ON THE HOMOGENIZATION OF AN OPTIMAL CONTROL PROBLEM IN A DOMAIN PERFORATED BY HOLES OF CRITICAL SIZE AND ARBITRARY SHAPE

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The paper studies the asymptotic behavior of the optimal control for the Poisson type boundary value problem in a domain perforated by holes of an arbitrary shape with Robin-type boundary conditions on the internal boundaries. The cost functional is assumed to be dependent on the gradient of the state and on the usual norm of the control. We consider the so-called "critical" relation between the problem parameters and the period of the structure $\varepsilon \to 0$. *Two "strange" terms arise in the limit. The paper extends, by first time in the literature, previous papers devoted to the homogenization of the control problem which always assumed the symmetry of the periodic holes.*

Keywords: homogenization of optimal control, perforated domain, arbitrary shape, critical case, strange term.

1 Introduction

The paper deals with the homogenization of the optimal control problem associated to the Poisson state equation in a domain perforated by holes of an arbitrary shape and with a cost functional which is assumed to be dependent on the gradient of the state and on the usual L^2 norm of the control. On the boundary of the holes we assume that a Robin-type boundary conditions holds. There are many papers devoted to similar purposes in the literature (see, e.g., [4], [7] for the critical scales). All of them were related to the special case in which the perforations are represented by balls. The general shape assumed here on the holes is a source of difficult questions and the so called "strange terms" (see. e.g. [1] and the general exposition made in the book [2]) must be correctly identified such as it was shown in [2], [3], [11]). Inspired in those papers, we use here some capacity type problems to study the optimal control problem and to prove the appearance of a "strange" terms in the limit of the cost functionals. Curiously enough, we prove that this strategy can be suitably adapted to get the convergence of energies in the case of the direct problem (i.e., without any associated control formulation). This allows to improve some previous estimates presented in [2]). The formulation can be also understood by replacing the holes by isolated particles, such as it appears in application in Chemical Engineering (see,e.g.[2] and its references).

2 Problem Statement: the adjoint problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary $\partial\Omega$. In the cube $Y = (-1/2, 1/2)^n$ consider a subdomain G_0, $\overline{G_0} \subset Y$, which is star-shaped with respect to a ball $T^0_\rho \subset Y$ of radius ρ with center at the origin. Let $\delta B = \{x : \delta^{-1}x \in B\}$, $\delta > 0$.

For $\varepsilon > 0$ let

$$\widetilde{\Omega_\varepsilon} = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\}.$$

Denote by $\mathbb{Z}^n$ the set of all vectors $j = (j_1, \dots, j_n)$ with integer coordinates j_k, $k = 1, \dots, n$. Consider the set

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G^j_\varepsilon,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : \overline{G^j_\varepsilon} \subset Y^j_\varepsilon = \varepsilon Y + \varepsilon j, G^j_\varepsilon \cap \widetilde{\Omega_\varepsilon} \neq \emptyset\}$, $a_\varepsilon = C_0 \varepsilon^\alpha$, $\alpha = \frac{n}{n-2}$. It easy to see that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = const > 0$. Note that $\overline{G^j_\varepsilon} \subset T^j_{a_\varepsilon} \subset T^j_{\varepsilon/4} \subset Y^j_\varepsilon$, where T^j_r is a ball in $\mathbb{R}^n$ radius r centered at the point $P^j_\varepsilon = \varepsilon j$.

We introduce

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \; S_\varepsilon = \partial G_\varepsilon, \; \partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega.$$

In Ω_ε we consider the optimal problem: for a given control $v \in L^2(\Omega_\varepsilon)$ and data $f \in L^2(\Omega)$, $a \in C^\infty(\overline{\Omega})$, $a(x) \geq a_0 > 0$ and under the critical assumption $\gamma = \frac{n}{n-2}$, $n \geq 3$, we denote by $u_\varepsilon(v) \in H^1(\Omega_\varepsilon, \partial\Omega)$ the unique weak solution of the problem

$$\begin{cases} -\Delta u_\varepsilon(v) = f + v, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = 0, & x \in S_\varepsilon, \\ u_\varepsilon(v) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where ν is the unit outward normal vector to S_ε.

