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STEINER SYMMETRIZATION FOR CONCAVE SEMILINEAR ELLIPTIC AND PARABOLIC EQUATIONS AND THE OBSTACLE PROBLEM

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Abstract. We extend some previous results in the literature on the Steiner rearrangement of linear second order elliptic equations to the semilinear concave parabolic problems and the obstacle problem.

1. **Introduction.** In this paper we extend some previous results in the literature on the Steiner rearrangement of second order semilinear parabolic problems of the type

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + h(t)g(u) = f, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $h \in W^{1,\infty}(0,T)$ is such that $h(t) \geq 0$ for all $t \in (0,T)$ and g is a concave continuous nondecreasing function such that $g(0) = 0$ satisfying that

$$\int_0^\tau \frac{d\sigma}{g(\sigma)} < \infty, \qquad \forall \tau > 0. \tag{H}$$

We recall that the existence and uniqueness of a weak solution $u \in C([0,T] : L^2(\Omega)) \cap L^2(\delta, T : H_0^1(\Omega))$ for any $\delta \in (0,T)$ can be obtained, for instance, by the application of the theory of maximal monotone operators in $L^2(\Omega)$ (see [6], [2] and [4]).

Let us start by recalling that given a general measurable function $v : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, with $n, m \geq 1$ and $n + m = N$, for a fixed $y \in \mathbb{R}^m$ we can define the function $\mu_v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by means of

$$\mu_v(t,y) = |\{x \in \mathbb{R}^n : |v(x,y)| > t\}|.$$

The Hardy-Littlewood-Polya decreasing rearrangement $v^* : [0,+\infty) \times \mathbb{R}^m \to \mathbb{R}$ is given as

$$v^*(s,y) = \sup\{t > 0 : \mu_v(t,y) > s\} = \inf\{t > 0 : \mu_v(t,y) \leq s\}.$$

It can be shown that, if ω represents a generic measurable subset of $\mathbb{R}^n \times \mathbb{R}^m$

$$\int_0^s v^*(\sigma, y)\, \mathrm{d}\sigma = \sup_{|\omega|=s} \int_\omega v(x,y)\, \mathrm{d}x, \quad \text{a.e. } y \in \mathbb{R}^m. \tag{1}$$

Finally we define the Steiner symmetrization of v with respect to x as

$$v^\#(x,y) = v^*(\omega_n |x|^n, y), \quad \text{a.e. } y \in \mathbb{R}^m,$$

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where ω_n is the measure of the n-dimensional ball (see details, for instance, in [10], [11]).

The basic idea underlying Steiner symmetrization is to consider the integral of the function over *slices*. We take very particular *slices* of the form

$$G(y) = \{x \in \mathbb{R}^m : u(x,y) > u^*(s,y)\}$$

where $|G(y)| = s$ (by construction of u^*). Variable s should formally be included in the definition but this will not lead to confusion.

We shall use the following notations:

$$\Omega = \Omega' \times \Omega''$$

is a product domain, where $(x,y) \in \Omega' \times \Omega''$. We shall denote by B a ball such that $|B| = |\Omega'|$ and then we introduce

$$\Omega^\# = B \times \Omega'' \qquad \Omega^* = (0, |\Omega'|) \times \Omega''.$$

Our main result is the following:

Theorem 1.1. *Let g be concave, verifying (H). Let $h \in W^{1,\infty}(0,T)$, such that $h(t) \geq 0$ for all $t \in (0,T)$, $f \in L^2(0,T : L^2(\Omega))$ with $f \geq 0$ in $(0,T)$ and let $u_0 \in L^2(\Omega)$ be such that $u_0 \geq 0$. Let $u \in C([0,T], L^2(\Omega)) \cap L^2(\delta, T : H_0^1(\Omega))$ and $v \in C([0,T] : L^2(\Omega^\#)) \cap L^2(\delta, T : H_0^1(\Omega^\#))$ be the unique solutions of*

