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MATHEMATICS

Homogenization of Variational Inequalities of Signorini Type for the *p*-Laplacian in Perforated Domains when $p \in (1, 2)$[1]

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Abstract—The asymptotic behavior, as $\varepsilon \to 0$, of the solution uε to a variational inequality with nonlinear constraints for the *p*-Laplacian in an ε-periodically perforated domain when $p \in (1, 2)$ is studied.

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Works [4, 6] are concerned with the investigation of the asymptotic behavior of the solution of the variational inequality for the *p*-Laplace operator, where $p \in [2, n)$ and ε-periodically perforated domain with nonlinear Robin type boundary condition. In the present work we investigate a similar homogenization problem for the *p*-Laplacian in the case when $p \in (1, 2)$. It is known (see [2]) that for this values of *p* the considered problems describe the motion of non-Newtonian fluids. This type of diffusion is also used to describe certain problems of Newtonian fluids in turbulent regime (see, e.g., [3]). The operator also has some interest in the context on non-linear elasticity.

Let Ω be a bounded domain in ?n, n ≥ 3, with a smooth boundary ∂Ω. Denote $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ and let G0 be the unit ball centered at the origin. For δ > 0 and a given set B ⊂ ?n we define δB = {x|δ–1x ∈ B}. We also define, for j ∈ ?n, G_j^ε = aεG0 + εj,

$$\tilde{\Omega}_\varepsilon = \{x \in \Omega \mid \rho(x, \partial\Omega) > 2\varepsilon\}, \quad G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j$$

(where 0 < ε ? 1), aε = C0εα, $\alpha = \frac{n}{n-p}$ and

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \bar{\tilde{\Omega}}_\varepsilon \neq \phi\}.$$

It is easy to check that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, where $d > 0$ is a constant. Finally, let us define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, $j \in \Upsilon_\varepsilon$ (where we point out that $\overline{G_\varepsilon^j} \subset Y_\varepsilon^j$ and that the center of the ball G_ε^j coincides with the center of Y_ε^j) and

$$\Omega_\varepsilon = \Omega \backslash \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cap S_\varepsilon.$$

In this setting we consider the following nonlinear diffusion problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $p \in (1, 2)$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$ and with ν the outward unit normal to S_ε and $\gamma = \alpha(p-1)$, $f \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, and σ the following maximal monotone graph

$$\sigma(\lambda) = \begin{cases} \sigma_0(\lambda), & \lambda > 0, \\ (-\infty, 0], & \lambda = 0, \\ \phi, & \lambda < 0, \end{cases} \tag{2}$$

where $\sigma_0 \in C^1(\mathbb{R})$, $\sigma_0(0) = 0$, $\sigma_0'(\lambda) \geq k_1 > 0$ and k_1 is a constant.

We note that boundary value problem (1) with a function such as σ(λ) in the boundary condition corresponds to the problem with the one-sided restrictions, i.e., Signorini type problem

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$$\begin{cases} u_\varepsilon \geq 0, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma_0(u_\varepsilon) \geq 0 \text{ and} \\ u_\varepsilon(\partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma_0(u_\varepsilon)) = 0, \quad \text{on} \quad S_\varepsilon. \end{cases}$$

Let us define the following functions

$$\hat{\psi}(\lambda) = \int_0^\lambda \sigma_0(\tau)d\tau, \tag{3}$$

$$\psi(\lambda) = \begin{cases} \hat{\psi}(\lambda), & \lambda \geq 0, \\ +\infty, & \lambda < 0. \end{cases} \tag{4}$$

This convex l.s.c. function ψ has σ as its sub differential, in the sense that

$$\begin{gathered} \psi(\lambda) - \psi(\mu) \leq \xi(\lambda - \mu), \\ \forall \lambda, \mu \in \mathbb{R}, \quad \xi \in \sigma(\lambda). \end{gathered} \tag{5}$$

This is typically denoted $\sigma = \partial\psi$. The weak solution of the problem (1) is defined as a function

$$u_\varepsilon \in K_\varepsilon = \{g \in W^{1,p}(\Omega_\varepsilon, \partial\Omega) : g \geq 0 \text{ a.e. on } S_\varepsilon\},$$

satisfying the integral inequality

$$\begin{gathered} \int_{\Omega_p} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\hat{\psi}(\phi) - \hat{\psi}(u_\varepsilon))ds \\ \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon)dx \end{gathered} \tag{6}$$

for any arbitrary function $\phi \in K_\varepsilon$.

