}_t^i - \Delta\mathbf{u}^i - \frac{1}{1-2\nu^i}\nabla(\operatorname{div}\mathbf{u}^i)\\ &- \frac{\rho^i g}{\mu^i}\nabla(\mathbf{u}^i\cdot\mathbf{e}_z) + \frac{\rho^i g}{\mu^i}\mathbf{e}_z\operatorname{div}\mathbf{u}^i\\ &= -\frac{\rho_i}{\mu_i}\nabla\phi^i + \mathbf{f}_u^i \quad \text{in } (0,T)\times\Omega_i,\end{aligned}\\ \mathbf{u}^i(0,\mathbf{x}) = \mathbf{u}_0^i(\mathbf{x}) \quad \text{in } \Omega_i,\\ \mathbf{u}_t^i(0,\mathbf{x}) = \mathbf{v}_0^i(\mathbf{x}) \quad \text{in } \Omega_i,\\ \mathbf{u}^i = 0 \quad \text{on } \partial_l\Omega_i, \quad i=1,\dots,p,\\ \mathbf{u}^i = \mathbf{u}^{i+1}, \quad \frac{\partial\mathbf{u}^i}{\partial z} = \frac{\partial\mathbf{u}^{i+1}}{\partial z} \quad \text{on } \partial_-\Omega_i = \partial_+\Omega_{i+1},\ i=1,\dots,p-1,\\ \mathbf{u}^1 = 0 \quad \text{on } \partial_+\Omega_1,\\ \mathbf{u}^p = 0 \quad \text{on } \partial_-\Omega_p\,.\end{gathered} \tag{4.5}$$

Definition 4.4. We assume the above mentioned regularity on the data f_ϕ and ϕ_0 and initial data. The function $\mathbf{u}$ is a weak solution of problem (4.5) if $\mathbf{u} \in H^1(0,T: V_u)$, $\mathbf{u}_{tt} \in L^2(0,T:V_u')$ and for any test function $\mathbf{w}$ such that $\mathbf{w} \in H^1(0,T:V_u)$ and $\mathbf{w} \in H^2(0,T:V_u')$ we have

$$\begin{aligned}&\sum_{i=1}^p\Big[\int_0^T(\rho^i\langle\mathbf{u}_{tt}^i,\mathbf{w}^i\rangle dt + \frac{1}{1-2\nu^i}\int_0^T\int_{\mathbf{\Omega_i}}\operatorname{div}\mathbf{u}^i\operatorname{div}\mathbf{w}^i\,d\mathbf{x}\,dt\\ &- \frac{\rho^i g}{\mu^i}\int_0^T\int_{\mathbf{\Omega_i}}\nabla(\mathbf{u}^i\cdot\mathbf{e}_z)\cdot\mathbf{w}^i\,d\mathbf{x}\,dt + \frac{\rho^i g}{\mu^i}\int_0^T\int_{\mathbf{\Omega_i}} e_z\operatorname{div}\mathbf{u}^i\mathbf{w}^i\,d\mathbf{x}\,dt\\ &+ \int_0^T\int_{\Omega_i}\nabla\mathbf{u}^i:\nabla\mathbf{w}^i\,d\mathbf{x}\,dt + \int_0^T\int_{\Omega_i}\gamma^i\nabla\mathbf{u}_t^i:\nabla\mathbf{w}^i d\mathbf{x}\Big]\\ &= \sum_{i=1}^p\Big[-\frac{\rho^i}{\mu^i}\int_0^T\int_{\Omega_i}\nabla\phi^i\cdot\mathbf{w}^i\,d\mathbf{x} + \int_0^T\langle\mathbf{f}_u^i(t,\mathbf{x}),\mathbf{w}^i(t,\mathbf{x})\rangle_{V_u'\times V_u}\Big].\end{aligned} \tag{4.6}$$

The following result is a non difficult adaptation of a more general result presented in [14, Theorem 6.1, Chapter 3].

Theorem 4.5. *Assumed the regularity on f_ϕ, the initial data and* (4.4) *there exists a weak solution,* $\mathbf{u}$*, of the problem* (4.5).

Proof. We define the two bilinear forms $a_u : V_u \times V_u \to \mathbb{R}$, $a^*_{u_t}(\mathbf{u}_t, \mathbf{w}) : V_u \times V_u \to \mathbb{R}$ and the linear form $L_u : V_u \to \mathbb{R}$ as follows:

$$\begin{aligned} a_u(\mathbf{u}, \mathbf{w}) := \sum_{i=1}^{p} \Big[\int_{\Omega_i} \Big\{ & \frac{1}{1-2\nu^i} \operatorname{div} \mathbf{u}^i \operatorname{div} \mathbf{w}^i - \frac{\rho^i g}{\mu^i} \nabla(\mathbf{u}^i \cdot \mathbf{e}_z) \cdot \mathbf{w}^i \\ & + \frac{\rho^i g}{\mu^i} \mathbf{e}_z \operatorname{div} \mathbf{u}^i \, \mathbf{w}^i + \nabla \mathbf{u}^i : \nabla \mathbf{w}^i + \gamma^i \nabla \mathbf{u}^i_t : \nabla \mathbf{w}^i \, d\mathbf{x} \Big\} \Big], \end{aligned} \tag{4.7}$$

$$a^*_{u_t}(\mathbf{u}_t, \mathbf{w}) := \sum_{i=1}^{p} \int_{\mathbf{\Omega_i}} \gamma^i \nabla \mathbf{u}^i_t : \nabla \mathbf{w}^i \, d\mathbf{x}, \tag{4.8}$$

$$\langle L_u(t), \mathbf{w} \rangle := \sum_{i=1}^{p} \Big[\frac{\rho^i}{\mu^i} \int_{\Omega_i} -\nabla \phi^i(t,\cdot) \cdot \mathbf{w}^i \, d\mathbf{x} + \langle \mathbf{f}^i_u(t, \ \cdot), \mathbf{w}^i(t, \ \cdot) \rangle_{V'_u \times V_u} \Big]. \tag{4.9}$$