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \Delta u + h(t) g(u) = f(t), & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

$$(P^\#) \begin{cases} \frac{\partial v}{\partial t} - \Delta v + h(t) g(v) = f^\#(t), & \text{in } \Omega^\# \times (0,T), \\ v = 0, & \text{on } \partial\Omega^\# \times (0,T), \\ v(0) = v_0, & \text{on } \Omega^\#, \end{cases}$$

where $v_0 \in L^2(\Omega^\#)$, $v_0 \geq 0$ is such that

$$\int_0^s u_0^*(\sigma, y) d\sigma \leq \int_0^s v_0^*(\sigma, y) d\sigma, \qquad \forall s \in [0, |\Omega'|] \text{ and a.e. } y \in \Omega''.$$

Then, for any $t \in [0,T]$ and $s \in [0, |\Omega'|]$

$$\int_0^s u^*(t, \sigma, y) d\sigma \leq \int_0^s v^*(t, \sigma, y) d\sigma \qquad \text{a.e. } y \in \Omega''.$$

The main idea of the proof is to use a generalization of the Trotter-Kato formula and to decompose the process in two different steps: the parabolic case without any absorption term ($g \equiv 0$) and the consideration of the auxiliary distributed ODE

$$\begin{cases} \xi_t + h(t) g(\xi) = 0, \\ \xi(0) = \xi_0. \end{cases}$$

Theorem 1 extends previous results in the literature on the comparison of Steiner rearrangements which until now were merely related to linear problems (see [1], [13], [7], [8], [9] and their references). The case in which g is convex is considered in [12].

2. Some definitions on the Steiner symmetrization.

We recall that Hardy's inequality and (1) provides us with the estimate

$$\int_{\Omega(y)} f(x,y)\, \mathrm{d}x \leq \int_0^s f^*(\sigma, y)\, \mathrm{d}\sigma, \qquad \text{a.e. } y \in \mathbb{R}^m.$$

Now, let u be a measurable function. We define the auxiliary function

$$F(s,y) = \int_0^s u^*(\sigma, y)\, \mathrm{d}\sigma, \qquad \text{a.e. } y \in \mathbb{R}^m.$$

From the definition of the rearrangement we have that

$$F(s,y)=\int_{\Omega(y)} u(x,y)\ \mathrm{d}x, \qquad \text{a.e. } y\in\mathbb{R}^m.$$

In [1] it was shown that:

Lemma 2.1. *Let F be defined as before and let u be regular enough. Then,*

$$\frac{\partial F}{\partial y_i}=\int_{\Omega(y)}\frac{\partial u}{\partial y_i}$$

$$\left(\frac{\partial^2 F}{\partial y_i\partial y_j}\right)\geq\left(\int_{\Omega(y)}\frac{\partial^2 u}{\partial y_i\partial y_j}\right)$$

in the sense of matrices.

The results in [1] where presented on the stationary case and without non linear perturbation (the integro-differential equation which results is very difficult to treat by maximum principle arguments). Now, we may consider t, the time variable as a first y component, we may extend all of the above to the evolutionary case. It, then, holds that

$$\frac{\partial F}{\partial t}=\int_{\Omega(t,y)}\frac{\partial u}{\partial t}$$

and the analogous for the second derivative, which we will not need. This is also a consequence of other results ([3], [14], [15]).

To conclude the definitions we define the concentration relation as

$$u\preccurlyeq v\equiv\int_0^s u^*(\sigma,y)\ \mathrm{d}\sigma\leq\int_0^s v^*(\sigma,y)\ \mathrm{d}\sigma, \qquad \text{a.e. } y\in\Omega'', \quad \text{for any } s\in[0,|\Omega'|].$$

Although it is not strictly a result on symmetrization the following lemma (see, e.g., [10]) is a very useful tool for what follows.

