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BOUNDED POSITIVE SOLUTIONS FOR DIFFUSIVE LOGISTIC EQUATIONS WITH UNBOUNDED DISTRIBUTED LIMITATIONS

Jesús Ildefonso Díaz

Instituto Matemático Interdisplinar
Depto. Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain

Jesús Hernández

Depto. Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain

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Dedicated to Georg Hetzer: elegant mathematician and very good friend in his 75th

Abstract. We establish the existence of bounded very weak solutions to a large class of stationary diffusive logistic equations with weights by constructing suitable sub and supersolutions. This class of problems corresponds to the case in which the absorption term dominates over the forcing term. The case of simultaneous singular nonlinearities and singular weights is also considered. This shows that if limitations in the growth of a population are distributed and unbounded, but satisfy some mild integrability assumption in terms of the distance to the boundary, solutions can still be bounded. The results extend several papers in the literature.

1. **Introduction.** The stationary diffusive logistic equation, also called as stationary Fisher-KPP equation, is a very well-known example of semilinear elliptic equation arising in applications, in particular in population dynamics. Moreover, the same type of nonlinear terms arise in other equations or systems, mostly in the Lotka-Volterra systems (predator-prey, competition, symbiosis) in mathematical biology.

The semilinear elliptic equation

$$\begin{cases} -\Delta u + q(x)u^p = \lambda m(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p > 1$, m and q are positive (or nonnegative) functions and λ is a real parameter, covers different variants of the logistic

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*Corresponding author: Jesús Ildefonso Díaz.

equation (see, *e.g.* the monograph [11], the paper [30] and their many references). A stochastic version was considered in [34].

The coefficient $m(x)$ represents the intrinsic growth rate of the population and it is thus positive (resp. negative) on the favorable (resp. unfavorable) domain. We point out that in this framework the forcing term $\lambda m(x)u$ dominates over the absorption nonlinearity term u^p for small values of u (remember that $p > 1$). Thus this type of problems are of a different nature than the case of problems in which the absorption dominates over the forcing (which corresponds to the case $p < 1$: see, *e.g.*, the survey [15]).

In the paper [30] the second author extended the study of (1), mostly dealing with bounded coefficients in the previous literature, to the case of $q \equiv 1$ and $m \in L^r(\Omega)$ where $r > \frac{N}{2}$, references to previous work can be found there. In this case one cannot expect to get classical solutions but it was proved that weak solutions in $H_0^1(\Omega)$, with some additional regularity, exist by using general existence theorems with sub and supersolutions. In particular, we obtain bounded positive solutions even if the coefficient $m(x)$ is unbounded. This shows that if limitations are distributed and unbounded, but satisfy some mild integrability assumptions, solutions can still be bounded.

The main goal of this paper is to extend and improve the results in [30] to the class of very weak solutions and provide existence results for other variants. In particular, we show that if the singular weights are of the kind $d(x)^{-\alpha}$, with $d(x) = d(x, \partial\Omega)$ and suitable $\alpha > 0$, we get bounded solutions as well.

In Section 2 we present a general result on the existence of very weak solutions improving the result on weak solutions given in [30]. An application to nonlinear problems improving the results in [30] and [3] is given, and the same happens in Section 4 dealing with the case of $m(x)$ changing sign, the special case of a superharmonic weight $m(x)$ and the case of a nonlinear diffusion. Finally, Section 5 describes some improvements of the results for the case of singular weights obtained in [31].

2. **Very weak solutions of a general semilinear problem.** In this section we study the existence and uniqueness of positive (and non-negative) solutions to the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for a.a. $x \in \Omega$.

We will present some improvement of the assumptions made in Section 2 of [30] dealing with the existence of weak solutions of (2). They are functions required to be in $H_0^1(\Omega)$ (the usual Sobolev space with the standard equivalent norm), so, by the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ with $2^* = \frac{2N}{N-2}$, the notion of weak solution ensures that the equation takes place on the dual space $H^{-1}(\Omega)$:

Definition 1. *We say that $u \in H_0^1(\Omega)$ is a (weak) solution to (2)) if $f(x,u) \in L^{\frac{2N}{N+2}}(\Omega)$ when $N > 3$ ($f(x,u) \in L^{1+\epsilon}(\Omega)$, for some $\epsilon > 0$ if $N = 2$, and $f(x,u) \in L^1(\Omega)$, if $N = 1$)*

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f(x,u)\varphi \tag{3}$$

for any $\varphi \in H_0^1(\Omega)$.

Definition 2. *We say that* $u_0 \in H^1(\Omega)$ *is a subsolution (resp. a supersolution) to (2) if*

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi - \int_\Omega f(x,u_0)\varphi \leq 0 \tag{4}$$

(resp. ≥ 0 *for* u^0*) for any* $\varphi \in H_0^1(\Omega)$ *such that* $\varphi \geq 0$ *and* $u_0 \leq 0$ *(resp.* $0 \leq u^0$*) on* $\partial\Omega$*, in the usual sense of traces for functions in* $H^1(\Omega)$*.*

A more general point of view started with a famous 1971 unpublished paper by H. Brezis (later collected in [7]) dealing with a larger class of solutions in which the equation takes place in the weighted space

$$L^1(\Omega,d) = \left\{ w : \Omega \to \mathbb{R} \text{ Lebesgue measurable} : \int_\Omega |w(x)|\, d(x) < \infty \right\},$$

where $d(x) = d(x,\partial\Omega)$. The key idea is that now the test functions must be more regular but in the natural space $W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ (or simply $C^2(\Omega) \cap C_0^{1,0}(\overline{\Omega})$).

Definition 3. *We say that* $u \in L^1(\Omega)$ *is a very weak solution to (2)) if* $f(\cdot,u)d(\cdot) \in L^1(\Omega)$ *and*

$$-\int_\Omega u\Delta\varphi = \int_\Omega f(x,u)\varphi \tag{5}$$

for any $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$*.*

Definition 4. *We say that* $u_0 \in L^1(\Omega)$ *is a subsolution (resp. a supersolution) to (2) if*

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi - \int_\Omega f(x,u_0)\varphi \leq 0$$

(resp. ≥ 0 *for* u^0*) for any* $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ *such that* $\varphi \geq 0$*.*

Notice that since any function $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ satisfies that $|\varphi(x)| \leq Cd(x)$ for any $x \in \overline{\Omega}$, for some $C > 0$, then the identity in (5) makes sense. The definition (4) can be extended to functions such that $u_0 \leq 0$ (resp. $0 \leq u^0$) on $\partial\Omega$, in a weak sense of traces which we will refer later (see Remark 2). Moreover, notice that any weak solution is a very weak solution.