That the bilinear form $a_u(\cdot,\cdot)\,\cdot$ is continuous and coercive, and that the lineal form $L_u(t)$ is continuous was proved in [3, Theorem 3]. On the other hand, the same type of arguments shows that the form $a^*_{u_t}(\mathbf{u}_t, \mathbf{w})$ is also continuous on $V_u \times V_u$. Then, we can apply the same arguments of the proof of [14, Theorem 6.1, Chapter 3], to the special case of no constraint, to the problem

$$\sum_{i=1}^{p} \langle \rho^i \mathbf{u}^i_{tt}, \mathbf{w}^i(t,\cdot) \rangle_{V'_u \times V_u} + a_u(\mathbf{u}, \mathbf{w}) + a^*_{u_t}(\mathbf{u}_t, \mathbf{w}) = \langle L_u(t), \mathbf{w} \rangle_{V'_u \times V_u}$$

□

(by using the transmission conditions) and we obtain the result.

4.3. **Coupled system.** To proof the existence and uniqueness of solutions of the coupled system an iterative method will be used. Firstly, we shall construct two sequences $\{\mathbf{u}^n(t,\mathbf{x})\}$ and $\{\phi^n(\mathbf{x})\}$ as follows. We start with $\phi^0(\mathbf{x})$ vector which has initial data, $\phi_0(\mathbf{x})$, as a first component and rest of components 0. With this vector and (4.5) problem the unique vector $\mathbf{u}^1(t,\mathbf{x})$ is obtained. Taking this vector as known, we solve (4.2) problem to get the solution associated to ϕ^1. In this way we build the next sequences (which allow us to introduce some notations which will be used in what follows):

$$\phi^0 = \begin{pmatrix} \phi_0 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \overset{P_2[\phi_0^i, \mathbf{f}_u^i]}{\longrightarrow} \mathbf{u}^1 = \begin{pmatrix} \mathbf{u}_1^1 \\ \mathbf{u}_2^1 \\ \cdot \\ \mathbf{u}_p^1 \end{pmatrix} \overset{P_1[\phi_0^1, \mathbf{u}^i, f_\phi^i]}{\longrightarrow} \phi^1 = \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \\ \cdot \\ \phi_p^1 \end{pmatrix}.$$

In general,

$$\phi^{n-1} = \begin{pmatrix} \phi_1^{n-1} \\ \phi_2^{n-1} \\ \cdot \\ \phi_p^{n-1} \end{pmatrix} \overset{P_2[\phi^i, \mathbf{f}_u^i]}{\longrightarrow} \mathbf{u}^n = \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^1 \\ \cdot \\ \mathbf{u}_p^n \end{pmatrix} \overset{P_1[\phi_0^1, \mathbf{u}^i, f_\phi^i]}{\longrightarrow} \phi^n = \begin{pmatrix} \phi_1^n \\ \phi_2^n \\ \cdot \\ \phi_p^n \end{pmatrix}.$$

We claim that it is possible to find some universal a priori estimates (i.e., independent on n) allowing to pass to the limit. Indeed, by multiplying the equation of $\mathbf{u}_i^n$ by $\rho^i \mathbf{u}_{i,t}^n(t,\mathbf{x})$ (where $\mathbf{u}_{i,t}^n(t,\mathbf{x}) = \frac{\partial \mathbf{u}_i^n(t,\mathbf{x})}{\partial t}$). Since, in general, we know that

$$
\begin{gathered}
\nabla \mathbf{u}^i \cdot \nabla \mathbf{u}_t^i = \frac{1}{2}\frac{\partial}{\partial t}(|\nabla \mathbf{u}^i|^2), \\
\int_0^T \int_{\Omega_i} \Delta \mathbf{u}^i \cdot \mathbf{u}_t^i = -\frac{1}{2}\int_0^T \frac{d}{dt}\int_{\Omega_i} |\nabla \mathbf{u}^i|^2. \\
\int_0^T \int_{\Omega_i} \operatorname{div} \mathbf{u}^i \operatorname{div} \mathbf{u}_t^i = \frac{1}{2}\int_0^T \frac{d}{dt}\int_{\Omega_i} (\operatorname{div} \mathbf{u}^i)^2 .
\end{gathered}
$$

Then, by using Green's formula, denoting the Poincaré's constant on Ω_i, by $C(\Omega_i)$, and applying Poincaré's and Young's inequalities ($ab \le \varepsilon a^2 + C_\varepsilon b^2$) we obtain the estimate

$$
\begin{aligned}
&\sup_{t\in[0,T]} \sum_{i=1}^p \rho^i \big(\frac{1}{2} - \frac{Tg\rho^i}{\mu^i}\big) \int_{\Omega_i} |\mathbf{u}_{i,t}^n|^2 d\mathbf{x} + \sum_{i=1}^p (1-\varepsilon)\rho^i\gamma^i \int_0^T \int_{\Omega_i} |\nabla \mathbf{u}_{i,t}^n|^2 \, d\mathbf{x}\, dt \\
&+ \sup_{t\in[0,T]} \sum_{i=1}^p \frac{\rho^i}{2}\big(1 - \frac{Tg\rho^i}{\mu^i}\big) \int_{\Omega_i} |\nabla \mathbf{u}_i^n|^2 d\mathbf{x} \\
&+ \sup_{t\in[0,T]} \sum_{i=1}^p \frac{\rho^i}{2}\big(\frac{1}{(1-2\nu^i)} - \frac{Tg\rho^i}{\mu^i}\big) \int_{\Omega_i} (\operatorname{div} \mathbf{u}_i^n)^2 \, d\mathbf{x} \\
&\le C_\varepsilon \sum_{i=1}^p \int_0^T \Big(\int_{\Omega_i} |\phi_i^{n-1}(t,\mathbf{x})|^2 d\mathbf{x} + \frac{\rho^i}{\gamma^i} \|\mathbf{f}_\mathbf{u}^i(t,\cdot)\|_{H^{-1}}^2 \Big) dt \\
&\quad + \sum_{i=1}^p \frac{\rho^i}{2} \int_{\Omega_i} |\mathbf{v}_0^i(\mathbf{x})|^2 d\mathbf{x} + \sum_{i=1}^p \gamma^i \rho^i \int_{\Omega_i} |\nabla \mathbf{v}_0^i(\mathbf{x})|^2 d\mathbf{x} \\
&\quad + \sum_{i=1}^p \frac{\rho^i}{2} \int_{\Omega_i} |\nabla \mathbf{u}_0^i(\mathbf{x})|^2 d\mathbf{x} + \sum_{i=1}^p \frac{\rho^i}{2(1-2\nu^i)} \int_{\Omega_i} \operatorname{div} \mathbf{u}_0^i(\mathbf{x})^2 d\mathbf{x} \\
&\quad + \int_0^T \int_{\partial_+\Omega_1} |\phi^0(t,\mathbf{s}) \frac{\partial}{\partial \mathbf{n}} \phi^0(t,\mathbf{s})| ds.
\end{aligned}
\tag{4.10}
$$

