Dynamical Systems, Differential
Equations and Applications
AIMS Proceedings, 2015

doi:10.3934/proc.2015.0066

pp. **66–74**

STABILIZATION OF A HYPERBOLIC/ELLIPTIC SYSTEM MODELLING THE VISCOELASTIC-GRAVITATIONAL DEFORMATION IN A MULTILAYERED EARTH

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Abstract. In the last 30 years several mathematical studies have been devoted to the viscoelastic-gravitational coupling in stationary and transient regimes either for static case or for hyperbolic case. However, to the best of our knowledge there is a lack of mathematical study of the stabilization as t goes to infinity of a viscoelastic-gravitational models crustal deformations of multilayered Earth. Here we prove that, under some additional conditions on the data, the difference of the viscoelastic and elastic solutions converges to zero, as t goes to infinity, in a suitable functional space. The proof of that uses a reformulation of the hyperbolic/elliptic system in terms of a nonlocal hyperbolic system.

1. **Introduction.** Volcanic eruptions are the outcome of significant physical and geological processes (see [5], [6] and [7]). In order to interpret geodetic anomalies such as displacements, gravity changes, etc. there is very extensive literature on deformation modeling coupling gravity effects. The techniques needed for calculation of displacements and gravity change, due to internal sources, have been developed during the last decades (see for example [10], [11] and [12]). The presence of incoherent materials and high temperatures needs consideration of anelastic properties. Either theoretical or computational methods for the calculation of viscoelastic-gravitational displacements have been described in ([7], [8], [13], and theirs references). The objective of this work is to study the stability of the viscoelastic-gravitational model (VGP), considering it as hyperbolic/elliptic system. We consider an Earth model composed by several viscoelastic-gravitational layers overlying a viscoelastic-gravitational half space. The viscoelastic-gravitational model (VGP) is given by the following system of partial diferential equations for each layer:

$$(VGP)\begin{cases} \rho^i \mathbf{u}^i_{tt}(t,\mathbf{x}) - \gamma^i \Delta \mathbf{u}^i_t(t,\mathbf{x}) - \Delta \mathbf{u}^i(t,\mathbf{x}) - \dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(t,\mathbf{x})\right) \\ -\dfrac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(t,\mathbf{x})\cdot\mathbf{e}_z\right) + \dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(t,\mathbf{x}) \\ = \dfrac{\rho^i}{\mu^i}\nabla\phi^i(t,\mathbf{x}) + \mathbf{f}^i_u(t,\mathbf{x}), & \text{in } (0,T)\times\Omega_i, \\ -\Delta\phi^i(t,\mathbf{x}) = 4\pi\rho^i G div\mathbf{u}^i(t,\mathbf{x}) + f^i_\phi(t,\mathbf{x}) & \text{in } (0,T)\times\Omega_i\,. \end{cases}$$

2010 *Mathematics Subject Classification.* 34H15, 37B25, 35Q74, 35Q84.

Key words and phrases. Stabilization, viscoelastic-gravitational, weak solution, hyperbolic/elliptic equations, LaSalle's Invariance principle, Lyapunov function.

where $\mathbf{u}$ denotes the displacement, ϕ gravitational perturbed potential, $\gamma^i \Delta \mathbf{u}_t^i$ is the term introduced due to the viscoelasticity of each layer, ν the Poisson's ratio, ρ the unperturbed density of the medium, g the externally imposed gravitational acceleration, μ the rigidity, G the universal gravitational constant, $\mathbf{e}_z$ the unit vector pointing in the positive z-direction (down into the medium) and $\mathbf{f}_u^i$ and f_ϕ^i the body forces. We consider a spatial domain, $\Omega, \Omega = \bigcup_{i=1 \cdot p} \Omega_i$, as it is shown in Figure 1. Each layer is given through a common horizontal

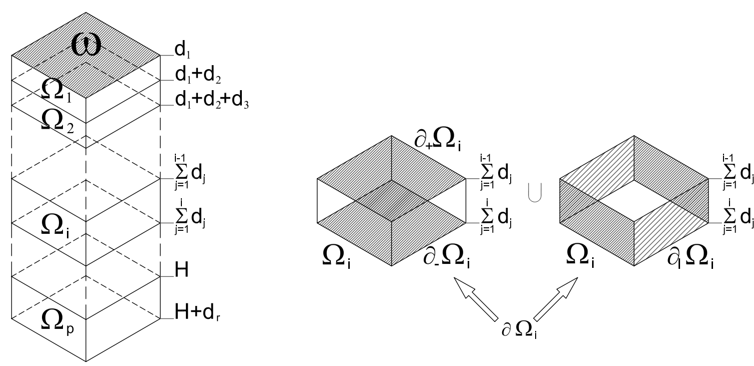

FIGURE 1. Layered Earth model. Illustration of the coordinate system and variation of the layer properties with depth.