We consider the cost functional $J_\varepsilon : L^2(\Omega_\varepsilon) \to \mathbb{R}$ given by

$$J_\varepsilon(v) \equiv \frac{\eta}{2} \|\nabla(u_\varepsilon(v) - u_T)\|^2_{L^2(\Omega_\varepsilon)} + \frac{N}{2} \|v\|^2_{L^2(\Omega_\varepsilon)}, \tag{2}$$

where u_T is a target given function, $u_T \in H^1_0(\Omega)$, and η, N are positive given constants. We point out that when the parameter η is large enough we get the approximate controllability property in $H^1(\Omega_\varepsilon, \partial\Omega)$ (in the sense that the associate state $u_\varepsilon(v)$ is so close as we want to the target function, i.e. $\|\nabla(u_\varepsilon(v) - u_T)\|_{L^2(\Omega_\varepsilon)} \leq \delta$ for any $\delta > 0$ arbitrary small (see, e.g. [5], Section 1.6). It is well known (see [6]) that there exist an unique optimal control $v_\varepsilon \in L^2(\Omega_\varepsilon)$ such that

$$J_\varepsilon(v_\varepsilon) = \inf_{v \in L^2(\Omega_\varepsilon)} J_\varepsilon(v). \tag{3}$$

The aim of this paper is to study the limit as $\varepsilon \to 0$ of the optimal control v_ε and of the cost functional $J_\varepsilon(v_\varepsilon)$.

Note that the function $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of the problem (1) if it satisfies an integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon(v)\nabla\phi dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} a(x)u_\varepsilon(v)\phi ds = \int\limits_{\Omega_\varepsilon}(f+v)\phi dx, \tag{4}$$

where ϕ is an arbitrary function from $H^1(\Omega_\varepsilon, \partial\Omega)$. By $H^1(\Omega_\varepsilon, \partial\Omega)$ we denote the closure in $H^1(\Omega_\varepsilon)$ of the set of infinitely differentiable functions in $\overline{\Omega}_\varepsilon$ such that vanish near the boundary $\partial\Omega$.

The adjoint problem which connected with the optimal control v_ε is formulated in the following terms

$$\begin{cases} \Delta p_\varepsilon = \Delta(u_\varepsilon(v_\varepsilon) - u_T), & x \in \Omega_\varepsilon, \\ \partial_\nu(p_\varepsilon - u_\varepsilon(v_\varepsilon) + u_T) + \varepsilon^{-k}a(x)p_\varepsilon = 0, & x \in S_\varepsilon, \\ p_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

(see [4] and [7]) where $p_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of this problem. It is well known that

$$v_\varepsilon = -\frac{\eta}{N}p_\varepsilon. \tag{6}$$

So the optimal pair $(u_\varepsilon, -\frac{\eta}{N}p_\varepsilon) \in H^1(\Omega_\varepsilon, \partial\Omega) \times H^1(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of the coupled system

$$\begin{cases} -\Delta u_\varepsilon = f - \frac{\eta}{N}p_\varepsilon, & x \in \Omega_\varepsilon, \\ \Delta p_\varepsilon = \Delta(u_\varepsilon - u_T), & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}a(x)u_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu(p_\varepsilon - u_\varepsilon + u_T) + \varepsilon^{-\gamma}a(x)p_\varepsilon = 0, & x \in S_\varepsilon, \\ u_\varepsilon = p_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

For $u_\varepsilon, p_\varepsilon$ we have the estimate (see [4], [7])

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon,\partial\Omega)} + \|p_\varepsilon\|_{H^1(\Omega_\varepsilon,\partial\Omega)} \le K(\|f\|_{L^2(\Omega)} + \|u_T\|_{H_0^1(\Omega)}), \tag{8}$$

where K here and below is independent of ε. From estimate (8) we conclude

$$\|P_\varepsilon u_\varepsilon\|_{H_0^1(\Omega_\varepsilon)} \le K, \;\; \|P_\varepsilon p_\varepsilon\|_{H_0^1(\Omega)} \le K, \tag{9}$$

where $P_\varepsilon : H^1(\Omega_\varepsilon, \partial\Omega) \to H_0^1(\Omega)$ is H^1 - extension operator, such that

$$\|P_\varepsilon u\|_{H_0^1(\Omega)} \le K\|u\|_{H^1(\Omega_\varepsilon,\partial\Omega)}, \;\; \|\nabla P_\varepsilon u\|_{L^2(\Omega)} \le K\|\nabla u\|_{L^2(\Omega_\varepsilon)}. \tag{10}$$

From estimations (8)-(10) as usual we derive that there is a subsequence (still denoted by ε) such that as $\varepsilon \to 0$

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0, \;\; P_\varepsilon p_\varepsilon \rightharpoonup p_0, \text{ weakly in } H_0^1(\Omega), \tag{11}$$

In order to formulate and to prove a homogenization result on the optimal problem we need to introduce some auxiliary functions which become explicit when the perforated by holes have not an arbitrary shape but they are symmetric balls.

