Monotone continuous dependence of solutions of singular quenching parabolic problems

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Abstract. We prove the continuous dependence, with respect to the initial datum of solutions of the "quenching parabolic problem" $\partial_t u - \Delta u + \chi_{\{u>0\}} u^{-\beta} = \lambda u^p$, with zero Dirichlet boundary conditions, when $\beta \in (0,1), p \in (0,1], \lambda \geq 0$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$. Notice that the absorption term $\chi_{\{u>0\}} u^{-\beta}$ is singular and monotone decreasing which does not allow the application of standard monotonicity arguments.
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1 Introduction.

It is well-known that monotonicty methods allow to prove the existence and uniqueness of semilinear parabolic problems

$$\partial_t u - \Delta u + f(x,u) = 0$$

in presence of additive monotone non-decreasing or Lipschitz-continuous nonlinear terms (see, e.g., the many references of the survey [16]). The extension to the case in which $f(x,u)$ is singular but monotone increasing as $f(x,u) = -a(x)/u^\beta$, for some $a(x) > 0$ and $\beta > 0$ was treated in numerous previous papers in the literature.

The situation changes radically when the term $f(x,u)$ is neither Lipschiz-continuous nor monotone-nondecreasing, for instance because $f(x,u)$ involves a singular decreasing dependence on u. This is the case of the family of problems which some authors refer as "quenching problems" since their solutions quench in a finite time ($u(x,t) \equiv 0$ for any $t \geq t^*$ and any x, for some $t^* > 0$). This type of problems will be the object of the present paper. More precisely, we consider the

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problem

$$\mathrm{P}_{(\lambda,u_0)}\begin{cases} \partial_t u - \Delta u + \chi_{\{u>0\}}u^{-\beta} = \lambda u^p & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0)=u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{1}$$

under the main structural assumptions

$$\beta \in (0,1), p\in(0,1], \lambda \geq 0,$$

and

$$0 \leq u_0 \in L^1(\Omega),$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points $(x,t)\in\Omega\times(0,T)$ where $u(x,t)>0$.

We recall that parabolic equations, involving as zero order term a negative power of the unknown, are quite common in the literature since 1960. The pioneering paper by Fulks and Maybee [25] was motivated by the study of the heat conduction in an electric medium. Perhaps, one of the first papers dealing with the equation (1) was [32] in the study of Electric Current Transient in Polarized Ionic Conductors (in fact for $\beta=1$). The literature on this type of problems increased then very quickly and models arising in other contexts were mentioned by different authors, specially when regarding the equation of (1) as the limit case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics (see [14, 19] for the elliptic case and [2, 36], [34] [41] for the parabolic equation). See also many other references in the survey [30] and the monograph [27]) and the extension to the case of quasilinear diffusion operators made in [28], [6] [7], [8] and [9]). We also mention that the equation also arises in the context of the study of space-charge problems ([38]), the Euler-Poisson system in Maxwell-Vlasov problems ([1]) and in hydrodynamic quantum fluids ([26]). What makes equations like (1) specially interesting is the fact that the solutions may raise to a free boundary defined as the boundary of the set $\{(x,t)\colon\ u(x,t)>0\}$ (see, e.g. [15]). In many contexts the boundary conditions are not zero but, for instance $u=1$ and thus the terminology of "quenching problem" was used in the literature to denote the appearance of blow-up result on $\partial_t u$ for the first time in which $u=0$ (see, e.g. [32, 35, 36]). It can be proved (see, e.g. [14], [12]) that the solution starts to growth near the boundary of its initial support (let us assume, for simplicity, that is is given as the whole domain Ω) as

$$u(t,x)\geq C\delta(x)^{\nu_0} \text{ with } \nu_0=\frac{2}{1+\beta}$$

where

$$\delta(x):=\operatorname{dist}(x,\partial\Omega)$$

(which we shall denote simply as δ). It is also well-known that the uniqueness of solution may fail (see Winkler [41]) except for the case in which there is not a free boundary (see [12]). In particular, it is known that the solution is not necessarily continuously dependent on the norm $\|u_0\|_{L^\infty(\Omega)}$.

