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EXISTENCE AND UNIQUENESS OF SINGULAR SOLUTIONS OF p-LAPLACIAN WITH ABSORPTION FOR DIRICHLET BOUNDARY CONDITION

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ABSTRACT. In this paper, we consider the existence and uniqueness of singular solutions of degenerate parabolic equations with absorption for zero homogeneous Dirichlet boundary condition. Moreover, we also get some estimates of the short time behavior of singular solutions.

1. INTRODUCTION

Given a smooth bounded $\Omega \subset \mathbb{R}^N$, we study the existence and uniqueness of singular solutions of the following equation:

$$\begin{cases} \partial_t u - \Delta_p u + u^q = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $p > 2$, $q > 1$. The singular solutions of (1) refer to the fundamental solutions, the very singular solutions, and the large solutions. Throughout this paper we assume, without loss of generality, that $0 \in \Omega$, and such a fundamental solution and a VSS have the singularity at $(0, 0)$.

By a fundamental solution of (1), we mean a continuous nonnegative function $u(x, t)$ satisfying (1) in the sense of distribution. Moreover,

$$u(x, 0) = 0, \quad \forall x \neq 0, \tag{2}$$

and there is a finite $c > 0$ such that

$$\lim_{t\to 0} u(x, t) = c\delta_0, \tag{3}$$

where δ_0 is a Dirac mass concentrated at 0. Since a VSS is more singular than any fundamental solution, then it also satisfies (2), and

$$\lim_{t\to 0} \int_\Omega u(x, t) = +\infty; \tag{4}$$

see e.g. [4], [17], [15], [14], [16].

It is well known that both kinds of solutions play a crucial role in studying the long time behavior of more general solutions; see [15], [14], and references therein.

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In spite of a large amount of papers dealing with VSS, as far as we know, most of them consider the case $\Omega = \mathbb{R}^N$. For instance, the existence of a self-similar VSS for the Cauchy problem associated to (1),

$$\partial_t u - \Delta_p u + u^q = 0, \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{5}$$

was proved by Peletier and Wang, [17], provided that $p-1 < q < p-1+\frac{p}{N}$.

The uniqueness of VSS required a different type of argument and it was proved by Kamin and Vazquez, [15] (see also Diaz and Saa [10] for a general quasilinear parabolic equation). In fact, these papers utilized the self-similarity, or scaling argument in $\mathbb{R}^N \times (0,\infty)$ in order to look for a solution, which is of the form:

$$W(x,t) = t^{\frac{-1}{q-1}} f(\eta), \quad \eta = |x| t^{-1/\beta}, \quad \beta = \frac{p(q-1)}{q+1-p}. \tag{6}$$

In order that W is a VSS, f must satisfy the ODE:

$$\begin{cases} (|f'|^{p-2}f')' + \frac{N-1}{\eta}|f'|^{p-2}f' + \frac{1}{\beta}\eta f' + \frac{1}{q-1}f - f^q = 0, & \text{in } (0,\infty), \\ f(\eta) \geq 0, \quad \text{on } [0,\infty), \\ f'(0) = 0, \quad \lim_{\eta\to\infty} \eta^{p/(q+1-p)} f(\eta) = 0. \end{cases} \tag{7}$$

Moreover, it was shown that

$$f(\eta) \begin{cases} > 0, & \text{for } 0 \leq \eta < \eta_0, \\ = 0, & \text{for } \eta \geq \eta_0, \end{cases}$$

(see also the treatment for $1 < p < 2$ in [5], [6]).

Concerning the uniqueness of VSS of (1), as far as we know, it has not been proved yet. Thus, our main result is as follows:

Theorem 1. *Let $p > 2$, and $p-1 < q < p-1+\frac{p}{N}$. Then, there exists a unique VSS, $u(x,t)$ of* (1)*. Furthermore, we have the short time behavior of u as $t \to 0$ at the singular point $x = 0$:*

$$\lim_{t\to 0} t^{\frac{1}{q-1}} u(0,t) = f(0). \tag{8}$$

Remark 2. The result (8) implies that the short time behavior of VSS for any bounded domain is similar to the one in $\mathbb{R}^N$ (compare to $W(x,t)$ in (6)).