3 Auxiliary capacity type problems

Let us define $w_\varepsilon^j(x) \in H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\varepsilon/4}), (j \in \Upsilon_\varepsilon)$, as a weak solution of the boundary value problem

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ \partial_\nu w_\varepsilon^j + \varepsilon^{-\gamma} a(x)(w_\varepsilon^j - 1) = 0, & x \in \partial G^j_\varepsilon, \\ w_\varepsilon^j = 0, & x \in \partial T^j_{\varepsilon/4}. \end{cases} \tag{12}$$

We define

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\bigcup_{j \in \Upsilon_\varepsilon} T^j_{\varepsilon/4}}. \end{cases} \tag{13}$$

Below we formulate some statements, proved in [2], [3], [11].

Lemma 1. *For $W_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ the estimates are valid*

$$\|\nabla W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|W_\varepsilon\|^2_{L^2(S_\varepsilon)} \le K, \|W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K\varepsilon^2, \tag{14}$$

$$0 \le W_\varepsilon \le 1,\ \forall x \in \Omega_\varepsilon. \tag{15}$$

Hence, as $\varepsilon \to 0$ for some subsequence we have

$$P_\varepsilon W_\varepsilon \rightharpoonup 0 \text{ weakly in } H^1_0(\Omega),\ P_\varepsilon W_\varepsilon \to 0 \text{ strongly in } L^2(\Omega). \tag{16}$$

Let $w_0(x, y)$ be a solution of the exterior problem

$$\begin{cases} \Delta_y w_0 = 0, & y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu w_0 + C_0 a(x)(w_0(x, y) - 1) = 0, & y \in \partial G_0, \\ w_0(x, y) \to 0, & |y| \to \infty, \end{cases} \tag{17}$$

where $x \in \Omega$ is a parameter. Following the paper [3], [11] and [2] we define by $\mathcal{C}$ the space of functions $\phi \in C^\infty(\overline{\mathbb{R}^n \setminus \overline{G_0}})$ for which there exists positive constant $R > 0$ such that $\phi = 0$ outside T^0_R, where T^0_R is a ball of radius R with the center at the origin of coordinates. For any $u \in \mathcal{C}$ we have

$$\||y|^{-1} u\|_{L^2(\mathbb{R}^n \setminus \overline{G_0})} \le K(n) \|\nabla u\|_{L^2(\mathbb{R}^n \setminus \overline{G_0})}. \tag{18}$$

In the space $\mathbb{C}$ we can introduce the norm

$$\|v\|_{\mathcal{C}} \equiv \|\nabla v\|_{L^2(\mathbb{R}^n\setminus\overline{G_0})}. \tag{19}$$

Denoting by $\mathcal{V}$ the closure of $\mathcal{C}$ with respect to this norm, it is easy to see that $\mathcal{V}$ is a Hilbert space.

The function $w_0 \in \mathcal{V}$ is a weak solution to the exterior problem (17) if it satisfies the integral identity

$$\int\limits_{\mathbb{R}^n\setminus\overline{G_0}} \nabla w_0 \nabla \phi dy + C_0 \int\limits_{\partial G_0} a(x)(w_0 - 1)\phi ds = 0, \tag{20}$$

for an arbitrary $\phi \in \mathcal{V}$.

We have (see [3], [11] and [2])

Lemma 2. *There exist a unique weak solution $w_0 \in \mathcal{V}$ to the problem (17) and it satisfies the estimate*

$$0 \le w_0 \le 1, \quad \max_{x\in\overline{\Omega}} |w_0(x,y)| \le \frac{K}{|y|^{n-2}}, \ \forall y \in \mathbb{R}^n \setminus \overline{G_0}. \tag{21}$$

The following lemma addresses the relation between the functions w_ε^j and $w_0(P_\varepsilon^j, \frac{x-P_\varepsilon^j}{a_\varepsilon})$ (see [3], [11] and [2]).

