Nodal solutions bifurcating from infinity for some singular p-Laplace equations: flat and compact support solutions.

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July 20, 2015

Abstract

We study the existence and multiplicity of nodal solutions with normal exterior derivative different or equal to zero (case of *flat solutions*) or having a *free boundary* (the boundary of the set where the solution vanishes) of some one-dimensional p-Laplace problems of eigenvalue type with a, possibly singular, nonlinear absorption terms.

Mathematics Subject classification: 35J65,35B32,35P30,35R35.

1 Introduction.

This paper deals with the study of the countable branches of nodal solutions bifurcating from the infinity for the one-dimensional nonlinear eigenvalue problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(|\dfrac{\mathrm{d}u}{\mathrm{d}x}|^{p-2}\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)+|u|^{m-1}u=\lambda|u|^{p-2}u \quad \text{in }]-1,1[, \\ u(-1)=u(1)=0, \end{cases} \tag{1}$$

where $p>1$, λ is a positive parameter, m and p are given real numbers such that

$$-1<m<p-1. \tag{2}$$

We point out that the above differential equation must be slightly modified for some values of the parameter m. So, for $m=0$ the equation should be understood in the framework of the multivalued maximal monotone graphs of $\mathbb{R}^2$ as

$$-(|u'|^{p-2}\,u')'+H(u)\ni\lambda|u|^{p-2}u$$

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where $H(u)$ is the multivalued maximal monotone graph of $\mathbb{R}^2$ given by

$$H(r) = \begin{cases} -1 & \text{if } r < 0, \\ [-1,1] & \text{if } r = 0, \\ +1 & \text{if } r > 0. \end{cases}$$

This case (at least for $p = 2$) is specially relevant in image processing (see [18], [19]). Moreover, if $m \in (-1, 0)$ then the equation should be replaced by

$$-(|u'|^{p-2}\, u')' + |u|^{m-1}u\chi_{\{u\neq 0\}} = \lambda |u|^{p-2}u$$

where

$$\chi_{\{u\neq 0\}}(x) = \begin{cases} 1 & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0, \end{cases}$$

and thus the whole expression $|u|^{m-1}u\chi_{\{u\neq 0\}}$ must be understood as

$$\left[|u|^{m-1}u\chi_{\{u\neq 0\}}\right](x) = \begin{cases} |u(x)|^{m-1}u(x) & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

We study this problem by using phase plane methods for ordinary differential equations. This kind of arguments were used by the authors in [7] and [11]. These methods provide a complete description of the solution set for (1).

The related problem for the equation

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(|\dfrac{\mathrm{d}u}{\mathrm{d}x}|^{p-2}\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) + |u|^{m-1}u = \lambda |u|^{q-1}u & \text{in }]-1,1[, \\ u(-1) = u(1) = 0, \end{cases} \tag{3}$$

with $0 < m < q < p-1$ was studied initially in [7], for $p = 2$, giving a complete description of both positive solutions and the continua with infinitely many compact support solutions obtained from the very especial solution satisfying condition (7) below by some "stretching" manipulations. These results were extended to the p-Laplacian allowing singular nonlinearities satisfying $-1 < m < q < p-1$ in [11] by using similar (more sophisticated) arguments. Then the case of a general bounded domain $\Omega \subset \mathbb{R}^N$ and $p = 2$ was treated by Il'yasov and Egorov in [17] combining variational and continuation methods, by using this time a variant of the Mountain Pass Theorem. The study of the stability of solutions was made in [10]. For related work see [16], [24] and [25].

The above problem (1) was also considered for $p = 2$ and $0 < m < 1$ in [8] again with ODE methods giving sharper results that the ones obtained through abstract bifurcation tools by Rabinowitz [22],[21], [23]. We point out that although there are several results dealing with the bifurcation from the infinity for the p-Laplace operator (see, e.g. Drabek et al. [12]) our results are new in this direction.

Besides many related problems in the literature (see, e.g., references in the monographs [4], [1]) a relevant motivation for this work was to provide some kind of "alternative" approach to the "ambiguity" raised by some purported solutions to the linear Schrödinger equation on the real line (see [5], [6] and [8] for more details and references). There is also an extensive literature on the

Schrödinger quasilinear equation associated to the p-Laplace diffusion operator (see e.g. You et al. [26], and its many references). The association of problems as (1) with the study of standing waves for the related semilinear Schrödinger equation and the semilinear wave equation with real coefficients was also largely considered in the literature (at least for $p = 2$: see, e.g. [2] and its references). Nevertheless, in both types of equations (Schrödinger and wave equation) the consideration of the case $p \neq 2$ leads to some new facts that seem not to be well presented in the previous literature (see Remark 4 below).