Clearly, the argument of the proof of the uniqueness of VSS in $\mathbb{R}^N \times (0,\infty)$ based on the self-similarity of solutions is not applicable to such a bounded domain Ω in $\mathbb{R}^N$. To prove Theorem 1, we show that there exist a minimal VSS and a maximal VSS. And both solutions are equal. Note that the existence of a minimal VSS is well known. This one is the convergence of the nondecreasing sequence of fundamental solutions and we use the large solutions to construct a maximal VSS. Thus, it is convenient for us to introduce the large solution in what follows.

In [7], M. Crandall, P. Lions, and P. Souganidis considered nonnegative solutions of the equation:

$$\begin{cases} \partial_t u - \Delta u + |\nabla u|^q = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{9}$$

with unbounded initial data of the form

$$u(0) = \begin{cases} +\infty, & \text{in } \mathcal{D}, \\ 0, & \text{in } \Omega \backslash \overline{\mathcal{D}}, \end{cases} \tag{10}$$

where $\mathcal{D}$ is an open subset of Ω. The initial data is comprehended as follows: $u(x,t) \to +\infty$, for any $x \in \mathcal{D}$, and $u(x,t) \to 0$, for any $x \in \overline{\Omega}\backslash\overline{\mathcal{D}}$.

This problem is motivated by studying the theory of large deviations of Markov diffusion processes. The authors showed that there is a unique solution to problem (9)–(10) when $q > 1$. Such a solution with initial data (10) is called a large solution. Many other references involve the study of such types of unbounded initial data (see, e.g., [1], [2], [3], and references therein). Roughly speaking, a large solution is more singular than any VSS. Inspired by their works, and also for our purposes later, to prove the uniqueness of VSS, we shall prove the existence and uniqueness of large solutions of problem (1). Moreover, the applications of the study of large solutions applies even to some unexpected contexts such as Control Theory (see [9]).

Theorem 3. *Let $p > 2$, and $q > p - 1$. Then, there is a unique large solution of* (1).

The paper is organized as follows: We will give some definitions, and preliminary results in the next section. Section 3 is devoted to proving the existence and uniqueness of large solutions. Finally, we prove the existence and uniqueness of VSS in Section 4.

2. Some definitions and preliminary results

Notation. We denote by $B(x,r)$ the open ball with center at x and radius $r > 0$. We also denote by $\mathcal{C}^{0,\beta}(\Omega)$, the β-Hölder continuous space, for $\beta \in (0,1]$.

Let us define a weak solution of problem (1).

Definition 4. u is called a weak solution of (1) if $u \in L^p_{loc}(0,\infty; W^{1,p}_0(\Omega)) \cap L^\infty_{loc}(\overline{\Omega} \times (0,\infty))$ satisfies (1) in the sense of distribution.

Next, we recall some results for the existence, uniqueness, and regularity of weak solutions of the following problem:

$$\left\{\begin{array}{ll} \partial_t v - \Delta_p v = f, & \text{in } \Omega \times (0,\infty), \\ v = 0, & \partial\Omega \times (0,\infty), \\ v(x,0) = v_0, & \Omega. \end{array}\right. \tag{11}$$

Theorem 5. *Let $p > 2$, $f \in L^\infty(\Omega \times (0,\infty))$, and $v_0 \in \mathcal{C}^{0,1}(\Omega)$. Then, there exists a weak bounded solution u of* (1)*. Moreover, there is a positive constant $\beta \in (0,1)$ such that*

$$|u(x,t) - u(y,s)| \le C\|u\|_{L^\infty(\Omega\times(0,T))} \left(|x-y| + \|u\|^{\frac{p-2}{p}}_{L^\infty(\Omega\times(0,T))} |t-s|^{\frac{1}{p}} \right)^\beta, \tag{12}$$

for $(x,t),(y,s) \in \overline{\Omega} \times [0,T]$, and $C > 0$ is a constant not depending on u.

Proof. We skip the proof, and refer to Theorem 1.2, [11] (see also [12], [13] for more regularity results). □

Next, we consider the large solutions of (1).

3. The existence and uniqueness of large solutions

We prove Theorem 3.