The main purpose of this work is to obtain a result on the continuous dependence (on the initial datum) of the solutions of $\mathrm{P}_{(\lambda,u_0)}$, extending and improving the results of [10] in which the existence of solutions (and some partial uniqueness result) was obtained in the class of solutions

$$\mathcal{M}(\nu) := \Big\{ u \in \mathcal{C}\left([0,T]; L^1(\Omega)\right) \ \big|\ \forall T' \in (0,T),\ \textit{there exists } C(T') > 0 \textit{ such that}: \\ \forall t \in (0,T'),\ u(t,x) \geq C(T')\delta(x)^{\nu} \quad \text{in } \Omega \Big\}, \quad (2)$$

when

$$\nu \in \left(0, \frac{2}{1+\beta}\right]. \quad (3)$$

To be more precise we introduce the notion of solution we shall use in this paper:

Definition 1 *A function $u \in \mathcal{C}([0,T); L^1(\Omega))$ is called a mild solution of* (1) *if $\chi_{\{u>0\}}u^{-\beta} \in L^1(\Omega \times (0,T))$ and u fulfills the identity*

$$u(\cdot,t) = S(t)u_0(\cdot) - \int_0^t S(t-s)(\chi_{\{u>0\}}u^{-\beta}(\cdot,s) - \lambda u^p)ds, \quad \textit{in } L^1(\Omega), \quad (4)$$

where $S(t)$ is the $L^1(\Omega)$-semigroup corresponding to the Laplace operator with homogeneous Dirichlet boundary conditions.

We recall some results already proved in the literature (see, e.g. [10]):

Theorem 2 *([10]) Let $0 \leq u_0 \in L^1(\Omega)$. Then,*

i) there exists the (global) maximal nonnegative mild solution u of (1)*, i.e. such that for any other mild solution v of* (1) *we have $0 \leq v \leq u$ in $\Omega \times [0,T]$. Moreover, for any $0 < \tau < T$, $u \in L^2(\tau,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (\tau,T))$.*

ii) there exists a finite time, $T^* > 0$ such that $u(\cdot,t)$ vanishes in $L^1(\Omega)$ for $t > T^*$. Moreover, T^* only depends on $\|u_0\|_{L^1(\Omega)}$, N and $|\Omega|$.

iii) u is the unique solution to problem (1) *in the class of mild solutions that have the same quentching time T^*.*

We also point out that it is possible to deal with a more general class of initial data $u_0 \in L^1(\Omega;\delta)$ by working in the class of very weak solutions (see, e.g. the general references [3], [18], [37]). Some regularity results involving the distance to the boundary, for a different class of equations including some gradient terms, can be found in [23] and [24].

Our main result gives the continuous dependence of solutions with respect to the initial data (implying, obviously, the uniqueness of solutions) as well as a smoothing effect with respect to the initial datum which improves, for $N \geq 3$, the standard smoothing effect associated to the linear heat equation (see Remark 8 below). We will use strongly some Hilbertian techniques, so we will consider initial data in $L^1(\Omega) \cap L^2(\Omega;\delta)$ and we will prove that if two solutions are in the class $u, v \in \mathcal{M}(\nu)$. i.e. with $\delta^{-\nu}u, \delta^{-\nu}v \geq C$ then we can estimate the $L^2(\Omega)$-norm of $\delta^{-\gamma}[u(t) - v(t)]_+$ for suitable $\gamma \in (0,1]$ in terms of the $L^2(\Omega;\delta)$-norm of $[u_0 - v_0]_+$. Notice that this implies, authomatically, an estimate on the $L^2(\Omega)$-norm of $[u(t) - v(t)]_+$ (see Remark 9 below). We conjecture, from the above comments, that the result below holds for initial data in $L^2(\Omega;\delta)$, but we will not develop this point of view in this paper.