Lemma 3. *We set $v_\varepsilon^j(x) = w_\varepsilon^j - w_0(P_\varepsilon^j, \frac{x-P_\varepsilon^j}{a_\varepsilon})$. Then*

$$\|\nabla v_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})} + \varepsilon^{-\gamma}\|v_\varepsilon^j\|^2_{L^2(\partial G_\varepsilon^j)} \le K\varepsilon^{n+2},\ \|v_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})} \le K\varepsilon^{n+2}, \tag{22}$$

$$|v_\varepsilon^j| \le \max_{x\in\partial T^j_{\varepsilon/4}} |w_0(P_\varepsilon^j, \frac{x-P_\varepsilon^j}{a_\varepsilon})|. \tag{23}$$

Define the function

$$H_0(x) = \int\limits_{\partial G_0} \partial_\nu w_0(x,y) ds_y = C_0|\partial G_0| a(x)(1 - \langle w_0(x,y)\rangle_{\partial G_0}), \tag{24}$$

where by $\langle w\rangle_{\partial G_0}$ we denote the mean-value of the function w on ∂G_0. Notice the dependence $H_0 = H_0(x : \partial G_0, a)$.

We also define another auxiliary function θ_ε^j as the solution of the following boundary-value problem of capacity type

$$\begin{cases} \Delta\theta_\varepsilon^j = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ \partial_\nu \theta_\varepsilon^j + \varepsilon^{-\gamma} a(x)(\theta_\varepsilon^j - w_\varepsilon^j) = 0, & x \in \partial G_\varepsilon^j, \\ \theta_\varepsilon^j = 0, & x \in \partial T^j_{\varepsilon/4}. \end{cases} \tag{25}$$

We introduce the extension function

$$\Theta_\varepsilon(x) = \begin{cases} \theta^j_\varepsilon(x), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}. \end{cases} \tag{26}$$

Is it easy to see that $\theta_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$.

Lemma 4. *The function Θ_ε defind in (26) satisfies the following estimates*

$$\|\nabla\Theta_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|\Theta_\varepsilon\|^2_{L^2(S_\varepsilon)} \le K,\ \ \|\Theta_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K\varepsilon^2, \tag{27}$$

$$0 \le \Theta_\varepsilon \le W_\varepsilon. \tag{28}$$

Proof. Taking θ^j_ε as a test function in the integral identity for the problem (25), we get

$$\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} |\nabla\theta^j_\varepsilon|^2 dx + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x)|\theta^j_\varepsilon|^2 ds = \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x)\theta^j_\varepsilon w^j_\varepsilon ds. \tag{29}$$

From here, we derive

$$\|\nabla\theta^j_\varepsilon\|^2_{L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})} + \varepsilon^{-\gamma}\|\theta^j_\varepsilon\|^2_{L^2(\partial G^j_\varepsilon)} \le K\varepsilon^{-\gamma}\|w^j_\varepsilon\|^2_{L^2(\partial G^j_\varepsilon)}. \tag{30}$$

Now, the estimations immediately follow from the Lemma 1.

Let us show that $\Theta_\varepsilon \le W_\varepsilon$. Taking in the integral identity for the weak solution of the problem (25) as a test-function $(\theta^j_\varepsilon - w^j_\varepsilon)^+ = \sup(\theta^j_\varepsilon - w^j_\varepsilon, 0) \in H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\varepsilon/4})$ we get

$$\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla\theta^j_\varepsilon \nabla(\theta^j_\varepsilon - w^j_\varepsilon)^+ dx + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x)\Big((\theta^j_\varepsilon - w^j_\varepsilon)^+\Big)^2 ds = 0.$$

From the integral identity for the function w^j_ε we obtain

$$\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w^j_\varepsilon \nabla(\theta^j_\varepsilon - w^j_\varepsilon)^+ dx = -\varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x)(w^j_\varepsilon - 1)(\theta^j_\varepsilon - w^j_\varepsilon)^+ ds \ge 0.$$

Hence, we have

$$0 \le \int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} |\nabla(\theta^j_\varepsilon - w^j_\varepsilon)^+|^2 dx + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x)\Big((\theta^j_\varepsilon - w^j_\varepsilon)^+\Big)^2 ds =$$

$$= \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x)(w^j_\varepsilon - 1)(\theta^j_\varepsilon - w^j_\varepsilon)^+ ds \le 0.$$

Thus, we derive that $(\theta^j_\varepsilon - w^j_\varepsilon)^+ = 0$ in $T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}$. The rest inequalities can be obtain in the similar way.

