Boundary control and homogenization: optimal climatization through smart double skin boundaries

J.I. Díaz [1], A.V. Podolskiy [2] and T.A. Shaposhnikova [2]

([1]) Instituto de Matematica Interdisciplinar (IMI)

Dpto. Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain

jidiaz@ucm.es,

([2]) Lomonosov Moscow State University

Department of Differential Equations

Faculty of Mechanics and Mathematics, Leninskie Gory, 119991, GSP-1, Moscow, Russian Federation

originalea@ya.ru

shaposh.tan@mail.ru

Dedicated to the unforgettable Olga A. Oleinik (1925-2001)

Abstract. We consider the homogenization of an optimal control problem in which the control v is placed on a part Γ_0 of the boundary and the spatial domain contains a thin layer of "small particles", very close to the controlling boundary, and a Robin boundary condition is assumed on the boundary of those "small particles". This problem can be associated with the climatization modeling of *Bioclimatic Double Skin Façades* which was developed in modern architecture as a tool for energy optimization. We assume that the size of the particles and the parameters involved in the Robin boundary condition are critical (and so they justify the occurrence of some "strange terms" in the homogenized problem). The cost functional is given by a weighted balance of the distance (in a H^1-type metric) to a prescribed target internal temperature u_T and the proper cost of the control v (given by its $L^2(\Gamma_0)$ norm). We prove the (weak) convergence of states u_ε and of the controls v_ε to some functions, u_0 and v_0, respectively, which are completely identified: u_0 satisfies an artificial boundary condition on Γ_0 and v_0 is the optimal control associated to a limit cost functional J_0 in which the "boundary strange term" on Γ_0 arises. This information on the limit problem makes much more manageable the study of the optimal climatization of such double skin structures.

Key words: optimal control, homogenization, thin layer of "small particles", critical case, cost functional convergence

AMS Subject Classification: 35B27, 93C20, 49N05

1. *Introduction*

A well-known energy optimization technique in modern architecture is the theory of *smart façade systems (* also called as *Bioclimatic Double Skin Façades)* in which climatization takes place by means of active glass windows (see, e.g., [4], [2] and [1]). Today, there are different types of active glass in the market: LCD Liquid Crystal, Gasochromic, SPO suspended particles, Electrochromic, etc. See, e.g., the case of fluids and windows in [2] and [8].

From the mathematical view point, many different climatization models have been proposed in the literature: see, for instance, Chapter 1 of the excellent book by Duvaut and Lions [14]. Some studies on internal climatization and homogenization can be found in [24]. In this paper, we will analyze a simplified formulation of *Double Skin Façades* in which there is an active flux control v_ε, located in a part Γ_0 of the boundary, and a kind of *celosia (* latticed windows called in this way in Spanish) traditionally made of masonry, wood, or a combination of these materials. We assume that the *celosía* is formed by a set of periodical small thermostats, of period $\varepsilon > 0$, located in an internal thin layer located very close to the controlling boundary. So, ε represents a small parameter related to the characteristic *celosia*.

Our simplified optimal control problem assumes that the state of the system u_ε (the internal temperature) satisfies a Poisson equation in the internal domain Ω_ε of $\mathbb{R}^n$, with $n \geq 3$, which is defined as the external domain to the set of periodical small thermostats (here represented by a set of ε-periodically balls) on whose contours S_ε a given climatization law (represented by a Robin boundary condition with a large parameter $\varepsilon^{-\gamma}$ as coefficient, where $\gamma = \frac{n-1}{n-2}$) takes place. Non-symmetrical shapes, and/or the case $n = 2$, can also be considered thanks to the techniques presented in [13], but for the sake of simplicity in the presentation we will not develop it here. We assume that any thermostat has a critical radius a_ε, where $a_\varepsilon = C_0\varepsilon^\alpha$ and $\alpha = \frac{n-1}{n-2}$. As in many other frameworks (see many examples and references in the monograph [9]), this critical size leads to the occurrence of strange terms in the homogenized problem (in contrast with what happens for other possible sizes).

It is assumed that the cost functional $J_\varepsilon(v_\varepsilon)$ is given by a weighted balance of the distance (in a $H^1(\Omega_\varepsilon)$ type metric) to a prescribed target internal temperature u_T and the proper cost of the control v (given by its $L^2(\Gamma_0)$ norm). Our main result proves the (weak) convergence of solutions u_ε and of the controls v_ε to some functions, u_0 and v_0, respectively, which are completely identified: u_0 satisfies the Poisson equation in the whole domain Ω (which we assume to be a bounded open set with $\partial\Omega$ of class C^1) and v_0 is the boundary optimal control but in an *artificial* boundary condition in which the thermostats effects are located on the own controllability boundary Γ_0. Moreover, we prove the convergence of the cost functional $J_\varepsilon(v_\varepsilon)$ to a new cost functional $J_0(v_0)$ in which the boundary *strange term* arises. This information on the limit problem makes much more manageable the study of the optimal climatization of such double skin structures.

In the last section, we consider the pure homogenization process (without any control, $v_\varepsilon \equiv 0$) and prove the convergence of the energies: an information which is stronger than the mere weak convergence $u_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$.

This paper complements the scope considered by previous papers in the literature concerning optimal control problems in which the controls are located in different parts of the spatial domain (see [22], [23], [21], [12] and [13]).

2. *Problem statement. Adjoint optimality problem*

Let $\Omega \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ be a bounded open set of $\mathbb{R}^n$, $n \geq 3$, with $\partial\Omega$ of class C^2, and $\partial\Omega = \Gamma_0 \cup \Gamma_1$, is assumed to be of class C^1, where $\Gamma_0 = \partial\Omega \cap \{x_n = 0\} \neq \emptyset$ is the $(n-1)$-dimensional domain on the plane $x_n = 0$ which represents the controlling boundary, $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma_0}$. Define $Y_0 = (-1/2, 1/2)^{n-1} \times (0,1)$, $Y_\varepsilon^j = \varepsilon Y_0 + \varepsilon j$, $j = (j_1, \dots j_{n-1}, 0)$, $j_i \in \mathbb{Z}, i = 1, \dots n-1$. We denote by P_ε^j the center of the cube Y_ε^j, $G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j$, where G_0 is the unit ball with the center coinciding with the center $(0, \dots 0, 1/2)$ of the cube Y_0 and $a_\varepsilon = C_0 \varepsilon^\alpha$ with $\alpha = (n-1)/(n-2)$. We define $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : j = (j_1, \dots, j_{n-1}, 0), Y_\varepsilon^j \subset \Omega\}$. It is easy to see (as in Chapter 6 of [9]) that $|\Upsilon_\varepsilon| \cong d\varepsilon^{1-n}$, $d = const > 0$, where $|\Upsilon_\varepsilon|$ denotes the cardinality of the set of isolated points Υ_ε.