Proof. **(i) Existence**

For any $n \geq 1$, let us put

$$\mathcal{D}_n = \{x \in \mathcal{D} : \text{dist}\left(x, \Omega \backslash \overline{\mathcal{D}}\right) > \frac{1}{n})\}$$

and construct a nondecreasing sequence of functions $\psi_n \in \mathcal{C}^{0,1}(\Omega)$ such that

$$\psi_n = \begin{cases} n, & \text{if } x \in \mathcal{D}_n, \\ 0, & \text{if } x \in \Omega \backslash \overline{\mathcal{D}}. \end{cases}$$

Now, we consider the following equation:

$$\begin{cases} \partial_t u_n - \Delta_p u_n + u_n^q = 0, & \text{in } \Omega \times (0, \infty), \\ u_n = 0, & \partial\Omega \times (0, \infty), \\ u_n(x, 0) = \psi_n(x), & \Omega. \end{cases} \tag{13}$$

Thanks to Theorem 5, there exists a unique bounded solution $u_n \in \mathcal{C}(\overline{\Omega} \times [0, \infty))$ of (13).

Clearly, $z(t) = (q-1)^{-\frac{1}{q-1}} t^{1-q}$ is a solution of the ODE:

$$\begin{cases} z'(t) + z^q(t) = 0, & t > 0, \\ z(0) = +\infty, \end{cases}$$

By the comparison principle, we get

$$u_n(x, t) \leq z(t), \quad \forall (x, t) \in \Omega \times (0, \infty). \tag{14}$$

We observe that $\{u_n\}_{n \geq 1}$ is nondecreasing. Thus, there is a function u such that $u_n \uparrow u$. The classical argument and regularity result imply that u is a weak continuous solution of (1).

Then, it remains to show that $u(0)$ fulfills condition (10). Indeed, for any $x \in \mathcal{D}$, there is a natural number $N_x \in \mathbb{N}$ such that $x \in \mathcal{D}_n$, $\forall n \geq N_x$. Therefore, the monotonicity of $\{u_n\}_{n \geq 1}$ yields

$$\liminf_{t \to 0} u(x, t) \geq \liminf_{t \to 0} u_n(x, t) = n.$$

The last inequality holds for any $n \geq N_x$, thereby proving $u(x, 0) = \infty$, for $x \in \mathcal{D}$.

Next, we show that $u(t)$ converges to 0 in $\Omega \backslash \overline{\mathcal{D}}$ as $t \to 0$. For any $y \in \Omega \backslash \overline{\mathcal{D}}$, let

$$\begin{cases} -\Delta_p \alpha(x) = 1, & B(y, r), \\ \alpha = 0, & \partial B(y, r), \end{cases} \tag{15}$$

where $r > 0$ is small enough such that $B(y, r) \subset \overline{\Omega} \backslash \overline{\mathcal{D}}$.

Put

$$w(x, t) = \lambda e^{Ct} e^{\frac{1}{\alpha(x)}},$$

for any $\lambda > 0$, and $C > 0$ is chosen later such that

$$\partial_t w - \Delta_p w + w^q \geq 0. \tag{16}$$

If this is done, since $w = \infty$ on the boundary $\partial B(y, r)$ and $u_n(x, 0) = 0$ in $B(y, r)$, then the comparison principle deduces

$$u_n(x, t) \leq w(x, t), \quad \forall (x, t) \in B(y, r) \times (0, \infty), \tag{17}$$

hence

$$u(x, t) \leq w(x, t), \quad \forall (x, t) \in B(y, r) \times (0, \infty).$$

This implies that

$$u(x,0) \le \lambda e^{\frac{1}{\alpha(x)}},$$

and the conclusion follows as $\lambda \to 0$.