In the above estimate we used the following inequalitites

$$
\begin{aligned}
&\sum_{i=1}^p \frac{\rho^i}{(\mu^i)^2\gamma^i} \int_0^T \int_{\Omega_i} \nabla \phi_i^{n-1}(t,\mathbf{x}) \cdot \mathbf{u}_{i,t}^n(t,\mathbf{x}) d\mathbf{x}\, dt \\
&= -\sum_{i=1}^p \frac{\rho^i}{(\mu^i)^2\gamma^i} \int_0^T \int_{\Omega_i} \phi_i^{n-1}(t,\mathbf{x}) \operatorname{div} \mathbf{u}_{i,t}^n(t,\mathbf{x}) d\mathbf{x}\, dt \\
&\le \sum_{i=1}^p \big(C_\varepsilon \int_0^T \int_{\Omega_i} |\phi_i^{n-1}(t,\mathbf{x})|^2 d\mathbf{x}\, dt + \epsilon \rho^i \gamma^i \int_0^T \int_{\Omega_i} |\nabla \mathbf{u}_{i,t}^n|^2 \, d\mathbf{x}\, dt\big),
\end{aligned}
$$

$$
\Big|\int_0^T \int_{\Omega_i} \nabla(\mathbf{u}_i^n \cdot \mathbf{e}_z) \cdot \mathbf{u}_{i,t}^n\Big| \le \frac{T}{2} \sup_{t\in[0,T]} \int_{\Omega_i} |\nabla \mathbf{u}_i^n|^2 + \frac{T}{2} \sup_{t\in[0,T]} \int_{\Omega_i} |\mathbf{u}_{i,t}^n|^2,
$$

$$
\Big|\int_0^T \int_{\Omega_i} \mathbf{e}_z \operatorname{div} \mathbf{u}_i^n \cdot \mathbf{u}_{i,t}^n\Big| \le \frac{T}{2} \sup_{t\in[0,T]} \int_{\Omega_i} |\operatorname{div} \mathbf{u}_i^n|^2 + \frac{T}{2} \sup_{t\in[0,T]} \int_{\Omega_i} |\mathbf{u}_{i,t}^n|^2.
$$

On the other hand, since $L^2(\Omega_i) \subset H^{-1}(\Omega_i)$, there exists $K > 0$ such that

$$\|\operatorname{div} \mathbf{u}_i^{n-1}\|_{H^{-1}(\Omega_i)} \le K \|\operatorname{div} \mathbf{u}_i^{n-1}\|_{L^2(\Omega_i)}.$$

Then, from the coerciveness of the bilinear form associated to the elliptic equation we know that

$$\begin{aligned} \int_{\Omega_i} |\nabla \phi_i^n(t,\mathbf{x})|^2 \, d\mathbf{x} \le K\Big(\int_{\Omega_i} |\operatorname{div} \mathbf{u}_i^{n-1}(t,\mathbf{x})|^2 \, d\mathbf{x} + \sum_{i=1}^p \|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2 \\ + \int_{\partial_+\Omega_1} \big|\phi^0(t,\mathbf{s}) \frac{\partial}{\partial \mathbf{n}} \phi^0(t,\mathbf{s})\big| ds \Big). \end{aligned} \tag{4.11}$$

Using twice Poincaré's inequality we obtain

$$\begin{aligned} &\sum_{i=1}^p \int_0^T \int_{\Omega_i} |\phi_i^n(t,\mathbf{x})|^2 d\mathbf{x}\, dt \\ &\le \sum_{i=1}^p \widehat{K}(C(\Omega_i)) \int_0^T \int_{\Omega_i} |\nabla \phi_i^n(t,\mathbf{x})|^2 \, d\mathbf{x}\, dt \\ &\le \sum_{i=1}^p \widehat{K} \Big\{ \int_0^T \int_{\Omega_i} |\mathbf{u}_i^{n-1}(T,\mathbf{x})|^2 \, d\mathbf{x}\, dt + \int_0^T \sum_{i=1}^p \|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2 dt \\ &\quad + \int_{\partial_+\Omega_1} \big|\phi^0(t,\mathbf{s}) \frac{\partial}{\partial \mathbf{n}} \phi^0(t,\mathbf{s})\big| ds \Big\} \\ &\le \sum_{i=1}^p \overline{K} \Big\{ T \sup_{t\in[0,T]} \int_{\Omega_i} |\nabla \mathbf{u}_i^{n-1}(t,\mathbf{x})|^2 \, d\mathbf{x} + \int_0^T \sum_{i=1}^p \|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2 dt \\ &\quad + \int_{\partial_+\Omega_1} \big|\phi^0(t,\mathbf{s}) \frac{\partial}{\partial \mathbf{n}} \phi^0(t,\mathbf{s})\big| ds \Big\}, \end{aligned}$$

