On the convergence of controls and cost functionals in some optimal control heterogeneous problems when the homogenization process gives rise to some strange terms

J.I. Díaz, A. V. Podolskiy and T.A. Shaposhnikova

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Abstract. We consider the convergence of solutions and cost functional in some optimal control problems arising in the study of the adsorption chemical phenomenon in which some microscopic reactant particles are placed over an internal manifold γ of the chemical reactor Ω. The chemical reaction is given by some Robin-type boundary condition on the boundary of the periodic set of particles. We consider the special case in which there is a critical relation between the coefficient of the reaction, the size of the particles and the dimension of the space. This gives rise to a "strange term", which is not occurring for other scales, and thus the limit cost functional must be suitably defined. In a last section, we use this type of technique to prove a similar "energy convergence" result (improving the $H_0^1(\Omega)-$weak convergence) for the problem without control for the critical scale case.

Keywords: optimal control heterogeneous problems, homogenization, perforated internal manifold, critical case, strange term

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1 Introduction

It is well-known that the extension $P_\varepsilon u_\varepsilon$ of the solution u_ε of some homogenization problems, given by a second order equation with some Robin-type boundary condition on the boundary of a set of periodic particles, merely converges in the weak topology of $H_0^1(\Omega)$, to the solution $u_0 \in H_0^1(\Omega)$ of associated homogenized problem (see, e.g. the exposition made in the monograph [2]). This fact creates some natural difficulties for the treatment of the convergence of some control problems in which a cost functional $J_\varepsilon(v)$ must be minimized. In order to know in which sense the homogenized problem is optimized, by taking a family of controls v_ε, we must show, not only the convergence of the controls v_ε to a macroscopic control function v_0, but also the convergence of the cost functional sequence $J_\varepsilon(v)$ to some global cost functional $J_0(v)$. This type of question is especially interesting in the case in which the "microscopic" cost function $J_\varepsilon(v)$ depends on the gradient of the "microscopic" states u_ε and when the scale of the particles is critical and some "strange term" arises in the homogenized equation ([6], [1] and [2]).

Among the possible formulations in which the above problem can be considered, our interest in this paper will be concentrated on the case in which the set of particles (or equivalently, of perforations) are placed along an internal manifold γ. This type of problem arises very often in many applied contexts, for instance in adsorption processes in chemical engineering in which the reactant medium is located merely on some kind of grill (or perforated surface); see, e.g. the presentation on the modeling made in [5] and [7].

The spatial domain is given by

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},$$

and we distinguish the different parts of the boundary by means of the notation

$$\partial\Omega_\varepsilon = \partial\Omega \bigcup S_\varepsilon, \ S_\varepsilon = \partial G_\varepsilon.$$

Let us indicate now the structure of the periodic distribution and size of the particles. We assume that Ω is a bounded domain in R^n, $n \geq 3$, with smooth boundary $\partial\Omega$, $\gamma = \Omega \cap \{x_1 = 0\}$ is an $(n-1)$ – dimensional domain in the plane $x_1 = 0$, $Y = (-1/2, 1/2)^n$ and G_0 is the unit ball $\{|x| < 1\}$. We set $\delta B = \{x : \delta^{-1}x \in B\}$, $\delta > 0$. We denote by Z' the set of n - dimensional vectors of the form $z = (0, z_2, \ldots, z_n)$, $z_j \in \mathbb{Z}$, $j = 2, \ldots, n$. Let ε be a small positive parameter. We set $a_\varepsilon = C_0\varepsilon^k$, and assume that k takes the critical value given in (1) below, with $C_0 > 0$ a given constant. We define the set $G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j$, where $G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j$, $j \in Z'$, $\Upsilon_\varepsilon = \{j \in Z' : \overline{G_\varepsilon^j} \subset \Omega$ and $\rho(\overline{G_\varepsilon^j}, \partial\Omega) \geq 2\varepsilon\}$, so that we have the estimate on the cardinality $|\Upsilon_\varepsilon| \cong d\varepsilon^{1-n}$, for some constant $d > 0$. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, $P_\varepsilon^j = \varepsilon j$, $j \in \Upsilon_\varepsilon$. Note that $\overline{G_\varepsilon^j} \subset Y_\varepsilon^j$ and the center of the cube Y_ε^j coincides with the center of the ball $G_\varepsilon^j = \varepsilon G_0 + \varepsilon j$. For a generic set $A \subset \mathbb{R}^n$, $A^+ = \{(x_1, \cdots, x_n) \in A : x_1 > 0\}$ (and similarly for A^-) and moreover, $A^0 = \{(x_1, \cdots, x_n) \in A : x_1 = 0\}$. So, $\gamma = \Omega \cap \{x_1 = 0\} = \Omega^0$. We will use the space

$$H^1(\Omega_\varepsilon, \partial\Omega) = \overline{\{u \in \mathcal{C}^\infty(\Omega_\varepsilon) : u \text{ vanishes on a neighborhood of } \partial\Omega\}}^{H^1(\Omega_\varepsilon)}.$$

