DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **22**, Number **5**, July **2017**

doi:10.3934/dcdsb.2017105

pp. **1757–1778**

STABILITY RESULTS FOR DISCONTINUOUS NONLINEAR ELLIPTIC AND PARABOLIC PROBLEMS WITH A S-SHAPED BIFURCATION BRANCH OF STATIONARY SOLUTIONS

Sabri Bensid

Department of Mathematics, Faculty of Sciences
University of Tlemcen, B.P. 119
Tlemcen 13000, Algeria

Jesús Ildefonso Díaz*

Instituto de Matemática Interdisciplinar
Depto. de Matemática Aplicada
Parque de Ciencias 3, 28040–Madrid, Spain

Abstract. We study stability of the nonnegative solutions of a discontinuous elliptic eigenvalue problem relevant in several applications as for instance in climate modeling. After giving the explicit expresion of the S-shaped bifurcation diagram $(\lambda, \|u_\lambda\|_\infty)$ we show the instability of the decreasing part of the bifurcation curve and the stability of the increasing part. This extends to the case of non-smooth nonlinear terms the well known 1971 result by M.G. Crandall and P.H. Rabinowitz concerning differentiable nonlinear terms. We point out that, in general, there is a lacking of uniquenees of solutions for the associated parabolic problem. Nevertheless, for nondegenerate solutions (crossing the discontinuity value of u in a transversal way) the comparison principle and the uniqueness of solutions hold. The instability is obtained trough a linearization process leading to an eigenvalue problem in which a Dirac delta distribution appears as a coefficient of the differential operator. The stability proof uses a suitable change of variables, the continuuity of the bifurcation branch and the comparison principle for nondegenerate solutions of the parabolic problem.

1. Introduction.

We consider in this paper the nonlinear eigenvalue problem associated to nonnegative solutions of the discontinuous elliptic equation

$$P(\lambda, f)\begin{cases} -u''(x) = \lambda f\big(u(x)\big) & \text{in } (0,1), \\ u'(0) = 0, \ u(1) = 0, \end{cases}$$

where $\lambda > 0$ and $f(u)$ is given by

$$f(u) = f_0 + (1 - f_0)H(u - \mu), \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35B35, 35J61, 35P30, 35K58, 86A10.

Key words and phrases. Nonlinear eigenvalue problem, discontinuous nonlinearity, S-shaped bifurcation curve, stability, free boundary, energy balance climate models.

The research of J.I. Díaz was partially supported by the project ref. MTM2014-57113 of the DGISPI (Spain) and the Research Group MOMAT (Ref. 910480) of the UCM. The results of this paper were started during the visit of the first author to the UCM, on October 2015, under the support of the University Aboubekr Belkaid of Tlemcen (Algeria). This author wants to thank this support as well as the received hospitality from the Instituto de Matemática Interdisciplar of the UCM.

* Corresponding author: Jesús Ildefonso Díaz.

for some $\mu > 0$, under the key assumption

$$f_0 \in (0,1). \tag{1.2}$$

Here $H(s)$ denotes the Heaviside discontinuous function

$$H(s) = 0 \quad \text{for} \quad s < 0, \qquad H(s) = 1 \quad \text{for} \quad s \geq 0.$$

Our interest is on nonnegative solutions. In fact, as we shall see later, nonnegative solutions must be strictly concave functions and thus such that $\max_{x\in[0,1]} |u(x)| = u(0)$ and $u > 0$ on $[0,1)$. A solution u_λ of problem $P(\lambda, f)$ is a function $u \in C^2((0,1)-\{x_{\mu,\lambda}\})\cap C^1([0,1))$, for some $x_{\mu,\lambda} \in [0,1)$ where $u(x_{\mu,\lambda}) = \mu$ (called as the free boundary associated to u) and with $u \geq 0$, $u \neq 0$, such that $-u''(x) = \lambda f(u(x))$, for any $x \in (0,1) - \{x_{\mu,\lambda}\}$, and $u'(0) = 0$, $u(1) = 0$.

Problem $P(\lambda, f)$ can be considered as a simplified version of some more general formulations arising in several different contexts. We emphasize that the assumption (1.2) is crucial since the nature of models and solutions for the case of $f_0 > 1$ is entirely different (see, e.g. [10], [39] and [6]). Problems dealing with the case $f_0 \in (0,1)$ arise, for instance, in the study of chemical reactors and porous media combustion (see e.g., [22], [25], [26], [24]), steady vortex rings in an ideal fluid ([27]), plasma studies ([41], [30], [19]), the primitive equations of the atmosphere in presence of vapor saturation ([8]), etc. Besides the above applications, our special main motivation was the consideration of problem $P(\lambda, f)$ as a simplified version of the so called diffusive energy balance models arising in climatology (see, e.g. [37], [34], [11], [39], [21] and a stochastic version in [17]). Although these models must be formulated on a Riemannian manifold without boundary representing the Earth atmosphere [21], the so called 1d-model corresponds to the case in which the surface temperature is assumed to depend only on the latitude component. By neglecting the term modeling the emitted terrestrial energy flux we lead to a formulation similar to $P(\lambda, f)$ in which the spatial domain $(0,1)$ must be associated to a semisphere, the discontinuous function represents the co-albedo (with a discontinuity which is associated to the radical change of the co-albedo when the temperature is crossing -10 centigrade degrees), the parameter λ the so-called solar constant, the boundary condition $u'(0) = 0$ formulates the simplified assumption of symmetry between both semispheres and the condition $u(1) = 0$ represents the renormalized temperature at the North pole (i.e. we are assuming that $u = T + T_N$ where $T_N < -10$ represents the North pole temperature and thus $\mu = -10 - T_N > 0$). Results on the asymptotic behaviour, when $t \to +\infty$, for the evolution energy balance model were obtained in [15] (see also [43]) where it was also proved the general multiplicity of stationary solutions according the value of λ. A sharper bifurcation diagram, as a S-shaped curve was rigorously obtained in [1]. Nevertheless the method of proof in [1] uses the information obtained trough suitable zero-dimensional energy balance models and thus there is lacking of a more detailed information about the associated free boundaries generated by the solutions (given as the spatial points where $T = T_N$). A next paper by the authors shall be devoted to the extension of the results of this paper to the case in which the absorption term, modeling the emitted terrestrial energy flux, is taken into account. This additional appears also in the simplification made by McKean [36] of the initial value problem for the FitzHugh–Nagumo equations which were introduced as a model for the conduction of electrical impulses in the nerve axon (see, e.g., Termam [42]).

In Section 2 of this paper we shall obtain an explicit S-shaped bifurcation curve for the nontrivial nonnegative solutions of $P(\lambda, f)$. Although there are several results in the previous literature that allow us to conclude easily that the bifurcation curve $(\lambda, \|u_\lambda\|_\infty)$ must be S-shaped (see, e.g. [40]), in order to carry out our stability study we shall present the explicit expression of such a curve as well as the information about the free boundary associated to solutions u_λ according the values of λ: i.e. the points $x_{\mu,\lambda} \in [0,1)$ where $u_\lambda(x_{\mu,\lambda}) = \mu$.

By denoting $\|u\|_\infty = \max_{x\in[0,1]} |u(x)|$, we shall prove, the following result:

Theorem 1.1. *i) If $\lambda < \lambda_1 := \frac{2\mu}{f_0}$ then there exists a unique solution u^*_λ without free boundary of $P(\lambda, f)$. Moreover*

$$\|u^*_\lambda\|_\infty = u^*_\lambda(0) = \frac{\lambda f_0}{2} < \mu, \tag{1.3}$$

i.e. the line $(\lambda, \gamma^*(\lambda))$

$$\gamma^*(\lambda) := \frac{\lambda f_0}{2}, \text{ if } \lambda \in (0, \lambda_1),$$

defines an increasing part of the bifurcation diagram.
ii) If $\lambda = \lambda_0 := 2(2 - f_0)\mu$ then there exists a unique solution u_{λ_0} of $P(\lambda_0, f)$ giving rise to a free boundary. Moreover u_{λ_0} is strictly concave and $u_{\lambda_0}(0) = \mu$.
iii) If $\lambda \in (\lambda_0, \lambda_1]$ then there exists $\underline{u}_\lambda$ solution of $P(\lambda, f)$ with a free boundary given by

$$\underline{x}_{\mu,\lambda} = \frac{1 - f_0}{2 - f_0} - \frac{1}{2 - f_0}\sqrt{1 - \frac{2\mu(2 - f_0)}{\lambda}}.$$

Moreover,

$$\|\underline{u}_\lambda\|_\infty = \underline{u}_\lambda(0) = \frac{\lambda}{2}\underline{x}^2_\mu + \mu := \underline{\gamma}(\lambda),$$

i.e.

$$\underline{\gamma}(\lambda) = \lambda\left(\frac{(1 - f_0)^2 + 1}{2(2 - f_0)^2}\right) - \frac{\sqrt{\lambda - 2\mu(2 - f_0)}}{\lambda(2 - f_0)} - \frac{\mu}{(2 - f_0)}.$$

iv) If $\lambda \in (\lambda_0, +\infty)$ then there exists $\overline{u}_\lambda$ solution of $P(\lambda, f)$ such that $\mu < \|\overline{u}_\lambda\|_\infty$ and $\|\underline{u}_\lambda\|_\infty < \|\overline{u}_\lambda\|_\infty$ if $\lambda \in (\lambda_0, \lambda_1]$. Moreover its free boundary is given by

$$\overline{x}_{\mu,\lambda} = \frac{1 - f_0}{2 - f_0} + \frac{1}{2 - f_0}\sqrt{1 - \frac{2\mu(2 - f_0)}{\lambda}}.$$

and

$$\|\overline{u}_\lambda\|_\infty = \overline{u}_\lambda(0) = \frac{\lambda}{2}\overline{x}^2_\mu + \mu := \overline{\gamma}(\lambda).$$

i.e.

$$\overline{\gamma}(\lambda) = \lambda\left(\frac{(1 - f_0)^2 + 1}{2(2 - f_0)^2}\right) + \frac{\sqrt{\lambda - 2\mu(2 - f_0)}}{\lambda(2 - f_0)} - \frac{\mu}{(2 - f_0)}.$$

Note that the behaviour of the branch near the two "turning points" $(\lambda_0, \underline{u}_{\lambda_0})$ and $(\lambda_1, u^*_{\lambda_1})$ is different and does not coincide with the results on the subject for the case of $f(u)$ smooth: in the first case $\lim_{\lambda\searrow\lambda_0} \frac{d\overline{\gamma}(\lambda)}{d\lambda} = +\infty$ and $\lim_{\lambda\nearrow\lambda_0} \frac{d\underline{\gamma}(\lambda)}{d\lambda} = -\infty$ but in the second one $0 < \lim_{\lambda\nearrow\lambda_1} \frac{d\overline{\gamma}(\lambda)}{d\lambda} < +\infty$ and $\lim_{\lambda\nearrow\lambda_1} \frac{d\overline{\gamma}(\lambda)}{d\lambda} < \frac{f_0}{2}$. Qualitatively, the associated bifurcation branch is similar to the one represented in Figure 1 below.