We introduce the sets

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,\ S_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} \partial G_\varepsilon^j,\ \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ \partial\Omega_\varepsilon = S_\varepsilon \bigcup \Gamma_0 \bigcup \Gamma_1.$$

The set G_ε represents the *celosía* or double skin. It is localized as a subset of $\Omega \cap \{x \in \mathbb{R}^n : x_n \in (0, \varepsilon)\}$ (see Figure 1).

For an arbitrary function $v \in L^2(\Gamma_0)$, we denote by $u_\varepsilon(v) \in H^1(\Omega_\varepsilon, \Gamma_1)$ the solution of the problem

$$\begin{cases} -\Delta u_\varepsilon(v) = f, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = 0, & x \in S_\varepsilon, \\ \partial_\nu u_\varepsilon(v) = v, & x \in \Gamma_0, \\ u_\varepsilon(v) = 0, & x \in \Gamma_1, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, $a(x) \in C^\infty(\overline{\Omega})$, $a(x) \geq a_0 = const > 0$, and the notation $\partial_\nu g$ represents the partial derivative along the outward unit normal vector ν to the boundary. Here, the space $H^1(\Omega_\varepsilon, \Gamma_1) = \{w \in H^1(\Omega_\varepsilon) \text{ such that } w = 0 \text{ on } \Gamma_1\}$.

We assume to be given a target function $u_T \in H^1(\Omega)$ and we consider the cost functional

$$J_\varepsilon : L^2(\Gamma_0) \to \mathbb{R},$$

given by

$$J_\varepsilon(v) = \frac{\eta}{2} \int_{\Omega_\varepsilon} B(x) \nabla(u_\varepsilon(v) - u_T) \nabla(u_\varepsilon(v) - u_T) dx + \frac{N}{2} \|v\|^2_{L^2(\Gamma_0)}, \tag{2}$$

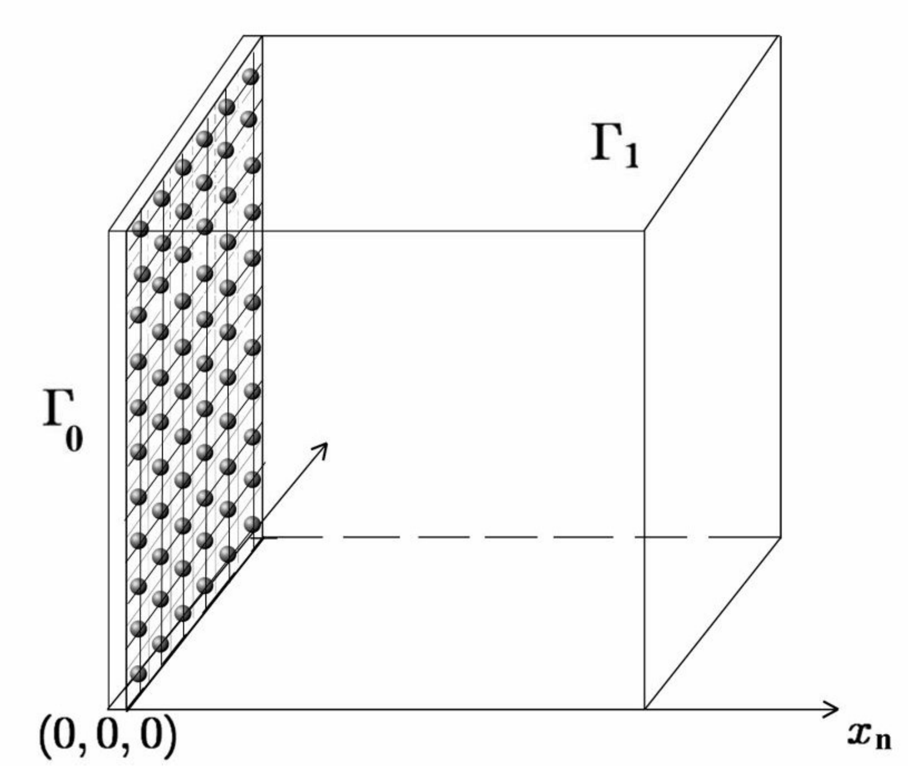

FIGURE 1. Example of a spatial domain with a *double skin boundary.*

where the weighted balance is defined through the arbitrary positive constants η, N, and the H^1-metric is defined by the $n \times n$ symmetric matrix $B(x) = (b_{ij}(x))$ such that

$$\lambda_1|\xi|^2 \leq b_{ij}(x)\xi_i\xi_j \leq \lambda_2|\xi|^2, \tag{3}$$

for a.e. $x \in \Omega$, $\lambda_i = const > 0$, $i = 1, 2$, $B \in C^1(\overline{\Omega})^{n\times n}$.

By well-known results (see e.g. [20], [15], [25]), there exists a unique optimal control $v_\varepsilon \in L^2(\Gamma_0)$

$$J_\varepsilon(v_\varepsilon) = \min_{v\in L^2(\Gamma_0)} J_\varepsilon(v). \tag{4}$$

One of the main goals of this paper is to find the limit as $\varepsilon \to 0$ of the optimal control v_ε, of the associate state $u_\varepsilon(v_\varepsilon)$ and of the cost functional value $J_\varepsilon(v_\varepsilon)$.

In order to characterize the optimal control v_ε, we will study the particularization of the abstract version of the Pontryagin maximum principle applied to elliptic PDEs mentioned in Section 1.3 of Lions [20].