open set, $\omega \subset \mathbb{R}^2$, and so

$$\Omega_1 := \omega \times (d_1, d_1 + d_2), \ \ \Omega_2 := \omega \times (d_1 + d_2, d_1 + d_2 + d_3), \ \text{etc.},$$

that is $\Omega_i := \omega \times \left(\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j \right) \subset \mathbb{R}^3$ when $\quad i = 1, \ldots, p-1$, and

$\Omega_p := \omega \times (H, H + d_r)$, when $H := \sum_{j=1}^{p-1} d_j$ and d_p can be equal to $+\infty$. Let $\mathbf{u}^i$:$[0,T] \times \Omega_i \longrightarrow \mathbb{R}^3$ be the displacement vector in each layer, $\mathbf{u}^i = \left(u_x^i, u_y^i, u_z^i\right)$, where T is an arbitrary time, and $\mathbf{f}_u^i$ and f_ϕ^i being the contribution of source terms which can represent magmatic intrusion, corresponding to body forces. Let us establish the boundary conditions of the problem. We identify the upper, lateral and bottom boundary for each layer: $\partial_+ \Omega_i = \omega \times \left\{ \sum_{j=1}^{i-1} d_j \right\}$, top boundary, $\partial_- \Omega_i = \omega \times \left\{ \sum_{j=1}^{i} d_j \right\}$, bottom boundary and $\partial_l \Omega_i = \partial \omega \times \left[\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j \right]$, lateral boundary.

Then $\partial \Omega_i = \partial_+ \Omega_i \cup \partial_- \Omega_i \cup \partial_l \Omega_i \forall i = 1, \ldots, p-1$. For the last layer, p-th, we have $\partial_+ \Omega_p = \omega \times \{H\}$ and $\partial_- \Omega_p = \omega \times \{H + d_p\}$. We shall add the boundary and transmission conditions for $i = 1, \ldots, p$ as follows. On the lateral boundary we have:

$$\mathbf{u}^i (t, \mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_l \Omega_i, t \in (0, T), \tag{1}$$

on upper boundary of the first layer $\partial_+ \Omega_1$:

$$\frac{\partial \mathbf{u}^1 (t, \mathbf{x})}{\partial z} = \mathbf{0}, \mathbf{x} \in \partial_+ \Omega_1, t \in (0, T), \tag{2}$$

and on bottom boundary, $\partial_- \Omega_p$:

$$\mathbf{u}^p (t, \mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_- \Omega_p, t \in (0, T). \tag{3}$$

In relation to gravitational perturbed potential we will assume that: on the lateral boundary $\partial_l \Omega_i$ for $i = 1, \ldots, p$:

$$\phi(t, \mathbf{x}) = 0, \mathbf{x} \in \partial_l \Omega_i, t \in (0, T), \tag{4}$$

on the upper boundary of the first layer $\partial_+ \Omega_1$:

$$\phi^1(t, \mathbf{x}) = \phi_0(t, \mathbf{x}), \mathbf{x} \in \partial_+ \Omega_1, t \in (0, T) \tag{5}$$

and on the bottom boundary, $\partial_- \Omega_p$:

$$\phi^p(t, \mathbf{x}) = 0, \mathbf{x} \in \partial_- \Omega_p, t \in (0, T). \tag{6}$$

We shall require "transmission conditions" since we can assure only that the first derivatives of $\mathbf{u}$ are continuous on the boundaries of the layers.Therefore, on $\partial_- \Omega_i = \partial_+ \Omega_{i+1}$ with $i = 1, \ldots, p-1$, the next conditions:

$$\mathbf{u}^i(t, \mathbf{x}) = \mathbf{u}^{i+1}(t, \mathbf{x}), \frac{\partial \mathbf{u}^i(t, \mathbf{x})}{\partial z} = \frac{\partial \mathbf{u}^{i+1}(t, \mathbf{x})}{\partial z}, \mathbf{x} \in \partial_- \Omega_i, t \in (0, T), \tag{7}$$

and

$$\phi^i(t, \mathbf{x}) = \phi^{i+1}(t, \mathbf{x}), \frac{\partial \phi^i(t, \mathbf{x})}{\partial z} = \frac{\partial \phi^{i+1}(t, \mathbf{x})}{\partial z}, \mathbf{x} \in \partial_- \Omega_i, t \in (0, T). \tag{8}$$

We have to add initial conditions in Ω:

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \mathbf{u}_t(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}). \tag{9}$$