□

Hence. we have

$$P_\varepsilon \Theta_\varepsilon \rightharpoonup 0, \text{ weakly in } H_0^1(\Omega). \tag{31}$$

Also we introduce $\theta(x,y)$ as a solution to the exterior problem

$$\begin{cases} \Delta_y \theta = 0, & y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu \theta + C_0 a(x)(\theta(x,y) - w_0(x,y)) = 0, & y \in \partial G_0, \\ \theta(x,y) \rightharpoonup 0, & |y| \to \infty, \end{cases} \tag{32}$$

where $x \in \Omega$ is a parameter.

This problem is similar to the (17), so we have that there exist a unique weak solution of the problem (32) in the space $\mathcal{V}$ that satisfies the estimation (21). Moreover, we have $0 \le \theta \le w_0$. In addition we note that statement similar to the Lemma 3 is also valed.

Lemma 5. *The function $h_\varepsilon^j = \theta_\varepsilon^j - \theta(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon})$ satisfies estimates*

$$\|\nabla h_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon})} + \varepsilon^{-\gamma} \|h_\varepsilon^j\|^2_{L^2(\partial G_\varepsilon^j)} \le K \varepsilon^{n+2}, \ \|h_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4} \setminus G^j_\varepsilon)} \le K \varepsilon^{n+2}. \tag{33}$$

We define a function

$$H_1(x) = \int\limits_{\partial G_0} \partial_\nu \theta ds_y = C_0 |\partial G_0| a(x) (\langle w_0(x,y) \rangle_{\partial G_0} - \langle \theta(x,y) \rangle_{\partial G_0}). \tag{34}$$

Once again, we point out the implicit dependence $H_1 = H_1(x : \partial G_0, a)$.

4 Main result

The next theorem gives the description of the limit functions u_0, p_0, obtained in (11).

Theorem 1. *Let $n \ge 3$, $\alpha = \gamma = \frac{n}{n-2}$ and let $(u_\varepsilon, p_\varepsilon)$ be a weak solution to the system (7). Then, the pair (u_0, p_0) defined in (11) is a weak solution of the coupled system*

$$\begin{cases} -\Delta u_0 + C_0^{n-2} H_0(x) u_0 = f - \frac{\eta}{N} p_0, & x \in \Omega, \\ -\Delta p_0 + C_0^{n-2} H_0(x) p_0 = -\Delta(u_0 - u_T) + C_0^{n-2} H_1(x) u_0, & x \in \Omega, \\ u_0 = p_0 = 0 & x \in \partial\Omega, \end{cases} \tag{35}$$

where $H_0(x)$, $H_1(x)$ are defined, respectively, by (24)and (34).

Remark 1. *If $G_0 = \{|y| < 1\}$, we can find the explicit formula for solutions $w_0(x,y)$ and $\theta(x,y)$ and then for the auxiliary functions $H_0(x)$, $H_1(x)$:*

$$w_0(x,y) = \frac{a(x)}{a(x) + \frac{n-2}{C_0}} |y|^{2-n}, \ \theta(x,y) = \Big(\frac{a(x)}{a(x) + \frac{n-2}{C_0}} \Big)^2 |y|^{2-n}, \tag{36}$$

$$H_0(x) = (n-2)\omega_n \frac{a(x)}{a(x) + \frac{n-2}{C_0}}, \ H_1(x) = (n-2)\omega_n \Big(\frac{a(x)}{a(x) + \frac{n-2}{C_0}} \Big)^2, \tag{37}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. This special case was considered in a different way in [7], for the volume perforated domain, and in [4] (for the critical scales) for the case in which the domain is perforated along a internal manifold.It can be proved that in the case in which the parameters are in the critical case then the auxiliary functions $H_0(x)$, $H_1(x)$ are not need to identify the optimal limit problem.

Proof. First we prove, that the function u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u_0 + C_0^{n-2} H_0(x) u_0 = f - \eta N^{-1} p_0, & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega. \end{cases} \tag{38}$$

Consider the function $\phi = \psi - W_\varepsilon \psi$, where W_ε defined in (13), $\psi \in C_0^\infty(\Omega)$, as a test-function in the integral identity for u_ε. We get

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\psi - W_\varepsilon\psi)dx + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) u_\varepsilon (\psi - W_\varepsilon \psi) ds = \int\limits_{\Omega_\varepsilon} (f - \eta N^{-1} p_\varepsilon)(\psi - W_\varepsilon\psi)dx. \tag{39}$$