Theorem 3 *Let $u_0, v_0 \in L^1(\Omega) \cap L^2(\Omega;\delta)$. Let u, v be weak solutions of $\mathrm{P}_{(\lambda,u_0)}$ and $\mathrm{P}_{(\lambda,v_0)}$, respectively such that $u, v \in \mathcal{M}(\nu)$ for some $\nu \in \left(0, \frac{2}{1+\beta}\right]$. Then, for any $T' \in (0,T)$ and for any $t \in (0,T')$, we have*

$$\left\|\delta^{-\gamma}[u(t)-v(t)]_+\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \left\|[u_0-v_0]_+\right\|_{L^2(\Omega;\delta)}, \tag{5}$$

with

$$\gamma := \min\{\nu(1+\beta), 1\} \tag{6}$$

and for some constant $C > 0$ independent of T' if constants $C(T')$ given in the condition $u, v \in \mathcal{M}(\nu)$ are independent of T'. In particular, $u_0 \leq v_0$ implies that for any $t \in [0,T]$,

$$u(t,\cdot) \leq v(t,\cdot) \quad a.e. \text{ in } \Omega$$

and

$$\left\|\delta^{-\gamma}\left(u(t)-v(t)\right)\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \left\|u_0-v_0\right\|_{L^2(\Omega;\delta)}. \tag{7}$$

As an application, we will prove the uniqueness of the positive solution for the following stationary problem with a singular absorption:

$$\mathrm{P}_{(\lambda,F,\beta,p)} \begin{cases} -\Delta u + \chi_{\{u>0\}} u^{-\beta} = \lambda_1 u^p + \lambda_2 F(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

where $\lambda_1, \lambda_2 \geq 0$ and

$$F \in L^1(\Omega), \quad F \geq 0 \text{ a.e. in } \Omega, \tag{8}$$

improving or completing different results in the literature (see, e.g. [19], [12], [29] and [17]).

2 On the continuous dependence

For the proof of Theorem 3 we shall need some well-known auxiliary results. The first one is a singular version of the Gronwall's inequality which is specially useful in the study of non-globally Lipschitz perturbations of the heat equation (see [4, p. 288], [5, Lemma 8.1.1, p. 125]).

Lemma 4 *Let $T > 0$, $A \geq 0$, $0 \leq a, b \leq 1$ and let f be a non-negative function with $f \in \mathrm{L}^p(0,T)$ for some $p > 1$ such that $p'.\max\{a,b\} < 1$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Consider a non-negative function $\varphi \in L^\infty(0,T)$ such that, for almost every $t \in (0,T)$,*

$$\varphi(t) \leq At^{-a} + \int_0^t (t-s)^{-b} f(s)\varphi(s)\, ds. \tag{9}$$

Then, there exists $C > 0$ only depending on T, a, b, p and $\|f\|_{L^p(0,T)}$ such that, for almost every $t \in (0,T)$,

$$\varphi(t) \leq ACt^{-a}. \tag{10}$$

We shall also use some regularizing effects properties satisfied by the semigroup $S(t)$ of the heat equation with zero Dirichlet boundary conditions.

Lemma 5

1. There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^2(\Omega)$,

$$\|\nabla S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{1}{2}}\|u_0\|_{L^2(\Omega)}. \tag{11}$$

2. There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^1(\Omega)$,

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{N}{4}}\|u_0\|_{L^1(\Omega)}. \tag{12}$$

3. There exists $C > 0$ such that, for any $t > 0$, any $m \in (0, 1]$ and any $u_0 \in L^2(\Omega; \delta^{2m})$,

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}}\|u_0\|_{L^2(\Omega,\delta^{2m})}. \tag{13}$$