The starting formulation of the optimal control problem is the following: For a given control $v \in L^2(\Omega_\varepsilon)$, and data $f \in L^2(\Omega)$, $a \in C^\infty(\overline{\Omega})$, $a(x) \geq a_0 = const > 0$ and under the crucial assumption

$$k = \frac{n-1}{n-2}, \tag{1}$$

we denote by $u_\varepsilon(v) \in H^1(\Omega_\varepsilon, \partial\Omega)$ to the *state* associated to this control as the unique weak solution of the problem

$$\begin{cases} -\Delta u_\varepsilon(v) = f + v, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-k} a(x) u_\varepsilon(v) = 0, & x \in S_\varepsilon, \\ u_\varepsilon(v) = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where ν is the unit outward normal vector to S_ε. We consider the cost functional $J_\varepsilon : L^2(\Omega_\varepsilon) \to \mathbb{R}$, given by

$$J_\varepsilon(v) = \frac{\eta}{2} \|\nabla u_\varepsilon(v) - \nabla u_T\|^2_{L^2(\Omega_\varepsilon)} + \frac{N}{2} \|v\|^2_{L^2(\Omega_\varepsilon)}, \tag{3}$$

where

$$u_T \in H^1_0(\Omega) \tag{4}$$

is a given *target* function and η, N are given positive parameters. It is well known (see, e.g., [8]) that there exist a unique *optimal control* $v_\varepsilon \in L^2(\Omega_\varepsilon)$ such that

$$J_\varepsilon(v_\varepsilon) = \inf_{v \in L^2(\Omega_\varepsilon)} J_\varepsilon(v). \tag{5}$$

A first goal of this paper is to study the limit, as $\varepsilon \to 0$, of the optimal control v_ε and of the limit *value* of the cost functional $J_\varepsilon(v_\varepsilon)$. We point out that when the parameter η is large enough we get the approximate controllability property in $H^1(\Omega_\varepsilon, \partial\Omega)$ (in the sense that the associated state $u_\varepsilon(v)$ is as close as we want to the *target function,* $\|\nabla u_\varepsilon(v) - \nabla u_T\|^2_{L^2(\Omega_\varepsilon)} \leq \delta$, for any $\delta > 0$ arbitrarily small: see, e.g. [3], Section 1.6).

Since the exponent k is critical we know (see, e.g. [14] and [2]) that in the absence of controls ($v = 0$) and if $P_\varepsilon u_\varepsilon$ is the extension of u_ε on $\Omega \setminus \overline{\Omega}_\varepsilon$, such that $P_\varepsilon u_\varepsilon \in H^1_0(\Omega)$ then $P_\varepsilon u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$, as $\varepsilon \to 0$, where $u_0 \in H^1_0(\Omega)$ is the weak solution of the problem involving some transmission conditions over the internal manifold γ :

$$\begin{cases} -\Delta u_0 = f & x \in \Omega^+ \cup \Omega^-, \\ \left[u_0\right] = 0, & x \in \gamma, \\ \left[\partial_{x_1} u_0\right] = A_n H_n(x) u_0 & x \in \gamma, \\ u_0 = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

where $H_n(x) = \frac{a(x)}{a(x)+C_n}$, $A_n = (n-2)C_0^{n-2}\omega_n$, $C_n = \frac{n-2}{C_0}$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Notice that we are using the following notation for the jump of a general function v across γ:

$$[v]_\gamma(x) = \lim_{h \to 0^+} \left(v(x + he_1) - v(x - he_1)\right),$$

where e_1 is the first element of the basis of $\mathbb{R}^n$. Notice also that now the notion of weak solution of (6) is given in the following terms (see, e.g. Section 5.1 of [2]): for any

$\psi \in C_0^\infty(\Omega)$

$$\int_\Omega \nabla u_0 \nabla \psi dx + A_n \int_\gamma H_n(\widehat{x}) u_0 \psi d\widehat{x} = \int_\Omega f \psi dx.$$

We point out that the assumption (1) is the main reason why the value of the function $H_n(x)$ is unexpected, corresponding to what in other similar frameworks is denoted as a "strange term"(see, e.g. [6], [1] and the monograph [2]). Some strong convergence results are also possible under additional assumptions (see, e.g., [14] and Section 4.7.1.4 of [2]) but the strong convergence needs to be stated with the help of certain auxiliary functions.

As a matter of fact, we will show the convergence of the extension of the optimal controls $\widetilde{v_\varepsilon} \rightharpoonup v_0$, weakly in $H_0^1(\Omega)$, where v_0 is the associated optimal control for the homogenized problem (where the right hand side f must be replaced by $f + v_0$) and v_0 is optimal in the sense that

$$J_0(v_0) = \inf_{v \in L^2(\Omega)} J_0(v)$$

where now in the cost function J_0 gives rise to a new term on γ:

$$J_0(v) = \frac{\eta}{2} \int_\Omega |\nabla u(v) - \nabla u_T|^2 dx + \frac{\eta A_n}{2} \int_\gamma H_n^2(\widehat{x}) u^2(v) d\widehat{x} + \frac{N}{2} \int_\Omega v^2 dx. \tag{7}$$

Notice that the target function u_T may correspond, for instance, to the case in which there is a desired distribution of the chemical products having some special transmission over the grill. The optimization of the transmission profile is possible thanks to the assumption $u_T \in H_0^1(\Omega)$ and the fact that the difference between u_0 and u_T is estimated in the norm of $H_0^1(\Omega)$. As a matter of fact, we will prove that