Proposition 1. *Let $u_T \in H^1(\Omega)$, $v_\varepsilon \in L^2(\Gamma_0)$ and $u_\varepsilon(v_\varepsilon) \in H^1(\Omega_\varepsilon, \Gamma_1)$ be the target state, the optimal control and the associate optimal state, respectively. Let $P_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1)$ be* *the unique solution of the problem*

$$\begin{cases} \Delta P_\varepsilon = div(B(x)\nabla(u_\varepsilon - u_T)), & x \in \Omega_\varepsilon, \\ \partial_\nu P_\varepsilon - (B(x)\nabla(u_\varepsilon - u_T), \nu) + \varepsilon^{-\gamma} a(x) P_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu P_\varepsilon - (B(x)\nabla(u_\varepsilon - u_T), \nu) = 0, & x \in \Gamma_0, \\ P_\varepsilon = 0, & x \in \Gamma_1. \end{cases} \tag{5}$$

Then, the optimal control v_ε is given by

$$v_\varepsilon = -\frac{\eta}{N} P_\varepsilon. \tag{6}$$

Proof. Since v_ε is the optimal control, we know that for any other control $v \in L^2(\Gamma_0)$

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)) = 0.$$

It is easy to see that if, for a given $\lambda \in \mathbb{R}$, we define

$$\theta_\varepsilon = \frac{1}{\lambda}(u_\varepsilon(v_\varepsilon + \lambda v) - u_\varepsilon(v_\varepsilon)),$$

then θ_ε is a weak solution to the problem

$$\begin{cases} \Delta \theta_\varepsilon = 0, & x \in \Omega_\varepsilon, \\ \partial_\nu \theta_\varepsilon + \varepsilon^{-\gamma} a(x) \theta_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu \theta_\varepsilon = v, & x \in \Gamma_0, \\ \theta_\varepsilon = 0, & x \in \Gamma_1. \end{cases} \tag{7}$$

Then, we have

$$0 = \lim_{\lambda \to 0} \frac{1}{\lambda}\Big(J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)\Big) = \eta \int_{\Omega_\varepsilon} B(x)\nabla(u_\varepsilon(v_\varepsilon) - u_T)\nabla\theta_\varepsilon dx + N \int_{\Gamma_0} v_\varepsilon v d\hat{x}. \tag{8}$$

We recall that if $P_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1)$ is a solution of (5), then we have the integral identity

$$\int_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla \phi dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon \phi ds = \int_{\Omega_\varepsilon} B(x)\nabla(u_\varepsilon - u_T)\nabla\phi dx, \tag{9}$$

for an arbitrary function $\phi \in H^1(\Omega_\varepsilon, \Gamma_1)$. Then, we can set $\phi = \theta_\varepsilon$ in it and use P_ε as a test function in the integral identity for the problem (7). Subtracting the one from the other, we get

$$\eta \int_{\Omega_\varepsilon} B(x)\nabla(u_\varepsilon(v_\varepsilon) - u_T)\nabla\theta_\varepsilon dx + N \int_{\Gamma_0} v_\varepsilon v d\hat{x} = \int_{\Gamma_0} (\eta P_\varepsilon + N v_\varepsilon) v d\hat{x} = 0,$$

from which we deduce that $v_\varepsilon = -\frac{\eta}{N} P_\varepsilon$.■

In order to get the homogenization (as $\varepsilon \to 0$) we will use the usual H^1-extensions of functions u_ε and P_ε to Ω which we denote by $\widetilde{u_\varepsilon}$ and $\widetilde{P_\varepsilon}$ (see, e.g., Section 3.1.1 of [9] and its references).

Then, from the properties of the extension operator (see [26]) and estimates (19), we have

Theorem 1. *Let $f \in L^2(\Omega)$, $u_T \in H^1(\Omega)$ and let $(u_\varepsilon, P_\varepsilon) \in H^1(\Omega_\varepsilon, \Gamma_1)^2$ be the weak solution of the coupled system*

$$\begin{cases} -\Delta u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ \Delta P_\varepsilon = div(B(x)\nabla(u_\varepsilon - u_T)), & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} a(x) u_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu P_\varepsilon - (B(x)\nabla(u_\varepsilon - u_T), \nu) + \varepsilon^{-\gamma} a(x) P_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu u_\varepsilon = -\dfrac{\eta}{N} P_\varepsilon, & x \in \Gamma_0, \\ \partial_\nu P_\varepsilon - (B(x)\nabla(u_\varepsilon - u_T), \nu) = 0, & x \in \Gamma_0, \\ u_\varepsilon = P_\varepsilon = 0, & x \in \Gamma_1. \end{cases} \tag{10}$$

Then,

$$\|\widetilde{u_\varepsilon}\|_{H^1(\Omega,\Gamma_1)} \le C, \;\; \|\widetilde{P_\varepsilon}\|_{H^1(\Omega,\Gamma_1)} \le C, \tag{11}$$

and thus there exists some subsequences (still denoted as the original ones) such that

$$\widetilde{u_\varepsilon} \rightharpoonup u_0, \;\; \widetilde{P_\varepsilon} \rightharpoonup P_0, \;\; \textit{weakly in } H^1(\Omega, \Gamma_1), \tag{12}$$

as $\varepsilon \to 0$, for some $(u_0, P_0) \in H^1(\Omega, \Gamma_1)^2$.

Proof. We start by getting some a priori estimates for u_ε and P_ε. From the integral identity for the function P_ε, we derive

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla P_\varepsilon|^2 dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon^2 ds = \int_{\Omega_\varepsilon} B(x)\nabla(u_\varepsilon - u_T)\nabla P_\varepsilon dx \le \\ \le \frac{1}{2}\|\nabla P_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + C\|\nabla(u_\varepsilon - u_T)\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{13}$$

The constant C doesn't depend on ε here and below. From here, we conclude

$$\|\nabla P_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|P_\varepsilon\|^2_{L^2(S_\varepsilon)} \le C\|\nabla(u_\varepsilon - u_T)\|^2_{L^2(\Omega_\varepsilon)}. \tag{14}$$

From the integral identity for the function u_ε, we have

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) u_\varepsilon P_\varepsilon ds = \int_{\Omega_\varepsilon} f P_\varepsilon dx - \frac{\eta}{N}\int_{\Gamma_0} P_\varepsilon^2 d\hat{x}. \tag{15}$$

From the integral identity for the function P_ε, we get

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon u_\varepsilon ds = \int_{\Omega_\varepsilon} B(x)\nabla(u_\varepsilon - u_T)\nabla u_\varepsilon dx. \tag{16}$$

Subtracting the equality (15) from (16), we get

$$\frac{\eta}{N}\int_{\Gamma_0} P_\varepsilon^2 d\hat{x} + \int_{\Omega_\varepsilon} B(x)\nabla(u_\varepsilon - u_T)\nabla(u_\varepsilon - u_T) dx = \int_{\Omega_\varepsilon} f P_\varepsilon dx - \int_{\Omega_\varepsilon} B(x)\nabla u_T \nabla(u_\varepsilon - u_T) dx. \tag{17}$$

This equality and the condition (3) imply

$$\|P_\varepsilon\|_{L^2(\Gamma_0))} \le C(\|\nabla u_T\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}), \; \|\nabla(u_\varepsilon - u_T)\|_{L^2(\Omega_\varepsilon)} \le C(\|\nabla u_T\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}). \tag{18}$$

From (14) and (18), we derive

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon,\Gamma_1)} \le C, \ \|P_\varepsilon\|_{H^1(\Omega_\varepsilon,\Gamma_1)} \le C. \tag{19}$$

Since $\widetilde{u_\varepsilon}$ and $\widetilde{P}_\varepsilon$, are the H^1-extensions of u_ε and P_ε to Ω, from the properties of the extension operator (see, e.g., [26]) and estimates (19), we get the estimates (11) and thus we conclude the weak convergences indicated in (12) for some subsequences. ■

In order to identify the limit problem satisfied by the pair (u_0, P_0), we need several auxiliary results.

