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MATHEMATICS

# A Time-Dependent Strange Term Arising in Homogenization of an Elliptic Problem with Rapidly Alternating Neumann and Dynamic Boundary Conditions Specified at the Domain Boundary: The Critical Case

**J. I. Díaz[a,*], D. Gómez-Castro[a], T. A. Shaposhnikova[b,**], and M. N. Zubova[b]**

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**Abstract**—A strange term arising in the homogenization of elliptic (and parabolic) equations with dynamic boundary conditions given on some boundary parts of critical size is considered. A problem with dynamic boundary conditions given on the union of some boundary subsets of critical size arranged ε-periodically along the boundary and with homogeneous Neumann conditions given on the rest of the boundary is studied. It is proved that the homogenized boundary condition is a Robin-type containing a nonlocal term depending on the trace of the solution $u(x, t)$ on the boundary $\partial\Omega$.

**Keywords:** homogenization, rapidly oscillating boundary conditions, dynamic boundary conditions

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In this paper, we identify a strange term arising in the homogenization of some elliptic (and parabolic) equations with dynamic boundary conditions given on some boundary parts of critical size. The case of dynamic boundary conditions specified on the boundary of cavities of critical radius in an ε-periodically perforated domain is studied in [5]. Unlike in the case when dynamic boundary conditions are set on the boundaries of cavities of diameters of order ε [1, 7], a new nonlocal term $H_u$ appears as an absorption term in the homogenized elliptic (or parabolic) equation; in this case, $H_u$ is a solution of an ordinary differential equation depending on the solution $u(x,t)$ of the homogenized model. A problem with dynamic conditions specified on the boundaries of cavities located ε-periodically along an $(n-1)$-dimensional manifold was considered in [6]. In this case, a nonlocal strange term $H_u(x,t)$ appears in transmission conditions posed on this manifold. In the present paper, we consider the case of dynamic boundary conditions specified on the union of outer boundary subsets of critical size arranged ε-periodically along the boundary; homogeneous Neumann conditions are set on the rest of the boundary. The main goal of this work is to prove that the homogenized boundary condition is a Robin-type one containing a nonlocal term depending on the trace of the solution $u(x,t)$ on the boundary $\partial\Omega$. This result is a generalization of the main theorem in [3] to the case of dynamic boundary conditions.

We would like to underline that one of the most important steps in obtaining the strange term (this is how the new term in the homogenized equation was called by Cioranescu and Murat in [2]) is the correct choice of the parameter values characterizing the perforation radius and the correct scaling of the absorption coefficient in the boundary condition.

The results of this work admit many generalizations. For a more detailed presentation of them, as well as numerous other works on this topic, the reader is referred to the monograph of the authors [4].

Let $\Omega$ be a bounded domain in $\mathbb{R}^2 \cap \{x_2 > 0\}$ with a smooth boundary consisting of two parts: $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial\Omega \cap \{x_2 > 0\}$ and $\Gamma_2 = \partial\Omega \cap \{x_2 = 0\} = [-l, l]$ for some $l > 0$.

We introduce the notation $Y_1 = \left\{(y_1, 0): -\frac{1}{2} < y_1 < \frac{1}{2}\right\}$ and $\hat{l}_0 = \{(y_1, 0): -l_0 < y_1 < l_0\} \subset Y_1$, where $l_0 \in \left(0, \frac{1}{2}\right)$. For a small parameter $\varepsilon > 0$ and a parameter

[a] *Instituto de Matematica Interdisciplinar, Universidad Complutense, Madrid, 28040 Spain*
[b] *Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia*
**e-mail: ji_diaz@mat.ucm.es*
***e-mail: shaposh.tan@mail.ru*

$0 < a_\varepsilon \ll \varepsilon$ whose value is "critical," that is, $a_\varepsilon = C_0\varepsilon\exp\left(-\frac{\alpha^2}{\varepsilon}\right)$, $C_0, \alpha > 0$, we introduce the set

$$\tilde{G}_\varepsilon = \bigcup_{j\in\mathbb{Z}'}(a_\varepsilon\hat{l}_0 + \varepsilon j) = \bigcup_{j\in\mathbb{Z}'} l_\varepsilon^j,$$