The stability of solutions $\underline{u}_\lambda$ and $\overline{u}_\lambda$ will be analyzed in Sections 3 and 4 respectively. We recall that in the case of smooth nonlinear functions $f(u)$ the instability of the decreasing part of the bifurcation curve (i.e. of solutions $\underline{u}_\lambda$) and the stability of the increasing part (i.e. of solutions u^*_λ and $\overline{u}_\lambda$) was shown in the famous paper Crandall and Rabinowitz [9] (for the application to the so called Sellers energy

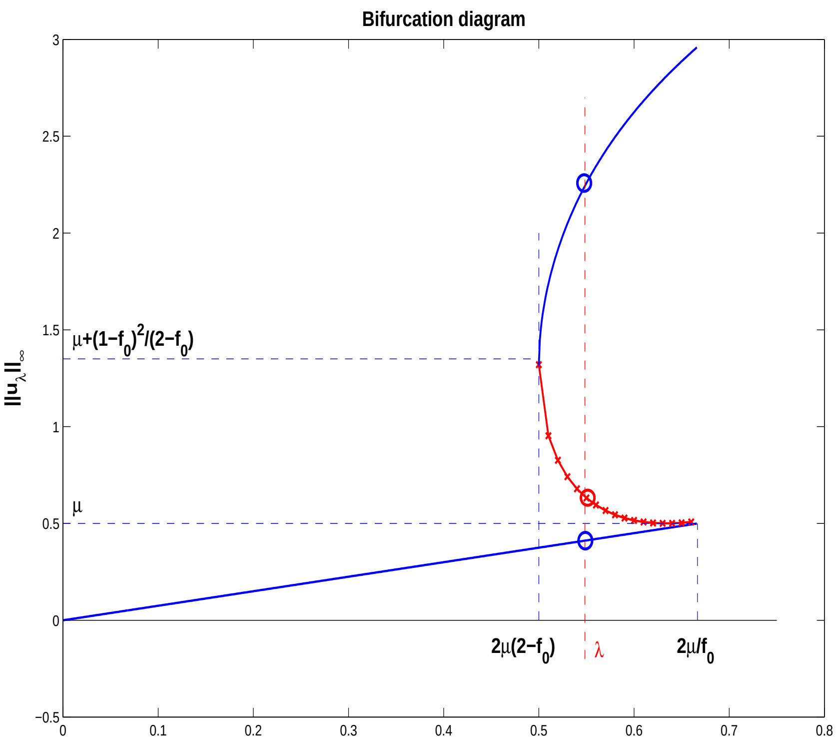

FIGURE 1. A qualitative description of the bifurcation curve

balance model with a smooth co-albedo function $f(u)$ see Hetzer [34]). One of our main goals in this paper is to prove that the same type of conclusions remains true for the case of non-smooth functions $f(u)$ by using some *ad hoc* methods. Before to present a brief idea of the rest of the Sections we point out that the solutions u_λ of $P(\lambda, f)$ are also solutions of the multivalued problem

$$P^*(\lambda,\beta)\begin{cases} -u''(x) \in \lambda\beta(u(x)) & a.e.\ x \in (0,1), \\ u'(0)=0,\ u(1)=0, \end{cases}$$

where β is the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(r) = \begin{cases} f(r) & \text{if } r \neq \mu, \\ [f_0, 1] & \text{if } r = \mu. \end{cases}$$

As explained later, both problems are equivalent in the class of nonnegative solutions thanks to the special case of those boundary conditions (as already mentioned, they lead to strictly concave functions). One reason to reformulate problem $P(\lambda, f)$ as $P^*(\lambda, f)$ is because the associated parabolic problem

$$PP^*(\lambda,\beta,v_0)\begin{cases} v_t - v_{xx} \in \lambda\beta(v) & x \in (0,1),\ t>0, \\ v_x(0,t)=0, \quad v(1,t)=0 & t>0, \\ v(x,0)=v_0(x), & x \in (0,1), \end{cases}$$

is well-posed (under suitable conditions on v_0) in contrast to what may happen with the associated parabolic version of the discontinuous problem (see, e.g. the comments presented in [45]).

In Section 3 we shall collect results on the solvability of problem $PP^*(\lambda, \beta, v_0,)$. We shall adapt to our framework some of the results of [11] and [21] showing that $PP^*(\lambda, \beta, v_0,)$ may have a multiplicity of solutions for some initial datum $v_0(x)$ and that, nevertheless, the solution is unique (and the comparison principle holds) in the class of "non degenerate solutions": i.e. solutions v such that

$$meas\{x \in (0,1), \mid v(t,x) - \mu \mid \leq \theta\} \leq C\theta,$$

for any $\theta \in (0, \theta_0)$ and for any $t > 0$, for some $C > 0$ and $\theta_0 > 0$. Here $meas(.)$ denotes the Lebesgue measure.

In Section 4, we shall prove the instability of stationary solutions $\underline{u}_\lambda$ of the decreasing part of the branch of solutions $(\lambda, \underline{\gamma}(\lambda))$. If we denote simply by $x_{\mu,\lambda} \in (0,1)$ the free boundary generated by $\underline{u}_\lambda$ (i.e. $\underline{u}_\lambda(x_{\mu,\lambda}) = \mu$) then we shall show that the instability of $\underline{u}_\lambda$ is an easy consequence of the study of the eigenvalue problem associated to the linearized equation

$$P_\nu(x_{\mu,\lambda} : f_0, \lambda) \begin{cases} -U''(x) - \lambda(1 - f_0)\delta_{\{x_{\mu,\lambda}\}} U(x) = \nu U(x), & x \in (0,1) \\ U'(0) = 0, \quad U(1) = 0. \end{cases}$$

Here $\delta_{\{x_{\mu,\lambda}\}}$ denotes the Dirac delta distribution at the free boundary point $x_{\mu,\lambda}$. We shall prove

Theorem 1.2. *If $x_{\mu,\lambda} = \underline{x}_{\mu,\lambda} \in (0,1)$ is the free boundary of $\underline{u}_\lambda$ then the principal eigenvalue ν_1 of problem $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ is negative. In particular, $\underline{u}_\lambda$ is instable.*

We point out that eigenvalue problems with some measures as coefficients of the operators (similarly to $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$) arise in several completely different contexts (see, e.g. the survey by Belloni and Robinett [3] on "quantum dots" for linear Schrödinger equations and some stability studies for KKP equations presented in Liang, Li and Matano [35]). For some nonlinear eigenvalue problems with measures in the operators see [18] and [12].

The stability of the increasing parts of the bifurcation curve, $(\lambda, \overline{\gamma}(\lambda))$ and $(\lambda, \gamma^*(\lambda))$, will be proved in Section 5 by a different technique to the linearization argument. We shall not construct, neither, any Lyapunov function. Our method of proof will use the comparison principle for the parabolic problem (in the class of non-degenerate solutions) and a suitable change of variables involving the parameter λ. This adapts to our framework some ideas of the papers [13] and [6]. We define

$$L(\lambda) = \sqrt{\lambda} := L, \quad \widetilde{x} = Lx, \quad \widetilde{t} = \lambda t,$$

and

$$V(\widetilde{x}, \widetilde{t}) = v\left(\frac{\widetilde{x}}{\sqrt{\lambda}}, \frac{\widetilde{t}}{\lambda}\right) = v(x,t),$$

with $v(t,x)$ solution of $PP^*(\lambda, \beta, v_0,)$. Then V satisfies

$$\widetilde{PP}(v_0, \lambda) \begin{cases} \frac{\partial V}{\partial \widetilde{t}} - \frac{\partial^2 V}{\partial \widetilde{x}^2} \in \beta(V) & \widetilde{x} \in \ (0, L), \quad \widetilde{t} \in (0, +\infty), \\ \frac{\partial V}{\partial \widetilde{x}}(0, \widetilde{t}) = 0, \quad V(L, \widetilde{t}) = 0 & \widetilde{t} > 0, \\ V(\widetilde{x}, 0) = v_0(\frac{\widetilde{x}}{\sqrt{\lambda}}) = V_0(\widetilde{x}), & \widetilde{x} \in (0, L). \end{cases}$$

A similar change of variable leads to the new stationary problem

$$\widetilde{P}(L,f)\begin{cases} -\frac{d^2}{d\widetilde{x}}W(\widetilde{x}) = f(W(\widetilde{x})) & \text{on} \quad (0,L), \\ \frac{d}{d\widetilde{x}}W(0)=0, \quad W(L)=0. \end{cases}$$

In this way, by proving the continuous dependence of solutions of $\widetilde{P}(L,f)$ with respect to L, we shall be able to show that given a $\widetilde{\lambda} \in (\lambda_0,+\infty)$ the solutions $\overline{u}_\lambda$ for λ near $\widetilde{\lambda}$ lead to sub and supersolutions of the parabolic problem $\widetilde{PP}(v_0,\lambda)$ which are closed enough to $\overline{u}_{\widetilde{\lambda}}$. A similar argument (even easier) applies to the part $(\lambda,\gamma^*(\lambda))$ of the branch. This implies the stability of the solutions in the increasing parts of the branch.

Theorem 1.3. *Solutions $\overline{u}_\lambda$ and u^*_λ are $L^\infty-$stable*

Finally, we point out that it seems possible to study the H^1-stability of solutions of $PP^*(\lambda,\beta,v_0)$ for different values of λ, by using other type of techniques (see, e.g. Arrieta, Rodríguez-Bernal and Valero [2], for a related discontinuous problem with λ prescribed and different boundary conditions, and Díaz, Hernández and Ilyasov [14], for a different related nonlinear free boundary eigenvalue problem). This goal will be presented in a different paper by the authors.