Let $H(\lambda)$ be the solution of the functional inclusion

$$B_0|H|^{p-2}H \in \sigma(\lambda - H), \tag{7}$$

where $B_0 > 0$ is a constant. In the case of σ as in (2), inclusion (7) has a unique solution of the form

$$H(\lambda) = \begin{cases} H_0(\lambda), & \lambda > 0, \\ \lambda, & \lambda \leq 0, \end{cases} \tag{8}$$

where $H_0(\lambda)$ is the solution of the functional equation

$$B_0|H_0|^{p-2}H_0 = \sigma_0(\lambda - H_0). \tag{9}$$

Note that $H_0(0) = 0$. If we decompose $u = u^+ - u^-$ where u^+, $u^- \geq 0$ are the positive and negative parts of u then we have

$$\begin{gathered} H(u) = H_0(u^+) - u^-, \\ |H(u)|^{p-2}H(u) = |H_0(u^+)|^{p-2}H_0(u^+) - |u^-|^{p-2}u^-. \end{gathered}$$

Also,

Lemma 1. *For every $s \neq 0$, $0 < H'(s) \leq 1$. In particular, H is a Lipschitz continuous function.*

Proof. If $H_0(s) \leq 0$, since $\sigma_0(0) = 0$, $\sigma_0'(s) \geq k_1 > 0$

$$0 \geq B_0|H_0(s)|^{p-2}H_0(s) = \sigma_0(s - H_0(s)) \geq k_1(s - H_0(s)),$$

then $s \leq 0$. So, for $s > 0$, $H(s) = H_0(s) > 0$. Hence, for $s > 0$, $B_0H_0^{p-1}(s) = \sigma_0(s - H_0(s))$. Differentiating with respect to s, for $s > 0$

$$B_0(p-1)H_0^{p-2}(s) = \sigma_0'(s - H_0(s))(1 - H_0'(s)),$$

$$H_0'(s) = \frac{\sigma_0'(s - H_0(s))}{B_0(p-1)H_0^{p-2}(s) + \sigma_0'(s - H_0(s))}.$$

It follows that $0 < H'(s) \leq 1$. for $s > 0$. Since, for $s < 0$, $H(s) = s$ we finish the proof.

Remark 1. *If σ is given by (2), $H(s) \leq s$ for all $s \in \mathbb{R}$. For $s \leq 0$ this is obvious and for $s > 0$ we point out that $H(0) = 0$ and $H'(s) \leq 1$.*

Let $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be a $W^{1,p}$—extension of u_ε, that satisfies the following inequalities

$$\begin{gathered} \|\tilde{u}_\varepsilon\|_{W^{1,p}(\Omega)} \leq K\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)}, \\ \|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)} \leq K\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}. \end{gathered} \tag{10}$$

Considering (6) it is easy to check that

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq K.$$

Hence, using this inequality and estimations (10) we conclude that there exists a subsequence (denote as the original sequence), such that as $\varepsilon \to 0$

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p}(\Omega). \tag{11}$$

We will use systematically that the function

$$\Phi_p\colon L^p(\Omega)^N \to L^{p'}(\Omega)^N, \quad \xi \mapsto |\xi|^{p-2}\xi \tag{12}$$

is continuous in the strong topology (see [8]).

The following theorem gives us the description of function u. What is remarkable in it is that a sequence of variational inequalities converges to the solution of a single-valued quasilinear equation with a Lipschitz absorption term.

Theorem 1. *Let $\alpha = \dfrac{n}{n-p}$, $\gamma = \alpha(p-1)$, $p \in (1, 2)$, $n \geq 3$. Suppose that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the weak solution of the problem (1), where $\sigma(\lambda)$ is given by formula (2) and $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ is a $W^{1,p}$-extension of u_ε satisfying (10). Then, the function u defined in (12) is a weak solution of the following problem*

$$\begin{cases} -\Delta_p u + \mathcal{A}(n,p)|H(u)|^{p-2}H(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{13}$$

where $H(\lambda)$ is given by formula (8), $H_0(\lambda)$ is a solution of the Eq. (9) for $B_0 = \left(\dfrac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$, $\mathcal{A}(n, p) = \left(\dfrac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

We will use the following auxiliary function W_ε defined as follows

$$W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \quad j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in \Upsilon_\varepsilon} T_\varepsilon^j, \end{cases}$$

where w_ε^j is the solution of the following boundary value problem

$$\Delta_p w_\varepsilon^j = 0, \quad x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j},$$
$$w_\varepsilon^j = 1, \quad x \in \partial G_\varepsilon^j,$$
$$w_\varepsilon^j = 0, \quad x \in \partial T_\varepsilon^j$$

and T_ε^j denotes the ball of radius $\varepsilon/4$ which center coincides with the center of cube Y_ε^j. It is easy to show that

$$\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^q dx \le K \varepsilon^{n(p-q)/(n-p)}, \tag{14}$$

where $1 \le q \le p$. $W_\varepsilon \to 0$ in $W_0^{1,q}(\Omega)$ at $\varepsilon \to 0$, for $q < p$. Also, the $W_0^{1,p}$ norm is bounded, so it has a weakly convergent subsequence. The limit of that sequence must be its $W_0^{1,q}$ limit, hence $W_\varepsilon \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$.

Proof of Theorem 1. Taking into account (3) and using the monotonicity of function $|\lambda|^{p-2}\lambda$ for $p > 1$, from inequality (6) we derive that u_ε satisfies the following inequality

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla(\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_0(\phi)(\phi - u_s) ds \ge \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \tag{15}$$

for any function $\phi \in K_\varepsilon$.