where $\mathbb{Z}' = \mathbb{Z}\times\{0\}$ is the set of vectors of the form $j = (j_1, 0)$, $j_1 \in \mathbb{Z}$. We set $\Upsilon_\varepsilon = \{j \in \mathbb{Z}' : \overline{l_\varepsilon^j} \subset [-l + 2\varepsilon, l - 2\varepsilon] \times \{0\}\}$. We introduce the sets $Y_\varepsilon^j = \varepsilon Y_1 + \varepsilon j$, $j \in \mathbb{Z}'$, and

$$l_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} l_\varepsilon^j.$$

It is straightforward to see that $\overline{l_\varepsilon^j} \subset Y_\varepsilon^j$. We introduce the notation $\gamma_\varepsilon = \Gamma_2\backslash\overline{l_\varepsilon}$. Note that $|l_\varepsilon^j| = 2a_\varepsilon l_0$ for an arbitrary $j \in \mathbb{Z}'$ and $|l_\varepsilon| \cong da_\varepsilon\varepsilon^{-1}$, $d = \text{const} > 0$.

In the cylinder $Q_T = \Omega\times(0,T)$, we consider the problem

$$\begin{gathered}
-\Delta_x u_\varepsilon = f(x,t), \quad (x,t)\in Q_T,\\
\beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \lambda\beta(\varepsilon)u_\varepsilon = \beta(\varepsilon)g(x,t),\\
(x,t)\in l_\varepsilon\times(0,T),\\
\partial_\nu u_\varepsilon = 0, \quad (x,t)\in\gamma_\varepsilon\times(0,T),\\
u_\varepsilon(x,t) = 0, \quad (x,t)\in\Gamma_1\times(0,T),\\
u_\varepsilon(x,0) = 0, \quad x\in l_\varepsilon,
\end{gathered} \tag{1}$$

where the coefficient $\beta(\varepsilon)$ has the critical value, that is, $\beta(\varepsilon) = \exp\left(\frac{\alpha^2}{\varepsilon}\right)$, $\lambda \ge 0$, $f \in C([0,T]; L^2(\Omega))$, $\partial_t f \in L^2(0,T; L^2(\Omega))$, and $g(x,t) \in C^1(\overline{\Omega}\times[0,T])$.

The weak solution of the initial boundary value problem (1) is a function $u_\varepsilon \in L^2(0,T; H^1(\Omega,\Gamma_1))$ that satisfies $\partial_t u_\varepsilon \in L^2(0,T; H^{-1/2}(l_\varepsilon,\Gamma_1))$ and the integral identity

$$\begin{aligned}
&\beta(\varepsilon)\int_0^T\langle\partial_t u_\varepsilon, v\rangle_{l_\varepsilon}dt + \int_0^T\int_\Omega\nabla u_\varepsilon\nabla v\,dxdt\\
&\quad + \lambda\beta(\varepsilon)\int_0^T\int_{l_\varepsilon} u_\varepsilon v\,dx_1dt\\
&= \beta(\varepsilon)\int_0^T\int_{l_\varepsilon} gv\,dx_1dt + \int_0^T\int_\Omega fv\,dxdt,
\end{aligned} \tag{2}$$

for an arbitrary function $v \in L^2(0,T; H^1(\Omega,\Gamma_1))$, where $\langle .,.\rangle_{l_\varepsilon}$ is the duality relation between the spaces $H^{1/2}(l_\varepsilon, \Gamma_1)$ and $H^{-1/2}(l_\varepsilon,\Gamma_1)$. As usual, $H^1(\Omega,\Gamma_1)$ denotes the closure in $H^1(\Omega)$ of the set of infinitely differentiable functions in $\overline{\Omega}$ that vanish in a neighborhood of $\Gamma_1$.

Using Galerkin approximations, we deduce the following theorem.

**Theorem 1.** *Problem* (1) *has a unique weak solution* $u_\varepsilon$, *which satisfies the estimate*

$$\begin{aligned}
&\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega,\Gamma_1))} + \beta(\varepsilon)\|u_\varepsilon\|^2_{L^2(0,T;L^2(l_\varepsilon))}\\
&\quad + \beta(\varepsilon)\|\partial_t u_\varepsilon\|^2_{L^2(0,T;L^2(l_\varepsilon))} \le K;
\end{aligned} \tag{3}$$

*here and below, K is a positive constant independent of* $\varepsilon$.

It follows from (3) that there is a subsequence (denoted in the same way as the original sequence) such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad L^2(0,T; H^1(\Omega,\Gamma_1)), \tag{4}$$

$$u_\varepsilon \rightharpoonup u \quad \text{strongly} \quad \text{in} \quad L^2(0,T; L^2(\Omega)) \tag{5}$$

as $\varepsilon \to 0$.