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0). \tag{8}$$

In a final Section, we will use this type of technique to prove a similar convergence result (improving the $H_0^1(\Omega)-$weak convergence) for the problem without control (2 [with $v = 0$ and $u_T \equiv 0$]) for the critical scale case (1). We will prove (see Theorem 2 below) that if $u_0 \in H_0^1(\Omega)$ is the weak limit of the extension $P_\varepsilon u_\varepsilon$ satisfying (6) then we have the "energy convergence"

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx \to \int_\Omega |\nabla u_0(x)|^2 dx + A_n \int_\gamma \left(\frac{a(\widehat{x})}{a(\widehat{x}) + C_n} \right)^2 u_0(\widehat{x})^2 d\widehat{x}. \tag{9}$$

As mentioned before, stronger convergence results to the mere weak convergence $P_\varepsilon u_\varepsilon \rightharpoonup u_0$, weakly in $H_0^1(\Omega)$ usually requires an additional assumption on the data in order to know that u_ε satisfies some additional regularity properties (see, e.g. the exposition made in Section 4.7.1.4 of [2] and its references).

We point out that this approach (building some artificial complementary system to get the energy convergence) can be applied to other homogenization problems (see, e.g., [10] and its references).

The structure of this paper is the following: Section 2 is devoted to the consideration of the above-mentioned control problem, while Section 3 contains the proof of the convergence result in absence of any control argument.

2 The optimal control problem

Although the existence of a unique optimal control $v_\varepsilon \in L^2(\Omega_\varepsilon)$ satisfying (5) is today a standard matter, the associate optimality conditions are not usually mentioned in the literature since, quite often, the cost functional is stated in terms of $\|u_\varepsilon(v) - u_T\|^2_{L^2(\Omega_\varepsilon)}$ and not in terms of the gradient of the difference. The following result shows a possible particularization of the abstract version of the Pontryagin maximum principle applied to elliptic PDEs mentioned in Section 1.3 of Lions [8].

Proposition 1. *Assume (4) and let $v_\varepsilon \in L^2(\Omega_\varepsilon)$ and $u_\varepsilon(v_\varepsilon) \in H^1(\Omega_\varepsilon, \partial\Omega)$ be the optimal control and the associate optimal state. Let $p_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ be the unique solution of the problem*

$$\begin{cases} -\Delta p_\varepsilon = -\Delta u_\varepsilon(v_\varepsilon) + \Delta u_T, & x \in \Omega_\varepsilon, \\ \partial_\nu(p_\varepsilon - u_\varepsilon(v_\varepsilon) + u_T) + \varepsilon^{-k} a(x) p_\varepsilon = 0, & x \in S_\varepsilon, \\ p_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{10}$$

Then the optimal control is given by

$$v_\varepsilon = -\frac{\eta}{N} p_\varepsilon. \tag{11}$$

Proof. Since v_ε is the optimal control we know that for any other control $v \in L^2(\Omega_\varepsilon)$

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)) = 0.$$

It is easy to see that if, for a given $\lambda \in \mathbb{R}$, we define

$$w_\varepsilon(v) = \frac{1}{\lambda}(u_\varepsilon(v_\varepsilon + \lambda v) - u_\varepsilon(v_\varepsilon))$$

then

$$|\nabla u_\varepsilon(v_\varepsilon + \lambda v) - \nabla u_T|^2 - |\nabla u_\varepsilon(v_\varepsilon) - \nabla u_T|^2 = 2\lambda(\nabla w_\varepsilon(v), \nabla u_\varepsilon(v_\varepsilon) - \nabla u_T) + o(\lambda),\ \lambda \to 0.$$

In consequence, we have

$$0 = \lim_{\lambda \to 0} \frac{1}{\lambda}(J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)) = \eta \int_{\Omega_\varepsilon} \nabla w_\varepsilon(v)(\nabla u_\varepsilon(v_\varepsilon) - \nabla u_T) dx + N \int_{\Omega_\varepsilon} v_\varepsilon v dx. \tag{12}$$

On the other hand, $w_\varepsilon = \frac{1}{\lambda}(u_\varepsilon(v_\varepsilon + \lambda v) - u_\varepsilon(v_\varepsilon)) \in H^1(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of the problem:

$$\begin{cases} -\Delta w_\varepsilon = v, & x \in \Omega_\varepsilon, \\ \partial_\nu w_\varepsilon + \varepsilon^{-k} a(x) w_\varepsilon = 0, & x \in S_\varepsilon, \\ w_\varepsilon = 0, & x \in \partial\Omega, \end{cases}$$

so that, for any test function $\psi \in H^1(\Omega_\varepsilon, \partial\Omega)$ we get

$$\int_{\Omega_\varepsilon} \nabla w_\varepsilon \nabla \psi dx + \varepsilon^{-k} \int_{S_\varepsilon} a(x) w_\varepsilon \psi ds = \int_{\Omega_\varepsilon} v\psi dx.$$