where $\widehat{K}$ and $\overline{K}$ are positive constants depending increasingly on the Poincaré's constants $C(\Omega_i)$. Thus, combining both inequalities we obtain

$$\begin{aligned} &\sup_{t\in[0,T]} \sum_{i=1}^p \rho^i \big(\frac{1}{2} - \frac{Tg\rho^i}{\mu^i}\big) \int_{\Omega_i} |\mathbf{u}_t^i|^2 d\mathbf{x} \\ &+ \sum_{i=1}^p (1 - \varepsilon C(\Omega_i)) \rho^i \gamma^i \int_0^T \int_{\Omega_i} |\nabla \mathbf{u}_t^i|^2 \, d\mathbf{x}\, dt \\ &+ \sup_{t\in[0,T]} \sum_{i=1}^p \frac{\rho^i}{2} \big(1 - \frac{Tg\rho^i}{\mu^i}\big) \int_{\Omega_i} |\nabla \mathbf{u}^i|^2 d\mathbf{x} \\ &+ \sup_{t\in[0,T]} \sum_{i=1}^p \frac{\rho^i}{2} \big(\frac{1}{(1-2\nu^i)} - \frac{Tg\rho^i}{\mu^i}\big) \int_{\Omega_i} (\operatorname{div} \mathbf{u}^i)^2 \, d\mathbf{x} + \sum_{i=1}^p \int_0^T \int_{\Omega_i} |\phi_i^n(t,\mathbf{x})|^2 d\mathbf{x} dt \\ &\le \sum_{i=1}^p \overline{K} \Big\{ T \sup_{t\in[0,T]} \int_{\Omega_i} |\nabla \mathbf{u}_i^{n-1}(t,\mathbf{x})|^2 \, d\mathbf{x} + \sum_{i=1}^p \int_0^T \sum_{i=1}^p \|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2 dt \\ &\quad + \sum_{i=1}^p \int_0^T \|\mathbf{f}_\mathbf{u}^i(t,\cdot)\|_{H^{-1}}^2 dt + + \sum_{i=1}^p \int_{\Omega_i} |\mathbf{v}_0^i(\mathbf{x})|^2 d\mathbf{x} + \sum_{i=1}^p \int_{\Omega_i} |\nabla \mathbf{v}_0^i(\mathbf{x})|^2 d\mathbf{x} \end{aligned}$$

$$+\sum_{i=1}^{p}\int_{\Omega_i}|\nabla \mathbf{u}_0^i(\mathbf{x})|^2 d\mathbf{x}\sum_{i=1}^{p}\int_{\Omega_i}\operatorname{div}\mathbf{u}_0^i(\mathbf{x})^2 d\mathbf{x} \tag{4.12}$$

$$+\int_0^T\int_{\partial_+\Omega_1}\big|\phi^0(t,\mathbf{s})\frac{\partial}{\partial \mathbf{n}}\phi^0(t,\mathbf{s})\big|ds\,dt\Big\}. \tag{4.13}$$

Summarizing, if we assume $T>0$ and ε small enough we conclude that any uniform estimate on the term

$$I_n=\sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}|\nabla \mathbf{u}_i^n(t,\mathbf{x})|^2\,d\mathbf{x}$$

allows us to make uniform all the above inequalities. But we can understood (4.13) in the form

$$I_n\le \delta I_{n-1}+A$$

with

$$\delta:=\frac{\overline{K}T}{\min_i \frac{\rho^i}{2}\big(1-\frac{Tg\rho^i}{\mu^i}\big)}$$

and A given trough the external and initial data. Thus, if $\delta\in(0,1)$, we obtain

$$\lim_n I_n\le A$$

and we obtain the uniform estimate $I_n\le 1+A$ for any $n\in\mathbb{N}$ large enough (which implies uniform estimates in (4.10) and (4.11)). But, as before, the condition $\delta\in(0,1)$ holds if we assume $T=T_0>0$ small enough and thus we obtain a set of uniform a priori estimates on the sequences $\{\mathbf{u}^n\}$ and $\{\phi^n\}$ which show that they converge weakly in $H^1(0,T_0:V_u)$ and $L^2(0,T_0:V_\phi)$, respectively, to a vectorial function $(\mathbf{u},\phi)$ which is a local (in time) weak solution of the coupled system.

To prove the uniqueness of the local weak solution (i.e. when $t\in[0,T_0]$ it suffices to show that if we assume as zero all the data then the unique solution is the trivial solution $(\mathbf{u},\phi)=(\mathbf{0},0)$. This is an special conclusion of the obtained continuous dependence estimate.

Moreover, such local time T_0 does not depend on any norm of the data (but only on the coefficients). In particular, if $T=T_0$ we obtain the estimate

$$\begin{aligned}
&\sum_{i=1}^{p}\int_0^T\int_{\Omega_i}|\nabla \mathbf{u}_t^i|^2\,d\mathbf{x}\,dt+\sum_{i=1}^{p}\int_0^T\int_{\Omega_i}|\phi_i^n(t,\mathbf{x})|^2 d\mathbf{x}\,dt\\
&\le\sum_{i=1}^{p}\overline{K}\{\sum_{i=1}^{p}\int_0^T\sum_{i=1}^{p}\|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2dt+\sum_{i=1}^{p}\int_0^T\|\mathbf{f}_\mathbf{u}^i(t,\cdot)\|_{H^{-1}}^2dt\\
&\quad+\sum_{i=1}^{p}\int_{\Omega_i}|\mathbf{v}_0^i(\mathbf{x})|^2d\mathbf{x}+\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\mathbf{v}_0^i(\mathbf{x})|^2d\mathbf{x}\\
&\quad+\sum_{i=1}^{p}\int_{\Omega_i}|\nabla\mathbf{u}_0^i(\mathbf{x})|^2d\mathbf{x}+\sum_{i=1}^{p}\int_{\Omega_i}\operatorname{div}\mathbf{u}_0^i(\mathbf{x})^2d\mathbf{x}\\
&\quad+\int_0^T\int_{\partial_+\Omega_1}\big|\phi^0(t,\mathbf{s})\frac{\partial}{\partial\mathbf{n}}\phi^0(t,\mathbf{s})\big|\,ds\,dt\},
\end{aligned}\tag{4.14}$$