The main results is stated in the following assertion.

**Theorem 2.** *Let* $u_\varepsilon$ *be a weak solution of problem* (1). *Then the function* $u \in L^2(0,T; H^1(\Omega,\Gamma_1))$ *defined by* (4) *and* (5) *is a weak solution of the nonlocal boundary value problem*

$$\begin{gathered}
-\Delta_x u = f(x,t), \quad (x,t)\in\Omega\times(0,T),\\
\partial_\nu u + \frac{\pi}{\alpha^2}u = H_u(x,t), \quad (x,t)\in\Gamma_2\times(0,T),\\
u(x,t) = 0, \quad (x,t)\in\Gamma_1\times(0,T),
\end{gathered} \tag{6}$$

*where* $H_u(x,t)$ *is a unique solution of the following Cauchy problem for the ordinary differential equation*:

$$\begin{gathered}
\frac{\partial H_u}{\partial t} + \left(\lambda + \frac{\pi}{2\alpha^2 l_0 C_0}\right)H_u = g(x,t) + \frac{\pi}{2\alpha^2 l_0 C_0}u(x,t),\\
(x,t)\in\Gamma_2\times(0,T),\\
H(x,0) = 0, \quad x\in\Gamma_2.
\end{gathered} \tag{7}$$

**Remark 1.** For $u \in L^2(0,T; L^2(\Gamma_2))$, the solution of the Cauchy problem has the form

$$\begin{aligned}
H_u(x,t) = \frac{\pi}{\alpha^2}\int_0^t&\left(g(x,\tau) + \frac{\pi}{2\alpha^2 l_0 C_0}u(x,\tau)\right)\\
&\times\exp\left(-\left(\lambda + \frac{\pi}{2\alpha^2 l_0 C_0}\right)(t-\tau)\right)d\tau.
\end{aligned} \tag{8}$$

**Proof**. We introduce auxiliary functions $w_\varepsilon^j$ and $q_\varepsilon^j$ as weak solutions of the boundary value problems

$$\begin{gathered}
\Delta w_\varepsilon^j = 0, \quad x\in T_{\varepsilon/4}^j\backslash\overline{T_{a_\varepsilon}^j},\\
w_\varepsilon^j = 1, \quad x\in\partial T_{a_\varepsilon}^j,\\
w_\varepsilon^j = 0, \quad x\in\partial T_{\varepsilon/4}^j;
\end{gathered} \tag{9}$$

$$\begin{aligned} \Delta q_\varepsilon^j &= 0, \quad x \in T_{\varepsilon/4}^j \backslash \overline{l_\varepsilon^j}, \\ q_\varepsilon^j &= 1, \quad x \in l_\varepsilon^j, \\ q_\varepsilon^j &= 0, \quad x \in \partial T_{\varepsilon/4}^j. \end{aligned} \tag{10}$$

Here, $T_r^j$ is a ball of radius $r$ centered at $(\varepsilon j, 0)$. Note that $w_\varepsilon^j$ and $q_\varepsilon^j$ are solutions of the problems

$$\begin{aligned} \Delta w_\varepsilon^j &= 0, \quad x \in (T_{\varepsilon/4}^j)^+ \backslash \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j &= 0, \quad x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ w_\varepsilon^j &= 1, \quad x \in \partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} w_\varepsilon^j &= 0, \quad x \in \{x_2 = 0\} \cap (T_{\varepsilon/4}^j \backslash \overline{T_{a_\varepsilon}^j}), \end{aligned} \tag{11}$$

where $A^+ = A \cap \{x_2 > 0\}$, $A \subset \mathbb{R}^2$, and

$$\begin{aligned} \Delta q_\varepsilon^j &= 0, \quad x \in (T_{\varepsilon/4}^j)^+, \\ q_\varepsilon^j &= 1, \quad x \in l_\varepsilon^j, \\ q_\varepsilon^j &= 0, \quad x \in (\partial T_{\varepsilon/4}^j)^+, \\ \partial_{x_2} q_\varepsilon^j &= 0, \quad x \in (T_{\varepsilon/4}^j \cap \{x_2 = 0\}) \backslash \overline{l_\varepsilon^j}, \end{aligned} \tag{12}$$

where $j \in \Upsilon_\varepsilon$ and $l_\varepsilon^j = a_\varepsilon \hat{l}_0 + \varepsilon j$. Note that

$$w_\varepsilon^j(x) = \frac{\ln\left(\frac{4r}{\varepsilon}\right)}{\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}. \tag{13}$$