Lemma 2.2. *Let $y,z\in L^1(0,M)$, $y,z\geq 0$ a.e., suppose y is non-increasing and*

$$\int_0^s y(\sigma)\ d\sigma\leq\int_0^s z(\sigma)\ d\sigma, \qquad \forall s\in[0,M].$$

Then, for every continuous non-decreasing function Φ we have

$$\int_0^s \Phi(y(\sigma))\ d\sigma\leq\int_0^s \Phi(z(\sigma))\ d\sigma \qquad \forall s\in[0,M].$$

Written in terms of the concentration relation the above property can be read as

$$y\preccurlyeq z\implies\Phi(y)\preccurlyeq\Phi(z)$$

for any function Φ convex and increasing.

Extending the above concentration relation we can define

Definition 2.3. Let $\Omega_1\equiv\Omega$ and $\Omega_2\equiv\Omega^{\#}$. Let

$$S_i : L^2(\Omega_i)\to\mathcal{C}([0,T]:L^2(\Omega_i))$$

We say that the pair (S_1,S_2) is *Steiner concentration monotone* if given $u_i\in L^2(\Omega_i)$ we have that

$$u_1\preccurlyeq u_2\implies S_1(t)u_1\preccurlyeq S_2(t)u_2, \qquad \text{for any } t\in[0,T].$$

It will be useful to recall that if $(u_n^i)\in L^2(\Omega_i)$ are two L^2- convergent sequences such that $u_n^i\to u^i$ and $u_n^1\preccurlyeq u_n^2$ then $u^1\preccurlyeq u^2$.

3. **Steiner comparison for linear parabolic equations and for a distributed nonlinear ODE.** We first compare the semigroup of a linear equation an its Steiner symmetrization, to show they are Steiner concentration monotone pairs. The following result can be proven by using as fundamental ingredient the proof for the elliptic case: see [8] for a detailed proof.

Proposition 1. *Let*

$$(A) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases},$$

$$(A^\#) \qquad \begin{cases} \frac{\partial v}{\partial t} - \Delta v = 0, & \text{in } (0,T)\times\Omega^\#, \\ v = 0, & \text{on } (0,T)\times\partial\Omega^\#, \\ v(0) = v_0, & \text{on } \Omega^\#, \end{cases}$$

and let S_Δ and $S_{\Delta^\#}$ be their associated L^2 semigroups on Ω and $\Omega^\#$ respectively. Then $(S_\Delta, S_{\Delta^\#})$ is a Steiner concentration monotone pair.That is, if

$$\int_0^s u_0^*(\sigma,y)d\sigma \le \int_0^s v_0^*(\sigma,y)d\sigma, \qquad \forall s\in[0,|\Omega'|],\ a.e.\ y\in\Omega'',$$

then we have

$$\int_0^s u^*(t,\sigma,y)\sigma \le \int_0^s v^*(t,\sigma,y)\sigma \qquad \forall t\in[0,T], \forall s\in[0,|\Omega'|], a.e.\ y\in\Omega'',$$

where $u = S_\Delta(\cdot)u_0$ and $v = S_{\Delta^\#}(\cdot)v_0$.

Concerning nonlinear distributed ODEs we have:

Proposition 2. *Let g be concave verifying (H) and let $h\in L^\infty(0,T)$, $h\ge 0$. Let u,v satisfy*

$$(B) \qquad \begin{cases} \frac{\partial u}{\partial t} + h(t)g(u) = 0, & \text{in } (0,T)\times\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases},$$

$$(B^\#) \qquad \begin{cases} \frac{\partial v}{\partial t} + h(t)g(v) = 0, & \text{in } (0,T)\times\Omega^\#, \\ v(0) = v_0, & \text{on } \Omega^\#. \end{cases}$$

Finally, let S_B and $S_{B^\#}$ be their associated evolution Green operators (i.e. the associated semigroups if $h(t)$ is constant). Then $(S_B, S_{B^\#})$ is a Steiner concentration monotone pair. That is, if

$$\int_0^s u_0^*(\sigma,y)d\sigma \le \int_0^s v_0^*(\sigma,y)d\sigma, \qquad \forall s\in[0,|\Omega'|],\ a.e.\ y\in\Omega'',$$

then we have

$$\int_0^s u^*(t,\sigma,y)d\sigma \le \int_0^s v^*(t,\sigma,y)d\sigma \qquad \forall t>0, s\in[0,|\Omega'|],\ a.e.\ y\in\Omega'',$$

for the solutions $u = S_B(\cdot)u_0$, $v = S_{B^\#}(\cdot)v_0$.