with $\overline{K}>0$ independent of T_0. Thus we can iterate the estimate on the intervals $[mT_0,(m+1)T_0]$ and the estimate (4.14) remains valid. So, no possible blow-up of the norms of $(\mathbf{u},\phi)$ on $(0,T)$ may arise if $T>0$ is any arbitrary number and part i) of Theorem 1 is completely proved (note that the resultant constant C in estimate (3.7) can be taken as $C=\overline{K}/[1-cT_0]^m$ if $T\in[mT_0,(m+1)T_0]$ for some natural number m and a suitable constant c such that $cT_0<1$.

5. Proof of Theorem 3.2 part (ii)

In contrast with the above set of estimates we can use now different arguments since we already know that there exists a solution of the iterative algorithm. We use now an idea which we can call as a "cancellation argument" (among the equations and boundary conditions) which essentially consists in differentiating the equation of ϕ^i with respect to t. If we neglect, for a while, the external data then we obtain

$$\frac{\partial}{\partial t}(-\Delta\phi^i)=4\pi\rho^i G\frac{\partial}{\partial t}\operatorname{div}\mathbf{u}^i=4\pi\rho^i G\operatorname{div}\mathbf{u}^i_t$$

and since the right hand side was controlled in the previous estimate we obtain that the left hand side is integrable in a suitable functional space. More precisely, by multiplying last expression by $\frac{\rho^i}{\mu^i}\phi^i$ and integrating over the space we obtain

$$\int_{\Omega_i}\frac{\rho^i}{\mu^i}\nabla\phi^i\cdot\nabla\phi^i_t\,d\mathbf{x}=\int_{\Omega_i}\frac{4\pi(\rho^i)^2}{\mu^i}G\operatorname{div}\mathbf{u}^i_t\,\phi^i\,d\mathbf{x}. \tag{5.1}$$

But the right hand side of (5.1) arises also when we multiply the equation of $\mathbf{u}_i$ by $\rho^i\mathbf{u}_{i,t}(t,\mathbf{x})$. Finally, if we apply such a process but now taking into account the contributions of the body forces f^i_ϕ and $\mathbf{f}^i_u$, the ones of the boundary data (when integrating by parts, specially on $\partial_+\Omega_1$), and the one of the initial data we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\sum_{i=1}^{p}\Big[\int_{\Omega_i}\Big(2\pi(\rho^i)^2G-\frac{T4\pi(\rho^i)^2Gg}{\mu^i}\Big)|\mathbf{u}^i_t|^2d\mathbf{x}\Big]\\
&+\int_0^T\int_{\Omega_i}4\pi\rho^iG\gamma^i|\nabla\mathbf{u}^i_t|^2\,d\mathbf{x}\,dt\\
&+\sup_{t\in[0,T]}\sum_{i=1}^{p}\Big[\int_{\Omega_i}\Big(2\pi\rho^iG-\frac{T2\pi(\rho^i)^2Gg}{\mu^i}\Big)|\nabla\mathbf{u}^i|^2d\mathbf{x}\\
&+\Big(\frac{2\pi\rho^iG}{1-2\nu^i}-\frac{T2\pi(\rho^i)^2Gg}{\mu^i}\Big)\int_{\Omega_i}(\operatorname{div}\mathbf{u}^i)^2\,d\mathbf{x}\Big]\\
&\le\sup_{t\in[0,T]}\sum_{i=1}^{p}\int_{\Omega_i}\frac{\rho^i}{2\mu^i}|\nabla\phi^i(t,\mathbf{x})|^2\,d\mathbf{x}\\
&\quad+\sum_{i=1}^{p}4\pi\rho^iG\rho^i\int_0^T\langle\mathbf{f}^i_u,\mathbf{u}^i_t\rangle dt+\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_0^T\langle\frac{\partial}{\partial t}(f^i_\phi),\phi^i\rangle dt\\
&\quad+\frac{4\pi(\rho^1)^2G}{\mu^1}\int_{\partial_+\Omega_1}|\phi_0(\mathbf{s})\mathbf{v}_0(\mathbf{s})\cdot\mathbf{n}|ds+\sum_{i=1}^{p}2\pi(\rho^i)^2G\int_{\Omega_i}|\mathbf{v}^i_0(\mathbf{x})|^2d\mathbf{x}\\
&\quad+\sum_{i=1}^{p}4\pi\rho^iG\gamma^i\int_{\Omega_i}|\nabla\mathbf{v}^i_0(\mathbf{x})|^2d\mathbf{x}+\sum_{i=1}^{p}2\pi\rho^iG\int_{\Omega_i}|\nabla\mathbf{u}^i_0(\mathbf{x})|^2d\mathbf{x}\\
&\quad+\sum_{i=1}^{p}\frac{2\pi\rho^iG}{1-2\nu^i}\int_{\Omega_i}\operatorname{div}\mathbf{u}^i_0(\mathbf{x})^2d\mathbf{x}+\frac{\rho^1}{\mu^1}\int_0^T\int_{\partial_+\Omega_1}\big|\phi^0(t,\mathbf{s})\frac{\partial^2}{\partial t\partial\mathbf{n}}\phi^0(t,\mathbf{s})\big|ds,
\end{aligned}$$

where we used that

$$\int_0^T \int_{\Omega_i} \frac{\rho^i}{\mu^i} \nabla\phi^i \cdot \nabla\phi_t^i = \frac{1}{2}\int_0^T \frac{d}{dt}\int_{\Omega_i} \frac{\rho^i}{\mu^i}|\nabla\phi^i|^2.$$