Then, if $p_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is the unique solution of (10) we know that for any test function $\phi \in H^1(\Omega_\varepsilon, \partial\Omega)$ we have

$$\int_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla \phi dx + \varepsilon^{-k} \int_{S_\varepsilon} a(x) p_\varepsilon \phi ds = \int_{\Omega_\varepsilon} (\nabla u_\varepsilon(v) - \nabla u_T) \nabla \phi dx.$$

Then, by taking $\psi = p_\varepsilon$ and $\phi = w_\varepsilon(v)$ we get that (12) is equivalent to the condition

$$0 = \int_{\Omega_\varepsilon} (\eta p_\varepsilon + N v_\varepsilon) v dx = 0 \ \forall v \in L^2(\Omega_\varepsilon).$$

So, we conclude that $v_\varepsilon = -\eta N^{-1} p_\varepsilon$.■

Concerning the homogenization (as $\varepsilon \to 0$) we will use the usual continuous extension operator $P_\varepsilon : H^1(\Omega_\varepsilon, \partial\Omega) \to H_0^1(\Omega)$ (see, e.g. Section 3.1.1 of [2] and its references).

Theorem 1. *Assume (4) and let $f \in L^2(\Omega)$ and $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon, \partial\Omega)^2$ be the weak solution of the coupled system*

$$\begin{cases} -\Delta u_\varepsilon = f - \eta N^{-1} p_\varepsilon & x \in \Omega_\varepsilon, \\ -\Delta p_\varepsilon = -\Delta u_\varepsilon + \Delta u_T & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-k} a(x) u_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu p_\varepsilon + \varepsilon^{-k} a(x) p_\varepsilon = \partial_\nu u_\varepsilon(v_\varepsilon) - \partial_\nu u_T, , & x \in S_\varepsilon, \\ u_\varepsilon = p_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{13}$$

Let $P_\varepsilon u_\varepsilon$ and $P_\varepsilon p_\varepsilon$ be the extensions of the functions u_ε and P_ε on $\Omega \setminus \overline{\Omega}_\varepsilon$, such that $P_\varepsilon u_\varepsilon, P_\varepsilon p_\varepsilon \in H_0^1(\Omega)$. Then

$$\|P_\varepsilon u_\varepsilon\|_{H_0^1(\Omega)} \le K \|u_\varepsilon\|_{H^1(\Omega_\varepsilon, \partial\Omega)}, \ \|\nabla P_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \le K \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{14}$$

$$\|P_\varepsilon p_\varepsilon\|_{H_0^1(\Omega)} \le K \|p_\varepsilon\|_{H^1(\Omega_\varepsilon, \partial\Omega)}, \ \|\nabla P_\varepsilon p_\varepsilon\|_{L^2(\Omega)} \le K \|\nabla p_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{15}$$

where the constant K here and below is independent of ε. Then as $\varepsilon \to 0$ we have

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0, \ \textit{weakly in } H_0^1(\Omega), \tag{16}$$

$$P_\varepsilon p_\varepsilon \rightharpoonup p_0, \ \textit{weakly in } H_0^1(\Omega), \tag{17}$$

for some $(u_0, p_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfying, in a weak sense, the system*

$$\begin{cases} -\Delta u_0 = f - \eta N^{-1} p_0 & x \in \Omega^+ \cup \Omega^-, \\ -\Delta p_0 = -\Delta u_0 + \Delta u_T & x \in \Omega^+ \cup \Omega^-, \\ \left[u_0\right] = \left[p_0\right] = 0, & x \in \gamma, \\ \left[\partial_{x_1} u_0\right] = A_n H_n(x) u_0 & x \in \gamma, \\ \left[\partial_{x_1}(p_0 - u_0 + u_T)\right] = A_n H_n(x)(p_0 - H_n(x) u_0), & x \in \gamma, \\ u_0 = p_0 = 0, & x \in \partial\Omega. \end{cases} \tag{18}$$

where $H_n(x) = \frac{a(x)}{a(x)+C_n}$, $A_n = (n-2)C_0^{n-2}\omega_n$, $C_n = \frac{n-2}{C_0}$ *with* ω_n *the surface area of the unit sphere in* $\mathbb{R}^n$.

On the other hand, if we consider the optimal control of the optimization problem

$$J_0(v_0) = \inf_{v \in L^2(\Omega)} J_0(v),$$

for the functional $J_0(v)$ given by (7), where, for a given control $v \in L^2(\Omega)$ the function $u(v)$ is the weak solution of the problem

$$\begin{cases} -\Delta u(v) = f + v, & x \in \Omega^+ \cup \Omega^-, \\ \left[u(v)\right] = 0, \ \left[\partial_{x_1} u(v)\right] = A_n H_n(x) u(v), & x \in \gamma, \\ u(v) = 0, & x \in \partial\Omega, \end{cases} \tag{19}$$

Then we can prove the following result.

Proposition 2. *Under the above assumptions, the optimal control* $v_0 \in L^2(\Omega)$ *is given by* $v_0 = -\eta N^{-1} p_0$. *Moreover, we have (8), i.e.* $\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0)$.