4. There exists $C > 0$ such that, for any $t > 0$, any $p \in [1, +\infty)$ and any $u_0 \in L^p(\Omega, \delta)$,

$$\|S(t)u_0\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{2p}}\|u_0.\|_{L^p(\Omega,\delta)} \tag{14}$$

Proof: Properties 1 and 2 are classical (see e.g. [40, remarque III.5] or [5, Proposition 3.5.7, Proposition 3.5.2]). Property 3 was established in [12, Proposition 1.12] in terms of the function $v_0 := u_0\delta^m$, *i.e.*

$$\|S(t)\delta^{-m}v_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}}\|v_0\|_{L^2(\Omega)}.$$

Property 4 was proved in [39, Remark 2.1-(c), p. 179]. □

Proof: *(of Theorem 3)* By the constant variations formula, we know that for any $t \in [0, T]$,

$$u(t) - v(t) = S(t)(u_0 - v_0) + \int_0^t S(t-s)\left(h(u(s)) - h(v(s))\right) ds \quad \text{in } \Omega, \tag{15}$$

where $h(u) := \lambda u^p - \chi_{\{u>0\}}u^{-\beta}$. By the convexity of the function $u \mapsto u^{-\beta}$, the concavity of the function $u \mapsto u^p$ and the assumption that $u, v \in \mathcal{M}(\nu)$, we deduce that

$$h(u) - h(v) \leq \left(C_1\delta^{-(\beta+1)\nu} + C_2\delta^{(1-p)\nu}\right)(u-v)_+ \leq C\delta^{-(\beta+1)\nu}(u-v)_+ \quad \text{in } \Omega. \tag{16}$$

Thus, if we denote $w := u - v$, we get for any $\tau, t \in [0, T]$ with $\tau \leq t$

$$w_+(t) \leq S(t-\tau)w_+(\tau) + C\int_\tau^t S(t-s)\delta^{-(\beta+1)\nu}w_+(s)\,ds. \tag{17}$$

Now we adapt to our framework some arguments of the proof of [12, Lemma 1.11, p. 1823] concerning other forcing nonlinear terms (see Remark 6). We multiply (17) by the weight $\delta^{-\gamma}$, with $\gamma \in [0, 1]$ to be chosen later, and take the L^2-norms. Then,

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma}S(t-\tau)w_+(\tau)\|_{L^2(\Omega)} + C\int_\tau^t \|S(t-s)\delta^{-[(\beta+1)\nu+\gamma]}w_+(s)\|_{L^2(\Omega)}\,ds.$$

Let us fix $s, t > 0$ and let us call $\psi := S(t-s)\delta^{-(\beta+1)\nu}w_+(s)$. Then, by Hölder inequality,

$$\|\delta^{-\gamma}\psi\|^2_{L^2(\Omega)} = \int_\Omega \frac{\psi^2}{\delta^{2\gamma}}\,dx \leq \left(\int_\Omega \frac{\psi^2}{\delta^2}\,dx\right)^\gamma \left(\int_\Omega \psi^2\,dx\right)^{1-\gamma}$$

(note that the limit cases $\psi \equiv 0$ and $\psi \equiv 1$ are allowed). Then, applying Hardy inequality,

$$\|\delta^{-\gamma}\psi\|_{L^2(\Omega)} \leq C\|\nabla\psi\|^{\gamma}_{L^2(\Omega)}\|\psi\|^{1-\gamma}_{L^2(\Omega)}.$$

By property 1 of Lemma 5, to $\frac{t-s}{2}$, we get

$$\|\delta^{-[(\beta+1)\nu+\gamma]}S(t-s)w_+(s)\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{\gamma}{2}}\|S\left(\frac{t-s}{2}\right)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)}. \tag{18}$$

Analogously, using property 4 of Lemma 5,

$$\|\delta^{-\gamma}S(t)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\frac{\gamma}{2}}\|S\left(\frac{t}{2}\right)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\left(\frac{\gamma}{2}+\frac{1}{4}\right)}\|w_+(0)\|_{L^2(\Omega;\delta)}. \tag{19}$$