3. Auxiliary statements

For $j \in \Upsilon_\varepsilon$, we introduce the function $w_\varepsilon^j(x)$ as being the unique solution to the capacity boundary value problem

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 1, & x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j, \end{cases} \tag{20}$$

where $T_{\varepsilon/4}^j$ is the ball of radius $\varepsilon/4$ with center in the point P_ε^j (the center of the cube Y_ε^j). It is easy to see (Section 3.1.5.1 of [9]) that

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{2-n} - \left(\frac{\varepsilon}{4}\right)^{2-n}}{a_\varepsilon^{2-n} - \left(\frac{\varepsilon}{4}\right)^{2-n}}.$$

Define the extension function

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon, \\ 1, & x \in \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j, \\ 0, & x \in \Omega \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T_{\varepsilon/4}^j}. \end{cases} \tag{21}$$

It is clear that we have $W_\varepsilon \rightharpoonup 0$, weakly in $H_0^1(\Omega)$, as $\varepsilon \to 0$.

Lemma 1. *Let $\phi \in C^1(\overline{\Omega}, \Gamma_1)$. Then,*

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (B\nabla W_\varepsilon, \nabla W_\varepsilon)\phi dx = \frac{C_0^{n-2}(n-2)\omega_n}{n} \int_{\Gamma_0} trB(0,\hat{x})\phi(0,\hat{x})d\hat{x}, \tag{22}$$

where $trB(x) = \sum_{j=1}^{n} b_{jj}(x)$ *is the trace of the matrix* $B(x)$ *and* ω_n *is the surface area of the unit sphere in* $\mathbb{R}^n$.

Proof. Note that if $\widetilde{B}$ is the matrix with the constant elements (with respect to the $y-$variable), then

$$\int_{S_1^0} (\widetilde{B}y, y)ds = \int_{T_1^0} div(\widetilde{B}y)dy = tr\widetilde{B}|T_1^0| = tr\widetilde{B}\frac{\omega_n}{n}, \tag{23}$$

where S_1^0, T_1^0 are the unit sphere and the unit ball with the center at the coordinate's origin.

Using equality (23), if $r=\sqrt{\sum\limits_{j=1}^{n}(x_i-P^j_{\varepsilon,i})^2}$, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(B(x)\nabla W_\varepsilon,\nabla W_\varepsilon)\phi(x)dx=\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\phi(P^j_\varepsilon)\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}(B(P^j_\varepsilon)\nabla w^j_\varepsilon,\nabla w^j_\varepsilon)dx=$$

$$=\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}a_\varepsilon^{2(n-2)}(n-2)^2\phi(P^j_\varepsilon)\sum_{k,l=1}^{n}\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}b_{kl}(P^j_\varepsilon)(x_l-P^j_{\varepsilon,l},x_k-P^j_{\varepsilon,k})r^{-2n}dr=$$

$$=\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}a_\varepsilon^{2(n-2)}(n-2)^2\phi(P^j_\varepsilon)\int_{a_\varepsilon}^{\varepsilon/4}r^{1-n}dr\int_{S_1^0}(B(P^j_\varepsilon)y,y)ds=$$

$$=\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}a_\varepsilon^{2(n-2)}(n-2)^2\frac{\omega_n}{n}\phi(P^j_\varepsilon)trB(P^j_\varepsilon)\int_{a_\varepsilon}^{\varepsilon/4}r^{1-n}dr=$$

$$=C_0^{n-2}(n-2)\frac{\omega_n}{n}\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\phi(P^j_\varepsilon)trB(P^j_\varepsilon)\varepsilon^{n-1}=\frac{C_0^{n-2}\omega_n(n-2)}{n}\int_{\Gamma_0}trB(0,\hat{x})\phi(0,\hat{x})d\hat{x},$$

which proves the result.■

We consider now the auxiliary boundary value problem

$$\begin{cases}\Delta h_\varepsilon=div(B\nabla W_\varepsilon), & x\in\Omega_\varepsilon,\\ \partial_\nu h_\varepsilon-(B\nabla W_\varepsilon,\nu)+\varepsilon^{-\gamma}a(x)h_\varepsilon=0, & x\in S_\varepsilon,\\ \partial_\nu h_\varepsilon-(B\nabla W_\varepsilon,\nu)=0, & x\in\Gamma_0,\\ h_\varepsilon=0, & x\in\Gamma_1.\end{cases}\tag{24}$$

As usual, we say that function $h_\varepsilon\in H^1(\Omega_\varepsilon,\Gamma_1)$ is a weak solution of problem (24) if it satisfies the integral identity

$$\int_{\Omega_\varepsilon}\nabla h_\varepsilon\nabla\phi dx+\varepsilon^{-\gamma}\int_{S_\varepsilon}a(x)h_\varepsilon\phi ds=\int_{\Omega_\varepsilon}(B\nabla W_\varepsilon,\nabla\phi)dx,\tag{25}$$

where ϕ is an arbitrary function from the space $H^1(\Omega_\varepsilon,\Gamma_1)$. Setting $\phi=h_\varepsilon$ in (20) and using the properties of function W_ε, we derive an estimate $\|\widetilde{h}_\varepsilon\|_{H^1(\Omega,\Gamma_1)}\le C$. Therefore, there exists a subsequence (we preserve the notation of the original one) such that

$$\widetilde{h}_\varepsilon\rightharpoonup h_0,\ \text{weakly in } H^1(\Omega,\Gamma_1),\ \varepsilon\to 0.\tag{26}$$

The following theorem identifies the limit function h_0.