The existence of solutions will be obtained by the method of sub and super solutions which was used by many authors in the last fifty years. It is well-known that such method has many applications to chemical reactions, Lotka-Volterra systems, combustion, ..., mostly in order to obtain positive classical solutions: see the book by Pao [39] and its many references. Weak solutions were studied as well, proving the existence of minimal and maximal solutions (see many references in [30]).

Here we will follow the approach given for very weak solutions (also called as L_d^1-solutions) showing that $f(\cdot,u)d(\cdot) \in L^1(\Omega)$ as made in Montenegro and Ponce [38] (see also [22]) getting minimal and maximal solutions (obtained by combining Schauder' s fixed point theorem and some equi-integrability argument).

Theorem 1 ([38]). *. Let* u_0 *and* u^0 *be a sub and a supersolution of (2), respectively. Assume that* $u_0 \leq u^0$ *a.e. on* Ω *and*

$$f(\cdot,v)d(\cdot) \in L^1(\Omega) \text{ for every } v \in L^1(\Omega) \text{ such that } u_0 \leq v \leq u^0 \text{ a.e. on } \Omega. \tag{6}$$

Then, there exists a minimal (resp. maximal) very weak solution $\underline{u}$ *(resp.*$\overline{u}$*) such that* $u_0 \leq \underline{u} \leq \overline{u} \leq u^0$ *a.e. on* Ω*.*

Remark 1. *It was proved also in [38] (see its Corollaries 5.2 and 5.3) that i) if u_0, $u^0 \in L^1(\Omega)$ and f is a Carathéodory function such that*

$$f(x,v) \in L^{\frac{2N}{N+2}}(\Omega) \text{ for every } v \in L^1(\Omega) \text{ such that } u_0 \leq v \leq u^0 \text{ a.e. on } \Omega \tag{7}$$

then (2) has a solution $u \in H_0^1(\Omega)$ such that $u_0 \leq u \leq u^0$ a.e. on Ω, and
ii) if u_0, $u^0 \in L^\infty(\Omega)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ then (2) has a solution $u \in C^{1,\gamma}(\overline{\Omega})$ such that $u_0 \leq u \leq u^0$ a.e. on Ω.
Notice that i) implies that, in fact, $\underline{u}$ and .$\overline{u}$ are weak solutions. This improves Theorem 1 of [30] where it was assumed that $f(x,v) \in L^\gamma(\Omega)$, for every $v \in L^1(\Omega)$ such that $u_0 \leq v \leq u^0$ a.e. on Ω, for some $\gamma > \frac{2N}{N+2}$, besides of other additional conditions. Notice that in that case we also get that $\underline{u}$ and $\overline{u}$ are in the Sobolev space $W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)$ and that if $\gamma > \frac{N}{2}$ (respect. $\gamma > N$) then $\underline{u}$ and $\overline{u}$ are in $C(\overline{\Omega})$ (respect. in $C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$).This can be proved by applying the classical L^p and C^ν regularity theorems collected in [28] and Sobolev imbeddings.

An useful auxiliary tool, used in the proof of Theorem 1 of [30], is to have the existence and uniqueness of solutions for the special case of problem

$$\begin{cases} -\Delta z + p(x,z) = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

when $p(x,z)$ is a Carathéodory function satisfying

$$p(x,0) = 0 \text{ a.a. } x \in \Omega, \tag{9}$$

$$\sup_{|w| \leq \alpha} |p(x,w)|\, d(x) \leq \phi_\alpha(x) \in L^1(\Omega), \ \forall \alpha > 0, \tag{10}$$

$$p(x,u)u \geq 0 \ u \text{ for a.a. } x \in \Omega. \tag{11}$$

Lemma 1 ([22]). *Assume (9), (10) and (11). Then, for any $h \in L^1(\Omega, d)$ there exists a unique very weak solution $z \in L^1(\Omega)$, with $p(x,z)d \in L^1(\Omega)$, to*

$$\begin{cases} -\Delta z + p(x,z) = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Moreover $z \in L^{N',\infty}(\Omega) \cap W_0^{1,r}(\Omega, d)$, $1 \leq r < \frac{2N}{2N-1}$.

Remark 2. *For the definition of the Lorentz space $L^{N',\infty}(\Omega)$ we send the reader to [36] and [22]. It is also proved there that, if in addition, $h \in L^1(\Omega, d^\alpha) = \{w \colon \Omega \to \mathbb{R}\ ,\ \int_\Omega |w(x)|\, d^\alpha(x) < \infty\}$ for some $\alpha \in [0,1)$ then $|\nabla z| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$. Notice that the above gradient estimates, jointly with the compact Sobolev imbeddings for the weighted spaces implies useful compactness arguments. In particular, the application $P : L^1(\Omega, d) \to L^2(\Omega)$ given by $P(h) = z$ is compact (which is an alternative to the compactness argument used in the proof of Theorem, 1 in [30]). Moreover the gradient estimates allow to give a sense to the trace of very weak solutions and thus the notion of sub and super very weak solutions given in Definition 4 can be extended to the case in which $u_0, u^0 \in W^{1,r}(\Omega, d)$, for some $1 \leq r < \frac{2N}{2N-1}$ and $u_0 \leq 0 \leq u^0$ on $\partial\Omega$ in the sense of traces (see [36]).*

Remark 3. *We recall that when we replace (10) by*

$$\sup_{|w| \leq \alpha} |p(x,w)| \leq \phi_\alpha(x) \in L^1(\Omega), \ \forall \alpha > 0, \tag{13}$$

then by Theorem 7 of Brezis and Browder [5]*) for any $h \in H^{-1}(\Omega)$ there exists a weak solution $z \in H_0^1(\Omega)$ with $p(x,z)$ and $p(x,z)z$ in $L^1(\Omega)$ satisfying the equation in $H^{-1}(\Omega)+L^1(\Omega)$, i.e. with test functions in $H_0^1(\Omega)\cap L^\infty(\Omega)$. On the other hand, sharper regularity results (in $L^s(\Omega)$ and in $W^{1,s}(\Omega)$) on very weak solutions for problems of the type (12), when $h \in L^r(\Omega,d)$, where obtained in Proposition 2.3 of* [41]*. In particular, if $p(x,z)\equiv 0$ then $z \in L^\infty(\Omega)$ if $r > \frac{N+1}{2}$.*

The uniqueness of (nontrivial) positive very weak solutions to (2) under the assumption (6) follows when $f(x,u)$ is decreasing a.a. $x \in \Omega$ ([22]). It is also well-known that under additional regularity conditions the uniqueness of positive weak solutions holds when, in one way or another, $f(x,u)$ satisfies some "concavity" condition (see, e.g. [8], [29] and [32]). The following result is a slight extension of some of the results in the Appendix II of [8]:

Theorem 2. *i) Assume that f is a Carathéodory function satisfying (6),*

$$f(x,u) \textit{ is locally Lipschitz continuous in } u \textit{ for a.a. } x \in \Omega, \tag{14}$$

and

$$f(x,u)/u \textit{ is (strictly) decreasing a.a. } x \in \Omega. \tag{15}$$

Then there exists a unique positive bounded very weak solution to (2) in the class of functions $v \in L^\infty(\Omega)$ such that

$$0 \le v(x) \le cd(x) \textit{ for some } c > 0 \textit{ and a.a. } x \in \Omega. \tag{16}$$

ii) Assume that f is a Carathéodory function satisfying (6) and (15). Then there exists a unique positive weak solution to (2) in the sense mentioned in Remark 3, i.e. in the class of functions $v \in H_0^1(\Omega)$ such that $f(\cdot,v) \in L^1(\Omega)$ and for which (3) holds for any test function in $H_0^1(\Omega)\cap L^\infty(\Omega)$.

Proof. i) We adapt to our framework the variant of Krasnoselkii's presented in [8]. The new tool used here is the *strong maximum principle* (or *uniform Hopf inequality*): any very weak solution $z \in L^1(\Omega)$ of

$$\begin{cases} -\Delta z = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

for some $h \ge 0$ with $h(x)d \in L^1(\Omega)$, satisfies

$$z(x) \ge C\left[\int_\Omega h(y)d(y)\right]d(x) \text{ for some } C = C(\Omega) > 0.$$

This is an unpublished result by J. M. Morel and L. Oswald presented in [21] and [6], and improved in [1]. Now, let u_1 and u_2 be two very weak solutions of (2). Let

$$\Lambda = \{t \in [0,1] : \ tu_1 \le u_2 \text{ on } \Omega\}.$$

Obviously $0 \in \Lambda$. Let us show that $1 \in \Lambda$ (then $u_1 \le u_2$ on Ω and by analogy $u_2 \le u_1$ on Ω, which gives the uniqueness). Suppose not, that

$$t_0 = \sup \Lambda < 1.$$

Then

$$-\Delta(u_2 - t_0u_1) = f(x,u_2) - t_0 f(x,u_1). \tag{18}$$

From condition (14) and the boundedness of u_1 and u_2 we know the existence of a positive constant K such that $f(x,s)+Ks$ is increasing, for a.a. $x\in\Omega$, for any $s\in[0,\|u_2\|_{L^\infty(\Omega)}]$. Then, using (15)

$$\begin{aligned}&-\Delta(u_2-t_0u_1)+K(u_2-t_0u_1)=f(x,u_2)+Ku_2-t_0\left[f(x,u_1)+Ku_1\right]\\&\geq f(x,t_0u_1)+Kt_0u_1-t_0\left[f(x,u_1)+Ku_1\right]=f(x,t_0u_1)-t_0f(x,u_1)\geq 0.\end{aligned}$$

Then by the above mentioned strong maximum principle, if $h(x)=f(x,t_0u_1)-t_0f(x,u_1)$ is not identically zero on Ω then

$$u_2-t_0u_1\geq\overline{C}d>0 \text{ on } \Omega, \text{ for some } \overline{C}>0.$$

On the other hand, since u_1 satisfies (16) there is some $\varepsilon>0$ such that $u_2-t_0u_1\geq\overline{C}d\geq\varepsilon cd\geq\varepsilon u_1$. Thus $t_0+\varepsilon\in\Lambda$ which is impossible. On the other hand, the case $u_2-t_0u_1\equiv 0$ is also impossible since from (18) $f(x,u_2)=t_0f(x,u_1)$ but from (15) we get that $f(x,t_0u_1)>t_0f(x,u_1)$, thus $u_1\leq u_2$ on Ω and the conclusion follows.
ii) We adapt to our framework the method II presented in [8]. We define an approximation of the $sign^+(s)$ function by taking a nondecreasing Lipschitz function $\theta\in W^{1,\infty}(\mathbb{R})$ such that $\theta(s)=0$ if $s\leq 0$ and $\theta(s)=0$ if $s\geq 1$ and then, for any $\varepsilon>0$, we take $\theta_\varepsilon(s)=\theta(s/\varepsilon)$. Then, if u_1 and u_2 are two weak solutions of (2) we get (in a weak sense)

$$-(\Delta u_1)u_2+(\Delta u_2)u_1=u_1u_2\left[\frac{f(x,u_1)}{u_1}-\frac{f(x,u_2)}{u_2}\right].$$

Then $\theta_\varepsilon(u_1-u_2)\in H_0^1(\Omega)\cap L^\infty(\Omega)$ can be taken as a test function and integrating by parts we get

$$\begin{aligned}&\int\left[(\nabla u_1)\,u_2-(\nabla u_2)\,u_1\right]\theta_\varepsilon'(u_1-u_2)\cdot\nabla\,(u_1-u_2)\\&\quad=\int u_1u_2\left[\frac{f(x,u_1)}{u_1}-\frac{f(x,u_2)}{u_2}\right]\theta_\varepsilon(u_1-u_2).\end{aligned}$$

It is clear that

$$\begin{aligned}&\int\left[(\nabla u_1)\,u_2-(\nabla u_2)\,u_1\right]\theta_\varepsilon'(u_1-u_2)\cdot\nabla\,(u_1-u_2)\\&\geq-\int(\nabla u_2)\,(u_1-u_2)\theta_\varepsilon'(u_1-u_2)\cdot\nabla\,(u_1-u_2)\\&\geq-\int_{\{0<u_1-u_2<\varepsilon\}}\left|\nabla u_2\cdot\nabla\,(u_1-u_2)\right|.\end{aligned}$$

Since $|\nabla u_2\cdot\nabla\,(u_1-u_2)|\in L^1(\Omega)$ and meas $\{0<u_1-u_2<\varepsilon\}\to 0$ as $\varepsilon\to 0$, using that $\dfrac{f(x,s)}{u}$ is strictly decreasing we get that

$$\int u_1u_2\left[\frac{f(x,u_1)}{u_1}-\frac{f(x,u_2)}{u_2}\right]_+\leq 0,$$

which implies that

$$\frac{f(x,u_1)}{u_1}\leq\frac{f(x,u_2)}{u_2}, \text{ i.e. } u_2\geq u_1 \text{ in } \Omega.$$

Analogously we get that $u_2\leq u_1$ in Ω, which proves the result. □

Remark 4. *We recall that the exposition made in Appendix II of [8] is devoted to solutions u such that $\Delta u\in L^\infty(\Omega)$, for a bounded domain Ω. We also point out that there are some special cases of function $f(x,u)$ for which it is possible to prove the* *uniqueness of the positive very weak solution of (2) without condition (16): see* [12] *and* [1].