Now, we demonstrate (16). Indeed, computation yields

$$w_t = Cw, \quad -\Delta_p w = -w^{p-1}\left(-\frac{\Delta_p \alpha}{\alpha^{2(p-1)}} + (p-1)|\nabla\alpha|^p\Big(\frac{1}{\alpha^{2p}} + \frac{2}{\alpha^{2p-1}}\Big)\right),$$

so

$$w_t - \Delta_p w + w^q = Cw + w^{p-2}\left(w^{q-(p-1)} - \frac{1}{\alpha^{2(p-1)}} - (p-1)|\nabla\alpha|^p\Big(\frac{1}{\alpha^{2p}} + \frac{2}{\alpha^{2p-1}}\Big)\right).$$

We observe that $|\nabla\alpha|$ is bounded, while $w(x,t) \to \infty$ faster than α^{-2p} as $x \to \partial B(y,r)$. Thus, there exists a real positive number $\delta > 0$ such that

$$\left(w^{q-(p-1)} - \frac{1}{\alpha^{2(p-1)}} - (p-1)|\nabla\alpha|^p\Big(\frac{1}{\alpha^{2p}} + \frac{2}{\alpha^{2p-1}}\Big)\right) > 0, \quad \text{on } \{r-\delta < |x-y| \le r\}.$$

It is important to note that we can choose δ being independent of C. It remains to take $C = C(\lambda) > 0$ large enough such that

$$Cw + w^{p-2}\left(w^{q-(p-1)} - \frac{1}{\alpha^{2(p-1)}} - (p-1)|\nabla\alpha|^p\Big(\frac{1}{\alpha^{2p}} + \frac{2}{\alpha^{2p-1}}\Big)\right) > 0, \quad \text{on } \{|x-y| \le r-\delta\}.$$

In brief, we get (16), likewise the existence of large solution follows.

(ii) Uniqueness

To prove the uniqueness, we use the scaling argument as in [7]. For any $\eta > 0$, we set

$$u_\eta(x,t) = \eta u(\eta^{\frac{q-(p-1)}{p-1}} x, \eta^{q-1} t).$$

Clearly, if u is a large solution of (1) with respect to $(\Omega, \mathcal{D})$, then u_η is a large solution of (1) with respect to $(\eta^{\frac{q-(p-1)}{p-1}}\Omega, \eta^{\frac{q-(p-1)}{p-1}}\mathcal{D})$.

By the routine argument, we obtain for any large solution v of (1) that

$$u_\eta(x,t) \ge v(x,t) \ge u_{\eta'}(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty), \tag{18}$$

for any $\eta > 1 > \eta' > 0$.

By the continuity of u, we can pass to the limit as $\eta \to 1^+$, $\eta' \to 1^-$ in (18) to get

$$u = v, \quad \text{in } \Omega \times (0,\infty).$$

This concluses the proof of Theorem 3. ☐

4. The existence and uniqueness of a very singular solution and its short time behavior

4.1. **The existence of a minimal VSS.** To construct a minimal VSS, our argument needs the uniqueness of fundamental solutions of (1). Then, we have the following lemma.

Lemma 6. *Let $p > 2$, and $0 < q < p - 1 + \frac{p}{N}$. Then, there exists a unique fundamental solution of* (1).

Proof. The existence of fundamental solutions is well known. We then focus on the uniqueness. It suffices to prove for the fundamental solution with initial data δ_0.

Let E be a unique solution of the equation (see [15]):

$$\begin{cases} \partial_t u - \Delta_p u + u^q = 0, & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(0) = \delta_0. \end{cases} \tag{19}$$

We first claim that if v is a fundamental solution of (1) with initial Dirac mass δ_0, then

$$v(x,t) \leq E(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty). \tag{20}$$

Indeed, for any $\tau > 0$, let

$$E_\tau(0) = \begin{cases} v(x,\tau), & \text{in } \Omega, \\ 0, & \text{outside } \Omega. \end{cases}$$

We consider the following problem:

$$\begin{cases} \partial_t u - \Delta_p u + u^q = 0, & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(0) = E_\tau(0). \end{cases} \tag{21}$$

Thanks to the classical result, this equation possesses a unique solution $E_\tau(x,t)$ (see e.g. [11]). Moreover, the strong comparison principle deduces:

$$v(x,t+\tau) \leq E_\tau(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

Clearly, $E_\tau(x,t)$ is bounded by the barrier function $z(t)$ for all $\tau > 0$. Then, there exists a nonnegative function $\overline{E}$ such that $E_\tau(x,t) \to \overline{E}(x,t)$ as $\tau \to 0$, uniformly on any compact set in $\mathbb{R}^N \times (0,\infty)$. Moreover, $\overline{E}$ is a weak solution of equation (19), and

$$v(x,t) \leq \overline{E}(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

If we can prove $\overline{E}(0) = \delta_0$, it follows then from the uniqueness of the fundamental solution of (19) that $\overline{E} = E$, hence we get claim (20).