2.1 Proof of the homogenization theorem

Before to start with the proof we point out the following convergence lemma already proved in [9] (see also [4]):

Lemma 1. *Let* P_ε^j *be the center of the ball* G_ε^j *and let* $T_{\varepsilon/4}^j$ *denote the ball of radius* $\varepsilon/4$ *with center* P_ε^j, $j \in \Upsilon_\varepsilon$. *Then, there exists a constant* $K > 0$ *such that*

$$\Big| \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} w ds - 2^{2-2n} \omega_n \int_\gamma w d\hat{x} \Big| \leq K\sqrt{\varepsilon} \|w\|_{H^1(\Omega)}, \ w \in H_0^1(\Omega). \tag{20}$$

The proof of the homogenization theorem will be divided in several steps.

2.1.1 Step1. Uniform in ε estimates of u_ε and p_ε

Let us prove the following a priori estimate

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon,\partial\Omega)} \le K,\ \|p_\varepsilon\|_{H^1(\Omega_\varepsilon,\partial\Omega)} \le K,$$

for a generic constant $K > 0$. Taking p_ε as test function in the variational formulation of problem (10) and applying Cauchy-Schwartz we obtain

$$\begin{aligned} \|\nabla p_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-k} \int_{S_\varepsilon} a(x) p_\varepsilon^2 ds &= \int_{\Omega_\varepsilon} \nabla p_\varepsilon (\nabla u_\varepsilon - \nabla u_T) dx \\ &\le \|\nabla p_\varepsilon\|_{L^2(\Omega_\varepsilon)} (\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_T\|_{L^2(\Omega_\varepsilon)}). \end{aligned} \tag{21}$$

Taking into account the variational form of the problem on $u_\varepsilon(v_\varepsilon)$ we have

$$\int_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla u_\varepsilon dx + \varepsilon^{-k} \int_{S_\varepsilon} a(x) u_\varepsilon p_\varepsilon ds = \int_{\Omega_\varepsilon} (f - \eta N^{-1} p_\varepsilon) p_\varepsilon dx \tag{22}$$

and using the function $u_\varepsilon(v_\varepsilon)$ as a test function in (10) we derive

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla p_\varepsilon dx + \varepsilon^{-k} \int_{S_\varepsilon} a(x) u_\varepsilon p_\varepsilon ds = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx - \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla u_T dx. \tag{23}$$

This, together with equation (22) and Hölder's inequality leads to

$$\begin{aligned} \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &= \int_{\Omega_\varepsilon} (f p_\varepsilon - \eta N^{-1} p_\varepsilon^2) dx + \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla u_T dx \le \\ &\le \int_{\Omega_\varepsilon} |f||p_\varepsilon| dx + \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla u_T\|^2_{L^2(\Omega_\varepsilon)} \end{aligned} \tag{24}$$

Hence, we get

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K(\int_{\Omega_\varepsilon} |f||p_\varepsilon| dx + \|\nabla u_T\|^2_{L^2(\Omega_\varepsilon)}), \tag{25}$$

for some $K > 0$. From (21), (25) we get

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla p_\varepsilon|^2 dx &\le C(\int_{\Omega_\varepsilon} |f||p_\varepsilon| dx + \|\nabla u_T\|^2_{L^2(\Omega_\varepsilon)}) \le \delta \int_{\Omega_\varepsilon} p_\varepsilon^2 dx + C_\delta \int_{\Omega_\varepsilon} f^2 dx + 2\|\nabla u_T\|^2_{L^2(\Omega_\varepsilon)} \le \\ &\le K\delta \int_{\Omega_\varepsilon} |\nabla p_\varepsilon|^2 dx + C_\delta \int_{\Omega_\varepsilon} f^2 dx + 2\|\nabla u_T\|^2_{L^2(\Omega_\varepsilon)}, \end{aligned}$$

where $\delta > 0$ is an arbitrary positive number, for some constant C, $C_\delta > 0$. Summarizing, we have derived the estimates

$$\|\nabla p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le K, \quad \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le K, \tag{26}$$

for some generic constant $K > 0$. From (25), (26) we conclude

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon,\partial\Omega)} \le K. \tag{27}$$

It follows from (26), (27) that the extensions $P_\varepsilon u_\varepsilon$, $P_\varepsilon p_\varepsilon$ of the functions $u_\varepsilon, p_\varepsilon$ to the entire domain Ω satisfy the estimates (14), (15).
Thus, from estimates (26) and (27) it follows that there is a subsequence (still denoted by $P_\varepsilon u_\varepsilon$ and $P_\varepsilon p_\varepsilon$) such that, as $\varepsilon \to 0$,

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega) \text{ and } P_\varepsilon u_\varepsilon \to u_0 \text{ strongly in } L^2(\Omega), \tag{28}$$

$$P_\varepsilon p_\varepsilon \rightharpoonup p_0 \text{ weakly in } H_0^1(\Omega) \text{ and } P_\varepsilon p_\varepsilon \to p_0 \text{ strongly in } L^2(\Omega). \tag{29}$$

2.1.2 Identification of the limit problem for u_0

Let us show that u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u_0 = f - \eta N^{-1} p_0, & x \in \Omega^+ \cup \Omega^-, \\ \big[u_0\big] = 0, \; \big[\partial_{x_1} u_0\big] = A_n H_n(x) u_0, & x \in \gamma, \\ u_0(x) = 0, & x \in \partial\Omega. \end{cases} \tag{30}$$