2. **On the bifurcation diagram of the stationary problem.** We recall that, given $\lambda>0$, by a solution of the multivalued problem $P^*(\lambda,\beta)$ we mean a non-negative function $u \geq 0$, $u \neq 0$, with $u \in W^{2,\infty}(0,1) \subset C^1([0,1])$ such that there exists $b \in L^\infty(0,1)$ with $b(x) \in \beta(u(x))$ a.e. $x \in (0,1)$ and such that $-u''=b$, on $L^\infty(0,1)$, and u satisfies the boundary conditions $u'(0)=0$, $u(1)=0$. Since any function $b \in L^1(0,1)$ with $b(x) \in \beta(u(x))$ a.e. $x \in (0,1)$ must satisfy that $b(x)>0$ for a.e. $x \in (0,1)$, we deduce that nonnegative and nontrivial solutions must be strictly concave functions and thus such that $\max_{x\in[0,1]} |u(x)| = u(0)$, $u>0$ and $u'<0$ on $[0,1)$. In particular, the free boundary (given as the boundary of the set $\{x \in [0,1)$ such that $u(x)=\mu\}$) must be reduced to a a single point $x_{\mu,\lambda} \in [0,1)$. Moreover, in fact, we deduce that $u \in C^2((0,1)-\{x_{\mu,\lambda}\})$ and thus u is solution of the discontinuous problem $P(\lambda,f)$ (once that the multivalued modification of $f(u)$ is reduced to a single point $\{x_{\mu,\lambda}\}$). Obviously, by similar reasons, any solution of $P(\lambda,f)$ is also a solution of the multivalued problem $P^*(\lambda,\beta)$. Note that under other type of boundary conditions, or in presence of some external source functions, the task of identifying solutions of the discontinuous and multivalued problems associated by means of the process of "filling in the jumps" may become much more complex (see, e.g. Rauch [38]).

The main result of this Section is Theorem 1.1 stated in the Introduction.

Proof of Theorem 1.1. i) First we consider the easier case of solutions such that $f(u(x))=f_0$ for any $x \in [0,1]$ (i.e. with absence of free boundary). Then, since $-u''(x)=\lambda f_0$ an easy calculation shows that

$$u(x) = -\frac{\lambda f_0}{2}x^2 + \frac{\lambda f_0}{2} \text{ for any } x \in [0,1].$$

Hence, denoting this solution by u^*, we obtain (1.3) since

$$\max_{x\in[0,1]} u^*(x) = u^*(0) = \frac{\lambda f_0}{2} < \mu \quad \text{if and only if} \quad \lambda < \frac{2\mu}{f_0}.$$

In the rest of the proof we shall search solutions u with a free boundary $x_{\mu,\lambda} \in [0,1)$ and we consider the corresponding problems verified by u on the different regions $(0, x_{\mu,\lambda})$ and $(x_{\mu,\lambda}, 1)$. On $(0, x_{\mu,\lambda})$ we get the linear problem

$$(P1)\begin{cases} -u''(x) = \lambda & \text{in } (0, x_{\mu,\lambda}), \\ u'(0) = 0, \; u(x_{\mu,\lambda}) = \mu, \end{cases}$$

and then

$$u(x) = \frac{\lambda}{2} x_{\mu,\lambda} - \frac{\lambda}{2} x^2 + \mu \text{ for any } x \in (0, x_{\mu,\lambda}). \tag{2.4}$$

On $(x_{\mu,\lambda}, 1)$ we get

$$(P2)\begin{cases} -u''(x) = \lambda f_0 & \text{in } (x_{\mu,\lambda}, 1), \\ u(x_{\mu,\lambda}) = \mu, \; u(1) = 0, \end{cases}$$

and thus

$$u(x) = -\frac{\lambda f_0}{2} x^2 + \left(\frac{\mu}{x_{\mu,\lambda} - 1} + \frac{\lambda f_0}{2}(1 + x_{\mu,\lambda})\right)x + \frac{\lambda f_0}{2} - \frac{\mu}{x_{\mu,\lambda} - 1} - \frac{\lambda f_0}{2}(x_{\mu,\lambda} + 1) \tag{2.5}$$

for any $x \in (x_{\mu,\lambda}, 1]$. The C^1-transmission condition (i.e. $u'(x_{\mu,\lambda}+) = u'(x_{\mu,\lambda}-)$) implies that necessarily

$$\frac{\lambda}{\mu} = \frac{1}{(1 - \frac{f_0}{2})(1 - x_{\mu,\lambda})(x_{\mu,\lambda} + B)} \tag{2.6}$$

with $B := \frac{f_0}{2 - f_0}$. In order to study this condition (giving information on the multiplicity and location of the free boundary) let us introduce the function

$$g(r) := \frac{1}{(1 - \frac{f_0}{2})(1 - r)(r + B)} \quad \text{for } r \in (0,1).$$

It is clear that a) g is monotone decreasing on $(\frac{f_0}{2-f_0}, \frac{1}{2-f_0})$, b) g is monotone increasing on $(\frac{1}{2-f_0}, 1)$, and c) g has a minimum (equal to $2(2 - f_0)$) at $\frac{1}{2-f_0}$.

Hence, equation (2.6), regarded as an equation in $x_{\mu,\lambda}$, has either one or two roots between $(0,1)$ when $\frac{\lambda}{\mu}$ is respectively equal or greater to $2(2 - f_0)$.

If, for $\lambda > 2(2-f_0)\mu$, we denote by $\underline{x}_{\mu,\lambda}, \overline{x}_{\mu,\lambda}$ the roots of (2.6) we get the explicit expressions

$$\underline{x}_{\mu,\lambda} = \frac{1 - f_0}{2 - f_0} - \frac{1}{2 - f_0}\sqrt{1 - \frac{2\mu(2 - f_0)}{\lambda}},$$

$$\overline{x}_{\mu,\lambda} = \frac{1 - f_0}{2 - f_0} + \frac{1}{2 - f_0}\sqrt{1 - \frac{2\mu(2 - f_0)}{\lambda}}.$$

We remark also that $\underline{x}_{\mu,\lambda} \in (\frac{f_0}{2-f_0}, \frac{1}{2-f_0})$ and $\overline{x}_{\mu,\lambda} \in (\frac{1}{2-f_0}, 1)$. If we denote by $\underline{u}_\lambda(x)$ and $\overline{u}_\lambda(x)$ the functions satisfying (2.4) and (2.5) and with free boundaries given respectively by $\underline{x}_{\mu,\lambda}$ and $\overline{x}_{\mu,\lambda}$ then we get the conclusions stated in iii) and iv). The proof of ii) is obvious since $u_{\lambda_0}(0) = \mu$ and $u'_{\lambda_0}(x) < 0$ for any $x \in 0, 1]$. Thus, the proof of Theorem 1.1 ends. $\square$

3. **On the multivalued parabolic problem.** Concerning the parabolic problem $PP^*(\lambda, \beta, v_0,)$, it is not too difficult to adapt to this setting some previous results in the literature (see, e.g. [24], [11] and [21]) concerning similar diffusion operators and other boundary conditions. We introduce the energy space

$$V = \{w \in H^1(0,1) \text{ such that } w(1) = 0\}.$$

It is well known that V is a closed subspace of the Hilbert space $H^1(0,1)$. Given $v_0 \in L^2(0,1)$ (more general initial data can be also considered: see, e.g. [7]) with $v_0 \geq 0$ a.e. on $(0,1)$, the notion of solution we shall use in this paper is the following:

Definition 3.1. We say that $v \in C([0,\infty) : L^2(0,1)) \cap L^2_{loc}(0,\infty : V)$ is a weak solution of $PP^*(\lambda, \beta, v_0)$ if there exists $z \in L^\infty((0,\infty) \times (0,1))$ with $z(t,x) \in \beta(u(t,x))$ for a.e. $(t,x) \in (0,\infty) \times (0,1)$, such that for every test function $\zeta \in L^2(0,\infty : V)$ with $\zeta_t \in L^2(0,\infty : V')$ and $\zeta(t,.) = 0$ with compact support on $(0,\infty)$, we have that $v(0,.) = v_0(.)$ in $L^2(0,1)$ and

$$-\int_0^\infty \langle v(t,.), \zeta'(t,.)\rangle_{V \times V'}\, dt + \int_0^\infty \int_0^1 [v_x(t,x)\zeta_x(t,x) - \lambda z(t,x)\zeta(t,x)]\, dxdt$$
$$= \int_0^1 v_0(x)\zeta(0,x)dx.$$

Concerning the existence of solutions we have:

Theorem 3.2. *i) Let $v_0 \in L^\infty(0,1)$. Then $\forall \lambda > 0$, there exists a weak solution v of $PP^*(\lambda, \beta, v_0)$ and $v_t \in L^2_{loc}(0,\infty : V')$. Moreover $v \in L^\infty((0,\infty) \times (0,1))$ and $v(t,.) \in H^2(0,1) \subset C^1([0,1])$ for any $t > 0$. If in addition $v_0 \in V$ then $v \in C([0,\infty) : V)$ and $v_t \in L^2_{loc}(0,\infty : L^2(0,1))$.*
ii) If $v_0 \in C([0,1])$ then $\forall \lambda > 0$, there exists a mild solution $v \in C([0,\infty) : C([0,1]))$ of $PP^*(\lambda, \beta, v_0)$ in the sense of semigroups on the space $X = C([0,1])$.

Proof. Parts i) and iii) are obvious adaptations of Theorem 1 and Corollary 1 of [11] (see also [21]). Part ii) was proved in [16] for the case of the diffusion operator associated to a Riemannian manifold without boundary (the particularization to our framework is a routine matter since the diffusion operator is now much more simple to be treated on the space $X = C([0,1])$. □

We point out that, since the initial datum v_0 does not need to be a strictly concave function, there is no equivalence, in general, among solutions of the multivalued and discontinuous parabolic equations associated trough the process of filling the jump.

Proposition 3.1. *Assume $\lambda = \lambda_1 := \frac{2\mu}{f_0}$. Let $v_0(x) = u^*_\lambda(x)$ be the unique solution of $P(\lambda_1, f)$ without free boundary. Then problem $PP^*(\lambda_1, \beta, v_0)$ admits at least two different solutions.*

Proof. Let $v^*(t,x) := u^*_\lambda(x)$. Obviously $v^*(t,x)$ satisfies.

$$(PP_L) \begin{cases} v_t - v_{xx} = \lambda_1 f_0 & x \in (0,1),\ t > 0, \\ v_x(0,t) = 0, \quad v(1,t) = 0 & t > 0, \\ v(x,0) = v_0(x), & x \in (0,1), \end{cases}$$

and since $v^*(x,t) < \mu$ for any $x \in (0,1]$, v^* is also a solution of $PP^*(\lambda, \beta, v_0)$. We recall that $v^*(0,t) = \mu$ for any $t \geq 0$. Now we shall construct a different solution of $PP^*(\lambda, \beta, v_0)$ by starting with the construction of a family of auxiliary functions $v^\omega(t,x)$, depending on a parameter $\omega > 0$. We introduce the partition $(0,1)\times[0,\omega] = Q_1^\omega \cup Q_2^\omega$ by

$$\begin{aligned} Q_1^\omega &= \{(x,t)\in(0,1)\times[0,\omega], x > t/\omega\} \\ Q_2^\omega &= \{(x,t)\in(0,1)\times[0,\omega], 0\le x\le t/\omega\}. \end{aligned}$$