Theorem 2. *The function h_0 defined in (26) is a weak solution of the boundary value problem*

$$\begin{cases}\Delta h_0=0, & x\in\Omega,\\ \partial_\nu h_0+\mathcal{A}_1\dfrac{a(x)}{a(x)+C_n}h_0=-\mathcal{A}_2\dfrac{trB(x)a(x)}{a(x)+C_n}, & x\in\Gamma_0,\\ h_0=0, & x\in\Gamma_1,\end{cases}\tag{27}$$

where $\mathcal{A}_1(n)=(n-2)C_0^{n-2}\omega_n$, $\mathcal{A}_2(n)=\dfrac{\mathcal{A}_1(n)}{n}$, $C_n=\dfrac{n-2}{C_0}$.

Proof. We take as test function in the integral identity (25) the function $\phi = W_\varepsilon \psi$, where $\psi \in C^\infty(\overline{\Omega}, \Gamma_1)$ (this is called an *oscillating test function* according to the general Tartar's method) and we get

$$\int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (h_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) h_\varepsilon \psi ds = \int_{\Omega_\varepsilon} (B(x) \nabla W_\varepsilon, \nabla W_\varepsilon) \psi dx + \beta_\varepsilon, \tag{28}$$

where $\beta_\varepsilon \to 0$, $\varepsilon \to 0$.
From (28), we derive

$$\begin{gathered} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_\nu w_\varepsilon^j h_\varepsilon \psi ds + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) h_\varepsilon \psi ds + \\ + \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} \partial_\nu w_\varepsilon^j h_\varepsilon \psi ds = \frac{C_0^{n-2}(n-2)\omega_n}{n} \int_{\Gamma_0} trB(\hat{x}) \psi(\hat{x}) d\hat{x}. \end{gathered} \tag{29}$$

Hence, it follows that

$$\begin{gathered} \varepsilon^{-\gamma} \int_{S_\varepsilon} \left(a(x) + \frac{n-2}{C_0} \right) h_\varepsilon \psi ds = \\ - \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} \partial_\nu w_\varepsilon^j h_\varepsilon \psi ds + \frac{C_0^{n-2}\omega_n (n-2)}{n} \int_{\Gamma_0} trB(\hat{x}) \psi(\hat{x}) d\hat{x} + \beta_\varepsilon. \end{gathered} \tag{30}$$

We set $\psi(x) = \dfrac{a(x)}{a(x) + C_n} v(x)$, where $v \in C^\infty(\overline{\Omega}, \Gamma_1)$, $C_n = \dfrac{n-2}{C_0}$, and get

$$\begin{gathered} \lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) h_\varepsilon v ds = \\ = C_0^{n-2}(n-2)\omega_n \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} h_0 v d\hat{x} + \frac{C_0^{n-2}(n-2)\omega_n}{n} \int_{\Gamma_0} \frac{trB(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} v(\hat{x}) d\hat{x}. \end{gathered} \tag{31}$$

Therefore, from (25) and (31), we obtain the integral identity for the function h_0

$$\int_\Omega \nabla h_0 \nabla \phi dx + \mathcal{A}_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} h_0 \phi d\hat{x} + + \frac{\mathcal{A}_1}{n} \int_{\Gamma_0} \frac{trB(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} \phi(\hat{x}) d\hat{x} = 0.$$

This implies the statement of the theorem. ∎

4. The main result

So, in order to obtain the characterization of u_0 and P_0, we have to pass to the limit in the identity (9).

Theorem 3. *Let $n \geq 3$, $\alpha = \gamma = \frac{n-1}{n-2}$ and let $(u_\varepsilon, P_\varepsilon)$ be a weak solution of the coupled system (10). Then, the pair (u_0, P_0) defined in (12) is a weak solution of the system*

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega, \\ \Delta P_0 = div(B\nabla(u_0 - u_T)), & x \in \Omega, \\ \partial_\nu u_0 + \mathcal{A}_1 \dfrac{a(x)}{a(x)+C_n} u_0 = -\dfrac{\eta}{N} P_0, & x \in \Gamma_0, \\ \partial_\nu P_0 - (B(x)\nabla(u_0 - u_T), \nu) + \mathcal{A}_1 \dfrac{a(x)}{a(x)+C_n} P_0 - \mathcal{A}_2 \dfrac{trB(x)a^2(x)}{(a(x)+C_n)^2} u_0 = 0, & x \in \Gamma_0, \\ u_0 = P_0 = 0, & x \in \Gamma_1, \end{cases} \tag{32}$$

where $trB(x) = \sum_{j=1}^{n} b_{jj}(x)$ is the trace of the matrix $B(x)$, $\mathcal{A}_1(n) = (n-2)C_0^{n-2}\omega_n$, $\mathcal{A}_2(n) = \dfrac{\mathcal{A}_1(n)}{n}$, $C_n = \dfrac{n-2}{C_0}$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. In addition, if we define the functional

$$J_0(v) = \frac{\eta}{2} \int_\Omega B\nabla(u_0(v) - u_T)\nabla(u_0(v) - u_T)dx + \frac{N}{2} \int_{\Gamma_0} v^2 d\hat{x}$$
$$+ \frac{\eta \mathcal{A}_1}{2n} \int_{\Gamma_0} trB(\hat{x}) \Big(\frac{a(\hat{x})}{a(\hat{x}) + C_n} \Big)^2 u_0^2(v) d\hat{x},$$

then

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0), \tag{33}$$

where $v_0 = -\dfrac{\eta}{N} P_0$ on Γ_0. In particular, $v_\varepsilon \rightharpoonup v_0$ weakly in $L^2(\Gamma_0)$, and v_0 is the optimal control of the problem

$$J_0(v_0) = \inf_{v \in L^2(\Gamma_0)} J_0(v), \tag{34}$$

associated to the state problem

$$\begin{cases} -\Delta u_0(v) = f, & x \in \Omega, \\ \partial_\nu u_0 + \mathcal{A}_1 \dfrac{a(x)}{a(x)+C_0} u_0(v) = v, & x \in \Gamma_0, \\ u_0(v) = 0, & x \in \Gamma_1. \end{cases}$$

Proof. Let us find the homogenized boundary value problem for the function u_0. We set $\phi = W_\varepsilon \psi$ in the integral identity for u_ε and get

$$\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) u_\varepsilon \phi ds = \mathcal{A}_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 \phi d\hat{x}.$$

Hence, $u_0 \in H^1(\Omega, \Gamma_1)$ satisfies integral identity

$$\int_\Omega \nabla u_0 \nabla \phi dx + \mathcal{A}_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 \phi d\hat{x} = \int_\Omega f \phi dx - \frac{\eta}{N} \int_{\Gamma_0} P_0 \phi d\hat{x}, \tag{35}$$

for any $\phi \in H^1(\Omega, \Gamma_1)$. From here, we derive that u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega, \\ \partial_\nu u_0 + \mathcal{A}_n \dfrac{a(x)}{a(x)+C_n} u_0 = -\frac{\eta}{N} P_0, & x \in \Gamma_0, \\ u_0 = 0, & x \in \Gamma_1. \end{cases}$$