Similarly, for $t \in [0,T]$, using the Poincaré's constants $C(\Omega_i)$

$$\begin{aligned}
&\sum_{i=1}^{p}\int_{\Omega_i} |\nabla\phi^i(t,\mathbf{x})|^2\, d\mathbf{x} \\
&\leq \sum_{i=1}^{p} C(\Omega_i) 4\pi\rho^i G \int_{\Omega_i} |\nabla\mathbf{u}^i(t,\mathbf{x})|^2 d\mathbf{x} + \sum_{i=1}^{p} \|\mathbf{f}_\phi^i(t,\cdot)\|_{H^{-1}}^2 \\
&\quad + 2\int_{\partial_+\Omega_1} \Big|\phi^0(t,\mathbf{s})\frac{\partial}{\partial\mathbf{n}}\phi^0(t,\mathbf{s})\Big| ds.
\end{aligned}$$

Adding both inequalities and by taking a suitable constant K (depending increasingly on the Poincaré's constant $C(\Omega_i)$) we obtain the result.

6. Associated stationary system

The above arguments can be also applied to prove the uniqueness of the associated stationary problem

$$\begin{gathered}
\rho^i\mathbf{u}_{tt}^i - \gamma^i\Delta\mathbf{u}_t^i - \Delta\mathbf{u}^i - \frac{1}{1-2\nu^i}\nabla(\operatorname{div}\mathbf{u}^i) - \frac{\rho^i g}{\mu^i}\nabla(\mathbf{u}^i\cdot\mathbf{e}_z) + \frac{\rho^i g}{\mu^i}\mathbf{e}_z \operatorname{div}\mathbf{u}^i \\
= -\frac{\rho_i}{\mu_i}\nabla\phi^i + \mathbf{f}_u(\mathbf{x}) \quad \text{in } \Omega_i, \\
-\Delta\phi^i = 4\pi\rho^i G \operatorname{div}\mathbf{u}^i(t,\mathbf{x}) + f_\phi(\mathbf{x}) \quad \text{in } \Omega_i,
\end{gathered} \tag{6.1}$$

under the same type of boundary conditions (but now with data independent of t):

$$\begin{gathered}
\mathbf{u}^i = 0, \quad \phi^i = 0, \quad \text{on } \partial_l\Omega_i \ \forall i = 1,\dots,p, \\
\mathbf{u}^i = \mathbf{u}^{i+1}, \quad \frac{\partial\mathbf{u}^i}{\partial z} = \frac{\partial\mathbf{u}^{i+1}}{\partial z} \quad \text{on } \partial_-\Omega_i = \partial_+\Omega_{i+1},\ i = 1,\dots,p-1, \\
\phi^i = \phi^{i+1}, \quad \frac{\partial\phi^i}{\partial z} = \frac{\partial\phi^{i+1}}{\partial z} \quad \text{on } \partial_-\Omega_i = \partial_+\Omega_{i+1},\ i = 1,\dots,p-1, \\
\mathbf{u}^1 = 0, \quad \phi^1 = \phi_0^1(\mathbf{x}) \quad \text{on } \partial_+\Omega_1, \\
\mathbf{u}^p = 0, \quad \phi^p = 0 \quad \text{on } \partial_-\Omega_p.
\end{gathered} \tag{6.2}$$

Note that in [3] the sign of the term in $\nabla\phi$ was the opposite. Nevertheless the techniques used there and the present papers can be easily adapted to this stationary formulation for proving the following result.

Theorem 6.1. *Under the same spatial regularity assumption on the data in Theorem 1 there exists a unique weak solution of problem* (6.1), (6.2).

Proof. The existence part follows the same strategy than the proof of part i) of Theorem 3.2 and, more specifically, the proof of [3, Theorem 1]. Obviously, the uncoupled problem for ϕ is exactly the same and the uncoupled problem for $\mathbf{u}$ is treated in same manner (since, at this moment, $\nabla\phi$ is assumed to be prescribed). We recall that the coerciveness of the bilinear form $a_u : V_u \times V_u \to \mathbb{R}$ given by (4.7) requires either some assumption on the coefficients (see (69) [also denoted as assumption $H(\rho,\mu,\nu)$] of [3]) or the application of the "dilatant argument": we make the changes of spatial variables $\mathbf{y} = \lambda\mathbf{x}$, we remark that the constant in the Poincaré's inequality can be assumed to depend linearly on λ (since such a constant only depends on the diameter of Ω) and then, by introducing the change of unknown

$$\mathbf{v}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) = \mathbf{u}(\frac{\mathbf{y}}{\lambda})$$

we see that the new condition in terms of the new coefficients always holds if we take λ large enough.

Moreover, the a priori estimates used to justify the passing to the limit process, $n \to +\infty$, remains valid word by word since the estimate of the right hand side of the equation for $\mathbf{u}^n$ only uses the norm of the vector $\|\nabla\phi^{n-1}\|$ and thus its sign information is not relevant to this purpose. The application of [3, Lemma 1] (note the resemblances with the argument on $\delta < 1$ used in the proof of part (i) of Theorem 3.2 of the present paper) ends the proof of the existence of solutions.

Concerning the proof of the uniqueness of solutions of (6.1), (6.2) we use what we call as "cancellation method" but now after multiplying the equation of $\mathbf{u}^i$ by $4\pi\rho^i\mathbf{u}^i$ and the one of ϕ^i by $(\rho^i/\mu^i)\phi^i$ (as done in [3]). Then adding the resultant equations, after applying Green's formula, we obtain

$$\sum_{i=1}^{p} 4\pi G\rho^i a(\mathbf{u}^i, \mathbf{u}^i) - \sum_{i=1}^{p} \frac{\rho^i}{\mu^i} \int_{\Omega_i} |\nabla\phi^i(\mathbf{x})|^2 \, d\mathbf{x} = 0$$