As mentioned before, similar results giving a complete description of non-negative (including positive) solutions together with the transition to compact support solutions were given in [8] for the case $p = 2$. For the case of a bounded domain $\Omega \subset \mathbb{R}^N$, a partial result (namely the existence of an unbounded continuum of non-negative solutions bifurcating from infinity at $\lambda_1 > 0$, the first eigenvalue to $-\Delta$ with Dirichlet boundary condtions) was obtained in [14] using asymptotic bifurcation arguments. Much later, Porretta proved in [20] the existence of (at least) a non-negative solution for any $\lambda > \lambda_1$ (for $p = 2$) and this result was improved in [9] using this time variational methods (Nehari manifolds) and a Pohozaev identity.

The contents of this paper is the following: in Section 2 we analize the existence of a branch of nonnegative solutions for a bounded interval of the parameter $\lambda \in]\lambda_1(p), \lambda_1^*(m,p)[$. We recall that $\lambda_1(p)$ is the first eigenvalue of the nonlinear equation

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(|\dfrac{\mathrm{d}u}{\mathrm{d}x}|^{p-2}\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) = \lambda |u|^{p-2}u \quad \text{in }]-1,1[, \\ u(-1) = u(1) = 0. \end{cases} \tag{4}$$

The first eigenvalue of the $p-$Laplacian for one dimensional domains was obtained by Otani in [13] and its value is given by

$$\lambda_1(p) = (p-1)\left(\frac{\pi}{p\sin(\dfrac{\pi}{p})}\right)^p. \tag{5}$$

Here $\lambda_1^*(m,p)$ is a certain value of the parameter whose exact definition depends crucially on p and m. The value of $\lambda_1^*(m,p)$ can be written in closed form as

$$\lambda_1^*(m,p) = (p-1)\left(\frac{\pi}{(p-m-1)\sin(\dfrac{\pi}{p})}\right)^p = \left(\frac{p}{p-m-1}\right)^p \lambda_1(p). \tag{6}$$

We show that the unique positive solution of (1) for $\lambda = \lambda_1^*(m,p)$ has a particular behaviour at $x = \pm 1$. Due to this behaviour this solution is called as "flat solution" (and also as "free boundary nonnegative solution" by other authors) since although $u(x) > 0$ for any $x \in (-1,1)$ it satisfies

$$\frac{\partial u}{\partial n}(\pm 1) = 0. \tag{7}$$

We also give some estimates on the convergence

$$\frac{\partial u_\lambda}{\partial n}(\pm 1) \nearrow 0 \text{ as } \lambda \searrow \lambda_1^*(m,p),$$

and make explicit some non-degeneracy estimates for $u_{\lambda_1^*(m,p)}$ (near the boundary of the interval $\Omega = (-1,1)$) extending the estimates given in ([5] for $p = 2$). The associated solution $u_{\lambda_1^*(m,p)}$ (when extended by zero to the real line $\mathbb{R}$) gives rise to a continuum of nonnegative solutions u_λ for $\lambda > \lambda_1^*(m,p)$ throught an appropriate rescaling. This kind of solutions have compact support included in $[-1,1]$.

In section 3 we obtain a qualitatively similar result for the branches of nodal solutions with a finite number of simple zeroes and that bifurcate from infinity. Here we will denote by λ_k the bifurcation value of the branch that bifurcates from the infinity and posseses $(k-1)$ sign-changes. Now the values of λ_k and $\lambda_k^*(m,p)$ can be written in closed form as

$$\lambda_k(p) = (p-1)\left(\frac{k\pi}{p\sin(\dfrac{\pi}{p})}\right)^p = k^p\lambda_1(p), \tag{8}$$

and

$$\lambda_k^*(m,p) = (p-1)\left(\frac{k\pi}{(p-m-1)\sin(\dfrac{\pi}{p})}\right)^p = \left(\frac{p}{p-m-1}\right)^p\lambda_k(p) = k^p\lambda_1^*(m,p). \tag{9}$$

We show that the behavior of the nodal solutions near the points where they vanish is of the same type that the behaviour of nonnegative solutions near the boundary.

2 The branch of nonnegative solutions.

In this Section we study the non-negative solutions of the equation (1) by using ordinary differential equations arguments. This allows us to obtain a complete description of the solution set as a function of the parameter λ. We generalize the results obtained by Díaz and Hernández in [8] for the case $p = 2$ and $m > 0$. Now we consider the equation (1) under the assumption (2).