We shall introduce a notion of a weak solution defining the energy spaces of test functions:

$$\begin{aligned}
&V_u := \{(\mathbf{u}^1, \phi^1), \ldots, (\mathbf{u}^p, \phi^p) \in \prod_{i=1}^{p} H^1(\Omega_i)^3 \times H^1(\Omega_i) \text{ such that } \mathbf{u}^i \\
&\quad \text{verifies (1) to (3) and (7)}\},\\
&V_\phi := \{((\mathbf{u}^1, \phi^1), \ldots, (\mathbf{u}^p, \phi^p)) \in \prod_{i=1}^{p} H^1(\Omega_i)^3 \times H^1(\Omega_i) \text{ such that } \phi^i \\
&\quad \text{verifies (4), (6), (8) and } \phi^i \equiv 0 \text{ on } \partial_+ \Omega_1\}.
\end{aligned}$$

The boundary data, ϕ_0, is extended to the interior of the domain Ω_1. So, there exists a function $\widehat{\phi}_0(t, \mathbf{x})$ for some $2 \leq q \leq +\infty$ such that

$$\widehat{\phi}_0 \in L^q(0, T : H^1(\Omega_1)), \ \widehat{\phi}_0(t, \mathbf{x}) = \phi_0(t, \mathbf{x}) \text{ in } (0, T) \times \partial_+ \Omega_1 \tag{10}$$

and

$$\widehat{\phi}_0(\mathbf{x}) = 0 \text{ in } (0, T) \times (\partial_- \Omega_1 \cup \partial_l \Omega_1). \tag{11}$$

The following regularity on the data is assumed:

$$\phi_0 \in L^2(0, T : \prod_{i=1}^{p} H^1(\Omega_i)) \text{ and verifies (10) and (11).} \tag{12}$$

$$\mathbf{f}_u \in L^2(0, T : \prod_{i=1}^{p} H^{-1}(\Omega_i)^3), \tag{13}$$

$$f_\phi \in L^q(0, T : \prod_{i=1}^{p} H^{-1}(\Omega_i)), \tag{14}$$

for some $2 \leq q \leq +\infty$ and

$$\mathbf{u}_0, \mathbf{v}_0 \in V_u. \tag{15}$$

We recall the weak solution of viscoelastic-gravitational problem (for more details see [1] and [2]).

Definition 1.1. We assume the regularity (12)-(15), on the functions $\mathbf{f}_u$, f_ϕ, ϕ_0, $\mathbf{u}_0$ and $\mathbf{v}_0$. We say that $(\mathbf{u},\phi)$ is a weak solution of the problem (VGP) with the boundary conditions (1)-(8) and (9) if $(\mathbf{u},\phi-\phi_0)\in L^2(0,T:V)$, $\mathbf{u}_{tt}\in L^2(0,T:V_u')$ and for any test function $(\mathbf{w},\theta)\in L^2(0,T:V)$, $\mathbf{v}\in H^1(0,T:V_u')$ the following identities hold:

$$\int_0^T\sum_{i=1}^p\Big[\left\langle\rho^i\mathbf{u}_{tt}^i(t,\boldsymbol{\cdot}),\mathbf{w}^i(t,\boldsymbol{\cdot})\right\rangle+\int_{\Omega_i}\frac{1}{1-2\nu^i}div\mathbf{u}^i(t,\mathbf{x})\,div\mathbf{w}^i(t,\mathbf{x})$$
$$-\frac{\rho^ig}{\mu^i}\nabla\left(\mathbf{u}^i(t,\mathbf{x})\cdot\mathbf{e}_z\right)\cdot\mathbf{w}^i(t,\mathbf{x})+\frac{\rho^ig}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(t,\mathbf{x})\;\mathbf{w}^i(t,\mathbf{x})$$
$$+\nabla\mathbf{u}^i(t,\mathbf{x}):\nabla\mathbf{w}^i(t,\mathbf{x})\;+\gamma^i\nabla\mathbf{u}_t^i(t,\mathbf{x}):\nabla\mathbf{w}^i(t,\mathbf{x})\;d\mathbf{x}\Big]\,dt$$
$$=-\int_0^T\left[\sum_{i=1}^p\frac{\rho^i}{\mu^i}\int_{\Omega_i}\nabla\phi^i(t,\mathbf{x})\cdot\mathbf{w}^i(t,\mathbf{x})\;d\mathbf{x}+\left\langle\mathbf{f}_u^i(t,\boldsymbol{\cdot}),\mathbf{w}^i(t,\boldsymbol{\cdot})\right\rangle_{V_u'\times V_u}\right]$$

and

$$\sum_{i=1}^p\int_{\Omega_i}\nabla\phi^i(t,\boldsymbol{\cdot})\cdot\nabla\theta^i(t,\boldsymbol{\cdot})\;d\mathbf{x}$$
$$=\sum_{i=1}^p\left[4\pi\rho^iG\int_{\Omega_i}div\mathbf{u}^i(t,\boldsymbol{\cdot})\;\theta^i(t,\boldsymbol{\cdot})\;d\mathbf{x}+\left\langle f_\phi(t,\boldsymbol{\cdot})\;\theta^i(t,\boldsymbol{\cdot})\right\rangle\right].$$

Theorem 1.2. *Assumed the regularity (12)-(15) on the data $\mathbf{f}_u$, f_ϕ, ϕ_0 $\mathbf{u}_0$ and $\mathbf{v}_0$. Then there exists a unique weak solution $\{\mathbf{u},\phi\}$ of the problem (VGP).*

This theorem has been proved in previous works ([2]: for the stationary case see [1]).