Using (16), we have

$$-\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(W_\varepsilon \psi)dx = -\int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon\psi)\nabla W_\varepsilon dx + \alpha_\varepsilon =$$

$$= -\sum_{j\in\Upsilon_\varepsilon} \int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla(u_\varepsilon\psi)\nabla w^j_\varepsilon dx + \alpha_\varepsilon = -\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}\cup\partial G^j_\varepsilon} \partial_\nu w^j_\varepsilon u_\varepsilon \psi ds + \alpha_\varepsilon =$$

$$= \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G^j_\varepsilon} a(x) w^j_\varepsilon u_\varepsilon \psi ds - \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G^j_\varepsilon} a(x) u_\varepsilon \psi ds - \tag{40}$$

$$-\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon u_\varepsilon\psi ds + \alpha_\varepsilon = -\varepsilon^{-\gamma}\int\limits_{S_\varepsilon} a(x) u_\varepsilon(\psi - W_\varepsilon\psi) ds - \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon u_\varepsilon \psi ds + \alpha_\varepsilon,$$

where $alpha_\varepsilon \to 0$ as $\varepsilon \to 0$.

From (39) and (40) we derive

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\psi dx - \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon u_\varepsilon \psi ds =$$

$$= \int\limits_{\Omega_\varepsilon} (f - \eta N^{-1} p_\varepsilon)(\psi - W_\varepsilon\psi)dx + \alpha_\varepsilon. \tag{41}$$

Using that (see [3], [11], [2])

$$\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon \psi u_\varepsilon ds =$$

$$= \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w_0(P^j_\varepsilon, \frac{x-P^j_\varepsilon}{a_\varepsilon})\psi u_\varepsilon ds = -C_0^{n-2} \int\limits_\Omega H_0(x) u_0 \psi(x) dx, \qquad (42)$$

we get (41) and (42) the integral identity for u_0

$$\int\limits_\Omega \nabla u_0 \nabla \psi dx + C_0^{n-2} \int\limits_\Omega H_0(x) u_0 \psi(x) dx = \int\limits_\Omega (f - \eta N^{-1} p_0)\psi dx. \qquad (43)$$

Next, we prove that the function p_0 is a weak solution of the problem

$$\begin{cases} -\Delta p_0 + C_0^{n-2} H_0(x) p_0 = -\Delta(u_0 - u_T) + C_0^{n-2} H_1(x) u_0, & x \in \Omega, \\ p_0 = 0, & x \in \partial\Omega. \end{cases} \qquad (44)$$

Let us take in the integral identity to the problem (5) as a test function $\phi = \psi - W_\varepsilon \psi$. We obtain

$$\int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla(\psi - W_\varepsilon\psi) dx + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x)(\psi - W_\varepsilon\psi) p_\varepsilon ds = \int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_T)\nabla(\psi - W_\varepsilon\psi) dx =$$

$$= \int\limits_\Omega \nabla(u_0 - u_T)\nabla\psi dx - \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(W_\varepsilon\psi) dx + \beta_\varepsilon, \qquad (45)$$

where $\beta_\varepsilon \to$, as $\varepsilon \to 0$.

We have that

$$-\int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla(W_\varepsilon\psi) dx = -\int\limits_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla(p_\varepsilon\psi) dx + \hat{\alpha}_\varepsilon =$$

$$= -\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G^j_\varepsilon} \partial_\nu w^j_\varepsilon p_\varepsilon \psi ds - \sum_{j\in\Upsilon_\varepsilon} \int\limits_{paT^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon p_\varepsilon \psi ds + \hat{\alpha}_\varepsilon =$$

$$= \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) p_\varepsilon \psi(x)(W_\varepsilon - 1) ds - \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon p_\varepsilon \psi ds + \hat{\alpha}_\varepsilon. \qquad (46)$$

From (45) and (46) we derive that the left hand side of (45) has the form

$$\int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla\psi dx - \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon p_\varepsilon \psi ds + \hat{\alpha}_\varepsilon,$$

where $\hat{\alpha}_\varepsilon \to 0$ as $\varepsilon \to 0$.

Hence, we have that the limit as $\varepsilon \to 0$ of the left hand side of (45) equal to

$$\int\limits_\Omega \nabla p_0 \nabla\psi dx + C_0^{n-2} \int\limits_\Omega H_0(x) p_0 \psi dx. \qquad (47)$$

From here and (45) we conclude that to obtain the integral identity for p_0 we have to find the limit of the following term

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (W_\varepsilon \psi) dx.$$

Applying the properties of the function W_ε we get that

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (W_\varepsilon \psi) dx = -\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) u_\varepsilon W_\varepsilon \psi ds + \kappa_\varepsilon, \tag{48}$$

where $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$.