3. A sublinear indefinite equation.

The sublinear indefinite equation

$$\begin{cases} -\Delta u = \lambda m(x) u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

when $0 < q < 1$ and m changing sign on Ω,

$$\begin{cases} |\Omega^+| = |\{x \in \Omega \mid m(x) > 0\}| > 0 \\ |\Omega^-| = |\{x \in \Omega \mid m(x) < 0\}| > 0, \end{cases} \tag{20}$$

can be considered in the class $m \in L^1(\Omega : d)$, and even in a larger class of functions $m \in L^1_{loc}(\Omega)$, generalizing the results presented in Section 3.1 of [30].

It is useful to start by considering previously the case of $m > 0$, and more specially, the linear eigenvalue problem with weight.

$$\begin{cases} -\Delta w = \lambda m(x) w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Such linear problem and its variants were intensively studied in the last thirty years since they arise, naturally, in the study of the linearized stability of solutions of problems

$$\begin{cases} -\Delta w = \lambda g(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

(see, e.g. [2], [24], [31], [16] and their many references). Notice that if we expect to have $w \in H^1_0(\Omega)$ then

$$\mu(\Omega, m) := \inf_{w \in H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla w|^2}{\displaystyle\int_\Omega m(x) w^2} > -\infty \tag{23}$$

which is related with the study of the best constant in the Hardy-Sobolev with weight. When

$$m \in L^r(\Omega) \text{ with } r > \frac{N}{2}. \tag{24}$$

we know that there is a first eigenvalue $\lambda_1 > 0$ which is simple and has a positive eigenfunction $\varphi_{1,m} \in C(\overline{\Omega})$. This follows from ([25]). Moreover, if $r > N$, $\varphi_{1,m} \in C^1(\overline{\Omega})$ and $\frac{\partial \varphi_{1,m}}{\partial n} < 0$ on $\partial\Omega$ (this follows from Theorem 2.3 of Brezis-Kato [9], $\varphi_{1,m} \in L^t(\Omega)$ for all $1 < t < \infty$ and this yields easily $m\varphi_{1,m} \in L^s(\Omega)$ for all $1 < s < r$, giving in turn $\varphi_{1,m} \in C(\overline{\Omega})$). In a similar way, if $r > N$, $\varphi_{1,m} \in C^1(\overline{\Omega})$ and $\frac{\partial \varphi_{1,m}}{\partial n} < 0$ on $\partial\Omega$ (see Gilbarg-Trudinger [28].

A more interesting case arises when $m(x)$ is singular near the boundary $\partial\Omega$. It was shown in [2] that if

$$m \in L^\infty_{loc}(\Omega),\ m(x) \geq m_0 > 0, \tag{25}$$

$$m(x) d(x)^2 \underset{d(x)\to 0}{\rightarrow} 0, \tag{26}$$

then condition (23) holds. So, this is the case if

$$0 < \liminf m(x)d(x)^{\gamma} \leq \limsup m(x)d(x)^{\gamma} < +\infty \quad \text{for some } 0 \leq \gamma < 2. \tag{27}$$

Moreover, it was proved in [37] that if Ω is convex $m(x) = \frac{c}{d(x)^2}$ then $\mu(\Omega, m) = 1/4c$, $\lambda = 1/4c$ is the infimum of the essential spectrum and problem (23) has no minimizer. Nevertheless, if $\mu(\Omega, m) < 1/4c$ then there exists a $\lambda_{\mu(\Omega,m)} \in (0, 1/4)$ which is the first eigenvalue of the problem (21), and so there is a positive solution w of such problem. Notice that such choice of m does not satisfies (26).

As mentioned before, we know that (24) implies that the first eigenfunction $\varphi_{1,m}$ of problem (21) is a bounded function: something that we know that may fail for some $m(x)$ more singular than condition (26) (see, [2], [24]). For our next result we will assume

$$\varphi_{1,m} \in L^{\infty}(\Omega), \tag{28}$$

which holds also for many cases in which $m \in L^1_{loc}(\Omega)$ and for which (24) fails.

Coming back to the sublinear problem, we point out that this case was considered in [3] by using an approximation method (see also the survey [35] for many other references). Here we are interested in to follow a different method: the construction of suitable super and subsolutions. This will be useful also for other related problems. We have the following generalization of Theorem 11 of [30].

Theorem 3. *Assume $0 < q < 1$, $m \in L^1(\Omega : d)$, $m > 0$ such that conditions (23) and (28) hold. Then, for any $\lambda > 0$ there exists a unique positive very weak solution to (19). Moreover, u is a weak solution and $u \in H^1_0(\Omega) \cap L^{\infty}(\Omega)$.*

Proof. It suffices to apply the method of super and subsolutions given in Theorem 1. As a subsolution we can take $u_0 \equiv c\varphi_{1,m}$, with $c > 0$ "small enough " since we have

$$\begin{cases} -\Delta\varphi_{1,m} = \lambda_1 m(x)\varphi_{1,m} & \text{in } \Omega, \\ \varphi_{1,m} = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

with $\varphi_{1,m} > 0$. We normalize $\varphi_{1,m}$ by $\|\varphi_{1,m}\|_{L^{\infty}(\Omega)} = 1$. Then

$$\begin{aligned} -\Delta u_0 - \lambda m(x)(u_0)^q &= \lambda_1 mc\varphi_{1,m} - \lambda mc^q(\varphi_{1,m})^q = \\ &= mc^q(\varphi_{1,m})^q(c^{1-q}\lambda_1(\varphi_{1,m})^{1-q} - \lambda) < 0 \end{aligned} \tag{30}$$

for $c > 0$ "small".

As a supersolution we take $u^0 \equiv C\widetilde{\psi}_1$ where $\widetilde{\psi}_1 > 0$ the eigenfunction of problem (21) when we take as spatial domain $\widetilde{\Omega}$, a smooth bounded domain such that $\widetilde{\Omega} \supset \overline{\Omega}$, and we extend m by zero out of $\overline{\Omega}$. Then, if $\widetilde{\lambda}_1 > 0$ is the first eigenvalue and $\beta = \min_{\overline{\Omega}} \widetilde{\psi}_1 > 0$ then we have

$$\begin{aligned} -\Delta u^0 - \lambda m(x)(u^0)^q &= \widetilde{\lambda}_1 mC\widetilde{\psi}_1 - \lambda mC^q(\widetilde{\psi}_1)^q = \\ &= mC^q(\widetilde{\psi}_1)^q(C^{1-q}\widetilde{\lambda}_1(\widetilde{\psi}_1)^{1-q} - \lambda) > 0 \end{aligned} \tag{31}$$

if $C > \left(\frac{\lambda}{\lambda_1 \beta^{1-q}}\right)^{\frac{1}{1-q}}$.