We introduce the functions

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+ \backslash \overline{(T_{a_\varepsilon}^j)^+}, \quad j \in \Upsilon_\varepsilon, \\ 1, & x \in (T_{a_\varepsilon}^j)^+, \quad j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \backslash \overline{\cup_{j \in \Upsilon_\varepsilon} (T_{\varepsilon/4}^j)^+}, \end{cases} \tag{14}$$

$$Q_\varepsilon(x) = \begin{cases} q_\varepsilon^j, & x \in (T_{\varepsilon/4}^j)^+, \quad j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \backslash \overline{\cup_{j \in \Upsilon_\varepsilon} (T_{\varepsilon/4}^j)^+}. \end{cases} \tag{15}$$

It is straightforward to see that $W_\varepsilon, Q_\varepsilon \in H^1(\Omega, \Gamma_1)$ and $W_\varepsilon \rightharpoonup 0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$. To compare these two functions, we will use the following lemma proved in [3].

**Lemma 1.** *Let $W_\varepsilon$ be the function defined by* (14), *and let $Q_\varepsilon$ be the function defined by* (15). *Then*

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \le K\sqrt{\varepsilon}. \tag{16}$$

It follows from (2) that for arbitrary test function $v \in L^2(0, T; H^1(\Omega, \Gamma_1))$ such that $\partial_t v \in L^2(0, T; L^2(l_\varepsilon))$ we have the integral inequality

$$\begin{aligned} &\beta(\varepsilon) \int_0^T \int_{l_\varepsilon} \partial_t v (v - u_\varepsilon) dx_1 dt + \int_0^T \int_\Omega \nabla v \nabla (v - u_\varepsilon) dx dt \\ &+ \beta(\varepsilon)\lambda \int_0^T \int_{l_\varepsilon} v(v - u_\varepsilon) dx_1 dt \ge \beta(\varepsilon) \int_0^T \int_{l_\varepsilon} g(v - u_\varepsilon) dx_1 dt \\ &+ \int_0^T \int_\Omega f(v - u_\varepsilon) dx dt - \frac{\beta(\varepsilon)}{2} \|v(x, 0)\|_{L^2(l_\varepsilon)}^2. \end{aligned} \tag{17}$$

For arbitrary functions $\eta \in C^1[0, T]$ and $\psi \in C^\infty(\overline{\Omega})$ that vanish in a neighborhood of the boundary $\Gamma_1$, we introduce the function $\phi(x, t) = \eta(t)\psi(x)$. Let $H_\phi(x, t)$ be a solution of the Cauchy problem

$$\begin{aligned} \frac{\partial H_\phi}{\partial t} + \left(\lambda + \frac{\pi}{2\alpha^2 l_0 C_0}\right) H_\phi &= g(x, t) + \frac{\pi}{2\alpha^2 l_0 C_0} \phi(x, t), \\ H_\phi(x, 0) &= 0. \end{aligned} \tag{18}$$

We set $H_{\varepsilon, j}(t) = H_\phi(P_\varepsilon^j, t)$. For $t \in (0, T)$, we define the function

$$Q_\phi^\varepsilon = \begin{cases} q_\varepsilon^j(x)(\phi(P_\varepsilon^j, t) - H_{\varepsilon, j}(t)), & x \in (T_{\varepsilon/4}^j)^+, \quad j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \backslash \overline{\cup_{j \in \Upsilon_\varepsilon} (T_{\varepsilon/4}^j)^+}. \end{cases} \tag{19}$$

In view of estimate (16), we obtain

$$Q_\phi^\varepsilon \rightharpoonup 0 \quad \text{weakly} \quad \text{in} \quad H^1(\Omega, \Gamma_1). \tag{20}$$