According to (16) and (17), we only need to find the limit as $\varepsilon \to 0$ of the following term in the variational form for the problem (2)

$$\varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) u_\varepsilon \phi ds, \; \forall \phi \in C_0^\infty(\Omega).$$

As in Section 5.7 of [2], we introduce the term

$$\mathcal{H}_\varepsilon \equiv \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (W_\varepsilon \phi) dx,$$

where

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \; j \in \Upsilon_\varepsilon, \\ 1, & x \in G^j_\varepsilon, \; j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \cup_{j \in \Upsilon_\varepsilon} \overline{T^j_{\varepsilon/4}}, \end{cases}$$

$T^j_{\varepsilon/4}$ is the ball of radius $\varepsilon/4$ with the center in the point P^j_ε, and with w^j_ε the solution to the cell problem

$$\begin{cases} \Delta w^j_\varepsilon = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ w^j_\varepsilon = 1, & x \in \partial G^j_\varepsilon, \\ w^j_\varepsilon = 0, & x \in \partial T^j_{\varepsilon/4}. \end{cases} \tag{31}$$

Since $W_\varepsilon \phi$ is a good test function for the condition of weak solution satisfied by u_ε we have

$$\mathcal{H}_\varepsilon = -\varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) u_\varepsilon \phi ds + \int\limits_{\Omega_\varepsilon} f \phi W_\varepsilon dx - \eta N^{-1} \int\limits_{\Omega_\varepsilon} p_\varepsilon W_\varepsilon \phi dx.$$

Taking into account that $W_\varepsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$, we get

$$\mathcal{H}_\varepsilon = -\varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) u_\varepsilon \phi ds + \kappa_\varepsilon, \; \kappa_\varepsilon \to 0, \; \varepsilon \to 0. \tag{32}$$

On the other hand we have

$$\mathcal{H}_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (u_\varepsilon \phi) dx + \kappa_{1,\varepsilon}, \;\; \kappa_{1,\varepsilon} \to 0, \, \varepsilon \to 0.$$

From the definition of W_ε, we derive

$$\mathcal{H}_\varepsilon = \varepsilon^{-k} C_n \int\limits_{S_\varepsilon} u_\varepsilon \phi ds - (n-2) C_0^{n-2} 2^{2n-2} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} u_\varepsilon \phi ds + m_\varepsilon, \tag{33}$$

where $m_\varepsilon \to 0$, as $\varepsilon \to 0$. Comparing expressions (32) and (33) we obtain

$$\varepsilon^{-k} \int\limits_{S_\varepsilon} (a(x) + C_n) u_\varepsilon \phi ds = (n-2) C_0^{n-2} 2^{2n-2} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} u_\varepsilon \phi ds + \tilde{\kappa}_\varepsilon, \tag{34}$$

where $\tilde{\kappa}_\varepsilon \to 0$, $\varepsilon \to 0$.

We set $\phi = \frac{a(x)}{a(x)+C_n}\psi(x)$ in (34) as a test function, where ψ is an arbitrary function from $C_0^\infty(\Omega)$. Then, passing to the limit we get

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) u_\varepsilon \psi ds &= (n-2) C_0^{n-2} \lim_{\varepsilon \to 0} 2^{2n-2} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \frac{a(x)}{a(x)+C_n} u_\varepsilon \psi ds \\ &= A_n \int\limits_\gamma H_n(x) u_0 \psi d\hat{x}. \end{aligned} \tag{35}$$

Note that the last equality follows from the convergence lemma 1. Therefore, from (35), the limit function u_0 satisfies the variational formulation

$$\int\limits_\Omega \nabla u_0 \nabla \psi dx + A_n \int\limits_\gamma H_n(x) u_0 \psi d\hat{x} = \int\limits_\Omega (f - \eta N^{-1} P_0) \psi dx, \;\; \forall \psi \in C_0^\infty(\Omega)$$

and thus, u_0 is the weak solution of (30).

2.1.3 Identification of the limit problem for p_0

Let us find the equation satisfied by p_0. Define

$$\mathcal{I}_\varepsilon \equiv \int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla (\phi W_\varepsilon) dx, \;\; \text{where } \phi \in C_0^\infty(\Omega).$$

From the variational formulation for p_ε it follows that

$$\mathcal{I}_\varepsilon + \varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) p_\varepsilon \phi ds = \int\limits_{\Omega_\varepsilon} \nabla (u_\varepsilon - u_T) \nabla (\phi W_\varepsilon) dx. \tag{36}$$

Taking into account the variational formulation for u_ε we obtain

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (\phi W_\varepsilon) dx = -\varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) u_\varepsilon \phi ds + \int\limits_{\Omega_\varepsilon} (f - \eta N^{-1} p_\varepsilon) \phi W_\varepsilon dx.$$

From (35) and properties of the function W_ε we derive

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\phi W_\varepsilon)dx = -A_n \int\limits_{\gamma} H_n(x) u_0 \phi d\hat{x}. \tag{37}$$