We recall that, as in [11], once we define

$$f(u) = \lambda|u|^{p-2}u - |u|^{m-1}u\chi_{\{u\neq 0\}},$$

(if $m \neq 0$) we can introduce the following notions of solution:

Definition 1: We say that $u \in W_0^{1,p}(-1,1)$ is a *positive strong solution* of problem (1) if $u > 0$ on $(-1,1)$, $f(u) \in L^1(-1,1)$, $(|u'|^{p-2}\,u')' \in L^1(-1,1)$, $-(|u'|^{p-2}\,u')'(x) = f(u(x))$ for a.e. $x \in (-1,1)$ and $\dfrac{\partial u}{\partial n}(\pm 1) < 0$.

Definition 2: We say $u \in W_0^{1,p}(-1,1)$ is a *flat solution* of problem (1) if u is as in the preceding definition but replacing the last condition by $\dfrac{\partial u}{\partial n}(\pm 1) = 0$.

In the case $m = 0$ the above definitions must be adapted in the sense that now it must exists $h \in L^1(-1,1)$, with $h(x) \in H(u(x))$ for a.e. $x \in (-1,1)$ such that $-(|u'|^{p-2}\,u')'(x) + h(x) = \lambda|u(x)|^{p-2}u(x)$ for a.e. $x \in\,]-1,1[$.

The main result of this section is the following:

Theorem 1 *Let $\lambda_1(p)$ and $\lambda_1^*(m,p)$ be given by (5) and (6) respectively. Then:*

a) If $\lambda \in]0, \lambda_1(p)[$ there is no nonnegative solution of (1).

b) If $\lambda \in]\lambda_1(p), \lambda_1^*(m,p)[$ there is a unique positive solution $u_\lambda$of (1). Moreover,

$$\frac{\partial u_\lambda}{\partial n}(\pm 1) < 0 \tag{10}$$

and

$$\underline{K}\, d(x, \partial\Omega) \leq u_\lambda(x) \leq \overline{K}\, d(x, \partial\Omega), \tag{11}$$

$\Omega = (-1, 1)$, *for some constants* $\overline{K} > \underline{K} > 0$.

c) If $\lambda = \lambda_1^*(m,p)$ there is a unique positive solution $u_{\lambda_1^*(m,p)}$ of (1). Moreover,

$$\frac{\partial u_{\lambda_1^*(m,p)}}{\partial n}(\pm 1) = 0, \tag{12}$$

$$\|u_{\lambda_1^*(m,p)}\|_{L^\infty[-1,1]} = \left(\frac{p}{(m+1)\lambda_1^*}\right)^{\frac{1}{p-m-1}},$$

and

$$\underline{K} d(x, \partial\Omega)^{p/(p-1-m)} \leq u_{\lambda_1^*(m,p)}(x) \leq \overline{K} d(x, \partial\Omega)^{p/(p-1-m)} \tag{13}$$

$\Omega = (-1, 1)$, *for some constants* $\overline{K} > \underline{K} > 0$.

d) If $\lambda > \lambda_1^*(m,p)$ then the function $u_{\lambda,\zeta} : [-1, 1] \to \mathbb{R}$ defined by

$$u_{\lambda,\zeta}(x) = \begin{cases} (\frac{\lambda_1^*(m,p)}{\lambda})^{\frac{1}{p-m-1}} u_{\lambda_1^*(m,p)}(\frac{x-\zeta}{\omega}) & \text{if } |x-\zeta| < \omega, \\ 0 & \text{if } x \in [-1, \zeta-\omega] \cup [\zeta+\omega, 1], \end{cases}$$

is a nonnegative solution of (1), where $\omega = \left(\frac{\lambda_1^*(m,p)}{\lambda}\right)^{\frac{1}{p}} < 1$, $\zeta \in [-1+\omega, 1-\omega]$ and $u_{\lambda_1^*(m,p)}$ is the unique positive solution of (1) for $\lambda = \lambda_1^*(m,p)$. Consequently, for each $\lambda > \lambda_1^*(m,p)$ there is a family of nonnegative solutions of (1), $u_{\lambda,\zeta}$, that depends arbitrarily on the parameter $\zeta \in [-1+\omega, 1-\omega]$. Moreover, the behavior of $u_{\lambda,\zeta}$ near the boundary of the points where $u_{\lambda,\zeta}$ vanishes is given by (13).

For the proof we shall need the following auxiliary Lemma .