Since $\widetilde{\psi}_1 > 0$ on $\overline{\Omega}$, it is easy to see that $u_0 \leq u^0$ in Ω and this ends the proof. Notice that condition (6) holds since u_0 and u^0 are bounded and $m(x)d(\cdot) \in L^1(\Omega)$.The uniqueness follows from the usual "concavity" argument. □

Remark 5. *If we assume $m \in L^r(\Omega)$ with $r > N$, then we can also take as supersolution (a multiple of) ψ, where ψ is the solution of*

$$\begin{cases} -\Delta\psi = m(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

Now $\psi \in C^1(\overline{\Omega})$, $\psi > 0$ on Ω and $\frac{\partial\psi}{\partial n} < 0$ on $\partial\Omega$.

For the case of m changing sign we have

Theorem 4. *Assume $0 < q < 1$, $m \in L^1(\Omega : d)$, with (20) such that conditions (23) and (28) hold for m^+as a weight. Assume also that $m^+ \in L^r(\Omega, d)$ with $r > \frac{N+1}{2}$. Then, for any $\lambda > 0$ there exists a positive very weak solution to (19) which, in fact, is a weak solution and such that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. Again, it suffices to apply Theorem 1. To built a subsolution we pick a smooth $\Omega^{'}$ such that $\overline{\Omega'} \subset \Omega^+$ and consider $\mu_1 > 0$ and $\psi_1 > 0$ the first eigenvalue and eigenfunction on $\Omega^{'}$ associated to the weight m^+, i.e., $\psi_1 = \varphi_{1,m^+}$, with φ_{1,m^+} the first eigenfunction of

$$\begin{cases} -\Delta\psi_1 = \mu_1 m^+(x)\psi_1 & \text{in } \Omega^{'}, \\ \psi_1 = 0 & \text{on } \partial\Omega^{'}. \end{cases} \tag{33}$$

Now, reasoning as in [3], it is possible to show that $u_0 \equiv c\widetilde{\psi}_1$, where $\widetilde{\psi}_1 = \psi_1$ on $\overline{\Omega'}$ and $\widetilde{\psi}_1 = 0$ in $\Omega - \overline{\Omega'}$, is a subsolution for $c > 0$ "small".
As supersolution we pick $u^0 \equiv C\psi$, $C > 0$, where ψ is the unique solution of

$$\begin{cases} -\Delta\psi = m^+(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $C > \left(\lambda \|\psi\|^q_{L^\infty(\Omega)}\right)^{\frac{1}{1-q}}$. Indeed, we have

$$\begin{aligned} -\Delta u^0 - \lambda m(x)(u^0)^q = Cm^+ - \lambda m C^q(\psi)^q \geq \\ \geq C^q m^+(C^{1-q} - \lambda\psi^q) > 0 \end{aligned} \tag{34}$$

if $C > \left(\lambda \|\psi\|^q_{L^\infty(\Omega)}\right)^{\frac{1}{1-q}}$.
Since $\mathrm{supp}(u_0) \subset \Omega$, it is easy to see that we can choose $u_0 \leq u^0$ and we have that condition (6) holds since $0 \leq u_0$, u^0 is bounded and $m(x)d(\cdot) \in L^1(\Omega)$. □

Remark 6. *Notice that the assumption on m^- is weaker than the conditions assumed on m^+.*

Remark 7. *One could think about replacing Ω' by Ω^+ in Theorem 4 by applying some approximation of Ω by interior subdomains Ω', but we don' t intend to pursue this matter here.*

Remark 8. *The result of Theorem 4 improves the results in [30] and [35]. But problem (19) has been treated from long time ago, mostly concerning existence of solutions with compact support if m is indefinite. See [31], the survey [35] and their bibliographies.*

4. The generalized logistic equation with indefinite unbounded weight.

4.1. **The generalized logistic equation with indefinite weight.** The study of the generalized logistic equation

$$\begin{cases} -\Delta u + u^p = \lambda m(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{35}$$

when $p > 1$ can be also treated in the framework of $m \in L^1(\Omega : d)$ as an alternative to Section 3.2 of [30].

A first auxiliary result shows that non-negative solutions to (35) are actually positive. This will allow later to use subsolutions with compact support in Ω getting however positive solutions.

Lemma 2. *If $u \geq 0$ is a bounded subsolution to (35) then $u > 0$ on Ω.*

Proof. In this case we can write

$$\begin{cases} -\Delta u + (u^{p-1} + \lambda m^-(x))u = \lambda m^+(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the Strong Maximum Principle for weak solutions (see, e.g., [28]) gives $u > 0$ on Ω (recall that $p > 1$). Indeed, notice that we have $-\Delta u + (M^{p-1} + \lambda m^-(x))u \geq 0$, if $0 \leq u \leq M$ on Ω, and thus the classical version of this principle for linear operators can be applied when $m^- \in C(\Omega)$. For the case in which $m \in L^1(\Omega : d)$ it suffices to introduce a regularization $m_n^- \in C(\overline{\Omega})$ of function m^- (for instance by convolution with a sequence of mollifiers) such that $m_n^- \to m^-$ uniformly on compact sets of Ω. Then $-\Delta u + (M^{p-1} + \lambda m_n^-(x))u + \lambda(m^-(x) - m_n^-(x))u \geq 0$ and on any compact subset K of Ω we get that $-\Delta u + \lambda(m_n^-(x) + \epsilon)u \geq 0$. Hence, the local proof given in [28] allows to conclude that $u > 0$ on Ω (notice that in the local argument used in [28] we can work with a local subsolution and thus, without loss of generality, that assume that $u \in C^2(K)$, $u \geq 0$ and $-\Delta u + (M^{p-1} + \lambda(m_n^-(x) + \epsilon))u = 0$ in K). □

Theorem 5. *Assume $p > 1$, $m \in L^1(\Omega : d)$, with (20) such that conditions (23) and (28) hold for m^+as a weight and on some smooth set Ω' such that $\overline{\Omega'} \subset \Omega^+$. Assume also that $m^+(x) \in L^{\frac{rp}{p-1}}(\Omega : d)$ with $r > \frac{(N+1)(p-1)}{2p}$. Then, for any $\lambda > \lambda_1(\Omega', m^+)$ there exists a bounded positive very weak solution to (35). If, in addition, $m \in L^r(\Omega)$, $r > \frac{N}{2}$ and*

$$r > \frac{Np}{2(p-1)} \tag{36}$$

the existence holds for any $\lambda > 0$, any solution is in $C(\overline{\Omega})$, and if

$$r > \frac{Np}{(p-1)} \tag{37}$$

then solutions are in $C^1(\overline{\Omega})$.