Taking in the integral identity for u_ε as a test function $\Theta_\varepsilon \psi$, we obtain

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (\Theta_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) u_\varepsilon \Theta_\varepsilon \psi ds = \int\limits_{\Omega_\varepsilon} (f - \eta N^{-1} p_\varepsilon) \Theta_\varepsilon \psi dx. \tag{49}$$

Using that $theta_\varepsilon^j$ is a weak solution of the problem (25) and applying the Green's formula we deduce

$$\int\limits_{T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}} \nabla \theta^j_\varepsilon \nabla (u_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x) \theta^j_\varepsilon u_\varepsilon \psi ds - \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu \theta^j_\varepsilon u_\varepsilon \psi ds =$$

$$= \varepsilon^{-\gamma} \int\limits_{\partial G^j_\varepsilon} a(x) w^j_\varepsilon u_\varepsilon \psi ds. \tag{50}$$

Summing up by $j \in \Upsilon_\varepsilon$ we get

$$\int\limits_{\Omega_\varepsilon} \nabla \Theta_\varepsilon \nabla (u_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) \Theta_\varepsilon u_\varepsilon \psi ds -$$

$$- \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu \theta^j_\varepsilon u_\varepsilon \psi ds = \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) u_\varepsilon W_\varepsilon \psi ds. \tag{51}$$

Taking into account (31) and subtracting from (51) the equality (49), we obtain

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) W_\varepsilon u_\varepsilon \psi ds = - \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu \theta^j_\varepsilon u_\varepsilon \psi ds + \widetilde{\alpha}_\varepsilon, \tag{52}$$

where $\widetilde{\alpha}_\varepsilon \to 0$ as $\varepsilon \to 0$.

From (48) and (52) we have

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla W_\varepsilon \psi dx = - \lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) W_\varepsilon u_\varepsilon \psi ds =$$

$$= \lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu \theta^j_\varepsilon u_\varepsilon \psi ds = -C_0^{n-2} \int\limits_{\Omega} H_1(x) u_0(x) \psi(x) dx. \tag{53}$$

Hence, the limit as $\varepsilon \to 0$ of the right hand side of (45) is equal to

$$\int\limits_{\Omega} \nabla(u_0 - u_T)\nabla\psi dx + C_0^{n-2} \int\limits_{\Omega} H_1(x)u_0\psi dx. \tag{54}$$

From (47), (54) we conclude that $p_0 \in H_0^1(\Omega)$ is a weak solution of the problem (44). □

5 Convergence of the cost functionals

Now we will find the limit of the cost functionals depending on ε

$$J_\varepsilon(v_\varepsilon) = J_\varepsilon(\eta N^{-1}p_\varepsilon) \equiv \frac{\eta}{2} \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon - \nabla u_T|^2 dx + \frac{\eta^2}{2N} \int\limits_{\Omega_\varepsilon} p_\varepsilon^2 dx. \tag{55}$$

Theorem 2. *The following limit holds*

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = \frac{\eta}{2} \int\limits_{\Omega} |\nabla(u_0 - u_T)|^2 dx + \frac{\eta C_0^{n-2}}{2} \int\limits_{\Omega} H_1(x)u_0^2 dx + \frac{N}{2} \int\limits_{\Omega} v_0^2 dx \equiv J_0(v_0), \tag{56}$$

where $v_0 = -\eta N^{-1}p_0$ is the optimal control for the problem

$$-\Delta u_0(v_0) + C_0^{n-2}H_0(x)u_0(v_0) = f + v_0,\ x \in \Omega,\ u_0(v_0) = 0,\ x \in \partial\Omega, \tag{57}$$

and

$$J_0(v_0) = \inf_{v\in L^2(\Omega)} J_0(v). \tag{58}$$

Proof. Taking in the integral identity for p_ε as a test function u_ε we obtain

$$\eta \int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_T)\nabla u_\varepsilon dx = \eta \int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla u_\varepsilon dx + \eta\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x)p_\varepsilon u_\varepsilon ds. \tag{59}$$

Similarly, using p_ε as atest function in the integral identity for u_ε, we get

$$\eta \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla p_\varepsilon dx + \eta\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x)u_\varepsilon p_\varepsilon ds = \eta \int\limits_{\Omega_\varepsilon} (f - \eta N^{-1}p_\varepsilon)p_\varepsilon dx. \tag{60}$$