Thus, from (36) we deduce

$$\mathcal{I}_\varepsilon = -\varepsilon^{-k}\int\limits_{S_\varepsilon} a(x) p_\varepsilon \phi ds - A_n \int\limits_{\gamma} H_n(x) u_0 \phi d\hat{x} + \hat{\kappa}_\varepsilon, \tag{38}$$

where $\hat{\kappa}_\varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand, using that w_ε^j is a weak solution to the problem (31), we get

$$\begin{gathered}\mathcal{I}_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla(p_\varepsilon \phi)dx + \alpha_\varepsilon \\ = \varepsilon^{-k} C_n \int\limits_{S_\varepsilon} p_\varepsilon \phi ds - (n-2)C_0^{n-2} 2^{2n-2} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} p_\varepsilon \phi ds + \beta_\varepsilon,\end{gathered} \tag{39}$$

where $\alpha_\varepsilon, \beta_\varepsilon \to 0$ as $\varepsilon \to 0$. Comparing (38) and (39) we derive

$$\begin{gathered}\varepsilon^{-k}\int\limits_{S_\varepsilon}(a(x)+C_n)p_\varepsilon \phi ds \\ = -A_n \int\limits_{\gamma} H_n(x) u_0 \phi d\hat{x} + (n-2)C_0^{n-2}2^{2n-2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} p_\varepsilon \phi ds + \widetilde{\alpha}_\varepsilon,\end{gathered} \tag{40}$$

where $\widetilde{\alpha}_\varepsilon \to 0$ as $\varepsilon \to 0$. Setting $\phi = H_n(x)\psi$ in (40), where ψ is an arbitrary function from $C_0^\infty(\Omega)$, passing to the limit as $\varepsilon \to 0$ and applying Lemma 1, we get

$$\lim_{\varepsilon\to 0}\varepsilon^{-k}\int\limits_{S_\varepsilon} a(x) p_\varepsilon \psi ds = -A_n \int\limits_{\gamma} H_n^2(x) u_0 \psi d\hat{x} + A_n \int\limits_{\gamma} H_n(x) p_0 \psi d\hat{x}. \tag{41}$$

Consequently, using (41) we get that p_0 satisfies the following identity

$$\int\limits_{\Omega} \nabla(p_0 - u_0 + u_T)\nabla\psi dx + A_n \int\limits_{\gamma} H_n(x) p_0 \psi d\hat{x} = A_n \int\limits_{\gamma} H_n^2(x) u_0 \psi d\hat{x}, \tag{42}$$

for $\forall\psi \in C_0^\infty(\Omega)$. Hence, p_0 is a weak solution of the problem

$$\begin{gathered}-\Delta p_0 = -\Delta u_0 + \Delta u_T,\ x \in \Omega^+ \cup \Omega^-, \\ \left[P_0\right] = 0,\ \left[\partial_{x_1}(p_0 - u_0 + u_T)\right] = A_n H_n(x)(p_0 - H_n(x)u_0),\ x \in \gamma, \\ p_0 = 0,\ x \in \partial\Omega.\end{gathered}$$

Thus, Theorem 1 is proved.∎

2.2 Proof of Proposition 2

The proof that the optimal control $v_0 \in L^2(\Omega)$ is given by $v_0 = -\eta N^{-1} p_0$ is entirely similar to the proof of Proposition 1. So, let us show the convergence result (8). For the function $v_\varepsilon = -\eta N^{-1} p_\varepsilon$, we have

$$J_\varepsilon(-\eta N^{-1} p_\varepsilon) = \frac{\eta}{2} \int\limits_{\Omega_\varepsilon} |\nabla(u_\varepsilon - u_T)|^2 dx + \frac{\eta^2}{2N} \int\limits_{\Omega_\varepsilon} p_\varepsilon^2 dx.$$

From Theorem 1 we already know that

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_T)\nabla u_T dx = \frac{1}{2} \int\limits_{\Omega} \nabla(u_0 - u_T)\nabla u_T dx.$$

On the other hand, from (22), (23) and Theorem 1, it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_T)\nabla u_\varepsilon dx = \lim_{\varepsilon\to 0} \frac{1}{2} \int\limits_{\Omega_\varepsilon} (f - \eta N^{-1} p_\varepsilon) p_\varepsilon dx =$$

$$= \frac{1}{2} \int\limits_{\Omega} (f - \eta N^{-1} p_0) p_0 dx = \frac{1}{2} \int\limits_{\Omega} (-\Delta u_0) p_0 dx.$$

Notice that, by an abuse in the notation we are identifying

$$\langle -\Delta u_0, p_0 \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \text{ with } \int\limits_{\Omega} (-\Delta u_0) p_0 dx.$$

By taking p_0 as test function in the variational formulation of the equation of u_0, and by taking u_0 as test function in the variational formulation of the equation of p_0, in (18), we get the cancellation of the term $A_n \int\limits_{\gamma} H_n(\widehat{x}) p_0 u_0 d\widehat{x}$, and thus

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_T)\nabla u_\varepsilon dx = \frac{1}{2} \int\limits_{\Omega} \nabla(u_0 - u_T)\nabla u_0 dx + \frac{A_n}{2} \int\limits_{\gamma} H_n^2(x) u_0^2 dx.$$