Proof. We apply Theorem 1). As a subsolution we take $u_0 \equiv c\widetilde{\psi}_1$ if $\lambda > \widetilde{\lambda}_1$ for $c > 0$ "small", which is the extension by zero on $\Omega - \overline{\Omega'}$ of ψ_1, the first eigenfunction corresponding to the first eigenvalue $\mu_1 > 0$ of the problem

$$\begin{cases} -\Delta \psi_1 = \mu_1 m^+(x) \psi_1 & \text{in } \Omega', \\ \psi_1 = 0 & \text{on } \partial\Omega'. \end{cases} \tag{38}$$

By (23) and (28) we know that $\psi_1 > 0$ on $\Omega^{'}$ and $u_0 \in L^\infty(\Omega)$. We have

$$\begin{aligned} -\Delta u_0 + (u_0)^p - \lambda m(x) u_0 &= c\widetilde{\lambda}_1 m^+(x)\widetilde{\psi}_1 + c^p\left(\widetilde{\psi}_1\right)^p \\ -\lambda c\widetilde{\lambda}_1 m^+(x)\widetilde{\psi}_1 &= c(\widetilde{\lambda}_1 - \lambda)m^+(x)\widetilde{\psi}_1 + c^p\left(\widetilde{\psi}_1\right)^p \\ &= c\widetilde{\psi}_1\left[(\widetilde{\lambda}_1 - \lambda)m^+(x) + c^{p-1}\left(\widetilde{\psi}_1\right)^{p-1}\right] \le 0 \text{ on } \Omega^{'}, \end{aligned}$$

if $\lambda > \widetilde{\lambda}_1$ for $c > 0$ "small". Moreover

$$-\Delta u_0 + (u_0)^p - \lambda m(x) u_0 = 0 \text{ on } \Omega - \overline{\Omega'}.$$

Thus u_0 is a very weak subsolution of problem (35).
As a supersolution we take $u^0 \equiv \psi$ with

$$\begin{cases} -\Delta\psi = H(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $H(x) = c_0(\lambda m^+(x))^{\frac{p}{p-1}}$. By the regularity results of Proposition 2.3 of ([41]), we have a better regularity on ψ than the one mentioned in Lemma 1 and Remark 3: $\psi \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$ since $r > \frac{(N+1)(p-1)}{2p}$. By the Hopf Strong Maximum Principle for weak solutions (see, e.g. [21],[6] and [1]) we know that $\psi > 0$. We have that, $u^0 \equiv \psi$ is a supersolution since

$$-\Delta\psi + \psi^p - \lambda m(x)\psi = H(x) + \psi^p - \lambda m^+(x)u - \lambda m^-(x)\psi$$

$$\ge H(x) + \psi^p - \lambda m^+(x)\psi \ge \begin{cases} 0 & \text{on } \Omega^+ \\ H(x) + \psi^p \ge 0 & \text{on } \Omega - \overline{\Omega^+}. \end{cases}$$

We need to show that $u_0 \le u^0$ and the proof depends on the smoothness of $u^0 \equiv \psi$. If ψ is continuous on $\overline{\Omega}$ (what happens when $r > \frac{(N+1)(p-1)}{2p} + \varepsilon$ for some $\varepsilon > 0$: see Proposition 2.3 of ([41])), since $supp(u_0) \subset \Omega$, the proof is immediate. For the general case it is enough to check that $u^0 \equiv \widetilde{\psi}$, the unique solution of

$$\begin{cases} -\Delta\psi = H(x) & \text{in } \Omega, \\ \psi = b > 0 & \text{on } \partial\Omega, \end{cases}$$

for any $b > 0$ is a supersolution satisfying $u_0 \le u^0$. This holds for $\lambda > \lambda_1(\Omega^{'}, m^+)$ for some $\Omega^{'}$ smooth with $\overline{\Omega'} \subset \Omega^+$ (notice that we cannot expect to have this for some $\lambda > \lambda_1^+(\Omega^+, m^+)$ and that we know that $\lambda_1(\Omega', m^+) < \lambda_1^+(m)$: see [25]), the respective principal eigenvalues for the linear part.
Condition (6) holds since $0 \le u_0$, u^0 are bounded and $m(x)d(\cdot) - u^p \in L^1(\Omega)$. The additional regularity is obtained as in Remark 1 (see also [30]). □

Remark 9. *In the above argument, when introducing $b > 0$, one could try to replace $\Omega^{'}$ by Ω by using some kind of approximation argument, but we don't intend to pursue this point here. In any case, the above result is not the best possible result (we include it in order to show how the method works in this case) and it could proved (in the classical framework) reasoning as in the paper [33] by Hess and Kato that there exists an unbounded continuum of positive solutions bifurcating from $\lambda_1^+(m)$, the unique positive principal eigenvalue of the linear part. It is clear that we have that $\lambda_1(\Omega', m^+) > \lambda_1^+(m)$ for all such domains $\Omega^{'}$.*

It is not difficult to show that positive solutions provided by Theorem 5 are unique and linearly stable.

Theorem 6. *Problem (35) has a unique positive solution which is linearly stable.*

Proof. Uniqueness is a particular case of Theorem 2 since $\lambda m(x) - u^{p-1}$ is strictly decreasing in u, a.a. $x \in \Omega$.
For the stability we can write

$$\begin{cases} -\Delta u = \lambda m(x)u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{39}$$

for a solution $u > 0$ and the linearized associated eigenvalue problem

$$\begin{cases} -\Delta w + pu^{p-1}w - \lambda m(x)w = \mu w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

This problem is well-defined and let μ_1 and $\psi_1 > 0$ the first eigenvalue and eigenfunction (see [31]). If we multiply (39) by ψ_1 and (40) by u and substract we obtain

$$\mu_1 = \frac{(p-1\int_\Omega u^p\psi_1}{\int_\Omega u\psi_1} > 0,$$

which gives the result. □

4.2. **The logistic equation with superharmonic weight.** We consider now the classical logistic equation (i.e. problem (35) with $p = 2$)

$$\begin{cases} -\Delta u = u(\lambda m(x) - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{41}$$

with $m(x) > 0$ and assume now, for instance, that

$$\begin{cases} -\Delta m = h \geq 0 & \text{in } \Omega, \\ m = 0 & \text{on } \partial\Omega. \end{cases} \tag{42}$$

with $h \in L^r(\Omega : d)$ such that $r > \frac{(N+1)}{2}$. Then, by the regularity results of ([41]) we know that $m \in L^\infty(\Omega)$ and by the Strong Maximum Principle for very weak solutions (see, e.g. [21], [6] and [1]) we have that $m > 0$ on Ω.