Lemma 1 *Let $\gamma : [r_F, +\infty[\to \mathbb{R}$ be the function defined by*

$$\gamma(\mu) = \int_0^\mu \sqrt[p]{\frac{p-1}{p}} \frac{1}{\sqrt[p]{F(\mu) - F(r)}} \mathrm{d}r,$$

where $r_F = \left(\frac{p}{m+1}\right)^{\frac{1}{p-m-1}}$, and $F(r) = (\frac{r^p}{p} - \frac{r^{m+1}}{m+1})$. The function γ is such that

i)$\gamma \in C([r_F, +\infty[) \cap C^1(]r_F, +\infty[)$.

ii) $\frac{d\gamma}{d\mu}(\mu) < 0$ *for all* $\mu \in]r_F, +\infty[$.

iii) $\lim_{\mu \to r_F^+} \frac{d\gamma}{d\mu}(\mu) = -\infty$ *if* $\frac{p}{p+1} \leq m+1$ *and* $\frac{d\gamma}{d\mu}(r_F^+)$ *is finite if* $m+1 < \frac{p}{p+1}$.

iv) $\gamma(r_F) = \frac{\pi \sqrt[p]{(p-1)}}{(p-m-1)\sin(\frac{\pi}{p})}, \qquad \lim_{\mu \to +\infty} \gamma(\mu) = \sqrt[p]{(p-1)} \left(\frac{\pi}{p \sin(\frac{\pi}{p})} \right).$

Remark 1 A curious fact is that in [15] (respectively [11]) it was shown the increasing nature of the curve γ (which also implies the uniqueness of solution*)* for problem (3) when m is near -1, $p = 2$ (respectively $p \neq 2$) and $q \in (m, 1)$ (respectively $q \in (m, p-1)$).

Proof of Lemma 1. We use the change of variables $\tau := r/\mu$ in the definition of γ. Then

$$\gamma(\mu) = \sqrt[p]{p-1} \int_0^\mu \frac{1}{\sqrt[p]{\mu^p - \frac{p}{m+1}\mu^{m+1} - (r^p - \frac{p}{m+1} r^{m+1})}} \mathrm{d}r =$$

$$\sqrt[p]{p-1} \int_0^1 \frac{1}{\sqrt[p]{1 - \tau^p - \frac{p}{(m+1)\mu^{p-m-1}}(1-\tau^{m+1})}} \mathrm{d}\tau. \tag{14}$$

We have

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\mu}(\mu) = -\sqrt[p]{p-1} \int_0^1 \frac{\frac{p-m-1}{(m+1)\mu^{p-m}}(1-\tau^{m+1})}{\sqrt[p]{\left(1 - \tau^p - \frac{p}{(m+1)\mu^{p-m-1}}(1-\tau^{m+1})\right)^{p+1}}} \mathrm{d}\tau. \tag{15}$$

For $\mu \in]r_F, \infty[$ it is not difficult to verify that the integral in (15) is convergent, and thus $\frac{\mathrm{d}\gamma}{\mathrm{d}\mu}(\mu) \in C(r_F, \infty)$ with $\frac{\mathrm{d}\gamma}{\mathrm{d}\mu}(\mu) < 0$. According to the assumption (2) the integral (15) is divergent for $\mu = r_F$, hence $\lim_{\mu \to r_F^+} \frac{\mathrm{d}\gamma}{\mathrm{d}\mu}(\mu) = -\infty$.

On the other hand, from assumption (2), the integral (14) is bounded as $\mu \downarrow r_F$, and in conclusion $\gamma \in C([r_F, \infty))$.

By virtue of the Fatou's lemma

$$\lim_{\mu \to +\infty} \gamma(\mu) = \lim_{\mu \to +\infty} \sqrt[p]{p-1} \int_0^1 \frac{1}{\sqrt[p]{1 - \tau^p - \frac{p}{(m+1)\mu^{p-m-1}}(1-\tau^{m+1})}} \mathrm{d}\tau =$$

$$\sqrt[p]{p-1} \int_0^1 \frac{1}{\sqrt[p]{1-\tau^p}} \mathrm{d}\tau = \frac{\sqrt[p]{p-1}}{p} \int_0^1 \sigma^{\frac{1}{p}-1}(1-\sigma)^{1-\frac{1}{p}-1} \mathrm{d}\sigma = \frac{\sqrt[p]{p-1}}{p} B(\frac{1}{p}, 1 - \frac{1}{p}).$$

Taking into account that $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ (see e.g. [3]) we obtain

$$\lim_{\mu\to+\infty}\gamma(\mu) = \frac{\sqrt[p]{p-1}}{p}\frac{\pi}{\sin(\frac{\pi}{p})}.$$

Finally

$$\gamma(r_F) = \sqrt[p]{p-1}\int_0^1 \frac{1}{\sqrt[p]{\tau^{m+1}-\tau^p}}\mathrm{d}\tau = \sqrt[p]{p-1}\int_0^1 \frac{1}{\tau^{\frac{m+1}{p}}\sqrt[p]{1-\tau^{p-m-1}}}\mathrm{d}\tau =$$
$$\frac{\sqrt[p]{p-1}}{p-m-1}\int_0^1 \sigma^{\frac{1}{p}-1}(1-\sigma)^{1-\frac{1}{p}-1}\mathrm{d}\sigma = \frac{\sqrt[p]{p-1}}{p-m-1}B(\frac{1}{p},1-\frac{1}{p}) = \frac{\sqrt[p]{p-1}}{(p-m-1)}\frac{\pi}{\sin(\frac{\pi}{p})}. \blacksquare$$