Next, let us obtain the limit problem satisfied by P_0. Let $\psi \in C^\infty(\overline{\Omega}, \Gamma_1)$. We take $\phi = W_\varepsilon \psi$ as a test function in the integral identity for the function P_ε and get

$$\int_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla (W_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon \psi ds = \int_{\Omega_\varepsilon} B(x) \nabla (u_\varepsilon - u_T) \nabla (W_\varepsilon \psi) dx. \tag{36}$$

In order to pass to the limit, as $\varepsilon \to 0$, in the right-hand side of the identity (35), we set $\phi = u_\varepsilon \psi$ in the integral identity (21) and $\phi = h_\varepsilon \psi$ in the integral identity for u_ε. Then, subtracting one from the other, we get

$$\begin{gathered} \int_{\Omega_\varepsilon} B(x) \nabla W_\varepsilon \nabla (u_\varepsilon \psi) dx = \int\limits_{\Omega_\varepsilon} \nabla h_\varepsilon \nabla (u_\varepsilon \psi) dx - \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla (h_\varepsilon \psi) dx + \\ + \int\limits_{\Omega_\varepsilon} f h_\varepsilon \psi dx - \frac{\eta}{N} \int\limits_{\Gamma_0} P_\varepsilon h_\varepsilon \psi dx. \end{gathered} \tag{37}$$

Identity (37) implies

$$\begin{gathered} \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} B \nabla W_\varepsilon \nabla (u_\varepsilon \psi) dx = \\ = \int_\Omega \nabla h_0 \nabla \psi u_0 dx - \int_\Omega h_0 \nabla u_0 \nabla \psi dx + \int_\Omega f h_0 \psi dx - \frac{\eta}{N} \int_{\Gamma_0} P_0 h_0 \psi d\hat{x}. \end{gathered} \tag{38}$$

We now use the fact that we already know the problems satisfied by u_0 and h_0. So, we get

$$\begin{gathered} \int_\Omega (\nabla h_0, \nabla \psi) u_0 dx - \int_\Omega h_0 \nabla u_0 \nabla \psi dx = \int_\Omega \nabla h_0 \nabla (u_0 \psi) dx - \int_\Omega \nabla u_0 \nabla (h_0 \psi) dx = \\ -\frac{\mathcal{A}_1}{n} \int_{\Gamma_0} \frac{trB(\hat{x}) a(\hat{x})}{a(\widehat{x}) + C_n} u_0 \psi d\hat{x} - \int_\Omega f h_0 \psi dx + \frac{\eta}{N} \int_{\Gamma_0} P_0 \psi d\hat{x} \end{gathered} \tag{39}$$

Comparing expressions (38) and (39), we conclude

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} B \nabla W_\varepsilon \nabla (u_\varepsilon \psi) dx = -\frac{\mathcal{A}_1}{n} \int_{\Gamma_0} \frac{trB(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} u_0 \psi d\hat{x}. \tag{40}$$

We take $W_\varepsilon \psi$, with $\psi \in C^\infty(\overline{\Omega}, \Gamma_1)$, as a test function in the integral identity for P_ε and get

$$\int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (P_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon \psi(x) ds = \int_{\Omega_\varepsilon} B \nabla W_\varepsilon \nabla (u_\varepsilon \psi) ds + \kappa_\varepsilon, \tag{41}$$

where $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$.

From the definition of the function W_ε and its properties, we can transform the left-hand side of equality (41) in the following way

$$\sum_{j\in\Upsilon_\varepsilon}\int_{\partial\ G^j_\varepsilon\cup\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon P_\varepsilon\psi ds+\varepsilon^{-\gamma}\int_{S_\varepsilon}a(x)P_\varepsilon\psi ds=-\frac{\mathcal{A}_1}{n}\int_{\Gamma_0}\frac{trB(\hat{x})a(\hat{x})}{a(\hat{x})+C_n}u_0\psi d\hat{x}+\kappa_\varepsilon,$$

where $\kappa_\varepsilon\to 0$ as $\varepsilon\to 0$.
The left-hand side of the last equality takes the form

$$\varepsilon^{-\gamma}\int_{S_\varepsilon}(a(x)+C_n)P_\varepsilon\psi ds+\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon P_\varepsilon\psi ds+\theta_\varepsilon,\tag{42}$$

where $\theta_\varepsilon\to 0$ as $\varepsilon\to 0$.
From (40)-(42), we deduce

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int_{S_\varepsilon}a(x)P_\varepsilon\varphi ds=\mathcal{A}_1\int_{\Gamma_0}\frac{a(\hat{x})}{a(\hat{x})+C_n}P_0\varphi d\hat{x}-\frac{\mathcal{A}_1}{n}\int_{\Gamma_0}trB(\hat{x})\Big(\frac{a(\hat{x})}{a(\hat{x})+C_n}\Big)^2u_0\varphi d\hat{x},\tag{43}$$

where φ is an arbitrary function from $C^\infty(\overline{\Omega},\Gamma_1)$. Passing to the limit in the integral identity for P_ε we get the theorem's statement.
Now we will find the limit as $\varepsilon\to 0$ of the cost functional

$$J_\varepsilon(v_\varepsilon)=\frac{\eta}{2}\int_{\Omega_\varepsilon}B\nabla(u_\varepsilon(v_\varepsilon)-u_T)\nabla(u_\varepsilon(v_\varepsilon)-u_T)dx+\frac{N}{2}\|v_\varepsilon\|^2_{L^2(\Gamma_0)}.$$

We take $\phi=u_\varepsilon$ in the integral identity for P_ε and $\phi=P_\varepsilon$ in the integral identity for u_ε. Then, we subtract the one from the other and pass to the limit as $\varepsilon\to 0$:

$$A\equiv\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}B(x)\nabla(u_\varepsilon-u_T)\nabla u_\varepsilon dx=\int_\Omega fP_0dx-\frac{\eta}{N}\int_{\Gamma_0}P_0^2d\hat{x}.\tag{44}$$