In order to apply the singular Gronwall's inequality, we must relate the weights $\delta^{-\gamma}$ and $\delta^{-(\beta+1)\nu}$ keeping in mind that $\gamma \in [0,1]$. To do that, we apply property 3 of Lemma 5 for some $m \in [0,1]$. We shall take

$$(\beta+1)\nu = \gamma + m. \tag{20}$$

Indeed, if $(\beta+1)\nu \in (1,2]$, then we take $\gamma = 1$, $m = (\beta+1)\nu - 1$ and we apply point 3 of Lemma 5 to the initial datum:

$$\|S\left(\frac{t-s}{2}\right)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t-s}{2}\right)\delta^{-(m+1)}w_+(s)\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{m}{2}}\|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}. \tag{21}$$

On the other hand, if $(\beta+1)\nu \in [0,1]$, we can take $\gamma = (\beta+1)\nu$ and thus, since $S(t-s)$ is a contraction in $L^2(\Omega)$, we get

$$\|S\left(\frac{t-s}{2}\right)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t-s}{2}\right)\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}, \tag{22}$$

which corresponds to (20) with $m = 0$. In other words,

$$\gamma = \min\{1, (\beta+1)\nu\}$$

and

$$m = \max\{(\beta+1)\nu - 1, 0\}.$$

Collecting the previous inequalities, we arrive to

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}}\|w_+(0)\|_{L^2(\Omega;\delta)} + C\int_0^t (t-s)^{-\frac{m}{2}}\|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}.$$

Thus, we can apply Lemma 4 with $a = \frac{2\gamma+1}{4} \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $b = \frac{m}{2}$ and $A = C\|w_+(0)\|_{L^2(\Omega;\delta)}$ to deduce that

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}}\|w_+(0)\|_{L^2(\Omega;\delta)}.$$

□

Remark 6 *The comparison principle was proved in [12] for a generalized version of equation* $\mathrm{P}_{(\lambda,u_0)}$ *in the sense that the right hand side term was replaced by a function* $f:[0,+\infty)\longrightarrow[0,+\infty)$ *satisfying:*

$$0\le f\in\mathcal{C}^2\left([0,+\infty)\right),\tag{23}$$

$$f \textit{ is concave, increasing and } \lim_{u\to+\infty}\frac{f(u)}{u}=0.\tag{24}$$

Note that $f(u):=u^p$ *does not satisfy* (23) *if* $p\in(0,1)$. *In fact, condition* (24) *is neither satisfied if* $p=1$. *It was also shown in [12] that for the mere purpose to get the comparison* $u\le v$, *the growth condition near* $\partial\Omega$ *is only required for the "supersolution"* v *(in fact, conclusion i) in [12, Lemma 1.11] remains true if* u *is a subsolution and* v *a supersolution i.e. we can replace the symbol "="* *in the equation of* $\mathrm{P}_{(\lambda,\mathrm{u}_0)}$ *by "*$\le$*" (in the case of* u*) and/or by "*$\ge$*" (in the case of* v*).*

Remark 7 *Sufficient conditions on the initial datum* u_0 *ensuring that there exists some weak solution of* $\mathrm{P}_{(\lambda,u_0)}$ *belonging to the class* $\mathcal{M}(\nu)$ *were given in [12, Theorem 1.10] under conditions* (23) *and* (24). *Similar results can be obtained in our framework (*$p\in(0,1)$*) by applying the super and subsolutions built in [14], [19] and [18].*

Remark 8 *The smoothing effect, with respect to initial datum, given by formula (5) is sharper than the one given in (12) for the solutions of the linear heat equation. Indeed, it is easy to check that* $2\gamma+1\le N$, *for* $N\ge 3$. *Moreover the exponent in (5) depends on the value of exponent* β *of the singular term.*