Proof. In a first step we assume, in addition that g is Lipschitz continuous and $g(0)=\varepsilon>0$. Let

$$\Phi(\xi) = \int_0^\xi \frac{d\sigma}{g(\sigma)}, \quad \Psi = \Phi^{-1}, \qquad H(t) = \int_0^t h(\sigma)d\sigma.$$

It is easy to check that

$$(S_B(t)u)(x,y) = \Psi(\Phi(u_0(x,y) - H(t)), \qquad (S_{B^\#}(t)v)(x,y) = \Psi(\Phi(v_0(x,y) - H(t)).$$

For these solutions

$$\begin{aligned}\mu_{S_B(t)u_0}(\tau,y) =&|\{x\in\mathbb{R}^n : |u(t,x,y)|>\tau\}|\\ =&|\{x\in\mathbb{R}^n : u_0(x,y)>\Phi(\Psi(\tau+H(t)))\}|\\ =&\mu_{u_0}(\Phi(\Psi(\tau)+H(t)),y).\end{aligned}$$

Since Φ, Ψ are monotone increasing then

$$\begin{aligned}\Big(S_B(t)u_0\Big)^*(s,y) =& \inf\{\tau>0 : \mu_{u_0}(\Phi(\Psi(\tau)+t),y)\le s\}\\ =& \inf\{\Phi(\Psi(\sigma)-H(t)) : \mu_{u_0}(\sigma,y)\le s\}\\ =&\Phi\Big(\Psi\big(\inf\{\sigma>0 : \mu_{u_0}(\sigma,y)\ge s\}\big)-H(t)\Big)\\ =&\Phi\Big(\Psi\big(u_0^*(s,y)\big)-H(t)\Big)=u^*(t,s,y).\end{aligned}$$

Therefore, u^* satisfies

$$\frac{\partial u^*}{\partial t}+h(t)g(u^*)=0.$$

Now let $w=e^{\lambda t}u$, then we have by the lemma $w^*=e^{\lambda t}u^*$, and so we have that w^* satifies

$$\frac{\partial w^*}{\partial t}+e^{\lambda t}h(t)g(e^{-\lambda t}w^*)-\lambda w^*=0.$$

We choose λ large enough so that $e^{\lambda t}h(t)g(e^{-\lambda t}w)-\lambda w$ be nonincreasing function on u for every $t\in(0,T)$. Analogous calculations provided information on $z=e^{\lambda t}v$. Let

$$\tilde T=\sup\left\{t : \int_0^s u^*(t,\sigma,y)\le\int_0^s v^*(t,\sigma,y)\sigma, \forall s\in[0,|\Omega'|]\right\}\ge 0.$$

Since $e^{\lambda t}h(t)g(e^{-\lambda t}w)-\lambda w$ is concave, for $t<\tilde T$, we apply lemma 2.2 and get

$$\begin{aligned}&\frac{d}{dt}\int_0^s(w^*(t,\sigma,y)-z^*(t,\sigma,y))d\sigma\\ =&\int_0^s h(t)(e^{\lambda t}h(t)g(e^{-\lambda t}z)-\lambda z-(e^{\lambda t}h(t)g(e^{-\lambda t}w)-\lambda w))d\sigma\le 0.\end{aligned}$$

So, we get

$$e^{\lambda t}\int_0^s(u^*(t,\sigma,y)-v^*(t,\sigma,y))d\sigma=\int_0^s(w^*(t,\sigma,y)-z^*(t,\sigma,y))d\sigma\le 0$$

and the result follows once g is Lipschitz continuous and $g(0)=\varepsilon$.