2. **Stabilization for $t\to+\infty$.** In this section, we study the stability of viscoelastic-gravitational problem (VGP). Therefore, we shall prove convergence of solutions of the hyperbolic problem to solutions of the elliptic problem as $t\to+\infty$. The main concern of this Section is to prove, that under some additional conditions on the data, the difference of the viscoelastic and elastic solutions converges to zero, as t goes to infinity, in a suitable functional space. $\{\mathbf{u}^*,\phi^*\}$ denotes the vectorial difference between the hyperbolic solution $\{\mathbf{u}(t,\mathbf{x}),\phi(t,\mathbf{x})\}$ and the elliptic solution $\{\mathbf{u}_\infty(\mathbf{x}),\phi_\infty(\mathbf{x})\}$. That is, defined

$$\mathbf{u}^*(t,\mathbf{x})=\mathbf{u}(t,\mathbf{x})-\mathbf{u}_\infty(\mathbf{x}),\phi^*(t,\mathbf{x})=\phi(t,\mathbf{x})-\phi_\infty(\mathbf{x}),$$

our goal is to prove

$$\mathbf{u}^*(t,\mathbf{x})\longrightarrow0\text{ in }V_u,\phi^*(t,\mathbf{x})\longrightarrow0\text{ in }V_\phi,\text{ as }t\to+\infty.$$

In order to simplify the study we shall not take into account convective terms (which formally corresponds to take $g=0$ in the problem (VGP)) and we shall assume also more regularity on the data than the one which is needed for the existence of solutions. To start with, we shall consider firstly the "autonomous case"

$$\mathbf{f}_u(t,.)=\mathbf{f}_{u,\infty}(.),\;f_\phi(t,.)=f_{\phi,\infty}(.)\text{ and }\widehat{\phi}_0(t,.)=\widehat{\phi}_{0,\infty}(.).$$

Let's state our main theorem concerning this case:

Theorem 2.1. *Under above mentioned hypothesis as well as*

$$\mathbf{v}_0^i,\mathbf{u}_0^i,\mathbf{u}_\infty^i\in H^2(\Omega_i),i=1,\dots,p,\tag{16}$$

we have that

$$\left\{\begin{array}{l}\mathbf{u}^*(t,\mathbf{x})\longrightarrow0\;\;in\;V_u\\\phi^*(t,\mathbf{x})\longrightarrow0\;\;in\;V_\phi\end{array}\right.\;\;as\;t\to+\infty.$$

Thanks to the classical theory of ordinary differential equations in Banach spaces, a sufficient condition to guarantee that a system is stable is the construction of a nonnegative Lyapunov function such that when applied to a solution is a continuously non-increasing function of t. As we shall see, the viscoelastic-gravitational problem can be considered as an infinite dimensional dynamic system. We recall here some well-known results (see e.g. [3]). Let (Z,d) be a complete metric space.

Definition 2.2. Let $\{S_t\}_{t\geq 0}$ be a dynamic system on the Banach space Z and let $z\in Z$. The ω-limit set associated to z is defined by

$$\omega\left(z\right):=\left\{y\in Z,\exists t_n\longrightarrow\infty, S_{t_n}z\longrightarrow y \text{ as } n\longrightarrow\infty\right\}.$$

Remark 1. The ω-limit set can be also written as:

$$\omega\left(z\right):=\bigcap_{s>0}\overline{\bigcup_{t\geq s}\left\{S_t z\right\}}.$$

Moreover, if $\bigcup\limits_{t\geq s}\left\{S_t z\right\}$ is relatively compact in Z then $S_t\left(\omega\left(z\right)\right)=\omega\left(z\right)\neq\emptyset$ (see [3]).