Remark 9 *It is clear that the estimate of the* $L^2(\Omega)$*-norm of* $\delta^{-\gamma}[u(t)-v(t)]_+$ *given in (5) implies, automatically an estimate on the* $L^2(\Omega)$*-norm of* $[u(t)-v(t)]_+$. *Indeed, it suffices to write*

$$\|[u(t)-v(t)]_+\|_{L^2(\Omega)}=\|\delta^\gamma\delta^{-\gamma}[u(t)-v(t)]_+\|_{L^2(\Omega)}\le$$
$$\|\delta^\gamma\|\,\|\delta^{-\gamma}[u(t)-v(t)]_+\|_{L^2(\Omega)}\le C(\Omega)\,\|\delta^{-\gamma}[u(t)-v(t)]_+\|_{L^2(\Omega)}\,.$$

As mentioned in the Introduction, we can apply the above result to the case of suitable elliptic problems:

Corollary 10 *Assume* $\beta\in(0,1)$, $p\in(0,1)$ *and either*

$$\lambda_1=0 \textit{ and } \lambda_2>0 \textit{ is large enough}\tag{25}$$

or

$$\lambda_1>0 \textit{ is large enough (and } \lambda_2\ge 0\textit{)}.\tag{26}$$

Then, problem $\mathrm{P}_{(\lambda,F,\beta,p)}$ *has at most a positive solution.*

Proof: Assume (25). By [19, Corollary 1, p. 1335], there exists $\lambda^*>0$ such that if $\lambda>\lambda^*$ then, there exists a positive solution u of the problem $\mathrm{P}_{(\lambda,F,\beta,p)}$. Precisely, if λ is large enough it was shown that any positive solution must verify

$$u(x)\ge\lambda\varphi_1(x)^{\frac{2}{1+\beta}}\quad\text{ in }\Omega,$$

with φ_1 the first eigenfunction of $-\Delta$ with homogeneous Dirichlet conditions. Then, assume that $\lambda > \lambda^*$ is large enough and that there are two different solutions u_∞ and v_∞ of $\mathrm{P}_{(\lambda,F,\beta,p)}$. By taking $u_0 = u_\infty$ and $v_0 = v_\infty$ as initial data in $\mathrm{P}_{(\lambda,u_0)}$ and $\mathrm{P}_{(\lambda,v_0)}$, since u_∞ and v_∞ are obviously respective solutions of the mentioned parabolic problems, we get that $C(T')$ is independent of T' and thus $u_\infty - v_\infty$ satisfies the same estimate as in the case $F(x) \equiv 0$, in particular we get

$$\left\|\delta^{-1}\left(u_\infty - v_\infty\right)_+\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|(u_\infty - v_\infty)_+\|_{L^2(\Omega,\delta)}.$$

Making $t \nearrow +\infty$ and reversing the role of u_∞ and v_∞, we get that $u_\infty = v_\infty$.
For the case (26), it was proved in [13, Lemma 3.11, p. 321] that any positive solution must satisfy the growth condition for some $\nu \in (1, \frac{2}{1+\beta})$ and thus, arguing as above (but now with $\gamma \in (0,1)$), we arrive at the same conclusion. $\square$

Remark 11 *The study of the limit case λ^* (concerning assumption (25)) is quite special since the maximal solution u^*_∞ is not always strictly positive. The growth estimate with $\nu = \frac{2}{1+\beta}$ (and thus the strict positivity) was proved in [13, Theorem 2.4, p. 307] under the additional condition*

$$\frac{3\beta + 1 + 2\sqrt{\beta^2+\beta}}{\beta+1} > \frac{N}{2}.$$

An example showing that this condition is almost optimal was given in [11, Theorem 3.2].

Remark 12 *In the one-dimensional case it is possible to give the exact multiplicity of positive solutions (see [31] for the case of* (25) *and $F(x) \equiv F_0 > 0$; see also [17] for* (26) *with $F(x) = 0$, $\beta \in (0, \frac{1}{q+1})$ and the q-Laplacian as diffusion operator).*

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