We define v^ω on Q_1^ω as the unique solution of the problem

$$P(Q_1^\omega)\begin{cases} v_t - v_{xx} = \lambda_1 f_0, & (x,t)\in Q_1^\omega, \\ v_x(1,t)=0, \quad v(\frac{t}{\omega},t)=\mu, & t\in[0,\omega], \\ v(x,0)=v_0(x) & x\in[0,1]. \end{cases}$$

The existence and uniqueness of a solution of $P(Q_1^\omega)$ can be obtained applying the results of Friedman [28]. Finally we define

$$v^\omega(x,t) = \mu + C^\omega(t)(x-t/\omega)(x+t/\omega) \text{ for all } (x,t)\in Q_2^\omega. \tag{3.7}$$

Let us show that it is possible to choose $C^\omega(t)$ in (3.7) such that
(i) $v^\omega \in C([0,1]\times[0,\omega])$, $v_x^\omega \in C((0,1)\times[0,\omega])$.
(ii) v^ω is a bounded weak solution of the auxilary problem

$$\begin{cases} v_t - v_{xx} = h^\omega(x,t) & x\in(0,1),\ t>0, \\ v_x(0,t)=0, \quad v(1,t)=0 & t>0, \\ v(x,0)=v_0(x), & x\in(0,1), \end{cases}$$

for some $h^\omega \in L^\infty((0,1)\times(0,\omega))$ such that $h^\omega \equiv \lambda_1 f_0$ in Q_1^ω and

$$h(x,t)\le\lambda_1(1-f_0) \text{ for } x\in(0,1) \text{ and } t\in(0,T_\omega), \text{ with } T_\omega \text{ small enough,} \tag{3.8}$$

(iii) $v^\omega(x,t) > \mu$ on Q_2^ω and $v^\omega < \mu$ on Q_1^ω.
Indeed, the continuity of v^ω follows from the continuity of the solution of $P(Q_1^\omega)$. Moreover, the solution v^ω of $P(Q_1^\omega)$ is regular on the segment $\{(t/\omega,t) : t\in(0,\omega)\}$ and the function

$$g^\omega(t) = v_x^\omega(t/\omega,t)$$

satisfies that $g^\omega \in C^1((0,\omega))$, $g^\omega(0)=(g^\omega)'(0)=0$ and from the strong maximum principle (see Friedman [28]) $g^\omega(t)<0$ if $t\in(0,\omega]$. Then choosing

$$C^\omega(t) = \frac{g^\omega(t)\omega}{2t},$$

we obtain that $v_x^\omega \in C((-1,1)\times[0,\omega])$. From the strong maximum principle we deduce (iii). To complete the proof we only need to show that $h(x,t)$ satisfies (3.8). A straightforward computation yields

$$h^\omega(x,t) = \omega x^2\left(\frac{tg'(t)-g(t)}{2t^2}\right) + \frac{g(t)-tg'(t)}{\omega}$$

(where g denotes g^ω) and thus, by choosing T_ω small enough we get that the function h^ω satisfies (3.8). Moreover, if we consider a regular approximation β_ϵ of β we can prove (as in [11], [21]) that the solutions v^ϵ of the regularized version of $PP^*(\lambda_1,\beta,v_0)$ satisfy that $v^\epsilon \ge v^\omega$ on $[0,1]\times[0,T_\omega]$. Using well known a priori estimates ([11], [21]) we have that $v^\epsilon \longrightarrow v$ weakly in $L^2(0,T:V)$ as $\epsilon\downarrow 0$, with v a bounded weak solution of $PP^*(\lambda_1,\beta,v_0)$ such that

$$v \ge v^\omega \text{ on } [0,1]\times[0,T_\omega], \text{ for any } \omega>0. \tag{3.9}$$

Finally, we observe that by the strong maximum principle $v^{\omega}(x,t) > v^{*}(x,t)$, for any $(x,t) \in (0,1) \times (0,T_{\omega}]$, for any $\omega > 0$ and thus the two solutions v and v^{*} cannot coincide. $\square$

Remark 1. Some different nonuniqueness results could be found by using selfsimilar special solutions as in Gianni and Hulshof [29].

Remark 2. It is not difficult to show that the solution v constructed in the above Proposition satisfies $v_x(x,t) > 0$ for any $x \in (0,1)$ and $t > 0$. Then by the *Implicit Function Theorem* there exists a continuous function $x_{\mu} : [0,T] \longrightarrow [0,1]$, defining *the free boundary* associated to v, i.e., such that $v(x_{\mu}(t),t) = \mu$ for any fixed $t \in [0,T]$. Clearly $x_{\mu} \in C^{1}((0,T])$. Moreover $x_{\mu}(t) \geq t/\omega$ for any $\omega > 0$. As $x_{\mu}(0) = 0$ we deduce that necessarily $x'_{\mu}(t) \uparrow +\infty$ as $t \downarrow 0$. For some additional regularity results on the free boundary see, e.g. [44], [20] and their references.

To avoid free boundaries such that $x'_{\mu}(t) \uparrow +\infty$ as $t \downarrow 0$ leading to non-uniqueness results we need to impose some kind of condition to the solutions v saying that when v is "crossing" the discontinuity value μ they do that in a "transversal way". This is the reason to introduce the following notion of "nondegeneracy property".

Definition 3.3. We say that a function $v(x)$ satisfies the nondegeneracy property at the level μ if there exists $C > 0$ and $\theta_0 > 0$ such that for any $\theta \in (0,\theta_0)$,

$$meas\{x \in (0,1) \text{ such that } |v(x) - \mu| \leq \theta\} \leq C\theta,$$

where $meas(.)$ denotes the Lebesgue measure on $(0,1)$.

The following result gives the comparison of solutions (and thus the uniqueness of solutions) in the class of solutions satisfying the non-degeneracy property.

Theorem 3.4. *Let $\underline{v}_0, v_0, \overline{v}_0 \in V$ such that*

$$\underline{v}_0(x) \leq v_0(x) \leq \overline{v}_0(x) \quad a.e. \quad x \in (0,1).$$

Assume that $\underline{v}(t,x), v(t,x)$ and $\overline{v}(t,x)$ are non-degenerate weak solutions of the corresponding problems $PP^{*}(\lambda,\beta,\underline{v}_0)$, $PP^{*}(\lambda,\beta,v_0)$ and $PP^{*}(\lambda,\beta,\overline{v}_0)$. Then

$$\underline{v}(t,x) \leq v(t,x) \leq \overline{v}(t,x) \quad \forall t > 0 \quad and\ a.e. \quad x \in (0,1).$$

The non-standard part of the proof is the following general a priori estimates (which, in particular imply that under the nondegeneracy property the multivalued term β generates a continuous operator from $L^{\infty}(0,1)$ into $L^{1}(0,1)$, and in fact also into $L^{q}(0,1)$, for any $q \in [1,\infty)$).

Lemma 3.5. *(i) Let $w, \hat{w} \in L^{\infty}(0,1)$ and assume that $w, \hat{w}$ satisfy the nondegeneracy property. Then there exists $\tilde{C} > 0$ such that for any $z, \hat{z} \in L^{\infty}(0,1)$, $z(x) \in \beta(w(x))$, $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in (0,1)$ we have*

$$\int_{\{x\in(0,1):w(x)>\hat{w}(x)\}} |z(x) - \hat{z}(x)|\, dx \leq (1 - f_0) \min\{\tilde{C} \parallel [w - \hat{w}]_{+} \parallel_{L^{\infty}(0,1)}, 1\}. \tag{3.10}$$

(ii) If $w, \hat{w} \in L^{\infty}(0,1)$ and satisfy the nondegeneracy property then

$$\int_0^1 (z(x) - \hat{z}(x))[w(x) - \hat{w}(x)]_{+} dx \leq (1 - f_0)C \parallel [w - \hat{w}]_{+} \parallel^2_{L^{\infty}(0,1)}. \tag{3.11}$$

Proof of Lemma 3.5. If $\parallel [w - \hat{w}]_{+} \parallel_{L^{\infty}(0,1)} > \theta_0$ then

$$\int_{\{x\in(0,1):w(x)>\hat{w}(x)\}} |z(x) - \hat{z}(x)|\, dx \leq (1 - f_0) \leq \frac{(1 - f_0)}{\theta_0} \parallel [w - \hat{w}]_{+} \parallel_{L^{\infty}(I)}.$$

Assume now that $\| [w-\hat{w}]_+ \|_{L^\infty(0,1)} \leq \theta_0$. Define the coincidence sets

$$A = \{x \in (0,1) : w(x) = \mu\} \quad \hat{A} = \{x \in (0,1) : \hat{w}(x) = \mu\},$$

as well as the decomposition

$$(0,1) = A \cup \Omega_+ \cup \Omega_- \quad (0,1) = \hat{A} \cup \hat{\Omega}_+ \cup \hat{\Omega}_-,$$

where

$$\Omega_+ = \{x \in (0,1) : w(x) > \mu\} \quad \Omega_- = \{x \in (0,1) : w(x) < \mu\}$$

and $\hat{\Omega}_+, \hat{\Omega}_-$ are defined similarly by replacing w by $\hat{w}$. Let $z, \hat{z}$ be defined as in the statement. Then over the subset $\widehat{I} = \{x \in (0,1) : w(x) > \hat{w}(x)\}$

$$\begin{array}{lll} |z(x) - \hat{z}(x)| \leq (1 - f_0) & \text{on} & A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-), \\ z(x) = \hat{z}(x) & \text{on} & (\Omega_+ \cap \hat{\Omega}_+) \cup (\Omega_- \cap \hat{\Omega}_-). \end{array}$$

Thus

$$\int_{\{x \in (0,1): w(x) > \hat{w}(x)\}} |z(x) - \hat{z}(x)|\, dx \leq (1 - f_0) \min\{|A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-)|, 1\}. \tag{3.12}$$

But we have

$$\Big[A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-)\Big] \cap \widehat{I} \subset B_{\theta_0} \equiv \{x \in \Omega : \mu - \theta_0 \leq w(x) \leq \mu + \theta_0\}.$$

Indeed, it is clear that $A \subset B_{\theta_0}$. Moreover,

$$\hat{w}(x) - \| [w-\hat{w}]_+ \|_{L^\infty((0,1))} \leq w(x) \leq \| [w-\hat{w}]_+ \|_{L^\infty((0,1))} + \hat{w}(x) \text{ a.e. } x \in \widehat{I} \subset (0,1).$$

Then the inclusion $\hat{A} \subset B_{\theta_0}$ is obvious. If $x \in \Omega_+ \cap \hat{\Omega}_-$, $\mu < w(x) \leq \theta_0 + \hat{w}(x) < \mu + \theta_0$ and so $x \in B_{\theta_0}$. Consequently, inequality (3.10) follows from the strong nondegeneracy assumption on w.