Theorem 2.3 (LaSalle's invariance principle)**.** *Let E be a Lyapunov function for $\{S_t\}_{t\geq 0}$, (i.e. such that $E\left(S_t z\right)\leq E\left(z\right)$ $\forall t\geq 0$ and $\forall z\in Z$), and let $z\in Z$ be such that $\bigcup\limits_{t\geq s}\left\{S_{t_n} z\right\}$ is relatively compact in Z. Then:*
(i) $\underset{t\to\infty}{lim} E\left(S_t z\right)=L$ exists,
(ii) $E\left(y\right)=L,\forall y\in\omega\left(z\right).$

Theorem 2.4. *([3]). Let E be a strict Lyapunov function for $\{S_t\}_{t\geq 0}$, (i.e. such that if $E\left(S_t z\right)=E\left(z\right)$ $\forall t\geq 0$ is verified then z is an equilibrium point for $\{S_t\}_{t\geq 0}$). Let $z\in Z$ be such that $\bigcup\limits_{t\geq 0}\left\{S_t z\right\}$ is relatively compact in Z. Let $\mathcal{E}$ be the set of equilibrium points of $\{S_t\}_{t\geq 0}$. Therefore,*
(i) $\mathcal{E}$ is a non-empty closed subset of Z,
(ii) $d(S_t z,\,\mathcal{E})\to 0$ as $t\to\infty$ (i.e. $\omega\left(z\right)\subset\mathcal{E}$).

Proof of the Theorem 2.1. In order to apply the above mentioned abstract results, we need to deal with the difference

$$\begin{aligned}\mathbf{u}^*\left(t,\mathbf{x}\right)&=\mathbf{u}\left(t,\mathbf{x}\right)-\mathbf{u}_\infty\left(\mathbf{x}\right),\\ \phi^*\left(t,\mathbf{x}\right)&=\phi\left(t,\mathbf{x}\right)-\phi_\infty\left(\mathbf{x}\right),\end{aligned}$$

Asterisk will be omitted to simplify the notation. We obtain the following coupled system:

$$\begin{cases}\rho^i\mathbf{u}^i_{tt}\left(t,\mathbf{x}\right)-\gamma^i\Delta\mathbf{u}^i_t\left(t,\mathbf{x}\right)-\Delta\mathbf{u}^i\left(t,\mathbf{x}\right)-\dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i\left(t,\mathbf{x}\right)\right)\\ =\dfrac{\rho^i}{\mu^i}\nabla\phi^i\left(t,\mathbf{x}\right) & \text{in } (0,+\infty)\times\Omega_i,\\ -\Delta\phi^i\left(t,\mathbf{x}\right)=4\pi\rho G div\mathbf{u}^i\left(t,\mathbf{x}\right) & \text{in } (0,+\infty)\times\Omega_i,\\ +\text{ Boundary conditions and non-zero initial conditions.}\end{cases}$$

In general, the challenges for this kind of problems are to be able to construct the dynamic system (hyperbolic/elliptic), and to find a Lyapunov function for that dynamic system and the space selected Z. For that, we consider the system as a unique evolution equation. We shall take the inverse Laplacian (with specified boundary conditions) on the second equation. So, we have that:

$$\phi^i\left(t,\mathbf{x}\right)=\left(-\Delta\right)^{-1}\left(4\pi\rho^i G div\mathbf{u}^i\left(t,\mathbf{x}\right)\right).$$

If we replace this term in the first equation of (VGP) we obtain a nonlocal equation which only involves the displacements:

$$\begin{aligned}&\rho^i \mathbf{u}_{tt}^i(t,\mathbf{x}) - \gamma\Delta\mathbf{u}_t^i(t,\mathbf{x}) - \Delta\mathbf{u}^i(t,\mathbf{x}) - \frac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(t,\mathbf{x})\right)\\ =&\frac{\rho^i}{\mu^i}\nabla\left((-\Delta)^{-1}\left(4\pi\rho^i G div\mathbf{u}^i(t,\mathbf{x})\right)\right) \qquad \text{in } (0,+\infty)\times\Omega_i,\end{aligned} \tag{17}$$

with the following boundary conditions:

$$\begin{aligned}\mathbf{u}^i(0,\mathbf{x}) =& \mathbf{u}_0^i(\mathbf{x}) - \mathbf{u}_\infty^i(\mathbf{x}) \text{ in } \Omega_i,\\ \mathbf{u}_t^i(0,\mathbf{x}) =& \mathbf{v}_0^i(\mathbf{x}) \text{ in } \Omega_i,\\ \mathbf{u}^i(t,\mathbf{x}) =& 0 \text{ on } \partial_l\Omega_i,\\ \mathbf{u}^i(t,\mathbf{x}) =& \mathbf{u}^{i+1}(t,\mathbf{x}),\\ \frac{\partial\mathbf{u}^i(t,\mathbf{x})}{\partial z} =& \frac{\partial\mathbf{u}^{i+1}(t,\mathbf{x})}{\partial z}, \text{ on } (0,+\infty)\times\partial_-\Omega_i \text{ for } i=1,\dots,p-1,\\ \mathbf{u}^1(t,\mathbf{x}) =& 0 \text{ on } (0,+\infty)\times\partial_+\Omega_1,\\ \mathbf{u}^p(t,\mathbf{x}) =& 0 \text{ on } (0,+\infty)\times\partial_-\Omega_p.\end{aligned} \tag{18}$$