(compare this with [3, equation (52)] in which the sign of the second term is the opposite one). Nevertheless, by using the standard estimate for elliptic equations (since $\operatorname{div}\mathbf{u}^i \in H^{-1}(\Omega_i)$), and Poincaré's inequality we arrive to the uniqueness conclusion if $C(\Omega_i)$ is small enough. More exactly, the conclusion holds if Ω is such that

$$4\pi G \sum_{i=1}^{p} \rho^i > \sum_{i=1}^{p} \frac{\rho^i C(\Omega_i) K}{\mu^i}. \tag{6.3}$$

Once again, we can apply the "dilatation argument" in the sense that if condition (6.3) does not hold then we make the dilatation $\mathbf{y} = \lambda\mathbf{x}$, the change of unknown $\mathbf{v}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) = \mathbf{u}(\frac{\mathbf{y}}{\lambda})$ and we see that the last term of the right hand side remains constant in λ but the third term of the left hand side appears multiplied by λ. Thus, the corresponding condition (similar to (6.3)) holds by taking λ large enough

$$\lambda > \frac{\sum_{i=1}^{p} \frac{\rho^i C(\Omega_i) K}{\mu^i}}{4\pi G \sum_{i=1}^{p} \rho^i}, \tag{6.4}$$

and thus the uniqueness in now proven without condition (6.3), which completes the proof. □

Discussion and conclusion

A rigorous well-posedness proof of the viscoelastic-gravitational model has been presented in this paper. The existence and uniqueness of solutions representing a layered Earth have been carried out. For that, some techniques of the weak solutions of partial differential equations theory have been applied. Moreover, we have given a constructive proof of the existence of the weak solutions. There is a clear geophysical need of this kind of models for interpretation of observed displacement and gravity changes at volcanic areas (see introduction and, e.g., [8, 19, 23]). In previous results obtained by other authors [19], introducing viscoelastic properties in all or part of the medium can extend the effects (displacements, gravity changes, etc.) considerably and therefore lower (and more realistic) pressure increases are required to model given observed effects. The viscoelastic effects seem to depend mainly on the rheological properties of the layer (zone) where the intrusion is located, rather than on the rheology of the whole medium. Those results should be confirmed with the described model. Additionally, normally purely elastic half-space models are used to interpret displacements and gravity data in active volcanic areas. Elastic-gravitational models allow the computation of gravity, deformation, and gravitational potential changes due to pressurized magma cavities and intruding masses together [11], taking into account the mass interaction with the self-gravitation of the Earth through coupling between model equations. In [11] a dimensional analysis of the elastic-gravitational model estimating the magnitude of intrusion mass and coupling effects at the space scale associated with volcano monitoring is performed. They show that the intrusion mass cannot be neglected in the interpretation of gravity changes while displacements are primarily caused by pressurization. Therefore the intrusion of mass, together with the associated pressurization of the magma chamber, produces distinctive changes in gravity that could be used to interpret gravity changes without ground deformation and vice versa, depending on what type of source plays the main role in the intrusion process. Their theoretical experiments indicate that mass and self-gravitation could produce changes in the magnitude and pattern of predicted gravity that may be above microgravity accuracy and the elastic-gravitational model is a refinement of purely elastic models which can better interpret gravity and deformation changes in active volcanic zones. Similar studies should be done for the viscoelastic-gravitational case described here checking the existence or not of effects as observed in the viscoelastic-gravitational problem for faulting (e.g., the introduction of a long-wavelength component into the time deformation and the need of a proper consideration of gravity for near-field computations and long time periods [17, 22]). The iterative scheme presented in this work can be useful to construct a numerical method to compute the coupled effects of gravity and viscoelastic deformations produced by possible sources embedded in an inhomogeneous Earth.

Acknowledgments. The authors thank to Professors J. Fernández and J. B. Rundle for different conversations on the modeling of the problem. The research of A. A. was partially supported by the National Research Fund of Luxembourg (AFR Grant 4832278). The research of J. I. D. was partially supported by the project ref. MTM2011- 26119 of the DGISPI (Spain) and the Research Group MOMAT (Ref. 910480) supported by UCM. He has received also support from the ITN FIRST of the Seventh Framework Program of the European Community (grant agreement number 238702).

References

[1] K. Aki, P. G. Richards; *Quantitative Seismology, University Science Books.* Sausalito, California, 2002.

[2] A. Arjona, J. I. Díaz; Stabilization of a Hyperbolic/Elliptic system modelling the viscoelastic-gravitational deformation in a multilayered earth. To appear in *Proceedings of The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications*, Madrid, Spain, 2014.

[3] A. Arjona, J. I. Díaz, J. Fernández, J. B. Rundle; On the Mathematical Analysis of an Elastic-gravitational Layered Earth Model for Magmatic Intrusion: The Stationary Case. *Pure Appl. Geophys.*, 165, 1465-1490, 2008.

[4] A. Arjona, T. Chacón, M. Gómez; Numerical modeling of volcanic ground deformation. *(Actas de XXII CEDYA (XI Congreso de Matemática Aplicada), Palma de Mallorca*, 2011.

[5] M. Battaglia, P. Segall; The interpretation of gravity changes and crustal deformation in active volcanic areas. *Pure Appl. Geophys.*, 161, 1453-1467.

[6] M. Battaglia, J. Gottsmann, D. Carbone, J. Fernández; 4D volcano gravimetry. *Geophysics*, 73, 6, WA3-WA18, 2008.

[7] M. Battaglia, D. P. Hill; Analytical modeling of gravity changes and crustal deformation at volcanoes: The Long Valley caldera, California, case study. *Tectonophysics*, 471. 45, 57. 2009.

[8] M. Bonafede, M. Mazzanti; Modelling gravity variations consistent with ground deformation in the Campi Flegrei caldera (Italy). *J. Volcanol. Geotherm. Res.*, 81, 137-157, 1998.

[9] H. Brézis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York, 2010.

[10] A. G. Camacho, P. J. González, J. Fernández, G. Berrino; Simultaneous inversion of surface deformation and gravity changes by means of extended bodies with a free geometry: Application to deforming calderas. *J. Geophys. Res.*, 116, B10401, doi: 10.1029/2010JB008165, 2011.