Let $w, \hat{w}$ satisfying the nondegeneracy property. As before we can assume that $\| [w-\hat{w}]_+ \|_{L^\infty((0,1))} \leq \theta_0$. Then remarking that

$$(z(x) - \hat{z}(x))(w(x) - \hat{w}(x)) = 0 \text{ if } x \in A \cap \hat{A}$$

and that if $w(x) \neq \mu$ (resp. $\hat{w}(x) \neq \mu$) and $x \in \hat{A}$ (resp. $x \in A$) we have that

$$x \in \{x \in (0,1) : 0 < |w(x) - \mu| \leq \theta_0\} \text{ (resp. } \{x \in (0,1) : 0 < |\hat{w}(x) - \mu| \leq \theta_0\}),$$

then, by the the nondegeneracy property, we obtain (3.11). □

Proof of Theorem 3.4. Let $T > 0$ arbitrary. Let us show the inequality $v(t,x) \leq \overline{v}(t,x) \quad \forall t > 0 \quad$ and a.e. $\quad x \in (0,1)$ (the other inequality is analogous). By multiplying by $[v(t) - \overline{v}(t)]_+$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\, [v(t) - \overline{v}(t)]_+ |^2 dx + \int_0^1 |\frac{\partial}{\partial x}[v(t) - \overline{v}(t)]_+ |^2 dx$$

$$\leq \lambda \int_{\{x \in (0,1): v(x,t) > \overline{v}(x,t)\}} (z(x,t) - \hat{z}(x,t))(v(x,t) - \overline{v}(x,t)) dx,$$

for some $z, \hat{z} \in L^\infty((0,1) \times (0,T))$ with $z(x,t) \in \beta(v(x,t))$, $\hat{z}(x,t) \in \beta(\overline{v}(x,t))$ for a.e. $(x,t) \in (0,1) \times (0,T)$. Then

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\, [v(t) - \overline{v}(t)]_+ |^2 dx + \int_0^1 |\frac{\partial}{\partial x}[v(t) - \overline{v}(t)]_+ |^2 dx \leq$$

$$\leq \lambda \left(\int_{\{x \in (0,1): v(x,t) > \overline{v}(x,t)\}} |z(x,t) - \hat{z}(x,t)|\, dx \right) \| [v(t) - \overline{v}(t)]_+ \|_{L^\infty(0,1)} .$$

By Poincaré and Sobolev inequalities

$$\| v \|_{L^\infty(0,1)} \leq C_1 \| v_x \|_{L^2(0,1)}, \quad \forall v \in V, \tag{3.13}$$

where $C_1 > 0$. Then by Lemma 3.1 we get

$$\begin{aligned}
&\lambda \left(\int_{\{x\in(0,1):v(x,t)>\overline{v}(x,t)\}} |z(x,t) - \hat{z}(x,t)|\, dx \right) \| [v(t) - \overline{v}(t)]_+ \|_{L^\infty(0,1)} \\
&- \int_0^1 |\frac{\partial}{\partial x} [v(t) - \overline{v}(t)]_+ |^2 dx \\
&\leq \| [v(t) - \overline{v}(t)]_+ \|^2_{L^\infty(0,1)} \left(\lambda C(1 - f_0) - \frac{1}{C_1^2} \right).
\end{aligned}$$

Then, if

$$\lambda C(1 - f_0) - \frac{1}{C_1^2} \leq 0, \tag{3.14}$$

we conclude that

$$\frac{d}{dt} \| [v(t) - \overline{v}(t)]_+ \|^2_{L^2(0,1)} \leq 0, \tag{3.15}$$

which leads to the conclusion.

If (3.14) does not hold we introduce the rescaling $y = \alpha x$ with $\alpha > 0$. Given a general function $h(x,t)$ we define $h(y,t)$ by $h(y,t) = h(\alpha x, t)$. Then the functions $v(y,t)$ and $\overline{v}(y,t)$ satisfy

$$\begin{aligned}
\frac{\partial v}{\partial t} - \alpha^2 v_{yy} &= \lambda z(y,t), \\
\frac{\partial \overline{v}}{\partial t} - \alpha^2 \overline{v}_{yy} &= \lambda \hat{z}(y,t),
\end{aligned}$$

in $(0,\alpha) \times (0,T)$. Then, as in ([11], [21]) it is easy to see that by taking α large enough we get to a new negative balance between the involved constants (as in (3.14)) and thus the conclusion holds. □

Remark 3. Several additional results on non-degeneracy solutions for the case of other boundary conditions can be adapte to the present framewor. In particular, it can be shown that if $v_0 \in V$ is non-degenerate then there exists a (unique) nondegenerate solution $v(x,t)$ (see, [11], [21], [20] and their references).

4. Instability of the lower solution.

Definition 4.1. Given the solution $u_\lambda(x)$ of problem $P(\lambda, f)$, we say that u_λ is $L^\infty-$stable if $\forall\epsilon > 0, \exists\delta > 0$ such that for any $v_0 \in L^\infty(0,1)$ generating a non-degenerate solution $v(t,.:v_0)$ of $PP^*(\lambda,\beta,v_0)$ and verifying that

$$\| v_0 - u_\lambda \|_{L^\infty(0,1)} < \delta,$$

then

$$\| v(t,.:v_0) - u_\lambda \|_{L^\infty(0,1)} < \epsilon.$$

Definition 4.2. Given the solution $u_\lambda(x)$ of problem $P(\lambda, f)$, we say that u_λ is $L^\infty-$unstable if is not $L^\infty-$stable.

It is well-known that in the case of Lipschitz or singular nonlinear functions $f(u)$ the stability and instability of a solution of the stationary problem u_λ is reduced to study the sign of the principal eigenvalue of the linearized problem (see, e.g.

Henry [32] and Hernández, Mancebo and Vega [33]). In our setting this would lead to consider the problem

$$(P_\nu)\begin{cases} -U''(x) - \lambda f'(u_\lambda(x))U(x) = \nu U(x), & x \in (0,1), \\ U'(0) = 0, \quad U(1) = 0. \end{cases}$$

In the case of the discontinuous function $f(u)$ given by (1.1) we observe that if $x_{\mu,\lambda} \in (0,1)$ denotes the free boundary generated by $\underline{u}_\lambda$ (i.e. $\underline{u}_\lambda(x_{\mu,\lambda}) = \mu$) then

$$< f'(u_\lambda(.)), \varphi >_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})} = (1-f_0) < \delta_{\{x_{\mu,\lambda}\}}, \varphi >_{\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})},$$

and thus

$$f'(u_\lambda(x)) = (1-f_0)\delta_{\{x_{\mu,\lambda}\}}, \tag{4.16}$$

where $\delta_{\{x_{\mu,\lambda}\}}$ denotes the Dirac delta distribution at the free boundary point $x_{\mu,\lambda}$. Then the linearized problem becomes problem $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ defined in the Introduction.

By a solution of the problem $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ we mean a function $U \in V$ with the condition that U' is discontinuous in $x_{\mu,\lambda}$ and $U'' \in V'$ and satisfying the equation in V'. We also recall that we say that an eigenvalue $\nu = \nu_1$ is the "principal eigenvalue" of $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ if that problem admits a strict positive solution on $(0,1)$.

We point out that the proof of the fact that if the principal eigenvalue of $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ is negative then the stationary solution is unstable (such as given, for instance, in [32]) remains valid even if $f'(u_\lambda(x))$ involves a measure as in (4.16). Essentially, the main idea of the proof is to consider solutions v of the parabolic problem $PP^*(\lambda, \beta, v_0)$ with $v_0 = u_\lambda + \eta w_0$ with w_0 smooth and η small enough and to approximate v, as $\eta \to 0$, by functions of the form $v(x,t) = u_\lambda(x) + \eta w(x,t)$ with $w(t,x) = e^{-\nu t}U(x)$, U solution of the eigenvalue problem $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$.

In this section we shall prove the instability of stationary solutions $\underline{u}_\lambda$ of the decreasing part of the branch of solutions $(\lambda, \underline{\gamma}(\lambda))$ by proving that the principal eigenvalue of $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ is negative. Before to present the proof of Theorem 1.2 we point out that, such as it was indicated in Fleishman and Mahar [26] for the case of Dirichlet boundary conditions, it is possible to characterize the solutions of $P_\nu(x_{\mu,\lambda} : f_0, \lambda)$ in terms of a suitable transmission problem.

Proposition 4.1. *Let U be a solution of $P_\nu(x_{\mu,\lambda} : \varepsilon, \lambda)$. Denote by $U_\pm(x_{\mu,\lambda})$ and $U'_\pm(x_{\mu,\lambda})$ the directional limits of U and U' on $x = x_{\mu,\lambda}$. Then U satisfies*

$$\begin{cases} -U'' = \nu U, & x \in (0, x_{\mu,\lambda}) \cup (x_{\mu,\lambda}, 1), \\ U'(0) = 0, \quad U(1) = 0, \\ U_-(x_{\mu,\lambda}) = U_+(x_{\mu,\lambda}),\ U'_-(x_{\mu,\lambda}) - U'_+(x_{\mu,\lambda}) = \lambda(1-f_0)U(x_{\mu,\lambda}). \end{cases}$$

Moreover, the converse implication is valid.

Proof. By well-known regularity results $U \in C^2((0,1) - \{x_{\mu,\lambda}\}) \cap C^0([0,1])$ and thus satisfies $-U''(x) = \nu U(x)$ when $x \neq x_{\mu,\lambda}$. To prove the jump condition on U' we consider $\zeta \in V$ be a test function function satisfying

$$\zeta(x) = \begin{cases} 1, & x \in (x_{\mu,\lambda} - l, x_{\mu,\lambda} + l), \quad 0 < l < 1. \\ 0, & x \in (0, x_{\mu,\lambda} - l - k) \cup (x_{\mu,\lambda} + l + k, 1), \quad 0 < k < 1. \end{cases}$$

We consider a regularization of the distribution $\delta_{\{x_{\mu,\lambda}\}}$ by a sequence of smooth functions $h_n(x)$ so that $h_n \in C^0([0,1])$ and $h_n \to \delta_{\{x_{\mu,\lambda}\}}$ in the set of bounded

Radon measures $M_b(0,1)$. Let U_n be the (unique) solution of the problem

$$(P_{\upsilon,n})\begin{cases} -U_n'' - \lambda(1-f_0)h_n(x)U_n = \nu U_n, & x\in(0,1),\\ U_n'(0)=0, \quad U_n(1)=0. \end{cases}$$

Since $(P_{\upsilon,n})$ is a linear eigenvalue problem we can renormalize the eigenfunctions by assuming, for instance, $\|U_n\|_\infty = 1$ for any $n\in\mathbb{N}$. Moreover, $U_n'' - \lambda(1-f_0)h_n(x)U_n \in V'$ and we have

$$< -U_n'' - \lambda(1-f_0)h_n(x)U_n, \zeta >_{V'\times V} = < \nu U, \zeta >_{V'\times V},$$

which implies (since by well-known regularity results $U_n\in H^2(0,1)$)

$$\begin{aligned} &-\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} U_n''(x)\zeta(x)dx - \lambda(1-f_0)\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} h_n(x)U_n(x)\zeta(x)dx\\ &=\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} \nu U_n(x)\zeta(x)dx. \end{aligned} \tag{4.17}$$