Therefore, the space of the states is assumed as $Z := V_u \times V_u$. Taking into account the proof described in [1] and [2] for uniqueness of weak solutions, we construct the Lyapunov function E in the following way:

$$\begin{aligned}E\left(\begin{pmatrix}\mathbf{u}\\ \mathbf{u}_t\end{pmatrix}\right) :=& \sum_{i=1}^p\Big[\frac{4\pi\left(\rho^i\right)^2 G}{2}\Big[\int_{\Omega_i}\left(\left|\mathbf{u}_t^i(t,\mathbf{x})\right|^2 + 4\pi\rho^i G\gamma^i\left|\nabla\mathbf{u}_t^i(t,\mathbf{x})\right|^2\right)d\mathbf{x}\\ &+ 2\pi\rho^i G\int_{\Omega_i}\Big(\left|\nabla\mathbf{u}^i(t,\mathbf{x})\right|^2 + \frac{4\pi\rho^i G}{1-2\nu}div\mathbf{u}^i(t,\mathbf{x})^2\Big)\\ &+ \int_\Omega\frac{\rho}{\mu}\left(\left|\nabla(-\Delta)^{-1}\left(4\pi\rho G div\mathbf{u}^i(t,\mathbf{x})\right)\right|^2\right)d\mathbf{x}\Big].\end{aligned}$$

We take $z := \begin{pmatrix}\mathbf{u}_0 - \mathbf{u}_\infty\\ \mathbf{v}_0\end{pmatrix}$ and the dynamic system given by

$$S_t z := \begin{pmatrix}\mathbf{u}(t,.)\\ \mathbf{u}_t(t,.)\end{pmatrix}$$

with $\mathbf{u}(t,.)$ being the solution of the above mentioned non local problem (17), (18). The main hypothesis we want to verify is that the function E is a strict Lyapunov function for $\{S_t\}_{t\geq 0}$. So, this coincides exactly with the argument used to prove the uniqueness of weak solutions of hyperbolic problem (see [1] and [2]). Now, we prove that $E(S_t z) \leq E(z) \quad \forall t \geq 0$ and $\forall z \in Z$:

$$\begin{aligned}&\sum_{i=1}^p\Big[2\pi\left(\rho^i\right)^2 G\Big[\int_{\Omega_i}\left(\left|\mathbf{u}_t^i(t,\mathbf{x})\right|^2 + 4\pi\rho^i G\gamma^i\left|\nabla\mathbf{u}_t^i(t,\mathbf{x})\right|^2\right)d\mathbf{x}\\ &+ 2\pi\rho^i G\int_{\Omega_i}\Big(\left|\nabla\mathbf{u}^i(t,\mathbf{x})\right|^2 + \frac{4\pi\rho^i G}{1-2\nu}div\mathbf{u}^i(t,\mathbf{x})^2\Big)\\ &+ \int_\Omega\frac{\rho}{\mu}\left(\left|\nabla(-\Delta)^{-1}\left(4\pi\rho G div\mathbf{u}^i(t,\mathbf{x})\right)\right|^2\right)d\mathbf{x}\Big]\\ &\leq \sum_{i=1}^p\Big[2\pi\left(\rho^i\right)^2 G\Big[\int_{\Omega_i}\left(\left|\mathbf{v}_0\right|^2 + 4\pi\rho^i G\gamma^i\left|\nabla\mathbf{v}_0^i\right|^2\right)d\mathbf{x}\end{aligned}$$

$$+ 2\pi\rho^i G \int_{\Omega_i} (\left|\nabla(\mathbf{u}_0^i - \mathbf{u}_\infty^i)\right|^2 + \frac{4\pi\rho^i G}{1-2\nu} div(\mathbf{u}_0^i - \mathbf{u}_\infty^i)^2)$$
$$+ \int_\Omega \frac{\rho^i}{\mu^i}\left(\left|\nabla\left(-\Delta\right)^{-1}\left(4\pi\rho G div(\mathbf{u}_0^i - \mathbf{u}_\infty^i)\right)\right|^2\right) d\mathbf{x}].$$

As $\mathbf{u}_\infty$ is a stationary solution the above inequality coincides with the continuous dependence estimate given in (see [1] and [2]). It is verified that E is a strict function due to definition of weak solution of the evolution problem.