As a test function in (17), we take $v = \phi(x, t) - Q_\phi^\varepsilon$. Then

$$\begin{aligned} &\beta(\varepsilon) \int_0^T \int_{l_\varepsilon} (\partial_t \phi - \partial_t Q_\phi^\varepsilon)(\phi - Q_\phi^\varepsilon - u_\varepsilon)\, dx_1 dt + \\ &+ \int_0^T \int_\Omega \nabla(\phi - Q_\phi^\varepsilon) \nabla(\phi - Q_\phi^\varepsilon - u_\varepsilon)\, dx dt + \\ &+ \lambda\beta(\varepsilon) \int_0^T \int_{l_\varepsilon} (\phi - Q_\phi^\varepsilon)(\phi - Q_\phi^\varepsilon - u_\varepsilon) dx_1 dt \\ &\ge \beta(\varepsilon) \int_0^T \int_{l_\varepsilon} g(\phi - Q_\phi^\varepsilon - u_\varepsilon) dx_1 dt \\ &+ \int_0^T \int_\Omega f(\phi - Q_\phi^\varepsilon - u_\varepsilon) dx dt \\ &- \frac{\beta(\varepsilon)}{2} \sum_{j \in \Upsilon_\varepsilon} \|\phi(x, 0) - \phi(P_\varepsilon^j, 0)\|_{L^2(l_\varepsilon^j)}^2. \end{aligned} \tag{21}$$

Note that $\partial_t Q_\phi^\varepsilon(x, t)\big|_{l_\varepsilon^j} = \partial_t \phi(P_\varepsilon^j, t) - (H_{\varepsilon, j}(t))'$ if $x \in l_\varepsilon^j$ and $t \in [0, T]$. Therefore,

$$\begin{aligned} &\beta(\varepsilon) \int_0^T \int_{l_\varepsilon} (\partial_t \phi - \partial_t Q_\phi^\varepsilon)(\phi - Q_\phi^\varepsilon - u_\varepsilon) dx_1 dt \\ &= \beta(\varepsilon) \sum_{j \in \Upsilon_\varepsilon} \int_0^T \int_{l_\varepsilon} (\partial_t \phi(x, t) - \partial_t \phi(P_\varepsilon^j, t))(\phi - Q_\phi^\varepsilon - u_\varepsilon) dx_1 dt \end{aligned}$$

$$+\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{l_\varepsilon^j}\left(\frac{d}{dt}H_{\varepsilon,j}\right)(\phi-Q_\phi^\varepsilon-u_\varepsilon)dx_1dt. \quad (22)$$

It is easy to see that

$$\lim_{\varepsilon\to0}\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{l_\varepsilon}(\partial_t\phi(x,t)-\partial_t\phi(P_\varepsilon^j,t)) \times(\phi-Q_\phi^\varepsilon-u_\varepsilon)dx_1dt=0. \quad (23)$$

Introducing the notation

$$I_\varepsilon\equiv\int_0^T\int_\Omega\nabla(\phi-Q_\phi^\varepsilon)\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt,$$

we have

$$I_\varepsilon=\int_0^T\int_\Omega\nabla(\phi-W_\phi^\varepsilon)\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt +\int_0^T\int_\Omega\nabla(W_\phi^\varepsilon-Q_\phi^\varepsilon)\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt,$$

where, for $t\in[0,T]$,

$$W_\phi^\varepsilon(x,t)=\begin{cases}w_\varepsilon^j(x)(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t)), & x\in(T_{\varepsilon/4}^j)^+,\quad j\in\Upsilon_\varepsilon,\\ 0,\quad x\in\Omega\setminus\cup_{j\in\Upsilon_\varepsilon}\overline{(T_{\varepsilon/4}^j)^+}.\end{cases}$$

Using Lemma 1, we deduce that

$$\lim_{\varepsilon\to0}\int_0^T\int_\Omega\nabla(W_\phi^\varepsilon-Q_\phi^\varepsilon)\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt=0. \quad (24)$$

In addition, we derive that

$$\lim_{\varepsilon\to0}\int_0^T\int_\Omega\nabla\phi\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt =\int_0^T\int_\Omega\nabla\phi\nabla(\phi-u)dxdt. \quad (25)$$

Taking into account the definition of $W_\phi^\varepsilon$ yields

$$-\int_0^T\int_\Omega\nabla W_\phi^\varepsilon\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt$$

$$=-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(T_{\varepsilon/4}^j)^+\setminus\overline{(T_{a_\varepsilon}^j)^+}}\nabla w_\varepsilon^j \times\nabla[(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t))(\phi-Q_\phi^\varepsilon-u_\varepsilon)]dxdt$$

$$=-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T_{\varepsilon/4}^j)^+}\partial_\nu w_\varepsilon^j(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t))(\phi-u_\varepsilon)dsdt$$

$$-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T_{a_\varepsilon}^j)^+}\partial_\nu w_\varepsilon^j(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t))(\phi-Q_\phi^\varepsilon-u_\varepsilon)dsdt.$$