Remark 2 Note that for $p=2$ and $m=0$ the function $\gamma : [2,+\infty[\to \mathbb{R}$ can be written in closed form as

$$\gamma(\mu) = \frac{\pi}{2} + \arcsin(\frac{1}{\mu-1}).$$

Proof of Theorem 1. Upon multiplication of equation (1) by u, integrating by parts and taking into account the variational definition of the first eigenvalue of (4) we get part a).

To show the qualitative behavior of solutions of the equation (1), we make the change of variables

$$y = Lx, \quad u(\frac{y}{L}) = (\frac{1}{L})^{\frac{p}{p-m-1}}w(y), \quad \text{with } \lambda = L^p. \tag{16}$$

in equation (1) to obtain

$$\mathcal{P}(L)\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}y}\left(|\dfrac{\mathrm{d}w}{\mathrm{d}y}|^{p-2}\dfrac{\mathrm{d}w}{\mathrm{d}y}\right) + |w|^{m-1}w = |w|^{p-2}w \ \text{ in }]-L,L[, \\ w(-L) = w(L) = 0. \end{cases} \tag{17}$$

If a positive solution of $\mathcal{P}(L)$ exists then necessarily it will have a maximum $\mu > 0$ at some point $\zeta \in (-L,L)$. So, let us consider

$$\mathcal{CP}\begin{cases} -(|u'|^{p-2}u')' = -|u|^{m-1}u + |u|^{p-2}u \\ u(\zeta) = \mu, \ u'(\zeta) = 0. \end{cases}$$

Hence, we have to consider only the case $\mu \in [r_F,\infty)$. Notice that, if u is a strong positive solution of $\mathcal{P}(L)$ then $(|u'|^{p-2}u')' \in L^1(-L,L)$ implies that $u' \in L^\infty(-L,L)$. Then, since by definition $f \in L^1(-L,L)$, the formula

$$-\int_\zeta^x (|u'|^{p-2}u')'u'd\tau = -\int_\zeta^x \frac{p-1}{p}\frac{\mathrm{d}}{\mathrm{d}y}(|u'|)^p(\tau)\mathrm{d}\tau = \int_\zeta^x f(u(\tau))u'(\tau)d\tau = \int_\zeta^x F'(u(\tau))\mathrm{d}\tau,$$

is well justified. Since $F'(s) = f(s) = -|s|^{m-1}s + |s|^{p-2}s > 0$ if $s > 1$ and $F(s) = \dfrac{s^p}{p} - \dfrac{s^{m+1}}{m+1} > 0$ if $s \in]r_F, +\infty[$ we have that for any $x \in (-L, L)$

$$|u'(x)| = \sqrt[p]{\frac{p}{p-1}(F(\mu) - F(u(x))}. \tag{18}$$

Notice that, in fact $u'(x) \leq 0$ for $x \in (-L, L)$ near ζ with $x > \zeta$ and that $u'(x) \geq 0$ for x near ζ and $x < \zeta$.

The solution of this equation is implicitly defined by

$$\int_{u(x)}^{\mu} \frac{dr}{\sqrt[p]{\frac{p}{p-1}(F(\mu) - F(r))}} = |\zeta - x|\,, \tag{19}$$

since the singularity at $r = \mu$ is integrable, i.e., that

$$\int_s^{\mu} \frac{dr}{(F(\mu) - F(r))^{1/p}} < \infty, \text{ for any } s \ \in (\mu - \varepsilon, \mu), \text{ for any } \varepsilon > 0 \text{ small enough.} \tag{20}$$

Indeed, it is easy to check that (20) always holds since $F(\mu) - F(r) \geq \delta(\mu - r)$ for some $\delta > 0$ and for any r near μ. Moreover, for a strong positive solution u of problem $\mathcal{CP}$, $u = 0$ only at $r = \pm L$.

To finish the proof we only need to justify that the function $u(x)$ defined implicitly by (19) satisfies that $f(u) \in L^1(-L,L)$. But we know that for $x \in (-L, -L + \delta)$, $u(x) \in [0, \varepsilon)$ for some $\varepsilon > 0$ small enough and we get that

$$-C \int_{-L}^{-L+\delta} f(u(x))dx \leq -\int_{-L}^{-L+\delta} f(u(x))u'(x)dx = \int_{-L}^{-L+\delta} |u'|^p \, dx < \infty$$

for some C near $\sqrt[p]{\frac{p}{p-1}F(\mu)}$, since $u \in C^1(-L,L)$, and the proof of the existence of part b) ends.