Observe that U_n is uniformly bounded in $C^0([0,1])$ and thus U_n' is uniformly bounded in $BV(0,1)$. So, there is subsequence, still denoted by U_n, and a function $U\in C^0([0,1])$ with $U_n'\in BV(0,1)$ such that $U_n''\rightharpoonup U''$ in $M_b(0,1)$ and $U_n'\to U'$ in $BV(0,1)$ and $U_n\to U$ in $C^0([0,1])$ as $n\to+\infty$. Integrating by parts in (4.17)

$$\begin{aligned} &-[U_n'(x_{\mu,\lambda}+l+k)\zeta(x_{\mu,\lambda}+l+k) - U_n'(x_{\mu,\lambda}-l-k)\zeta(x_{\mu,\lambda}-l-k)]\\ &+\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} U_n'(x)\zeta'(x)dx\\ &=\lambda(1-f_0)\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} h_n(x)U_n(x)\zeta(x)dx + \nu\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} U_n(x)\zeta(x)dx\\ &=\lambda(1-f_0)\lambda(1-f_0)\int_{x_{\mu,\lambda}-l-k}^{x_{\mu,\lambda}+l+k} h_n(x)U_n(x)\zeta(x)dx + O(l+k), \end{aligned}$$

Taking limits, first as $l\to0, k\to0$, and then as $n\to+\infty$ we obtain the stated jump conditions (since $U_n'\to U'$ in $BV(0,1)$ implies the convergence of the directional derivatives in the point $x_{\mu,\lambda}$). The proof of the reciprocal implication reduces merely to write the definition of U'' as an element of V'. □

Remark 4. The instability of the stationary solutions of the decreasing part of the branch of solutions when we replace our boundary conditions by the Dirichlet boundary conditions was announced, without any proof, in the paper Fleishman and Mahar [26]. They claim that a proof of the negativeness of the principal eigenvalue of the associated problem could be obtained by using a characterization of the eigenvalue problem in terms of a transmission problem similar to (P_ν), as indicated in Proposition 4.1. Nevertheless, as far as we know such a proof was never published by them. In fact, our proof of Theorem 1.2 will not use this fact and so it is absolutely independent of their claim (no study of the associated parabolic problem is done in the mentioned reference).

Proof of Theorem 1.2. Since we search to prove that for $\nu<0$ there is a positive solution we write $\nu=-\tau^2$. Since necessarily

$$\begin{cases} -U'' = -\tau^2 U, & \text{in } (0,\underline{x}_{\mu,\lambda}),\\ U'(0)=0, \end{cases}$$

we can assume that

$$U(x) = U_-(x) = A\cosh(\tau x) \quad \text{if} \quad x \in [0, \underline{x}_{\mu,\lambda}) \quad \text{for some} \quad A \neq 0.$$

Moreover, since

$$\begin{cases} -U'' = -\tau^2 U, & \text{in } (\underline{x}_{\mu,\lambda}, 1), \\ U(1) = 0, \end{cases}$$

we can assume that

$$U(x) = U_+(x) = B\sinh(\tau(1-x)) \quad \text{if} \quad x \in (\underline{x}_{\mu,\lambda}, 1] \quad \text{for some} \quad B > 0.$$

The required continuity on U (first of the two transmission conditions on $\underline{x}_{\mu,\lambda}$) requires the identity

$$A\cosh(\tau\underline{x}_{\mu,\lambda}) = B\sinh(\tau(1-\underline{x}_{\mu,\lambda})),$$

so that

$$A = B\frac{\sinh(\tau(1-\underline{x}_{\mu,\lambda}))}{\cosh(\tau\underline{x}_{\mu,\lambda})}. \tag{4.18}$$

On the other hand, since

$$U_+'(\underline{x}_{\mu,\lambda}) = -B\tau\cosh(\tau(1-\underline{x}_{\mu,\lambda})) \text{ and } U_-'(\underline{x}_{\mu,\lambda}) = A\tau\sinh(\tau\underline{x}_{\mu,\lambda}),$$

the second transmission condition leads to

$$B\tau\cosh(\tau(1-\underline{x}_{\mu,\lambda})) + A\tau\sinh(\tau\underline{x}_{\mu,\lambda}) = \lambda(1-f_0)B\sinh(\tau(1-\underline{x}_{\mu,\lambda})).$$

Using (4.18) we get

$$\tau\left[\cosh(\tau(1-\underline{x}_{\mu,\lambda})) + \frac{\sinh(\tau(1-\underline{x}_{\mu,\lambda}))}{\cosh(\tau\underline{x}_{\mu,\lambda})}\sinh(\tau\underline{x}_{\mu,\lambda})\right] = \lambda(1-f_0)\sinh(\tau(1-\underline{x}_{\mu,\lambda})).$$

Thus, we haave eliminated B and by dividing by $\cosh(\tau(1-\underline{x}_{\mu,\lambda}))$ we get

$$\tau\left[1 + \tanh(\tau(1-\underline{x}_{\mu,\lambda}))\tanh(\tau\underline{x}_{\mu,\lambda})\right] = \lambda(1-f_0)\tanh(\tau(1-\underline{x}_{\mu,\lambda}))$$

i.e.

$$\tau = \lambda(1-f_0)\frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))} := \Psi_{x_{\mu,\lambda},f_0}(\tau). \tag{4.19}$$

Now, we recall that we know that $\underline{x}_{\mu,\lambda} \in (\frac{f_0}{2-f_0}, \frac{1}{2-f_0})$. Let us start by assuming, additionally, that $\underline{x}_{\mu,\lambda} \in (0, \frac{1}{2})$, i.e.

$$\underline{x}_{\mu,\lambda} < 1 - \underline{x}_{\mu,\lambda}.$$

We need the following lemma

Lemma 4.3. *Let τ_1 be a solution of the equation (4.19). Then if $x_{\mu,\lambda} \in (0, \frac{1}{2})$ we have*

$$\Psi_{x_{\mu,\lambda},f_0}(\tau) > \lambda\frac{(1-f_0)}{2}\tanh\tau.$$

Proof of Lemma 4.3. Recalling that

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1+\tanh a\tanh b}$$

we get

$$\tanh\tau = \tanh(\tau\underline{x}_{\mu,\lambda} + \tau(1-\underline{x}_{\mu,\lambda})) =$$

$$= \frac{\tanh(\tau\underline{x}_{\mu,\lambda})}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))} + \frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}.$$

Since $\underline{x}_{\mu,\lambda} < 1-\underline{x}_{\mu,\lambda}$, then $\tanh(\tau\underline{x}_{\mu,\lambda}) < \tanh(\tau(1-\underline{x}_{\mu,\lambda}))$ and thus

$$\frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))} = \tanh\tau - \frac{\tanh(\tau\underline{x}_{\mu,\lambda})}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}$$

$$> \tanh\tau - \frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}.$$

i.e.

$$\frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))} > \frac{1}{2}\tanh\tau,$$

as wanted. □

Proof of Theorem 1.2 (continuation). Define $\Psi_{f_0}(\tau) := \lambda\frac{(1-f_0)}{2}\tanh\tau$. For $\tau > 0$, this function is concave: indeed,

$$\Psi''_{f_0}(\tau) = -\lambda(1-f_0)\frac{sinh(\tau)}{cosh^3(\tau)} < 0.$$

Then, there exists $\tau^* > 0$ such that

$$\tau^* = \lambda\frac{(1-f_0)}{2}\tanh\tau^*.$$

Hence, from Lemma 4.3, we conclude that

$$\tau_1 > \tau^* > 0.$$

This leads to the conclusion if we have $\underline{x}_{\mu,\lambda} \in (0, \frac{1}{2})$. Now, it remains to prove that $\tau_1 > 0$ also in the case $\underline{x}_{\mu,\lambda} \in (\frac{1}{2}, \frac{1}{2-f_0})$. In that case it is clear that there exists $K(f_0) > 1$ such that

$$\tanh(\tau\underline{x}_{\mu,\lambda}) < K(f_0)\tanh(\tau(1-\underline{x}_{\mu,\lambda})) \quad \forall\underline{x}_{\mu,\lambda} \in (\frac{1}{2}, \frac{1}{2-f_0}).$$

Then

$$\frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))} = \tanh\tau - \frac{\tanh(\tau\underline{x}_{\mu,\lambda})}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}$$

$$> \tanh\tau - \frac{K(f_0)\tanh(\tau(1-\underline{x}_{\mu,\lambda})}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}.$$

So that

$$\frac{\tanh(\tau(1-\underline{x}_{\mu,\lambda}))}{1+\tanh(\tau\underline{x}_{\mu,\lambda})\tanh(\tau(1-\underline{x}_{\mu,\lambda})} > \frac{1}{(1+K(f_0))}\tanh\tau.$$

If we take now $\tau^*_{f_0}$ defined as a positive solution of the equation

$$\tau^*_{f_0} = \frac{1}{(1+K(f_0))}\tanh\tau^*,$$

then, from Lemma 4.3, we conclude that $\tau_1 > \tau^*_{f_0} > 0$. Summarizing, since we have proved that the solution τ_1 of (4.19) is strictly positive, $\tau_1 > 0$, to end the proof of Theorem 1.2 it is enough to take $B > 0$, arbitrary, and then $A > 0$ given by (4.18). In this way we are building a positive solution $U \in C^0((0,1))$ and thus the principal eigenvalue of the problem (P_ν) satisfies $\nu_1 = -\tau_1^2 < 0$ and, hence, the lower branch $\underline{\gamma}(\lambda)$ is an unstable branch of solutions. □

5. **Stability of the upper solution.** In this section we shall prove the stability of the increasing parts, $(\lambda, \overline{\gamma}(\lambda))$ and $(\lambda, \gamma^*(\lambda))$, of the bifurcation branch of solutions.