Theorem 7. *Let m satisfying (42). Then, for any $\lambda > 0$ there exists a unique positive very weak solution of (41). Moreover $0 < u(x) \leq \lambda m(x)$ in Ω.*

Proof. It suffices to check that $u^0 \equiv \lambda m(x)$ is a super supersolution since

$$-\lambda\Delta m \geq 0 = \lambda m(\lambda m - \lambda m).$$

□

Let us see what happens in the general case of problem (35) with $p > 1$. We have then

$$\begin{cases} -\Delta u = u(\lambda m(x) - u^{p-1}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{43}$$

Now the "candidate" to a supersolution is $u^0 \equiv (\lambda m(x))^{\frac{1}{p-1}}$. If we write $\alpha = \frac{1}{p-1} > 0$ we have

$$-\Delta m^{\alpha} = -\alpha(\alpha-1)\,(m(x))^{\alpha-2}\,|\nabla m|^2 - \alpha\,(m(x))^{\alpha-1}\,\Delta m \geq 0$$

if (42) holds and $0 < \alpha < 1$. It is clear that $0 < \alpha < 1$ if and only if $p > 2$.

Theorem 8. *Let m satisfying (42) and let $p > 2$. Then, for any $\lambda > 0$ there exists a unique positive very weak solution of (41) such that $0 < u(x) \leq (\lambda m(x))^{\frac{1}{p-1}}$ in Ω.*

Proof. Again, it suffices to check that $u^0 \equiv (\lambda m(x))^{\frac{1}{p-1}}$ is a super supersolution since

$$-\Delta\left[(\lambda m(x))^{\frac{1}{p-1}}\right] \geq 0 = (\lambda m(x))^{\frac{1}{p-1}}\,(\lambda m(x) - \lambda m(x)).$$

□

Remark 10. *The above argument still works for $-\Delta\beta(u)$ diffusion operators if we assume that $-\Delta(\beta^{-1}(\lambda m(x)) \geq 0$.*

4.3. **The logistic equation with nonlinear diffusion.** We study now a generalization of the logistic equation with nonlinear diffusion

$$\begin{cases} -\Delta w^n = \lambda m(x) w - w^k & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{44}$$

where Ω is as above,

$$n > 1 \text{ and } k > 1, \tag{45}$$

and $m > 0$ which was intensively studied in the literature (see, e..g, [20] and its references). With the change of variable $u = w^n$ equation (44) is transformed into

$$\begin{cases} -\Delta u = \lambda m(x) u^q - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{46}$$

where $q = \frac{1}{n}$, $p = \frac{k}{n}$. Then we study (46) with

$$0 < q < 1,\ 0 < q < p.$$

We have the following existence result.

Theorem 9. *Assume $0 < q < 1$, $p > 1$, $m \in L^1(\Omega : d)$, $m(x) \geq m_0 > 0$ in Ω, for some m_0, such that conditions (23) and (28) hold. Then, for any $\lambda > 0$ there exists a bounded positive very weak solution to (46). Moreover, if (24) holds then for any $\lambda > 0$ there exists a positive weak solution $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$.*

Proof. We take as a subsolution $u_0 \equiv c\varphi_{1,m}$, $c > 0$, for $c > 0$ "small" (thanks to assumptions (23) and (28)) since

$$\begin{aligned} &-\Delta u_0 - \lambda m(x)(u_0)^q + (u_0)^p = \lambda_1 m c\varphi_{1,m} - \lambda m c^q(\varphi_{1,m})^q + c^p(\varphi_{1,m})^p = \\ &= m c^q(\varphi_{1,m})^q(\lambda_1 c^{1-q}(\varphi_{1,m})^{1-q} - \lambda + \tfrac{1}{m} c^{p-q}(\varphi_{1,m})^{p-q}) \leq 0 \end{aligned} \tag{47}$$

for $c > 0$ "small". □

As a supersolution one can take the positive very weak solution to (19) given by Theorem 3, and show as before that we have $u_0 \leq u^0$.

Remark 11. *Since $0 < q \leq \min(1, p)$, it is possible to get a monotone continuous dependence of solutions u with respect to the weight $m(x)$ (even if it is changing sign on Ω). Indeed, it was proved in [14] (see Lemma 2.4 and its variants) that*

$$\left\| [u^{p-q} - \widehat{u}^{p-q}]_+ \right\|_{L^2(\Omega)} \leq \| [m - \widehat{m}]_+ \|_{L^2(\Omega)}\,, \tag{48}$$

if $\widehat{u}$ is the solution of problem (46) corresponding to the weight $\widehat{m}$.

Remark 12. *We remark that in the subcase in which $0 < q < p < 1$ in spite of to having a "strong absorption with respect to the diffusion" we have that $u > 0$ on Ω. This is in contrast with the reversed balance in which the absorption dominates over the forcing ($0 < p < q < 1$): in that case positive flat solutions and nonnegative solutions with compact support may arise (see, e.g., the survey [15] and its many references). The situation may change radically if $m(x)$ changes sign (see, e.g., Remark 2.8 of [14]).*

5. **A "generalized logistic" equation with singular weights.** We consider now the case of a " generalized logistic" equation with singular weights

$$\begin{cases} -\Delta u + k(x)u^p = \lambda m(x)u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{49}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{2,\gamma}$ boundary for some $\gamma > 0$, and $k, m \in C^1(\Omega)$, $k, m > 0$ on Ω and satisfy

$$q < p$$
$$|m(x)| \leq k_1 d(x)^{-\beta},\ |k(x)| \leq k_2 d(x)^{-\beta'} \tag{50}$$

where $d(x) = d(x, \partial\Omega)$, with $k_1, k_2 > 0$ and

$$0 < \beta < 1 + q \text{ and } 0 < \beta' < 1 + p.$$

Let us concentrate in the singular forcing case in which

$$m(x)u^q = \frac{g(x)}{u^a d(x)^b}, \text{ with } a > 0 \text{ and } b \geq 0$$

(so that $q = -a$ and $\beta = b$) with

$$g \in L^\infty(\Omega),\ g > 0,$$

$$\begin{cases} -\Delta u + k(x)u^p = \lambda \frac{g(x)}{u^a d(x)^b} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{51}$$

We will take as supersolution $u^0 = z$ where $z > 0$ is the unique solution of the problem (19), i.e.

$$\begin{cases} -\Delta z = \lambda \frac{g(x)}{z^a d(x)^b} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