Finally, using this expression, we derive

$$\lim_{\varepsilon\to 0} J_\varepsilon(-\eta N^{-1} P_\varepsilon) = \lim_{\varepsilon\to 0} \frac{\eta}{2} \Big(\int\limits_{\Omega_\varepsilon} |\nabla(u_\varepsilon - u_T)|^2 dx + \frac{\eta}{N} \int\limits_{\Omega_\varepsilon} p_\varepsilon^2 dx \Big) =$$

$$= \frac{\eta}{2} \int\limits_{\Omega} |\nabla(u_0 - u_T)|^2 dx + \frac{\eta A_n}{2} \int\limits_{\gamma} H_n^2(\widehat{x}) u_0^2 d\widehat{x} + \frac{\eta^2}{2N} \int\limits_{\Omega} p_0^2 dx = J_0(-\eta N^{-1} p_0). \blacksquare$$

Remark 1. *It seems possible to generalize the above results to some variants of the above optimal control problem: by including some transport terms and possible different diffusion coefficients on* Ω^- *and* Ω^+ *(see a formulation in [5]), by modifying the cost functional including other gradient expressions which are not necessarily the ones given by the diffusion coefficients (in the style of the paper [13] for the case of Dirichlet boundary conditions on the internal boundary* S_ε*), etc.*

3 Stronger convergence for the problem without control

In this last Section we will prove the energy convergence (by including in the limit energy a part of the strange term of the homogenized problem) by introducing the artificial complementary system (10) which corresponds formally to the case $v \equiv 0$ and $u_T \equiv 0$.

Theorem 2. *Let u_ε be the solution of (2) with $v \equiv 0$ at the critical scale (1). Let $u_0 \in H_0^1(\Omega)$ be the weak limit of the extension $P_\varepsilon u_\varepsilon$. Then we have the convergence*

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \to \int\limits_{\Omega} |\nabla u_0|^2 dx + A_n \int\limits_{\gamma} \Big(\frac{a(\widehat{x})}{a(\widehat{x}) + C_n}\Big)^2 u_0^2(\widehat{x}) d\widehat{x}. \tag{43}$$

Proof. It is known that u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega^- \cup \Omega^+, \\ \Big[u_0\Big] = 0, & x \in \gamma, \\ \Big[\partial_{x_1} u_0\Big] = A_n H_n(x) u_0, & x \in \gamma, \\ u_0(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $H_n(x) = \frac{a(x)}{a(x)+C_n}$, $A_n = (n-2)C_0^{n-2}\omega_n$, $C_n = \frac{n-2}{C_0}$.
Let us introduce the weak solution p_ε of the problem

$$\begin{cases} \Delta p_\varepsilon = \Delta u_\varepsilon & x \in \Omega_\varepsilon, \\ \partial_\nu p_\varepsilon - \partial_\nu u_\varepsilon + \varepsilon^{-k} a(x) p_\varepsilon = 0 & x \in S_\varepsilon, \\ p_\varepsilon = 0 & x \in \partial\Omega. \end{cases}$$

As in the proof of Theorem 1 we get that $P_\varepsilon p_\varepsilon \rightharpoonup p_0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$ with p_0 the weak solution of the problem

$$\begin{cases} -\Delta p_0 = -\Delta u_0 & x \in \Omega^+ \cup \Omega^-, \\ \big[p_0\big] = 0, & x \in \gamma, \\ \Big[\partial_{x_1} p_0\Big] = A_n H_n(x)(p_0 - H_n(x) u_0) + \Big[\partial_{x_1} u_0\Big], & x \in \gamma, \\ p_0 = 0, & x \in \partial\Omega. \end{cases}$$

From the variational formulation to the problem (2), with $v = 0$, we have

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla p_\varepsilon dx + \varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) u_\varepsilon p_\varepsilon ds = \int\limits_{\Omega_\varepsilon} f p_\varepsilon dx.$$

Similarly from the variational formulation to the problem on p_ε we derive

$$\int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla u_\varepsilon dx + \varepsilon^{-k} \int\limits_{S_\varepsilon} a(x) p_\varepsilon u_\varepsilon ds = \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.$$

Thus we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} f p_\varepsilon dx \to \int_{\Omega} f p_0 dx =$$

$$= \int_{\Omega} \nabla u_0 \nabla p_0 dx + A_n \int_{\gamma} H_n(\widehat{x}) u_0 p_0 d\widehat{x} =$$

$$= \int_{\Omega} |\nabla u_0|^2 dx + A_n \int_{\gamma} H_n^2(\widehat{x}) u_0^2 d\widehat{x},$$

which ends the proof. ■

Remark 2. *It is well known (see, e.g. Section 4.7.1.4 of [2] and the references indicated there) that if we know that $u_0 \in W^{1,\infty}(\Omega)$ then we can get some results implying the strong convergence of u_ε plus a suitable correction. Notice that the conclusion presented in this paper* (9)*, follows completely different arguments.*

Remark 3. *In the case of Dirichlet boundary conditions some similar kind of convergence was established already in the pioneering work [1]and in some of their multiple generalizations (see, e.g. [13] and its references). But it always was stated in terms of a "strange term" given by a measure μ. The interest of the convergence (9) is that there is no measure at all but a strange term given by a completely identified function (in our case given by the explicit expression $A_n H_n(x) u_0$ on γ).*

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