In the general case since g is associated to a maximal monotone graph of $\mathbb{R}^2$ we can approximate it by its Yosida approximation (which is still concave and satisfies (H)) and we get the result by passing to the limit. Finally we make $\varepsilon\downarrow 0$ and use the continuity of solutions with respect to g (see [5] and [4]). □

4. **Proof of the main theorem.** The special case $f=0$ and $h(t)\equiv h$ independent on t is easier. Since we know

$$\int_0^s u_0^*(\sigma,y)\,d\sigma\le\int_0^s v_0^*(\sigma,y)\,d\sigma, \qquad \forall s,\forall y$$

applying Proposition 1 and Proposition 2 inductively we get

$$\begin{aligned}&\int_0^s\left[\left(S_A\left(\frac{t}{n}\right)S_B\left(\frac{t}{n}\right)\right)^n u_0\right]^*(\sigma,y)\,d\sigma\\ \le&\int_0^s\left[\left(S_{A\#}\left(\frac{t}{n}\right)S_{B\#}\left(\frac{t}{n}\right)\right)^n v_0\right]^*(\sigma,y)\,d\sigma\end{aligned}$$

where S_A is the semigroup associated to problem (A) and S_B is the semigroup associated to problem (B) and analogously for $S_{A^\#}$ and $S_{B^\#}$.

Taking limits, applying the Trotter-Kato formula (see [5]) and applying convergence under the integral sign we get

$$\int_0^s [S_P(t)u_0]^*(\sigma,y)d\sigma \leq \int_0^s [S_{P^\#}(t)v_0]^*(\sigma,y)d\sigma$$

for any $t\in[0,T]$, for any $s\in[0,|\Omega'|]$ and a.e. $y\in\Omega''$. For the case $f\neq 0$ and $h(t)$ time dependent the Trotter-Kato formula can be also applied (see, e.g. [16]). In fact, to deal with the affine case $f(t)\neq 0$ we shall use a "reduction of order technique" argument which can be found on [4]. We point out that by an approximation argument and posterior passing to the limit process we can assume, without loss of generality, that in fact $f\in H^1(0,T;L^2(\Omega))$. Now, let us define $f(t+\cdot)\in H^1(0,T;L^2(\Omega)) : s\mapsto f(t+s)$ and $U(t)=(u(t),f(t+\cdot))\in L^2(\Omega)\times H^1(0,T;L^2(\Omega))$, $V(t)=(v(t),f^\#(t+\cdot))\in L^2(\Omega^\#)\times H^1(0,T;L^2(\Omega^\#))$. Let us note that U is the unique solution of

$$\begin{cases}\frac{\partial U}{\partial t}+\hat{L}U=0, & t\in(0,T)\\ U(0)=(u_0,f)\end{cases}\qquad\qquad \begin{cases}\frac{\partial V}{\partial t}+\hat{L}V=0, & t\in(0,T)\\ V(0)=(v_0,f^\#)\end{cases}$$

where

$$\hat{L}(u,\xi)=(-\Delta u+h(t)g(u)-\xi(0),\xi').$$

We can use a decomposition $\hat{L}=\hat{L}_1+\hat{L}_2$ in the following way:

$$\hat{L}_1(u,\xi)=(-\Delta u+h(t)g(u),0),\qquad \hat{L}_2(u,\xi)=(-\xi(0),\xi').$$