Indeed, using the test function $\varphi \in \mathcal{C}_c^\infty\big(\mathbb{R}^N \backslash \{0\}\big)$ for (21), we have

(22)

$$\begin{aligned} \int_{\mathbb{R}^N} E_\tau(x,t)\varphi dx + \int_0^t \int_{\mathbb{R}^N} \big(|\nabla E_\tau|^{p-2}\nabla E_\tau \cdot \nabla\varphi + E_\tau^q \varphi\big) dxds &= \int_{\mathbb{R}^N} E_\tau(x,0)\varphi dx \\ &= \int_\Omega v(x,\tau)\varphi dx. \end{aligned}$$

Passing to the limit as $\tau \to 0$ yields:

$$\int_{\mathbb{R}^N} \overline{E}(x,t)\varphi dx + \int_0^t \int_{\mathbb{R}^N} \big(|\nabla \overline{E}|^{p-2}\nabla \overline{E} \cdot \nabla\varphi + \overline{E}_\tau^q \varphi\big) dxds = 0.$$

After that, letting $t \to 0$ deduces:

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \overline{E}(x,t)\varphi dx = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty\big(\mathbb{R}^N \backslash \{0\}\big). \tag{23}$$

Next, using the test function $\psi \in \mathbb{C}_c^\infty\big(\mathbb{R}^N\big)$ instead of φ in (22), and passing $\tau \to 0$ yields

$$\int_{\mathbb{R}^N} \overline{E}(x,t)\psi dx + \int_0^t \int_{\mathbb{R}^N} \big(|\nabla \overline{E}|^{p-2}\nabla \overline{E} \cdot \nabla\psi + \overline{E}_\tau^q \psi\big) dxds = \lim_{\tau\to 0} \int_\Omega v(x,\tau)\psi dx = \psi(0),$$

which implies

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\overline{E}(x,t)\psi dx = \psi(0). \tag{24}$$

By (24) and (23), we get $\overline{E}(0)=\delta_0$, so claim (20) follows.

Now, let v_1 and v_2 be two fundamental solutions of (1). Note that the well-known L^1-contraction principle does not hold for such a bounded domain in general. Thus, we use the L^1-contraction principle for the following truncation:

$$\|S_1(v_1-v_2)(t)\|_{L^1(\Omega)} \le \|S_1(v_1-v_2)(s)\|_{L^1(\Omega)}, \quad \text{for } 0<s<t<\infty, \tag{25}$$

where

$$S_k(r)=\int_0^r T_k(s)ds, \quad \text{and } T_k(s)=\begin{cases} s, & \text{if } |s|\le k,\\ k\,\mathrm{sign}(s), & \text{if } |s|>k,\end{cases}$$

for any $k>0$. Note that inequality (25) can be obtained by using the test function $T_1(v_1-v_2)$ to the difference between the equations satisfied by v_1 and v_2.

Next, we note that $0\le S_1(r)\le |r|$, for any $r\in\mathbb{R}$. It then follows from (25) and the triangle inequality that

$$\begin{aligned}\|S_1(v_1-v_2)(t)\|_{L^1(\Omega)} &\le \|(v_1-v_2)(s)\|_{L^1(\Omega)} \le \|(v_1-E)(s)\|_{L^1(\Omega)}+\|(E-v_2)(s)\|_{L^1(\Omega)}\\ &= \int_\Omega \big(E(x,s)-v_1(x,s)\big)dx + \int_\Omega \big(E(x,s)-v_2(x,s)\big)dx.\end{aligned} \tag{26}$$

Passing $s\to 0$ in (26) yields

$$\|S_1(v_1-v_2)(t)\|_{L^1(\Omega)}=0, \quad \text{for } 0<t<\infty,$$

which implies $v_1(x,t)=v_2(x,t)$, in $\Omega\times(0,\infty)$. This concludes the proof of Lemma 6. □