Finally, we shall check that $\bigcup_{t\geq 0}\{S_t z\}$ is relatively compact in Z to complete the proof. By using the time derivative integration of $E\left(S_t z\right)$, we obtain that

$$\sup_{t\in[0,T]}\sum_{i=1}^p\left[\int_{\Omega_i}\left|\mathbf{u}_t^i\right|^2 d\mathbf{x}\right] + \int_0^T\int_{\Omega_i}\left|\nabla\mathbf{u}_t^i\right|^2 d\mathbf{x}dt + \sup_{t\in[0,T]}\sum_{i=1}^p\int_{\Omega_i}\left|\nabla\mathbf{u}^i\right|^2 d\mathbf{x}$$
$$+ \sup_{t\in[0,T]}\sum_{i=1}^p\int_{\Omega_i}(div\mathbf{u}^i)^2 d\mathbf{x} + \sup_{t\in[0,T]}\sum_{i=1}^p\int_{\Omega_i}\left|\nabla\phi^i(t,\mathbf{x})\right|^2 d\mathbf{x}$$
$$\leq C[\int_0^T\sum_{i=1}^p\left(\left\|\mathbf{f}_u^i(t,.)\right\|_{H^{-1}}^2 + \left\|\mathbf{f}_\phi^i(t,.)\right\|_{H^{-1}}^2 + \left\|\frac{\partial}{\partial t}(\mathbf{f}_\phi^i)(t,.)\right\|_{H^{-1}}^2\right) dt$$
$$+ \int_{\partial_+\Omega_1}\left|\phi_0\left(\mathbf{s}\right)\mathbf{v}_0\left(\mathbf{s}\right)\cdot\mathbf{n}\right| ds + \sum_{i=1}^p\int_{\Omega_i}\left|\mathbf{v}_0^i\left(\mathbf{x}\right)\right|^2 d\mathbf{x}$$
$$+ \sum_{i=1}^p\int_{\Omega_i}\left|\nabla\mathbf{v}_0^i\left(\mathbf{x}\right)\right|^2 d\mathbf{x} + \sum_{i=1}^p\int_{\Omega_i}\left|\nabla\mathbf{u}_0^i\left(\mathbf{x}\right)\right|^2 d\mathbf{x} + \sum_{i=1}^p\int_{\Omega_i} div\mathbf{u}_0^i\left(\mathbf{x}\right)^2 d\mathbf{x}$$
$$+ \int_0^T\int_{\partial_+\Omega_1}\left|\phi^0\left(t,\mathbf{s}\right)\frac{\partial}{\partial\mathbf{n}}\phi^0\left(t,\mathbf{s}\right)\right| ds + \int_0^T\int_{\partial_+\Omega_1}\left|\phi^0\left(t,\mathbf{s}\right)\frac{\partial^2}{\partial t\partial\mathbf{n}}\phi^0\left(t,\mathbf{s}\right)\right| ds],$$

for a suitable positive constant C (depending on T, Ω_i and the constants ρ^i, μ^i, ν^i,γ^i and G). Now, we prove that $\{\mathbf{u}_t\left(t,.\right)\}$ is relatively compact in V_u. Firstly, we derive the viscoelastic-gravitational problem with respect to time. We obtain the following problem with the above boundary condition:

$$\begin{aligned}&\rho^i\mathbf{u}_{ttt}^i\left(t,\mathbf{x}\right) - \gamma^i\Delta\mathbf{u}_{tt}^i\left(t,\mathbf{x}\right) - \Delta\mathbf{u}_t^i\left(t,\mathbf{x}\right) - \frac{1}{1-2\nu^i}\nabla div\mathbf{u}_t^i\left(t,\mathbf{x}\right)\\ =&\frac{\rho^i}{\mu^i}\nabla\phi_t^i\left(t,\mathbf{x}\right)\ \text{ in } (0,\infty)\times\Omega_i,\end{aligned} \tag{19}$$

with

$$\begin{aligned}\mathbf{u}_t^i\left(0,\mathbf{x}\right) =&\mathbf{v}_0^i\left(x\right) \text{ in }\Omega_i,\\ \mathbf{u}_{tt}^i\left(0,\mathbf{x}\right) =&\gamma\Delta\mathbf{v}_0^i\left(\mathbf{x}\right) + \Delta\mathbf{u}_0^i\left(\mathbf{x}\right) - \Delta\mathbf{u}_\infty^i\left(\mathbf{x}\right) + \frac{1}{1-2\nu}\nabla div\mathbf{u}_0^i\left(\mathbf{x}\right) \text{ in }\Omega_i.\end{aligned}$$

By the regularity on data (12)-(15), we can infer that $\mathbf{u}_t \in L^\infty(0,\infty : H^1(\Omega))$. If we multiply the problem (19) by $\mathbf{u}_{tt}$ and integrate, next inequality holds:

$$\begin{aligned}&\rho^i\frac{d}{dt}\int_{\Omega i}\left(\left|\mathbf{u}_{tt}^i(t,\mathbf{x})\right|^2 + \left|\gamma^i\nabla\mathbf{u}_{tt}^i(t,\mathbf{x})\right|^2 + \left|\nabla\mathbf{u}_t^i(t,\mathbf{x})\right|^2\right.\\ &+\left.\frac{1}{1-2\nu^i}\left|div\mathbf{u}_t^i(t,\mathbf{x})\right|^2\right) d\mathbf{x}\\ \leq&\frac{C(\epsilon)\rho^i}{\mu^i}\int_{\Omega i}\left(\left|\nabla\phi_t^i(t,\mathbf{x})\right|^2 + \left|\mathbf{u}_{tt}^i(t,\mathbf{x})\right|^2\right) d\mathbf{x} + C,\end{aligned} \tag{20}$$