Taking into account the integral identity for the limit of the adjoint problem, we get

$$\begin{aligned}\mathcal{A}&=\int_\Omega\nabla u_0\nabla P_0dx+\mathcal{A}_1\int_{\Gamma_0}\frac{a(\widehat{x})}{a(\widehat{x})+C_n}u_0P_0dx=\\&=\int_\Omega B\nabla(u_0-u_T)\nabla u_0dx+\frac{\mathcal{A}_1}{n}\int_{\Gamma_0}trB(\hat{x})\Big(\frac{a(\hat{x})}{a(\hat{x})+C_n}\Big)^2u_0^2d\hat{x}.\end{aligned}\tag{45}$$

From here, we obtain (33). Since the trace is a continuous operator on $H^1(\Omega_\varepsilon,\Gamma_1)$ we get that $v_\varepsilon\rightharpoonup v_0$ weakly in $L^2(\Gamma_0)$. Since we have that $v_0=-\frac{\eta}{N}P_0$ on Γ_0 and this is the optimality condition associated to the control problem (34) (the proof is an easy variation of Proposition 1), then v_0 is the unique optimal control problem associated to the convex cost functional $J_0(v)$.■

Remark 1. *It is not too difficult to prove that the coupled system (32) has only one weak solution* (u_0,P_0). *For instance, one indirect proof can be obtained through the strict convexity of the functional* J_0. *In particular, this implies that the weak convergence obtained* *in Theorem 1 holds for any subsequences of the original ones (since the limit* (u_0, P_0) *is unique).*

Remark 2. *Such as it is detailed explained in the book* [9]*, it can be proved that the choice of the scales and parameters* $\alpha = \gamma = \frac{n-1}{n-2}$ *is the reason to get an anomalous boundary behavior on* Γ_0 *(for bigger size of the elements of the lattice we get different coefficients). This phenomenon was called in the literature as the appearance of a "strange term" and in several papers it was associated to a certain "measure"* μ*. One of the merits of Theorem 3 is to show that the "strange term" is a certain completely identified function on* Γ_0.

Remark 3. *Similar problems arise in many different applications, especially in the field of Chemical Engineering (see, e.g.,* [17] *and Chapter 5 of* [9]*). Some models in Climatology also use the identification as a final boundary condition the limit of a thin layer on which there are some suitable balances of differential equations and transmissions conditions (see the so called energy balance models coupled with a deep ocean in* [11]*). Problems quite similar to the one considered in this paper arise also in Elasticity (see, e.g.,* [5]*).*

Remark 4. *Many generalizations and applications seem possible and some of them will be developed in some future works by the authors: i) Non-symmetrical shapes can also be treated thanks to the techniques presented in* [13]*, ii) The case of non-periodic lattices (under the assumption of "stationary and ergodic" random media) can be considered as in the framwork traeted in* [3]*,* [18]*,* [6] *and* [19] *(see many other references in Appendice C of the book* [9]*). iii) Optimal control problems for semilinear equations and/or nonlinear boundary conditions could be approached as, for instance, in* [7]*. iv) The extension of the techniques of this paper can be also applied to the consideration of several parabolic problems (see, e.g., Appendix A of* [9]*) and its references. v) by passing to the limit when parameter* $N \to +\infty$ *it is possible to get some results on the approximate controllability with internal observation (the* H^1 *norm of* $(u_\varepsilon(v) - u_T)$ *can be made as small as wanted): see* [16].

5. Convergence of the energy for the problem without control

In this last Section, we consider the boundary value problem without any control (i.e. problem (1) with $v \equiv 0$)

$$\begin{cases} -\Delta u_\varepsilon(v) = f, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = 0, & x \in S_\varepsilon, \\ \partial_\nu u_\varepsilon(v) = 0, & x \in \Gamma_0, \\ u_\varepsilon(v) = 0, & x \in \Gamma_1, \end{cases} \tag{46}$$

where $f \in L^2(\Omega)$, $a(x) \in C^\infty(\overline{\Omega})$, $a(x) \geq a_0 = const > 0$. The homogenization techniques of previous sections can be easily adapted to prove that $\widetilde{u_\varepsilon} \rightharpoonup u_0$ weakly in $H^1(\Omega_\varepsilon, \Gamma_1)$ and that u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u_0(v) = f, & x \in \Omega, \\ \partial_\nu u_0 + \mathcal{A}_1 \dfrac{a(x)}{a(x)+C_0} u_0(v) = 0, & x \in \Gamma_0, \\ u_0(v) = 0, & x \in \Gamma_1. \end{cases}$$

Our main goal now is to prove that the consideration of an artificial complementary system (formally corresponding to the case $v \equiv 0$ and $u_T \equiv 0$ and $B = I$) allows to prove the convergence of the corresponding energies.

Theorem 4. *Let u_ε be the solution of (46) with $v \equiv 0$. Let $u_0 \in H_0^1(\Omega)$ be the weak limit of the extension $P_\varepsilon u_\varepsilon$. Then, we have the convergence of the energy*

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \to \int_\Omega |\nabla u_0|^2 dx + \mathcal{A}_1 \int_{\Gamma_0} \Big(\frac{a(\hat{x})}{a(\hat{x})+C_n} \Big)^2 u_0^2 d\hat{x}. \tag{47}$$

Proof. We consider the auxiliary problem

$$\begin{cases} \Delta P_\varepsilon = \Delta u_\varepsilon & x \in \Omega_\varepsilon, \\ \partial_\nu P_\varepsilon - \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} a(x) P_\varepsilon = 0, & x \in S_\varepsilon, \\ \partial_\nu P_\varepsilon - \partial_\nu u_\varepsilon = 0, & x \in \Gamma_0, \\ P_\varepsilon = 0, & x \in \Gamma_1. \end{cases} \tag{48}$$

As in the proof of Theorem 3, we get that $\widetilde{P_\varepsilon} \rightharpoonup P_0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$, with P_0 the weak solution of the problem

$$\begin{cases} \Delta P_0 = \Delta u_0 & x \in \Omega, \\ \partial_\nu P_0 - \partial_\nu u_0 + \mathcal{A}_1 \dfrac{a(x)}{a(x)+C_n} P_0 - \mathcal{A}_1 \dfrac{a^2(x)}{(a(x)+C_n)^2} u_0 = 0, & x \in \Gamma_0, \\ P_0 = 0, & x \in \Gamma_1, \end{cases} \tag{49}$$

where $A_1(n) = (n-2)C_0^{n-2}\omega_n$, $C_n = \dfrac{n-2}{C_0}$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

From the variational formulation of the problem (46), taking P_ε as a test function, we have

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) u_\varepsilon P_\varepsilon ds = \int_{\Omega_\varepsilon} f P_\varepsilon dx.$$