Let us define the problems

$$(C)\begin{cases}\frac{\partial U}{\partial t}+\hat{L}_1U=0,\\ U(0)=(u_0,f),\end{cases},\qquad (C^\#)\begin{cases}\frac{\partial V}{\partial t}+\hat{L}_1V=0,\\ V(0)=(v_0,f^\#),\end{cases}$$

$$(D)\begin{cases}\frac{\partial U}{\partial t}+\hat{L}_2U=0,\\ U(0)=(u_0,f),\end{cases},\qquad (D^\#)\begin{cases}\frac{\partial V}{\partial t}+\hat{L}_2V=0,\\ V(0)=(v_0,f^\#),\end{cases}$$

and the correspondent solution operators

$$S_C(t)(u_0,f)=(S_P(t)u_0,f),\quad S_{C^\#}(t)(v_0,f^\#)=(S_P(t)u_0,f),$$

$$S_D(t)(u_0,f)=\left(u_0+\int_0^t f(s)ds,f\right),\quad S_{D^\#}(t)(v_0,f^\#)=\left(v_0+\int_0^t f^\#(s)ds,f^\#\right).$$

Let Q be the projection operator such that $u(t)=QU(t)$. Let us study QS_C and QS_D. Since

$$\int_0^s u_0^*(\sigma,y)\ d\sigma\leq\int_0^s v_0^*(\sigma,y)\ d\sigma,\qquad \text{for all } s\in[0,|\Omega'|] \text{ and a.e. } y\in\Omega''$$

we have, by the above explicit formulas (for the first component we apply the similar proof as in the case $f=0$)

$$\int_0^s [Q\ S_C(t)u_0]^*(\sigma,y)d\sigma\leq\int_0^s [Q\ S_{C^\#}(t)v_0]^*(\sigma,y)d\sigma,$$

$$\int_0^s [Q\ S_D(t)u_0]^*(\sigma,y)d\sigma\leq\int_0^s [Q\ S_{D^\#}(t)v_0]^*(\sigma,y)d\sigma.$$

By applying an induction argument again we get

$$\int_0^s\left[Q\left(S_C\left(\frac{t}{n}\right)S_D\left(\frac{t}{n}\right)\right)^n u_0\right]^*(\sigma,y)d\sigma$$
$$\leq\int_0^s\left[Q\left(S_{C^\#}\left(\frac{t}{n}\right)S_{D^\#}\left(\frac{t}{n}\right)\right)^n v_0\right]^*(\sigma,y)d\sigma.$$

Finally, since all the operators are maximal monotone on their respective Hilbert spaces, we can take limits, apply the Trotter-Kato formula to justify the convergence of the limits and the result holds.

5. **Remarks and applications.** We point out that the main result applies to the parabolic obstacle problem:

$$\begin{cases} \left.\begin{array}{l} \frac{\partial u}{\partial t} - \Delta u - f(t,x) \geq 0,\ u \geq 0 \\ (\frac{\partial u}{\partial t} - \Delta u - f(t,x))u = 0 \end{array}\right\} & \text{in } (0,T)\times\Omega, \\ \qquad u = 0, & \text{on } (0,T)\times\partial\Omega, \\ \qquad u(0) = u_0, & \text{on } \Omega, \end{cases}$$

assumed $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ and $f \in L^2(0,T : L^2(\Omega))$. The main argument is it can be reformulated in terms of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \beta(u) \ni f(t,x) + 1, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

where β is the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(r) = \begin{cases} \emptyset, & r < 0, \\ (-\infty, 1], & r = 0, \\ \{1\}, & r > 0, \end{cases}$$

and that $\beta(u)$ can be approximated by a sequence $\beta_\lambda(u)$ of non decreasing concave functions satisfying (H) (take, for instance, $\beta_\lambda(u)$ such that $\beta_\lambda(u) = u^{\frac{1}{\lambda}}$ if $u \geq 0$). It is well known that the correspondent solutions u_λ converge strongly in $C([0,T] : L^2(\Omega))$ to the solution u of the obstacle problem and so the comparison of the associated Steiner rearrangements is mantained after passing to the limit.

Finally, we also mention that the associated nonlinear elliptic equation

$$\begin{cases} -\Delta u + g(u) = f(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

can be considered in this framework (since we can write $u(x) = \lim_{t\to\infty} u(t,x)$ for some suitable $u(t,x)$ solution of a nonlinear parabolic problem for which we can apply Theorem 1.1).

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