That problem was intensively studied in the literature since the pioneering paper by Crandall, Rabinowitz and Tartar [12] (see also [4], when $b = 0$ and $g \in L^s(\Omega)$ for different $s > 1$, [18] and their many references). The following conclusions were proved in [18]:
i) Assume

$$a + b > 1 \text{with } b \in [0, 2). \tag{53}$$

Then there exists a positive very weak solution z of (52). Moreover $z \in C(\overline{\Omega}) \cap W^{2,s}_{loc}(\Omega)$ for any $s \in [1, +\infty)$.

ii) Assume

$$a+b<1. \tag{54}$$

Then there exists a very weak solution z of (52). Moreover $z \in W_0^1(\Omega, |\cdot|_{N(\gamma),\infty}) \cap W^{2,s}_{loc}(\Omega)$, for any $\gamma \in]0,1[$ and for any $s \in [1,+\infty)$ with $N(\gamma)=\frac{N}{N-1+\gamma}$.

iii) Let

$$a>0$$

and denote $h(x)=\frac{g(x)}{d(x)^b}$. Assume h be such that there exist $C_h>0$ and $\gamma \in [0,1[$ such that

$$h(x) \geq C_h \text{ a.e. } x \in \Omega \text{ and } h \in L^1(\Omega : d^{\gamma-\frac{2a}{1+a}}). \tag{55}$$

Then there exists a positive very weak solution z of (52). Moreover $z \in W_0^1(\Omega, |\cdot|_{N(\gamma),\infty})$ with $N(\gamma)=\frac{N}{N-1+\gamma}$.

Remark 13. *We point out that if $b \geq 2$ no very weak solution of (51) may exist (see [6] and [26]) and that the uniqueness of solutions holds. A sharper study was made in [1] (specially for the case $a=1$ and $b=0$). Many other papers where devoted to the case in which the forcing term is of the form $\lambda(\frac{g(x)}{z^a d(x)^b}+h(x,z))$: see, e.g. [26] and its references. On the other hand, we mention that there are also many studies on different reaction-diffusion with singular terms in which the absorption term is more singular than the forcing term (a balance which is not being regarded in this paper), i.e., when $q \geq p$ and $0>p$ (see, e.g., [21], [23], [40],[27], [13], [17] and [19]).*

Concerning the subsolution to problem (49) we can take $u_0 \equiv c\varphi_0$, $c>0$, with φ_0 given by

$$\begin{cases} -\Delta \varphi_0 = \lambda_0 k(x) \varphi_0 & \text{in } \Omega, \\ \varphi_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{56}$$

so that $\varphi_0>0$ and we will assume $k(x)$ such that φ_0 is bounded (we can normalize φ_0 by $\|\varphi_0\|_{L^\infty(\Omega)}=1$). Then we assume that absorption dominates on the forcing in the following way:

$$-1<q<1,\ p>q \text{ and } k(x)<k_1\frac{m(x)}{d(x)^{p-1}} \text{ for a.a. } x \in \Omega. \tag{57}$$

The argument to check that, for $\lambda>0$ fixed and $c>0$ "small", u_0 is a subsolution is the following

$$\begin{aligned} &-\Delta u_0 + k(x)(u_0)^p - \lambda m(x)(u_0)^q = c^q(\varphi_0)^q(\lambda_0 m(x) c^{1-q}(\varphi_0)^{1-q} + c^{p-q}k(x)(\varphi_0)^{p-q} \\ &-\lambda m(x)) \leq c^q(\varphi_0)^q m(x)(\lambda_0 c^{1-q}(\varphi_0)^{1-q} + c\tfrac{k(x)}{m(x)}(\varphi_0)^{p-q} - \lambda) \leq 0 \end{aligned} \tag{58}$$

for $\lambda>0$ fixed and $c>0$ "small", thanks to the assumption (57). As a supersolution we take $u^0=z$ where $z>0$ is the unique solution of

$$\begin{cases} -\Delta u = \lambda m(x) u^q & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{59}$$

(see also Theorem 3.1 in [32]). We suppose now that the absorption term dominates on the forcing term

$$a > 0,\ p > -a,\ \ b \in [0,2) \text{ and } k(x) < k_1 \frac{m(x)}{d(x)^{p-1}} \frac{g(x)}{d(x)^b} \text{ for any } x \in \Omega. \tag{60}$$

Then by using (60) and the boundary estimates given in ([18]) we have that $f(x,u)d = (\lambda \frac{g(x)}{u^a d(x)^b} - k(x)u^p)d \in L^1(\Omega)$ and that $u_0 \leq u^0$. Then, as conclusion, we have the following result which generalizes Theorem 3.17 in [32].

Theorem 10. *Assume that conditions (23) and (28) hold for $k(x)$ as a weight, (60) and that one of the following conditions holds: (54), (53) or (55). Then, for any $\lambda > 0$ there exists a positive very weak solution to (51).*

Remark 14. *If we assume that Ω is a bounded domain in $\mathbb{R}^N$ with $C^{2,\gamma}$ boundary for some $\gamma > 0$, and $k, m \in C^1(\Omega)$, $k, m > 0$ on Ω and satisfy*

$$|m(x)| \leq k_1 d(x)^{-\beta},\ \ |k(x)| \leq k_2 d(x)^{-\beta'} \tag{61}$$

with $k_1, k_2 > 0$ and

$$0 < \beta < 1 + q \text{ and } 0 < \beta' < 1 + p.$$

Let us define $\delta_2 = \min\{\delta_1, 1 + p - \beta'\}$, where $\delta_1(\gamma) \in (0,1)$ according the regularity of $\partial\Omega$. Then, if we assume (60) we can conclude that for any $\lambda > 0$ there exists a positive solution to (49) $u \in C^{1,\delta}(\overline{\Omega})$ with $0 < \delta < \delta_2$. If $p > 1$ this solution is unique for λ "large enough" and the mapping $\lambda \to u(x,\lambda) \in C_0^{1,\delta}(\overline{\Omega})$ is C^∞ and strictly increasing. That conclusion was proved in [32].

Remark 15. *For uniqueness, the quantity $H_\lambda(x,u)$ in Assumption 2.14 in Theorem 3.17* [32] *is relevant. In particular, to get the uniqueness we use in Assumption 2.14 in* [32] *the auxiliary function*

$$\begin{aligned} H_\lambda(x,u) &= \lambda(1-q)m(x)u^q - (1-p)k(x)u^p = \\ &= u^p((1-q)m(x)u^{q-p} + (p-1)k(x)) > 0 \end{aligned}$$

if $p > 1$.

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E-mail address: jidiaz@ucm.es

E-mail address: jesus.hernande@telefonica.net