As a byproduct we obtain that u is an even function. Equation (19), when particularized for $x = -L$ and $x = L$, gives the identity $|\zeta + L| = |\zeta - L|$, which implies $\zeta = 0$, particularizing again (19) for x and $-x$ we obtain $u(x) = u(-x)$, i. e. u is an even function.

In the case of $\lambda = \lambda^*(m)$ the associated function v is such that $\mu = r_F$ and in consequence $v'(\pm L) = 0$. Moreover, since

$$\frac{1}{m+1}s^{1+m} \geq \frac{s^{m+1}}{m+1} - \frac{s^p}{p} \geq \frac{(p-1-m)}{p(1+m)}s^{1+m} \quad \text{for } s \in (0, 1),$$

we get that there exist two positive constants $\underline{M} < \overline{M}$ such that

$$\underline{M}\tau^{\frac{p-1-m}{p}} \leq \frac{1}{\sqrt[p]{\frac{p}{p-1}}} \int_0^{\tau} \frac{dr}{\sqrt[p]{-F(r)}} \leq \overline{M}\tau^{\frac{p-1-m}{p}} \tag{21}$$

for any $\tau \in (0, 1)$ which leads to conclusion (13).

The dependence on λ of the value of normal derivative of u_λ at the boundary can be obtained from (18).

$$|\frac{\mathrm{d}u_\lambda}{\mathrm{d}x}(\pm 1)| = \left(\frac{1}{\lambda}\right)^{\frac{m+1}{p(p-m-1)}} \sqrt[p]{\frac{p}{p-1}F(\gamma^{-1}(\lambda^{\frac{1}{p}}))}. \tag{22}$$

Since the function F is increasing in $[1,+\infty[$, $\gamma^{-1}(]\gamma(+\infty),\gamma(r_F)] = [r_F,+\infty[\subset [1,+\infty]$, and the functions γ^{-1} and $\left(\frac{1}{s}\right)^{\frac{m+1}{p(p-m-1)}}$ are decreasing, $|\frac{\mathrm{d}u_\lambda}{\mathrm{d}x}(\pm 1)|$ is a decreasing function of λ. In addition $F(\gamma^{-1}((\lambda_1^*(m,p))^{\frac{1}{p}})) = 0$ which implies that $\frac{\mathrm{d}u_\lambda}{\mathrm{d}x}(\pm 1) = 0$ if and only if $\lambda = \lambda_1^*(m,p)$. (Notice that $F\circ\gamma^{-1}$ is differentiable only if $m+1 < \frac{p}{p+1}$).

The proof of part d) is similar to the the proof of part v) of Theorem 1 in [11].

According to (19) the unique positive solution of $\mathcal{P}(L)$ satisfies the identity $\gamma(w(0)) = L$, where γ is as defined in Lemma 1. Since γ is injective and continuous, its inverse $\gamma^{-1} :]\lim_{\mu\to+\infty}\gamma(\mu),\gamma(r_F)] \to \mathbb{R}$ is well defined, and gives the bifurcation diagram of the equation $\mathcal{P}(L)$. Hence the bifurcation diagram of (1), when written in terms of $\|u\|_{L^\infty([-1,1])}$ and λ, is given by

$$\|u\|_{L^\infty([-1,1])} = \frac{\gamma^{-1}(\lambda^{\frac{1}{p}})}{\lambda^{\frac{1}{p-m-1}}}.\blacksquare$$

Remark 3 For $p = 2$ and $m = 0$ the bifurcation diagram of (1) can be written in closed form as

$$\|u\|_{L^\infty([-1,1])} = \frac{1}{\lambda}\left(1 + \frac{1}{\sin(\sqrt{\lambda} - \frac{\pi}{2})}\right),$$

where $\lambda \in]\lambda_1(2), \lambda_1^*(0,2)] \equiv]\frac{\pi^2}{4}, \pi^2]$.

Remark 4 For both types of wave equations (the Schrödinger and the semilinear wave equation) mentioned in the Introduction the consideration of the case $p \neq 2$ leads to some new facts that seem to be not well mentioned in the previous literature. Indeed: if we consider, for instance, $m \in (0,1)$, the study of standing wave solutions leads to the consideration of the problem $-(|u'|^{p-2}u')' + |u|^{m-1}u = \lambda u$, and thus, by the results of [11], the difference with respect to the case $p = 2$ is that there is an additional strictly positive solution (that does not exist for $p = 2$) which is stable (since this part of the bifurcation diagram of solutions is increasing in λ).

3 The branches of nodal solutions.

In this section we consider solutions of (1) that change sign. Let us define precisely what we mean by nodal solutions.