Proof of Theorem 1.3. As in [13], we shall introduce some suitable change of variables involving the parameter λ. We define

$$L(\lambda) = \sqrt{\lambda} := L, \qquad \widetilde{x} = Lx, \quad \text{and} \quad \widetilde{t} = \lambda t.$$

Then, if $v(x,t)$ is a solution of $PP^*(\lambda, \beta, v_0)$ we define the function

$$V(\widetilde{x}, \widetilde{t}) := v\left(\frac{\widetilde{x}}{\sqrt{\lambda}}, \frac{\widetilde{t}}{\lambda}\right) = v(x,t).$$

We have that V satisfies

$$\widetilde{PP}(\lambda, \beta, v_0) \begin{cases} \frac{\partial V}{\partial \widetilde{t}} - \frac{\partial^2 V}{\partial \widetilde{x}^2} \in \lambda\beta(V) & \widetilde{x} \in (0, L), \quad \widetilde{t} \in (0, +\infty), \\ \frac{\partial V}{\partial \widetilde{x}}(0, \widetilde{t}) = 0, \quad V(L, \widetilde{t}) = 0 & \widetilde{t} > 0, \\ V(\widetilde{x}, 0) = v_0(\frac{\widetilde{x}}{\sqrt{\lambda}}) := V_0(\widetilde{x}), & \widetilde{x} \in (0, L). \end{cases}$$

The same change of variable leads to the new stationary problem

$$\widetilde{P}(L, f) \begin{cases} -\frac{d^2}{d\widetilde{x}} W(\widetilde{x}) = f(W(\widetilde{x})) & \text{in } (0, L), \\ \frac{d}{d\widetilde{x}} W(0) = 0, \quad W(L) = 0. \end{cases}$$

It is useful to reformulate $\widetilde{P}(L, f)$ by using a shooting method in terms of the parameter

$$\gamma := \|u_\lambda\|_\infty = W(0).$$

We introduce the ODE Cauchy problem

$$ODE(\gamma) \begin{cases} -u''(x) = \lambda \widetilde{f}(u(x)) & \text{in } \ (0, +\infty), \\ u(0) = \gamma, \ u'(0) = 0, \end{cases}$$

where $\widetilde{f}(u)$ is the prolongation of $f(u)$ on the negatives values, i.e.

$$\widetilde{f}(r) = \begin{cases} f_0 & \text{if } \ r \in (-\infty, \mu], \\ 1 & \text{if } \ r \in (\mu, +\infty). \end{cases}$$

We have a first result on the qualitative behaviour of solutions of $ODE(\gamma)$:

Lemma 5.1. *For any $\gamma > \mu$, there exists a unique function u_γ satisfying $ODE(\gamma)$ and such that $u_\gamma = \mu$ in a single point $x_\gamma > 0$. Moreover, u_γ is nondegenerate and $u_\gamma(x) \geq 0$ if and only if $x \in [0, \widehat{L}(\gamma)]$ with $\widehat{L}(\gamma)$ given by*

$$\widehat{L}(\gamma) = \frac{(1-f_0)\sqrt{2(\gamma - \mu)}}{f_0} + \sqrt{\frac{2(1-f_0)^2(\gamma-\mu) + 2f_0((2-f_0)(\gamma-\mu)+\mu)}{f_0^2}}. \tag{5.20}$$

Proof of Lemma 5.1. Given $\gamma > \mu$, then necessarily $-u''_\gamma(x) = 1$ near $x = 0$, i.e.

$$u_\gamma(x) = -\frac{x^2}{2} + \gamma, \quad \text{for} \quad x > 0 \quad \text{small enough.}$$

Then

$$u_\gamma(x_\gamma) = \mu \text{ if and only if } x_\gamma = \sqrt{2(\gamma - \mu)},$$

and thus

$$u'_\gamma(x_\gamma) = -\sqrt{2(\gamma-\mu)}.$$

Since, we search a C^1 function with a single free boundary, then for $x > x_\gamma$, u_γ must satisfy

$$\begin{cases} -u''_\gamma = f_0 & \text{in } (x_\gamma, +\infty), \\ u_\gamma(x_\gamma) = \mu,\ u_\gamma{}'(x_\gamma) = -\sqrt{2(\gamma-\mu)}, & \end{cases}$$

i.e.

$$u_\gamma(x) = -\frac{f_0}{2}x^2 - (1-f_0)\sqrt{2(\gamma-\mu)}x + (2-f_0)(\gamma-\mu) + \mu.$$

In particular,

$$u(x_\gamma) \geq 0 \text{ if and only if } x \in [0, \widehat{L}(\gamma)],$$

with

$$-\frac{f_0}{2}(\widehat{L}(\gamma))^2 - (1-f_0)\sqrt{2(\gamma-\mu)}\widehat{L}(\gamma) + (2-f_0)(\gamma-\mu) + \mu = 0,$$

i.e., $\widehat{L}(\gamma)$ is given by (5.20). On the other hand, it is clear that $u_\gamma(x)$ is not degenerate. Indeed,

$$|u_\gamma(x)-\mu| = \begin{cases} -\frac{x^2}{2} + \gamma - \mu & \text{if } x \in [0, x_\gamma], \\ -\frac{f_0}{2}x^2 - (1-f_0)\sqrt{2(\gamma-\mu)}x + (2-f_0)(\gamma-\mu), & \text{if } x \in [x_\gamma, +\infty), \end{cases}$$

and since $u_\gamma \in C^1$ and is strictly concave

$$(u_\gamma(x)-\mu) = u'_\gamma(x_\gamma)(x-x_\gamma) + O(|x-x_\gamma|^2).$$

Thus, the set of points where $|u_\gamma(x)-\mu| \leq \theta$ is contained in the interval

$$\left(x_\gamma - \frac{\theta}{\sqrt{2(\gamma-\mu)}},\ x_\gamma + \frac{\theta}{\sqrt{2(\gamma-\mu)}}\right),$$

of measure $2\theta/\sqrt{2(\gamma-\mu)}$. □

We recall that we have an explicit description of the function $\overline{\gamma}(\lambda)$. Then, by the change of variable we get

$$L(\lambda) = \sqrt{\overline{\lambda}(\gamma)}, \quad \text{where} \quad \overline{\lambda}(\gamma) \quad \text{is the inverse function of} \quad \overline{\gamma}(\lambda).$$

From now, we shall use the identifying notation

$$L(\overline{\lambda}(\gamma)) = \widehat{L}(\gamma).$$

Then we get a very precise information on solutions $u_\gamma(x)$ in terms of γ and λ. In particular, given $\lambda \in (2(2-f_0)\mu, +\infty)$, we know that $\lambda = \overline{\lambda}(\gamma)$ for some $\gamma > \gamma_0$ and, in particular, the study made in Section 2 gives a proof to the following Lemma:

Lemma 5.2. *The function $\gamma \to u_\gamma(x)$ is strictly increasing and continuous for any fixed $x \in [0,\infty)$. In particular, for any $\gamma > \gamma_0$ fixed and for any $\epsilon > 0$ fixed, there exists a $\delta > 0$ and $h > 0$ such that*

$$\delta < u_{\gamma+h}(x) - u_\gamma(x) < \epsilon, \quad \forall x \in [0, \widehat{L}(\gamma)]$$

$$-\epsilon < u_{\gamma-h}(x) - u_\gamma(x) < -\delta, \quad \forall x \in [0, \widehat{L}(\gamma)].\ \square$$

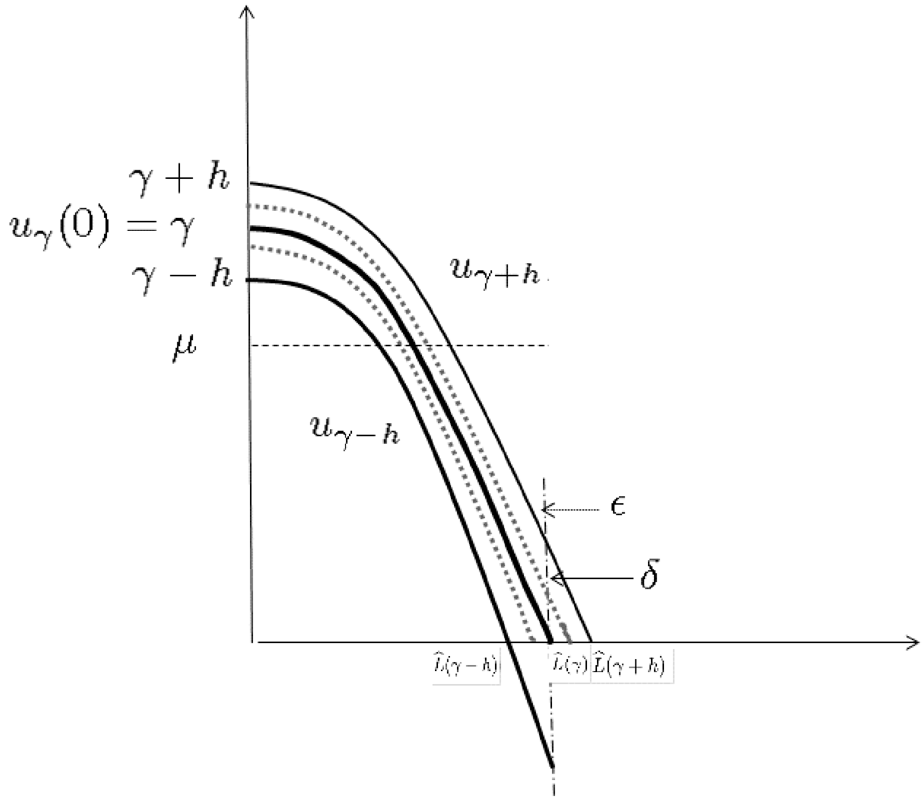

FIGURE 2. Security neighborhood

End of the proof of Theorem 1.3. Let $\overline{v}_1(\lambda) := u_{\gamma+h}(\widehat{L}(\overline{\gamma}(\lambda)))$ and $\underline{v}_1(\lambda) := u_{\gamma-h}(\widehat{L}(\overline{\gamma}(\lambda)))$. Given $\lambda \in (2(2-f_0)\mu, +\infty)$ and $\epsilon > 0$, let $\overline{u}_\gamma$ be the unique solution of $P(\lambda, f)$ in the increasing part of the branch $(\lambda, \overline{\gamma}(\lambda))$. Let $\delta > 0$ given Lemma 5.2 corresponding to $\gamma = \overline{\gamma}(\lambda)$. Then, for any $v_0 \in V$, v_0 non-degenerate, we define

$$V_0(\widetilde{x}) = v_0(\frac{\widetilde{x}}{\sqrt{\lambda}}).$$

Thus $V_0 \in L^\infty(0, L(\lambda))$. It is clear also that any neighborhood of $v_0(.)$ in $L^\infty((0,1))$ of radius δ is equivalent to a neighborhood of $V_0(.)$ in $L^\infty(0, L(\lambda))$ of radius also δ. Let $V(\widetilde{x}, \widetilde{t})$ be the non-degenerate solution of $\widetilde{PP}(\lambda, \beta, v_0)$. Then we have

$$u_{\gamma-h}(\widetilde{x}) \leq V(\widetilde{x}, \widetilde{t}) \leq u_{\gamma+h}(\widetilde{x}), \quad \forall \widetilde{x} \in (0, L(\lambda)) \text{ and } \forall \widetilde{t} > 0,$$

where $h > 0$ is given in Lemma 5.2. Indeed, $u_{\gamma+h}(\widetilde{x})$ verifies that

$$\begin{cases} \frac{\partial u_{\gamma+h}}{\partial \widetilde{t}} - \frac{\partial^2 u_{\gamma+h}}{\partial \widetilde{x}^2} \in \lambda\beta(u_{\gamma+h}) & \widetilde{x} \in \ (0, L(\lambda)), \quad \widetilde{t} \in (0, +\infty), \\ \frac{\partial u_{\gamma+h}}{\partial \widetilde{x}}(0, \widetilde{t}) = 0, \quad u_{\gamma+h}(L(\lambda), \widetilde{t}) = \overline{v}_1(\lambda) & \widetilde{t} > 0, \\ u_{\gamma+h}(\widetilde{x}, 0) = u_{\gamma+h}(\widetilde{x}) > V_0(\widetilde{x}), & \widetilde{x} \in (0, L(\lambda)). \end{cases}$$