Now, let v_c^Ω be the unique solution of (1) with initial Dirac $c\delta_0$, for $c>0$. Clearly, $\{v_c^\Omega\}_{c>0}$ is the nondecreasing sequence, and it is bounded by $z(t)$. Thus, there is a function, say $v_{\min}^\Omega$, such that v_c^Ω converges to $v_{\min}^\Omega$ as $c\to\infty$. We will show that $v_{\min}^\Omega$ is the minimal VSS. It is equivalent to proving that for any VSS $u(x,t)$ of (1), it holds that

$$u(x,t)\ge v_c^\Omega(x,t), \quad \forall (x,t)\in\Omega\times(0,\infty), \tag{27}$$

for any $c>0$.

To prove (27), we wish to construct a sequence $\{u_n^{(c)}\}_{n\ge 1}$ such that

$$\begin{cases} u_n^{(c)}(x)\le u(x,\frac{1}{n}), \quad \forall x\in\Omega,\\[2mm] \|u_n^{(c)}\|_{L^1(\Omega)}=c,\\[2mm] \lim\limits_{n\to\infty} u_n^{(c)}=c\delta_0.\end{cases} \tag{28}$$

Indeed, since $\lim_{t\to 0}\int_\Omega u(x,t)dx=\infty$, there is a natural number $n_0\in\mathbb{N}$, such that $\int_\Omega u(x,\frac{1}{n})dx>c$, for any $n\ge n_0$. Moreover, $u(.,\frac{1}{n})$ is a continuous function in Ω. Then, there is a positive number $h=h(n,c)$ such that

$$\|T_h\big(u(.,\frac{1}{n})\big)\|_{L^1(\Omega)}=c.$$

Thus, $u_n^{(c)}=T_h\big(u(.,\frac{1}{n})\big)$ is a desired function satisfying (28).

Consider the following equation:

$$\begin{cases} \partial_t v - \Delta_p v + v^q = 0, & \text{in } \Omega \times (0,\infty), \\ v = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(0) = u_n^{(c)}, & \text{in } \Omega. \end{cases} \tag{29}$$

By the classical result, there exists a unique solution of (29), say $v_{n,c}(x,t)$. Thanks to the strong comparison principle, we get

$$v_{n,c}(x,t) \le u(x, t + \frac{1}{n}), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

It is not difficult to observe that $v_{n,c}(x,t)$ converges to $v_c^\Omega(x,t)$ as $n \to \infty$, the unique fundamental solution in Lemma 6. Thus, conclusion (27) follows. In other words, $v_{\min}^\Omega$ is a minimal VSS.

Remark 7. By the construction, the sequence $\{v_{\min}^{B_R}\}_{R>0}$ is nondecreasing, and it converges to V as $R \to \infty$, a self-similar VSS of equation (1) in $\mathbb{R}^N \times (0,\infty)$.

4.2. **The existence of a maximal VSS.** We have the following result.

Theorem 8. *Let $p > 2$, and $p-1 < q < p-1+\frac{p}{N}$. Then, there is a maximal VSS of* (1)*.*

Proof. Let u_ε be a unique large solution of problem (1) with initial data

$$u(0) = \begin{cases} +\infty, & \text{in } B(0,\varepsilon), \\ 0, & \text{in } \Omega \backslash \overline{B(0,\varepsilon)}. \end{cases}$$

It is clear that $\{u_\varepsilon\}_{\varepsilon>0}$ is a nondecreasing sequence. Then, there is a function $v_{\max}$ such that $u_\varepsilon \downarrow v_{\max}$ as $\varepsilon \to 0$, which is a weak solution of (1). Moreover, $v_{\max}$ is continuous on any compact of $\overline{\Omega} \times [0,\infty) \backslash \{(0,0)\}$ by Theorem 5.

Now, we show that $v_{\max}$ is a maximal VSS.