Definition 3: Let Z be the set $\{x_1, \ldots, x_k : -1 = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < x_{k+1} = 1\}$. A solution of (1), $u_\lambda : [-1,1] \to \mathbb{R}$ is a nodal solution with k nodes if u_λ satisfies the conditions of Definition 1 except the positivity condition, $u_\lambda(x) = 0$ if $x \in Z$ and $u_\lambda(x) \neq 0$ if $x \in]-1,1[$ and $x \notin Z$.

The main result of this section is the following:

Theorem 2 *Let $\lambda_k(p)$ and $\lambda_k^*(m,p)$ be the positive real numbers defined by (8)-(9) with $k \geq 2$. Then:*

a) If $\lambda \in]0, \lambda_k(p)[\equiv]0, k^p\lambda_1(p)[$ there is no nodal solution of (1) with $(k-1)$ nodes.

b) If $\lambda \in]\lambda_k(p), \lambda_k^*(m,p)[\equiv]k^p\lambda_1(p), k^p\lambda_1^*(m,p)[$ there is a unique nodal solution (1) with $(k-1)$ nodes. Moreover

$$\frac{\partial u_\lambda}{\partial n}(x_i) \neq 0, \quad \text{for } i \in \{0, 1, \ldots, k+1\}$$

and

$$\underline{K}\,|x - x_i| \leq |u_\lambda(x)| \leq \overline{K}\,|x - x_i|\,, \tag{23}$$

if $|x - x_i| = d(x, Z)$, for some constants $\overline{K} > \underline{K} > 0$.

c) If $\lambda = \lambda_k^*(m,p) \equiv k^p\lambda_1^*(m,p)$ there is only one nodal solution, $u_{\lambda_k^*(m,p)}$ (1) with $(k-1)$ nodes. Moreover

$$\frac{\partial u_{\lambda_k^*(m,p)}}{\partial n}(x_i) = 0, \quad \text{for } i \in \{0, 1, \ldots, k+1\}.$$

and

$$\|u_{\lambda_k^*(m,p)}\|_{L^\infty[-1,1]} = \left(\frac{p}{(m+1)\lambda_k^*}\right)^{\frac{1}{p-m-1}}.$$

Moreover, the behavior of $u_{\lambda_k^*(m,p)}$ near the boundary of the points where $u_{\lambda_k^*(m,p)}$ vanishes is of the same type than (13),i.e.

$$\underline{K}\,|x - x_i|^{p/(p-1-m)} \leq \left|u_{\lambda_k^*(m,p)}(x)\right| \leq \overline{K}\,|x - x_i|^{p/(p-1-m)} \tag{24}$$

if $|x - x_i| = d(x, Z)$, for some constants $\overline{K} > \underline{K} > 0$.

d) If $\lambda > \lambda_k^*(m,p)$ then the function $u_\lambda : [-1,1] \to \mathbb{R}$ defined by

$$u_\lambda(x) = \sum_{i=1}^{i=k} s(i) u_{\lambda,\zeta_i}(x)$$

is a solution of (1) where

$$u_{\lambda,\zeta_i}(x) = \begin{cases} \left(\frac{\lambda_1^*(m,p)}{\lambda}\right)^{\frac{1}{p-m-1}} u_{\lambda_1^*(m,p)}\left(\dfrac{x-\zeta_i}{\omega}\right) & \text{if } |x - \zeta_i| < \omega, \\ 0 & \text{if } x \in [-1, \zeta_i - \omega] \cup [\zeta_i + \omega, 1], \end{cases}$$

$\omega = \left(\dfrac{\lambda_1^*(m,p)}{\lambda}\right)^{\frac{1}{p}}$, $\zeta_i \in [-1+\omega, 1-\omega]$ with $\zeta_i \in [-1+\omega, 1-\omega]$ for $i = 1, \ldots, k$ and $\zeta_{i+1} - \zeta_i > 2\omega$ for $i = 1, \ldots, k-1$, $u_{\lambda_1^*(m,p)}$ is the unique positive solution of (1) for $\lambda = \lambda_1^*(m,p)$, and consider the

set of functions $s : \{1, 2, \ldots, k\} \to \{-1, 1\}$ such that $s(1) = 1$ and that exists $i \in \{2, \ldots, k\}$ such that $s(i) = -1$. Therefore, for each $\lambda > \lambda_k^*(m, p)$ there are, up to a change sign, $2^{(k-1)} - 1$ continua of nonnegative-nonpositive solutions that depend arbitrarily on the parameters ζ_i with $i \in \{1, \ldots, k\}$. The number of functions s defined above, up to a change sign, is $2^{(k-1)} - 1$, where $k > 1$. Moreover the behavior of u_λ near the boundary of the points where u_λ vanishes is given by (24).