Similarly, from the variational formulation of the problem (49) on P_ε, taking u_ε as a test function, we derive

$$\int_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla u_\varepsilon dx + \varepsilon^{-k} \int_{S_\varepsilon} a(x) P_\varepsilon u_\varepsilon ds = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.$$

Thus, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} f P_\varepsilon dx \to \int_\Omega f P_0 dx =$$
$$= \int_\Omega \nabla u_0 \nabla P_0 dx + \mathcal{A}_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 P_0 d\widehat{x} =$$
$$= \int_\Omega |\nabla u_0|^2 dx + \mathcal{A}_1 \int_{\Gamma_0} \Big(\frac{a(\hat{x})}{a(\hat{x}) + C_n}\Big)^2 u_0^2 d\hat{x},$$

which ends the proof. ■

Remark 5. *It can be proved (for instance, by adapting the arguments presented in Section 4.7.1.4 of* [9]*) that if we know that* $u_0 \in W^{1,\infty}(\Omega)$ *then we can get some results implying the strong convergence of* u_ε *plus a suitable "correction term". Notice that the conclusion presented in the proof of Theorem 4 follows different ideas.*

Ackowledgements. The research of J.I. Díaz was partially supported the projects MTM2017-85449-P and PID2020-112517GB-I00 of the DGISPI, Spain and the Research Group MOMAT (Ref. 910480) of the UCM.

References

[1] A.Azad, T. Ngo and B.Samali. Control of wind-Induced Motion of Tall Buildings Using Smart Façade Systems, *Electronic Journal of Structural Engineering* **14** 1 (2015), 33-40

[2] M. Ben Bonham, *Bioclimatic Double Skin Façades*, Taylor and Francis, Ney York, 2020.

[3] X. Blanc, C. Le Bris, and P.-L. Lions. Stochastic homogenization and random lattices. *J. Math.Pures Appl.* **88**.1 (2007), 34–63.

[4] W. W. Braham, Active Glass Walls. A Typological and Historical Account, *Departmental Papers (Architecture)*, 2005.

[5] H. Brezis, L. Caffarelli and A. Friedman. Reinforcement problems for elliptic equations and variational inequalities, *Ann. Mat. Pura et Appl.* **123** (1980), 219-246.

[6] L. A. Caffarelli and A. Mellet. Random homogenization of an obstacle problem. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **26** 2 (2009), 375–395.

[7] C. Conca, P. Donato, E. C. Jose, and I. Mishra. Asymptotic analysis of optimal controls of a semilinear problem in a perforated domain. *J. Ramanujan Math. Soc.* **31** 3 (2016), 265–305.

[8] L.J. Claros-Marfil, B. Lauret and J.F. Padial, A new and inexpensive open source data acquisition and controller for solar research: Application to a water-flow glazing, *Renewable Energy* **92** (2016), 450-461.

[9] J.I. Díaz, D. Gómez-Castro and T.A. Shaposhnikova. *Nonlinear Reaction-Diffusion Processes for Nanocomposites. Anomalous improved homogenization.* De Gruyter Series in Nonlinear Analysis and Applications **39**, Walter de Gruyter GmbH & Co KG, Berlin, 2021.

[10] J.I. Díaz, D. Gómez-Castro, T.A. Shaposhnikova and M.N. Zubova. Change of homogenized adsorption term in diffusion processes with reaction on the boundary of periodically distributed asymmetric particles of critical size. *Electronic Journal of Differential Equations.* **178** (2017), 1-25.

[11] J. I. Díaz, A. Hidalgo and L. Tello. Multiple solutions and numerical analysis to the dynamic and stationary models coupling a delayed energy balance model involving latent heat and discontinuous albedo with a deep ocean. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **470** 2170 (2014), p. 20140376.

[12] J.I.Díaz, A.V. Podolskiy and T.A. Shaposhnikova. On the convergence of controls and cost functionals in some optimal control heterogeneous problems when the homogenization process gives rise to some strange term. *J. Math. Anal. Appl.* **506** 1 (2021), 125559.

[13] J.I.Díaz, A.V. Podolskiy and T.A. Shaposhnikova. On the homogenization of an optimal control problem in a domain perforated by holes of critical size and arbitrary shape. To appear in *Doklady Mathematics.*

[14] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique.* Paris. Dunod, 1972.

[15] A. V. Fursikov. *Optimal Control of Distributed Systems. Theory and Applications*, Translations of Mathematical Monographs, American Mathematical Society. Providence, RI. 1999.

[16] R. Glowinski, J.-L. Lions, J. He. *Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach*, Cambridge University Press, 2008.

[17] D. Gómez, M. Lobo, E. Pérez, and E. Sánchez-Palencia. Homogenization in perforated domains: a Stokes grill and an adsorption process. *Appl. Anal.* **97**.16 (2018), 2893–2919.

[18] V. V. Jikov, S. Kozlov, and O. A. Oleinik. *Homogenization of differential operators and integral functionals.* Springer Verlag, Berlin, 1994.

[19] E. Y. Khruslov and L. A. Khil'kova. A model of stationary diffusion with absorption in domains with a fine-grained random boundary. *Ukr. Math. J.* **71** 5 (2019), 692–705.

[20] J. L. Lions. *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles*, Dunod, Paris 1968.

[21] A.V. Podolskiy, T.A. Shaposhnikova. Optimal Control and Strange Term Arising from Homogenization of the Poisson Equation in the Perforated Domain with the Robin-type Boundary Condition in the Critical Case. *Doklady Mathematics*, 102 (2020), 497-501.

[22] J.Saint Jean Paulin and H. Zoubairi. Optimal control and "strange term" for a Stokes problem in perforated domains. *Portugaliae Mathematica*, **9**. 2 (2002),161-178.

[23] M. H. Strömqvist. Optimal Control of the obstacle Problem in a Perforated Domain. *Appl. Math. Optim.* **66** (2012), 239-255.

[24] C. Timofte, Homogenization results for climatization problems, *Ann Univ Ferrara* **53** (2007), 437–448.

[25] F. Troltzsch, *Optimal control of partial differential equations.* American Mathematical Society, Providence, RI. 2010

[26] M.N. Zubova and T.A. Shaposhnikova. Homogenization of boundary value problems in perforated domains with the third boundary condition and the resulting change in the character of the nonlinearity in the problem. *Diff. Eq.* **47** 1 (2011), 1-13.

[27] M.N. Zubova and T.A. Shaposhnikova. Homogenization of a boundary value problem in a domain perforated by cavities of arbitrary shape with nonlinear boundary condition on their boundaries: the case of critical values of the parameters. *Journal of Mathematical Sciences,* **244** 2,(2020), 235-253.