Indeed, for any $x \in \Omega \backslash \{0\}$, there is a real number $\varepsilon_x > 0$ such that $u_\varepsilon(x,t) \to 0$ as $t \to 0$, for any $\varepsilon \in (0, \varepsilon_x)$. It follows then from the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ that $v_{\max}(x,0) = 0$. Or $v_{\max}$ fulfills (2).

It remains to prove that for any VSS v of (1), it holds that

$$v \le u_\varepsilon, \quad \text{in } \Omega \times (0,\infty), \text{ for any } \varepsilon > 0. \tag{30}$$

On the one hand, proceeding as the proof of (17) yields, for any $\tau > 0$,

$$v(x,\tau) \le \lambda e^{C\tau} e^{\frac{1}{\alpha(x)}}, \quad \forall x \in \Omega, |x| \ge \varepsilon/2,$$

where $\alpha(x)$ is the solution of (15) in $B(x, \varepsilon/4)$. Thus,

$$v(x,\tau) \le m_\varepsilon \lambda e^{C\tau}, \quad \forall x \in \Omega, |x| \ge \varepsilon/2, \tag{31}$$

with $m_\varepsilon = \sup_{y \in B(x,\varepsilon/4)} \{e^{\frac{1}{\alpha(y)}}\}$.

On the other hand, since $u_\varepsilon(x,t) \to +\infty$ uniformly on any compact of $B(0,\varepsilon)$ as $t \to 0$, there is a time $s_\tau = s(\tau) > 0$ such that

$$v(x,\tau) \le u_\varepsilon(x,s), \quad \text{for any } x \in B(0,\varepsilon/2), \quad \forall s \in (0, s_\tau). \tag{32}$$

A combination of (31) and (32) deduces:

$$v(x,\tau) \le m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(x,s), \quad \text{for any } x \in \Omega, \quad \forall s \in (0, s_\tau).$$

Clearly, $\big(m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(.,.+s)\big)$ is the super solution of (1). Therefore, the strong comparison principle yields

$$v(x,t+\tau) \leq m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(x,t+s), \quad \forall (x,t) \in \Omega \times (0,\infty). \tag{33}$$

Letting $s \to 0$ in (33) gives us

$$v(x,t+\tau) \leq m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

The last inequality holds for any $\tau > 0$, so we obtain, after $\tau \to 0$,

$$v(x,t) \leq m_\varepsilon \lambda + u_\varepsilon(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

Finally, passing $\lambda \to 0$ yields conclusion (30), or we get Theorem 8. □

Remark 9. We denote by $v^{\Omega}_{\max}$, the maximal VSS of equation (1) in $\Omega \times (0,\infty)$. By the construction, the sequence $\{v^{B_R}_{\max}\}_{R>0}$ is nondecreasing. Note that this sequence is also bounded by $z(t)$. Thus,

$$v^{B_R}_{\max}(x,t) \uparrow W(x,t), \tag{34}$$

for any $(x,t) \in \mathbb{R}^N \times (0,\infty)$, as $R \to \infty$. It is not difficult to verify that W is a self-similar VSS of the Cauchy problem associated with (1) in $\mathbb{R}^N \times (0,\infty)$.

Now, we complete the proof of Theorem 1.

Since W and V are two self-similar solutions of the Cauchy equation (1), they must satisfy (7). It follows from the uniqueness result of (7) (see [15]) that

$$W = V, \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{35}$$

Next, we claim that for any $c > 0$,

$$v^{B_R}_c(x,t) \leq v^{\Omega}_c(x,t) + m_\varepsilon \lambda e^{Ct}, \quad \text{in } B_R \times (0,\infty), \tag{36}$$

for any $R > 0$ large enough such that $\Omega \subset\subset B(0,R)$, and m_ε is in (31). Recall that v^{Ω}_c (resp. $v^{B_R}_c$) is the unique solution of equation (1) with initial data $c\delta_0$ in $\Omega \times (0,\infty)$ (resp. $B(0,R) \times (0,\infty)$).

Indeed, we first observe that $(v^{\Omega}_c + m_\varepsilon \lambda e^{Ct})$ is a super-solution of equation (1) in $\Omega \times (0,\infty)$. Moreover, using the same analysis as (31) yields

$$v^{B_R}_c(x,t) \leq m_\varepsilon \lambda e^{Ct}, \quad \text{for } x \in B(0,R), |x| \geq \varepsilon/2, \text{and } t > 0. \tag{37}$$

Note that this inequality can be done by using a smoothing effect to the initial data and the uniqueness result in Lemma 6.