The following two lemmata show that the structure of these branches can be obtained from the branch of positive solutions.

Lemma 2 *Let $u_\mu : [-1, 1] \to \mathbb{R}$ be a nodal solution of (1) for $\lambda = \mu$ with k nodes. The function $u^+_{\mu,i} : [-\frac{x_i - x_{i-1}}{2}, \frac{x_i - x_{i-1}}{2}] \to \mathbb{R}$ defined by $u^+_{\mu,i}(x) = |u_\mu(x - \frac{x_i + x_{i-1}}{2})|$ is a positive solution of*

$$-\frac{d}{dx}\left(|\frac{du}{dx}|^{p-2}\frac{du}{dx}\right) + |u|^{m-1}u = \mu|u|^{p-2}u, \; in \;]-\frac{x_i - x_{i-1}}{2}, \frac{x_i - x_{i-1}}{2}[, \qquad (25)$$
$$u(-\frac{x_i - x_{i-1}}{2}) = u(\frac{x_i - x_{i-1}}{2}) = 0,$$

where $i \in \{1, \ldots, k, k+1\}$. Moreover, the zeros of u_λ are equispaced in $[-1, 1]$, i.e., $x_i - x_{i-1} = \dfrac{2}{k+1}$ and the function $w_\lambda : [-1, 1] \to \mathbb{R}$ defined by

$$w_\lambda(y) = (k+1)^{\frac{p}{p-m-1}} u^+_{\mu,i}(y\frac{x_i - x_{i-1}}{2}) \quad for \; y \in [-1, 1],$$

is the unique positive solution of (1) for $\lambda = \dfrac{\mu}{(k+1)^p}$.

Proof of Lemma 2. Since the differential equation (1) is invariant under translations, under the transformation $(u) \to (-u)$ and u_μ satisfies $u_\mu(x_{i-1}) = u_\mu(x_i) = 0$, the function $u^+_{\mu,i}$ is a solution of (25) for all $i \in \{1, \ldots, k, k+1\}$. In addition, u_μ is continuous and by (18) $u \in C^1(]-1, 1[)$, consequently $\dfrac{\mathrm{d}u^+_{\mu,i}}{\mathrm{d}x}(x_i^-) = -\dfrac{\mathrm{d}u^+_{\mu,i+1}}{\mathrm{d}x}(x_i^+)$. As we have proved in Lemma 1 , $u^+_{\mu,i}$, for all $i \in \{1, \ldots, k, k+1\}$, are even functions and according to (18) the maxima of each of them are equal. Therefore, the uniqueness for positive solutions implies the identity $x_i - x_{i-1} = x_{i+1} - x_i$, i.e, the zeroes of a nodal solution divide the interval $]-1, 1[$ in $k+1$ subintervals of the same length, in other words $x_i - x_{i-1} = \dfrac{2}{k+1}$ where $i \in 1, \ldots, k$ and $x_0 = -1$.

Finally, if we make the change of independent variable $y = \dfrac{2x}{x_i - x_{i-1}} \equiv (k+1)x$, $w_\lambda(y) = (k+1)^{\frac{p}{p-m-1}} u^+_{\mu,i}\left(\dfrac{y}{k+1}\right)$ in (25) we obtain that w_λ is a solution of (1) for $\lambda = \dfrac{\mu}{(k+1)^p}$.■

Lemma 3 *Let $u_\lambda : [-1, 1] \to \mathbb{R}$ be a nodal solution of (1) with k nodes. If $\dfrac{du_\lambda}{dx}(x_i) \neq 0$ then u_λ is such that*

$$u_\lambda(x + \frac{2i}{k+1}) = -u_\lambda(-x + \frac{2i}{k+1}) \quad for \; all \; x \in [0, \frac{2}{k+1}] \quad and \; all \; i \in \{1, \ldots, k\}. \qquad (26)$$

Proof of Lemma 3. According to Theorem 1 the function u_λ is $C^1(]-1,1[)$, $u^+_{\lambda,i}$ is solution of (25) and is symmetric with respect to $\dfrac{x_i + x_{i-1}}{2}$. Since the positive solution is unique for (25), is invariant under translations, for the transformations $u \to -u$ and $x \to -x$ the identity (26) follows.■

Proof of Theorem 2. Combining the results in Theorem 1, Lemma 2 and Lemma 3, Theorem 2 follows.■

Acknowledgements

The research of J.I. Díaz and J. Hernández was partially supported by the projects MTM2011-26119 and MTM2014-57113 of the DGISPI (Spain). The research of J.I. Díaz was partially supported by the UCM Research Group MOMAT (Ref. 910480).

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