Then, since both $V(\widetilde{x},\widetilde{t})$ and $u_{\gamma+h}(\widetilde{x})$ are non-degenerate and since both functions are continuous on $(0,L(\lambda))$, we conclude from Theorem 3.4 that

$$V(\widetilde{x},\widetilde{t}) \le u_{\gamma+h}(\widetilde{x}), \quad \widetilde{t} > 0.$$

Analogously, we have also the comparison from above, i.e.

$$u_{\gamma-h}(\widetilde{x}) \le V(\widetilde{x},\widetilde{t}).$$

Then, by Lemma 5.2, we get that

$$\left\| V(.,\widetilde{t}) - u_\lambda(.) \right\|_{L^\infty(0,L(\lambda))} \le \epsilon,$$

which ends the proof after making, again, the change of variables $\widetilde{x} = \sqrt{\lambda}x$.

The proof of the stability of the increasing part of the branch $(\lambda,\gamma^*(\lambda))$ is similar, and even easier than before since in this part of the branch the solutions do not have free boundary (since $\|u^*\|_{L^\infty(0,1)} < \mu$) and we can take a neighborhood of u^* in $L^\infty(0,1)$ such that the solutions with initial datum in it do not have free boundary. In consequence they are solutions of a linear problem for which the $L^\infty-$stability is well-known. □

Remark 5. The above arguments have some common points with the ones used in the paper by Bertsch and Klaver [6]. Nevertheless, we point out that their stability results only apply (for the one-dimensional case) when $f_0 > 1$ (see their Theorem 7.1) and thus assumption (1.2) is not satisfied. Their stability results apply to the case in which the elliptic equation contains a discontinuous but monotone nonlinear term. The associated multivalued equation is of the type $-u_{xx} + \lambda\beta(u) \ni 0$ as, for instance, it arises in the study of the stationary Stefan problem with an absorption term (see [6] and [10]).

REFERENCES

[1] D. Arcoya, J. I. Díaz and L. Tello, S-Shaped bifurcation branch in a quasilinear multivalued model arising in Climatology, *Journal of Differential Equations*, **150** (1998), 215–225.

[2] J. Arrieta, A. Rodríguez-Bernal and J. Valero, Dynamics of a reaction–diffusion equation with a discontinuous nonlinearity, *International Journal of Bifurcation and Chaos*, **16** (2006), 2965–2984.

[3] M. Belloni and R. W. Robinett, The infinite well and Dirac delta function potentials as pedagogical, mathematical and physical models in quantum mechanics, *Physics Reports*, **540** (2014), 25–122.

[4] S. Bensid and S. M. Bouguima, On a free boundary problem, *Nonlinear Anal. T.M.A*, **68** (2008), 2328–2348.

[5] S. Bensid and S. M. Bouguima, Existence and multiplicity of solutions to elliptic problems with discontinuities and free boundary conditions, *Electronic Journal of Differential Equations,* **2010** (2010), 1–16.

[6] M. Bertsch and M. H. A. Klaver, On positive solutions of $-\Delta u + f(u) = 0$ with f discontinuous, *J. Math. Anal. Appl.*, **157** (1991), 417–446.

[7] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Differential Equations,* **1** (1996), 73–90.

[8] M. Coti Zelati, A. Huang, I. Kukavica, R. Temam and M. Ziane, The primitive equations of the atmosphere in presence of vapor saturation, Nonlinearity (2015), `http://dx.doi.org/10.1088/0951-7715/28/3/625`, in press.

[9] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal*, **8** (1971), 321–340.

[10] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries,* Pitman, London, 1985.

[11] J. I. Díaz, Mathematical analysis of some diffusive energy balance climate models, In *Mathematics, Climate and Environment (J. Díaz and J. -L. Lions, eds.) Masson,* Paris, **27** (1993), 28–56.

[12] J. I. Díaz, A. C. Fowler, A. I. Muñoz and E. Schiavi, Mathematical analysis of a model of river channel formation, *Pure appl. geophys.,* **165** (2008), 1663–1682.

[13] J. I. Diaz and J. Hernández, Global bifurcation and continua of nonegative solutions for a quasilinear elliptic problem, *Comptes Rendus Acad. Sci. Paris*, **329** (1999), 587–592.

[14] J. I. Díaz, J. Hernández and Y. Ilyasov, Stability criteria on flat and compactly supported ground states of some non-Lipschitz autonomous semilinear equations, To appear in Chinese Annals of Mathematics.

[15] J. I. Díaz, J. Hernandez and L. Tello, On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in Climatology, *Jour. Mathematical Analysis and Applications*, **216** (1997), 593–613.

[16] J. I. Díaz and G. Hetzer, A Functional Quasilinear Reaction-Diffusion Equation Arising in Climatology, É quations aux dérivées partielles et applications. Articles dédi és à J.-L. Lions, Elsevier, Paris, (1998), 461–480.

[17] J. I. Díaz, J. A. Langa and J. Valero, On the asymptotic behaviour of solutions of a stochastic energy balance climate model, *Physica D*, **238** (2009), 880–887.

[18] J. I. Díaz, J. F. Padial and J. M. Rakotoson, On some Bernouilli free boundary type problems for general elliptic operators, *Proceedings of the Royal Society of Edimburgh*, **137** (2007), 895–911.

[19] J. I. Díaz and J. M. Rakotoson, On a nonlocal stationary free boundary problem arising in the confinement of a plasma in a Stellarator geometry, *Archive for Rational Mechanics and Analysis*, **134** (1996), 53–95.

[20] J. I. Díaz and S. Shmarev, Langragian approach to level sets: Application to a free boundary problem arising in climatology, *Archive for Rational Mechanics and Analysis*, **194** (2009), 75–103.

[21] J. I. Diaz and L. Tello, On a nonlinear parabolic problem on a Riemannian manifold without boundary arising in Climatology, *Collectanea Mathematica,* **50** (1999), 19–51.

[22] I. H. Farina and R. Aris, Transients in distributed chemical reactors, Part 2: Influence of diffusion in the simplified model, *Chem. Engng J.,* **4** (1972), 149–170.

[23] E. Fereisel, A note on uniqueness for parabolic problems with discontinuous nonlinearities. *Nonlinear Analysis*, **16** (1991), 1053–1056.

[24] E. Feireisl and J. Norbury, Some existence, uniqueness and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities, *Proc. Roy. Soc. Edinburgh Sect.*, **119** (1991), 1–17.

[25] B. A. Fleishman and T. J. Mahar, Analytic methods for approximate solution of elliptic free boundary problems, *Nonlinear Anal.*, **1** (1977), 561–569.

[26] B. A. Fleishman and T. J. Mahar, A step function model in chemical reactor theory: Multiplicity and stability of solutions, *Nonl. Anal.*, **5** (1981), 645–654.

[27] L. E. Fraenkel and M. S. Berger, A global theory of steady vortex rings in an ideal fluid, *Acfa Math.*, **132** (1974), 13–51.

[28] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[29] R. Gianni and J. Hulshof, The semilinear heat equation with a Heaviside source term, *Euro. J. of Applied Mathematics*, **3** (1992), 367–379.

[30] A. A. Guetter, A free boundary problem in plasma containment, *SIAM J. Appl. Math.*, **49** (1989), 99–115.

[31] A. A. Guetter, On solutions of an elliptic boundary value problem with a discontinuous nonlinearity, *Nonl. Anal. T.M.A,*, **23** (1994), 1413–1425.

[32] D. Henry, *Geometric theory of semilinear parabolic equations*, in Lecture Notes in Mathematics No. 840, Springer-Verlag, New York, 1981.

[33] J. Hernandez, F. J. Mancebo and J. M. Vega, On the linearization ofsome singular nonlinear elliptic problems and applications, *Ann. Inst. H. Poincaré Anal.,* **19** (2002), 777–813.

[34] G. Hetzer, The structure of the principal component for semilinear diffusion equations from energy balance climate models, *Houston Journal of Mathematics,* **16** (1990), 203–216.

[35] X. Liang, X. Lin and H. Matano, A variational problem associated with the minimal speed of travelling waves for spatially periodic reaction-diffusion equations, *Transactions of the American Mathematical Society*, **362** (2010), 5605–5633.

[36] H. P. McKean, Nagumo's equation, *Advances in Math.*, **4** (1970), 209–223.

[37] G. R. North and J. A. Coakley, Differences between seasonal and mean annual energy balance model calculations of climate and climate sensitivity, *Journal of the Atmospheric Sciences*, **36** (1979), 1189–1203.

[38] J. Rauch, Discontinuous semilinear differential equations and multiple valued maps, *Proceedings of the American Mathematical Society*, **64** (1977), 277–282.

[39] I. Stakgold, Free boundary problems in climate modeling. In, Mathematics, Climate and Environment, J. I. Díaz, and J.-L.Lions (Edits.), Research Notes in Applied Mathematics n^o 27, Masson, Paris, 1993, 179–188.

[40] C. A. Stuart, The number of solutions of boundary value problems with discontinuous nonlinearities, *Archive for Rational Mechanics and Analysis*, **66** (1977), 225–235.

[41] R. Temam, A nonlinear eigenvalue problem: The shape at equilibrium of a confined plasma, *Arch. Rational. Mech. Anal*, **60** (1975), 51–73.

[42] D. Terman, A free boundary problem arising from a bistable reaction–diffusion equation, *SIAM J. Math. Anal.*, **14** (1983), 1107–1129.

[43] J. Valero, Attractors of parabolic equations without uniqueness, *Journal of Dynamics and Differential Equations*, **13** (2001), 711–744.

[44] X. Xu, Existence and regularity theorems for a free boundary problem governing a simple climate model, *Aplicable Anal.*, **42** (1991), 33–57.

[45] K. Yosida, *Functional Analysis*, Springer, 1965.

Received January 2016; revised June 2016.

E-mail address: edp_sabri@yahoo.fr
E-mail address: jidiaz@ucm.es