Since $w_\varepsilon^j$ is a solution of problem (9), we infer the relations

$$\partial_\nu w_\varepsilon^j|_{(\partial T_{\varepsilon/4}^j)^+}=\frac{4}{-\alpha^2+\varepsilon\ln(4C_0)},$$
$$\partial_\nu w_\varepsilon^j|_{(\partial T_{a_\varepsilon}^j)^+}=\frac{\exp(\alpha^2/\varepsilon)}{C_0\alpha^2-C_0\varepsilon\ln(4C_0)}. \quad (26)$$

It follows that

$$-\int_0^T\int_\Omega\nabla W_\phi^\varepsilon\nabla(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt$$

$$=\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T_{\varepsilon/4}^j)^+}(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t))(\phi-u_\varepsilon)dsdt$$

$$-\frac{\beta(\varepsilon)}{\alpha^2C_0}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T_{a_\varepsilon}^j)^+}(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t)) \times(\phi-Q_\phi^\varepsilon-u_\varepsilon)dsdt+\alpha_\varepsilon, \quad (27)$$

where $\alpha_\varepsilon\to0$ as $\varepsilon\to0$.

Relations (20)–(27) yield the inequality

$$\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{l_\varepsilon}(H'_{\varepsilon,t}+\lambda H_{\varepsilon,j}-g(P_\varepsilon^j,t))(\phi-Q_\phi^\varepsilon-u_\varepsilon)dx_1dt$$

$$-\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T_{a_\varepsilon}^j)^+}\frac{1}{C_0\alpha^2}(\phi(P_\varepsilon^j,t) -H_{\varepsilon,j}(t))(\phi-u_\varepsilon)dsdt+\alpha_\varepsilon$$

$$+\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T_{\varepsilon/4}^j)^+}(\phi(P_\varepsilon^j,t)-H_{\varepsilon,j}(t))(\phi-u_\varepsilon)dsdt$$

$$+\int_0^T\int_\Omega\nabla\phi\nabla(\phi-u_\varepsilon)dxdt\ge\int_0^T\int_\Omega f(\phi-Q_\phi^\varepsilon-u_\varepsilon)dxdt, \quad (28)$$

$\alpha_\varepsilon\to0$ as $\varepsilon\to0$.

Here, we used the fact that

$$\lim_{\varepsilon\to0}\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\left\|\phi(x,0)-\phi(P_\varepsilon^j,0)\right\|^2_{L^2(l_\varepsilon^j)}=0.$$

In what follows, we will need the following lemma of [3].

**Lemma 2.** *Let* $h \in H^1(\Omega, \Gamma_1)$. *Then*

$$\left|\frac{\beta(\varepsilon)\pi}{2l_0}\int_{l_\varepsilon} h dx_1 - \beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_{(\partial T^j_{a_\varepsilon})^+} h ds\right| \le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega,\Gamma_1)}. \tag{29}$$

Estimate (29) implies that, as $\varepsilon \to 0$,

$$\left|\frac{\beta(\varepsilon)}{\alpha^2 C_0}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T^j_{a_\varepsilon})^+}(\phi(P^j_\varepsilon,t) - H_{\varepsilon,j}(t))(\phi - Q^\varepsilon_\phi - u_\varepsilon)dsdt - \frac{\beta(\varepsilon)\pi}{2\alpha^2 C_0 l_0}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{l_\varepsilon}(\phi(P^j_\varepsilon,t) - H_{\varepsilon,j}(t)) \times(\phi - Q^\varepsilon_\phi - u_\varepsilon)dx_1dt\right| \to 0. \tag{30}$$

In view of (28)–(30), we derive the inequality

$$\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{l^j_\varepsilon}\left(\frac{dH_{\varepsilon,j}}{dt} + \lambda H_{\varepsilon,j} - g(P^j_\varepsilon,t) - \frac{\pi}{2\alpha^2 C_0 l_0}\phi(P^j_\varepsilon,t) + \frac{\pi}{2\alpha^2 C_0 l_0}H_{\varepsilon,j}\right)(\phi - Q^\varepsilon_\phi - u_\varepsilon)dx_1dt + \int_0^T\int_\Omega \nabla\phi\nabla(\phi - u)dxdt + \frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T^j_{\varepsilon/4})^+}(\phi(P^j_\varepsilon,t) - H_{\varepsilon,j}(t))(\phi - u_\varepsilon)dsdt \ge \int_0^T\int_\Omega f(\phi - Q^\varepsilon_\phi - u_\varepsilon)dxdt + \tilde{\alpha}_\varepsilon, \tag{31}$$

where $\tilde{\alpha}_\varepsilon \to 0$ as $\varepsilon \to 0$.