From (59), (60) we derive

$$\lim_{\varepsilon\to 0} \frac{\eta}{2} \int\limits_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_T)\nabla u_\varepsilon dx = \frac{\eta}{2} \int\limits_{\Omega} (f - \eta N^{-1}p_0)p_0 dx =$$

$$= \frac{\eta}{2} \int\limits_{\Omega} \nabla u_0 \nabla p_0 dx + \frac{\eta}{2} C_0^{n-2} \int\limits_{\Omega} H_0(x)u_0 p_0 dx =$$

$$= \frac{\eta}{2} \int\limits_{\Omega} \nabla(u_0 - u_T)\nabla u_0 dx + \frac{\eta}{2} C_0^{n-2} \int\limits_{\Omega} H_1(x)u_0^2 dx. \tag{61}$$

Taking into account that

$$\lim_{\varepsilon\to 0}\frac{\eta}{2}\int\limits_{\Omega}\nabla(u_\varepsilon-u_T)\nabla u_T dx=\frac{\eta}{2}\int\limits_{\Omega}\nabla(u_0-u_T)\nabla u_T dx,$$

and using (61) we obtain the statement of the Theorem 2. □

6 Convergence of the energy for the problem without control

In this last section we will use this type of ideas to prove the energy convergence for a direct problem, without any control formulation. As we will see, we must include in the limit energy some strange term associated to the homogenized problem. To do that we introduce the artificial complementary problem (5), which corresponds formally to the case $v=0$ and $u_T=0$. This improves the convergence result given in Section 4.7.1.4 of [2].

Theorem 3. *Let u_ε be the solution of (1) with $v\equiv 0$ at the critical scale. Let $u_0\in H_0^1(\Omega)$ be the weak limit of the extension $P_\varepsilon u_\varepsilon$. Then we have the convergence*

$$\int\limits_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2 dx\to\int\limits_{\Omega}|\nabla u_0|^2 dx+C_0^{n-2}\int\limits_{\Omega}H_1(x)u_0^2 dx. \tag{62}$$

Proof. It is well known (see [3], [11] and [2]) that $u_0\in H_0^1(\Omega)$ is the unique weak solution of the problem

$$\begin{cases}-\Delta u_0+C_0^{n-2}H_0(x)u_0=f, & x\in\Omega,\\ u_0(x)=0, & x\in\partial\Omega.\end{cases}\tag{63}$$

Let us introduce the weak solution p_ε of the problem

$$\begin{cases}\Delta p_\varepsilon=\Delta u_\varepsilon, & x\in\Omega_\varepsilon,\\ \partial_\nu(p_\varepsilon-u_\varepsilon)+\varepsilon^{-\gamma}a(x)p_\varepsilon=0, & x\in S_\varepsilon,\\ p_\varepsilon=0, & x\in\partial\Omega.\end{cases}\tag{64}$$

It is known that $P_\varepsilon p_\varepsilon\rightharpoonup p_0$ weakly in $H_0^1(\Omega)$ as $\varepsilon\to 0$ and p_0 is a weak solution of the problem

$$\begin{cases}-\Delta p_0+C_0^{n-2}H_0(x)p_0=-\Delta u_0+C_0^{n-2}H_1(x)u_0, & x\in\Omega,\\ p_0=0, & x\in\partial\Omega,\end{cases}\tag{65}$$

where H_0 and H_1 defined by formulas (24) and (34) respectively.

From the variational formulation to the problem (1) with $v]equiv0$, we have

$$\int\limits_{\Omega_\varepsilon}\nabla u_\varepsilon\nabla p_\varepsilon dx+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}a(x)u_\varepsilon p_\varepsilon ds=\int\limits_{\Omega_\varepsilon}fp_\varepsilon dx.$$

Similarly from the variational formulation to the problem on $p_{]ee}$ we deduce

$$\int\limits_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla u_\varepsilon dx + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} a(x) p_\varepsilon u_\varepsilon ds = \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.$$

Thus we have

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int\limits_{\Omega_\varepsilon} f p_\varepsilon dx \to \int\limits_{\Omega} f p_0 dx =$$

$$= \int\limits_{\Omega} \nabla u_0 \nabla p_0 dx + C_0^{n-2} \int\limits_{\Omega} H_0(x) u_0 p_0 dx = \int\limits_{\Omega} |\nabla u_0|^2 dx + C_0^{n-2} \int\limits_{\Omega} H_1(x) u_0^2 dx,$$

which ends the proof.

□

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