[11] M. Charco, J. Fernández, F. Luzón, J. B. Rundle; On the relative importance of self-gravitation and elasticity in modeling volcanic ground deformation and gravity changes. *Journal of Geophysical Research*, 111, B03404, doi: 10.1029/2005JB003754, 2006.

[12] M. Charco, P. Galán del Sastre; Efficient inversion of 3D Finite Element models of volcano deformation. *Geophys. J. Int.*, 196, 1441-1454, doi:10.1093/gji/ggt490, 2014.

[13] J. I. Díaz, R. Quintanilla; Spatial and continuous dependence estimates in linear viscoelasticity. *J. Math. Anal. And Applic.*, 273, 1-16, 2002.

[14] G. Duvaut, J. L. Lions; *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972.

[15] J. Fernández, J. B. Rundle; Gravity changes and deformation due to a magmatic intrusion in a two-layered crustal model. *J.Geophys.Res.*, 99, 2737-2746, 1994a.

[16] J. Fernández, J. B. Rundle; FORTRAN program to compute displacement, potential and gravity changes due to a magma intrusion in a multilayered Earth models. *Comput.Geosci.* , 23, 231-249, 1994b.

[17] J. Fernández, Ting-To Yu, J. B. Rundle; Horizontal viscoelastic-gravitational displacement due to a rectangular dipping thrust fault in layered Earth model. *J. Geophys.Res.*, 101, 6, 13.581-13.594. (correction: Journal of Geophysical Research, 103, B12, 30283-30286, 1996.

[18] J. Fernández, J. M. Carrasco, J. B. Rundle, V. Araña; Geodetic methods for detecting volcanic unrest: a theoretical approach. *Bull. Volcanol.* , 60, 534-544, 1999.

[19] J. Fernández, K. F. Tiampo, J. B. Rundle; Viscoelastic displacement and gravity changes due to point magmatic intrusions in a gravitational layered solid earth. *Geophys. J. Int.*, 146, 155-170, 2001a.

[20] J. Fernández, M. Charco, K. F. Tiampo, G. Jentzsch, J. B. Rundle; Joint interpretation of displacement and gravity data in volcanic areas. A test example: Long Valley Caldera, California. *J. Volcanol.Geotherm. Res.*, 28, 1063-1066, 2001b.

[21] J. Fernández, K. F. Tiampo, G. Jentzsch, M. Charco, J. B. Rundle; Inflaction or deflation?. New results for Mayon volcano applying elastic-gravitational modeling. *Geophys.Res.Lett., 28, 12, 2349-2352, 2001c.*

[22] J. Fernández, J. B. Rundle; Postseismic visoelastic-gravitational half space computations: Problems and solutions. *Geophys. Res. Lett.*, 31, 2004.

[23] A. Folch, J. Fernández, J. B. Rundle, J. Martí; Ground deformation in a viscoelastic medium composed of a layer overlying a half space: A comparison between point and extended sources. *Geophys.J.Int.*, 140, 37-50, 2000.

[24] Y. C. Fung; *Foundations of Solid Mechanics.* Prentice-Hall, Englewood Cliffs, N.J., 1965.

[25] D. Gilbarg, N. S. Trudinger; *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, 1977.

[26] D. F. McTigue; Elastic stress and deformation near a finite spherical magma body: Resolution of the point source paradox. *J. Geophys.Res.*, 92, 12931-12940, 1987.

[27] K. Mogi; Relations between the eruptions of various volcanoes and the deformation of the ground surfaces around them. *Bull.Earth.Res.*, 92, 12 931-12 940, 1958.

[28] Y. Okada; Surface deformation due to shear and tensile faults in a half-space. *Bull. Seism. Soc. Am.*, 75, 1135-1154, 1958.
[29] S. Okubo; Gravity and potential changes due to shear and tensile faults in a half-space. *J. Geophys.Res.*, 97, 7117-7144, 1992.
[30] F. F. Pollitz; Gravitational viscoelastic postseismic relaxation in a layered spherical earth. *J. Geophys.Res.*, 102, 17921-17941, 1997.
[31] J. B. Rundle; Static elastic-gravitational deformation of a layared half space by point couple sources. *J. Geophys.Res.*, 85, 5355-5363, 1980.
[32] J. B. Rundle; Numerical Evaluation of static elastic-gravitational deformation of a layared half space by point couple sources. *Rep., Sand*, 80-2048, 1981a.
[33] J. B. Rundle; Deformation gravity and potential changes due to volcanic loading of the crust. *J. Geophys.Res.*, 87, 10, 729-10, 744, 1981b.
[34] J. B. Rundle; Viscoeslastic-Gravitational Deformation by a Rectangular Thrust Fault in a Layered Earth. *J. Geophys.Res.*, 87, 9, 7787-7796, 1981b.
[35] J. B. Rundle; Correction to "Deformation, gravity and potential changes due to volcanic loading of the crust". *J. Geophys.Res.*, 88, 10, 647-10, 653, 1983.
[36] X. Yang, P. M. Davis, J. H. Dieterich; Deformation from inflation of a dipping finite prolate spheroid in an elastic half-space as a model for volcanic stressing. *J.Geophys.Res.*, 93, 4249-4257, 1998.
[37] T. T. Yu, J. B. Rundle, J. Fernández; Surface deformation due to strike-slip fault in an elastic gravitational layer overlying a viscoelastic gravitational half-space. *J. Geophys. Res.*, 101, 3199-3214. (Correction: Journal of Geophysical Research, 104, 15313-15316.), 1996.

Alicia Arjona
European Center for Geodynamics and Seismology, Rue Josy Welter, 19, L-7256 Walferdange, Gran-Duchy of Luxembourg
E-mail address: alicia.arj@gmail.com

Jesús Ildefonso Díaz
Departamento de Matemática Aplicada, Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Plaza de las Ciencias, 3, 28040 Madrid, Spain
E-mail address: jidiaz@ucm.es