Taking into account that $H_{\varepsilon,j}(t)$ is a solution of Cauchy problem (18), we deduce that the first sum on the right-hand side of inequality (31) is zero.

The assertion proved in [3, 6] implies that

$$\lim_{\varepsilon\to 0}\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{(\partial T^j_{\varepsilon/4})^+}(\phi(P^j_\varepsilon,t) - H_{\varepsilon,j}(t))(\phi - u_\varepsilon)dsdt = \frac{\pi}{\alpha^2}\int_0^T\int_{\Gamma_2}(\phi(x,t) - H_\phi(x,t))(\phi - u)dx_1dt, \tag{32}$$

where the function $H_\phi(x,t)$ is defined by (8).

It follows from (31) and (32) that $u$ satisfies the integral inequality

$$\int_0^T\int_\Omega \nabla\phi\nabla(\phi - u)dxdt + \frac{\pi}{\alpha^2}\int_0^T\int_{\Gamma_2}(\phi - H_\phi)(\phi - u)dx_1dt \ge \int_0^T\int_\Omega f(\phi - u)dxdt. \tag{33}$$

Since $u$ satisfies inequality (33), we conclude that $u$ is a solution of the integral identity

$$\int_0^T\int_\Omega \nabla u\nabla v dxdt + \frac{\pi}{\alpha^2}\int_0^T\int_{\Gamma_2}(u - H_u)v dx_1dt = \int_0^T\int_\Omega f v dxdt,$$

where $v(x,t) = \eta(t)\psi(x)$, $\eta \in C^1[0,T]$, $\psi \in C^\infty(\bar{\Omega})$, and $\psi$ is zero on some neighborhood of the boundary $\Gamma_1$. In view of the fact that the set of functions of this type is everywhere dense in $L^2(0,T;H^1(\Omega,\Gamma_1))$, we infer that $u$ is a weak solution of problem (6), (7).

We will prove the uniqueness of the weak solution of problem (6), (7). Let $u_1$ and $u_2$ be two weak solutions of problem (6), (7). Then $w = u_1 - u_2$ is a weak solution of the problem

$$\Delta_x w = 0, \quad (x,t) \in \Omega\times(0,T),$$
$$\partial_\nu w + \frac{\pi}{\alpha^2}w = H_w(x,t), \quad (x,t) \in \Gamma_2\times(0,T),$$
$$w(x,t) = 0, \quad (x,t) \in \Gamma_1\times(0,T),$$
$$\frac{\partial H_w}{\partial t} + \left(\lambda + \frac{\pi}{2\alpha^2 l_0 C_0}\right)H_w = \frac{\pi}{2\alpha^2 l_0 C_0}w, \quad (x,t) \in \Gamma_2\times(0,T),$$
$$H_w(x,0) = 0, \quad x \in \Gamma_2.$$

Since $H_w$ is a solution of Cauchy problem (7), it satisfies the estimate

$$\int_0^t\int_{\Gamma_2}|H_w(x,\tau)|^2 dx_1d\tau \le Ct\int_0^t\int_{\Gamma_2}|w(x,\tau)|^2 dx_1d\tau.$$

The integral identity for $w$ yields

$$\int_0^t\int_\Omega|\nabla w|^2 dxd\tau + \frac{\pi}{\alpha^2}\int_0^t\int_{\Gamma_2}w^2 dx_1d\tau = \int_0^t\int_{\Gamma_2}wH_w dx_1d\tau \le C_1\sqrt{t}\int_0^t\int_{\Gamma_2}w^2 dx_1d\tau \le C_2\sqrt{t}\int_0^t\int_\Omega|\nabla w|^2 dxd\tau.$$

For $C_2^2 t < 1$, we immediately obtain $w \equiv 0$. By applying iterations with respect to time, we conclude that $w = 0$ a.e. in $\Omega\times(0,T)$.

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