where $C(\epsilon)$ is the constant of the inequality of Young. If now we derive the second equation of viscoelastic-gravitational model with respect to time, we get for $C > 0$ that:

$$\sum_{i=1}^{p}\left(\int_{\Omega_i}\left|\nabla\phi_t^i(t,\mathbf{x})\right|^2 d\mathbf{x}\right)^{1/2} \leq C\sum_{i=1}^{p}\left\|\mathbf{u}_t^i(t,\cdot)\right\|_{L^2(\Omega_i)},$$

and using Poincaré inequality we get

$$\int_{\Omega_i}\left|\mathbf{u}_{tt}^i(t,\mathbf{x})\right|^2 d\mathbf{x} \leq C\int_{\Omega_i}\left|\nabla\mathbf{u}_{tt}^i(t,\mathbf{x})\right|^2 d\mathbf{x}.$$

Finally, we take ϵ small enough and integrate in time we arrive to a $L^\infty-$estimate. The supposed additional regularity $\mathbf{v}_0^i$, $\mathbf{u}_0^i$, $\mathbf{u}_\infty^i \in H^2(\Omega_i)$, $i = 1,\ldots,p$ allows us to conclude that $\mathbf{u}_t^i \in L^\infty\left(0,\infty : L^2(\Omega^i)\right)$ which implies that $\mathbf{u}_{tt}^i, div\mathbf{u}_t^i \in L^\infty\left(0,\infty : L^2(\Omega^i)\right)$ and $\phi_t^i \in L^\infty\left(0,\infty : H^1(\Omega^i)\right)$. □

Now we shall consider the non-autonomous case. Under some additional regularity it is possible to apply the main idea of proof of the above result. We suppose now that

$$\phi_0 \in W^{1,\infty}(0,+\infty : \prod_{i=1}^{p} H^1\left(\Omega_i\right)) \text{ and verifies (10) and (11).} \tag{21}$$

$$\mathbf{f}_u \in W^{1,\infty}(0,+\infty : \prod_{i=1}^{p} L^2\left(\Omega_i\right)^3), \tag{22}$$

$$f_\phi \in W^{1,\infty}(0,+\infty : \prod_{i=1}^{p} L^2\left(\Omega_i\right)), \tag{23}$$

for some $2 \leq q \leq +\infty$ and

$$\mathbf{f}_u(t,.) \to \mathbf{f}_{u,\infty}(.) \text{ in } \prod_{i=1}^{p} L^2\left(\Omega_i\right) \text{ as } t \to +\infty, \tag{24}$$

$$f_\phi(t,.) \to f_{\phi,\infty}(.) \text{ in } \prod_{i=1}^{p} L^2\left(\Omega_i\right) \text{ as } t \to +\infty, \tag{25}$$

$$\widehat{\phi}_0(t,.) \to \widehat{\phi}_{0,\infty}(.) \text{ in } H^1(\Omega_1) \text{ as } t \to +\infty. \tag{26}$$

Theorem 2.5. *Assume the above conditions and (16). Then the conclusion of Theorem 2.1 remains valid.*

Proof. We adapt to this framework the main ideas of [4]. Thanks to the regularity assumed on the data, as in the last part of the proof of Theorem 2.1, we obtain the additional regularity $\mathbf{u}_{tt}^i \in L^\infty\left(0,\infty : L^2(\Omega^i)\right)$ and $\mathbf{u}_t^i \in L^\infty\left(0,\infty : H^1(\Omega^i)\right)$, $\phi_t^i \in L^\infty\left(0,\infty : H^1(\Omega^i)\right)$. This implies that the ω-limit set is not empty. In fact, there exists a subsequence, $t_n \to +\infty$ such that the convergence (given in the definition of the ω-limit set) takes place strongly in the space $Z := V_u \times V_u$ (since the compactness arguments of the proof of of Theorem 2.1 remain valid under the additional regularity assumed on the data). Then any element of the ω-limit set must be a solution of the associated stationary system. Moreover, since we have uniqueness of solutions for the associated stationary system, the convergence takes place independently of the subsequence t_n and the conclusion holds. □

Acknowledgments. The research of AA was partially supported by the National Research Fund of Luxembourg (Marie Curie Post-doctoral AFR Grant 4832278). The research of JID was partially supported by the project ref. MTM2011- 26119 of the DGISPI (Spain) and the Research Group MOMAT (Ref. 910480) supported by UCM. He has received also support from the ITN FIRST of the Seventh Framework Program of the European Community (grant agreement number 238702).

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Received September 2014; revised December 2014.

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