In particular, we get

$$v^{B_R}_c(x,t) \leq m_\varepsilon \lambda e^{Ct}, \quad \forall x \in \partial\Omega.$$

Thus, the comparison principle (after a smoothing effect to initial data and the uniqueness result) deduces

$$v^{B_R}_c \leq v^{\Omega}_c + m_\varepsilon \lambda e^{Ct}, \quad \text{in } \Omega \times (0,\infty). \tag{38}$$

Thus, claim (36) follows from (37) and (38).

Next, passing to the limit as $c \to \infty$ in (36) yields

$$v^{B_R}_{\min} \leq v^{\Omega}_{\min} + m_\varepsilon \lambda e^{Ct}, \quad \text{in } B_R \times (0,\infty). \tag{39}$$

Letting $R \to \infty$ in (39) deduces:

$$W = V \leq v^{\Omega}_{\min} + m_\varepsilon \lambda e^{Ct}, \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{40}$$

By combining (34) and (40), we obtain

$$v_{\min}^{\Omega} \leq v_{\max}^{\Omega} \leq W \leq v_{\min}^{\Omega} + m_{\varepsilon}\lambda e^{Ct}, \quad \text{in } \Omega \times (0,\infty). \tag{41}$$

Thanks to the L^1-contraction in (25), we have, for any $t > s > 0$,

$$\|S_1\big(v_{\max}^{\Omega} - v_{\min}^{\Omega}\big)(t)\|_{L^1(\Omega)} \leq \|S_1\big(v_{\max}^{\Omega} - v_{\min}^{\Omega}\big)(s)\|_{L^1(\Omega)} \leq \|\big(v_{\max}^{\Omega} - v_{\min}^{\Omega}\big)(s)\|_{L^1(\Omega)} \tag{42}$$

It follows from (41) and (42) that

$$\int_{\Omega} S_1\big(v_{\max}^{\Omega}(t) - v_{\min}^{\Omega}(t)\big)dx \leq \int_{\Omega} m_{\varepsilon}\lambda e^{Cs}dx = |\Omega| m_{\varepsilon}\lambda e^{Cs}.$$

Passing $s \to 0$ in the last inequality yields

$$\int_{\Omega} S_1\big(v_{\max}^{\Omega}(t) - v_{\min}^{\Omega}(t)\big)dx \leq |\Omega| m_{\varepsilon}\lambda.$$

Then, the uniqueness result follows when $\lambda \to 0$.

To end this part, we prove the short time behavior for the unique VSS u of equation (1).

Thanks to (41) and the uniqueness, we have

$$u(0,t) \leq W(0,t) = t^{\frac{-1}{q-1}} f(0) \leq u(0,t) + m_{\varepsilon}\lambda e^{Ct}, \quad \forall t > 0;$$

or

$$t^{\frac{1}{q-1}} u(0,t) \leq f(0) \leq t^{\frac{1}{q-1}} u(0,t) + \lambda m_{\varepsilon} t^{\frac{1}{q-1}} e^{Ct}. \tag{43}$$

The result follows by passing $t \to 0$ in (43).

Consequently, we have

Corollary 10. *Let u_L be the unique large solution of equation* (1)*. Then, $u_L(x,t)$ has the rate $t^{\frac{-1}{q-1}}$ as $t \to 0$ for any $x \in \mathcal{D}$.*

Proof. It is sufficient to show that the result holds for $x = 0 \in \mathcal{D}$. Let u be the unique VSS of equation (1). Then, we have

$$u(0,t) \leq u_L(0,t) \leq (q-1)^{\frac{-1}{q-1}} t^{\frac{-1}{q-1}}.$$

This inequality and (43) lead to the conclusion. □

Remark 11. Our argument in this paper can be applied to obtain the same results for the porous medium equation

$$\begin{cases} \partial_t u - \Delta(u^m) + u^q = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega \times (0,\infty); \end{cases}$$

see our forthcoming paper [8].

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