Let $v \in C_0^\infty(\Omega)$ and let us consider $\phi = v - W_\varepsilon H(v)$ as a test function, where $H(\lambda)$ is defined by (8). Notice that $\phi|_{S_\varepsilon} = v - H(v) \ge 0$ due to Remark 1, and hence $\phi \in K_\varepsilon$. Let us define $\psi_\varepsilon = \phi - \tilde{u}_\varepsilon$, and rewrite (15) as $I_\varepsilon^1 + I_\varepsilon^2 \ge I_\varepsilon^3$ where

$$I_\varepsilon^1 = \int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon H(v))|^{p-2} \nabla(v - W_\varepsilon H(v)) \nabla \psi_\varepsilon dx,$$

$$I_\varepsilon^2 = \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(v - H(v)) \psi_\varepsilon ds, \quad I_\varepsilon^3 = \int_{\Omega_\varepsilon} f \psi_\varepsilon dx.$$

Let us define

$$\xi_1 = \Phi_p(\nabla(v - W_\varepsilon H(v))),$$
$$\xi_2 = \Phi_p(\nabla v), \quad \xi_3 = \Phi_p(\nabla(W_\varepsilon H(v))).$$

We write $I_\varepsilon^1 = J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3$, where

$$J_\varepsilon^1 = \int_{\Omega_\varepsilon} (\xi_1 - (\xi_2 - \xi_3)) \cdot \nabla \psi_\varepsilon dx,$$

$$J_\varepsilon^2 = \int_{\Omega_\varepsilon} \xi_2 \cdot \nabla \psi_\varepsilon dx, \quad J_\varepsilon^3 = -\int_{\Omega_\varepsilon} \xi_3 \cdot \nabla \psi_\varepsilon dx.$$

Lemma 3 below implies the inequality $|\xi_1 - (\xi_2 - \xi_3)| \le C(|\xi_2||\xi_3|)^{\frac{p-1}{2}}$. Hence, we can write

$$|J_\varepsilon^1| \le K \int_{\Omega_\varepsilon} |\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} \times (|\nabla(W_\varepsilon H(v))| + |\nabla v| + |\nabla u_\varepsilon|) dx$$
$$\le K \int_{\Omega_\varepsilon} \Big(|\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p+1}{2}} + |\nabla v|^{\frac{p+1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}}$$
$$+ |\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} |\nabla u_\varepsilon|\Big) dx$$
$$\le K \int_{\Omega_\varepsilon} \Big(|\nabla W_\varepsilon|^{\frac{p+1}{2}} + |\nabla u_\varepsilon| |\nabla W_\varepsilon|^{\frac{p-1}{2}}\Big) dx.$$

Applying Hölder's inequality for p on the second term

$$|J_\varepsilon^1| \le K \left\{ \|\nabla W_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega)}^{\frac{2}{p+1}} + \|\nabla u_\varepsilon\|_{L^p(\Omega)} \left(\int_\Omega |\nabla W_\varepsilon|^{\frac{p}{2}} dx \right)^{\frac{p-1}{p}} \right\} \to 0$$

as $\varepsilon \to 0$, by taking into account that $\frac{p}{2}, \frac{p+1}{2} < p$ and estimate (14). Moreover, convergence (11) implies

$$\lim_{\varepsilon \to 0} J_\varepsilon^2 = \int_\Omega |\nabla v|^{p-2} \nabla v \nabla(v - u) dx. \tag{16}$$

Consider J_ε^3. Splitting $\nabla(W_\varepsilon H(v)) = W_\varepsilon \nabla H(v) + H(v) \nabla W_\varepsilon$, since $W_\varepsilon \nabla H(v) \to 0$ in $L^p(\Omega)^N$, Φ_p is continuous and ψ_ε is bounded in $W^{1,p}$ we have that

$$\lim_{\varepsilon \to 0} J_\varepsilon^3 = -\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Phi_p(H(v) \nabla W_\varepsilon) \cdot \nabla \psi_\varepsilon dx.$$

On the other hand, it is easy to check that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon\cdot\nabla(|H(v)|^{p-2}H(v)\psi_\varepsilon)$$

$$=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Phi_p(H(v)\nabla W_\varepsilon)\cdot\nabla\psi_\varepsilon dx.$$

Hence

$$J_\varepsilon^3=-\int_{\Omega_\varepsilon}|\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon\cdot\nabla[|H(v)|^{p-2}\times H(v)(v-W_\varepsilon H(v)-u_\varepsilon)]dx+\alpha_\varepsilon,$$

where $\alpha_\varepsilon\to 0$ as $\varepsilon\to 0$. From Green's formula we derive that $J_\varepsilon^3=K_\varepsilon^1+K_\varepsilon^2$

$$K_\varepsilon^1=-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}\partial_{\nu_p}w_\varepsilon^j|H(v)|^{p-2}H(v)(v-H(v)-u_\varepsilon)ds,$$

$$K_\varepsilon^2=-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}\partial_{\nu_p}w_\varepsilon^j|H(v)|^{p-2}H(v)(v-u_\varepsilon)ds+\alpha_\varepsilon.$$

Taking into account that $\gamma=\alpha(p-1)$, $u_\varepsilon\ge 0$ on S_ε and

$$\partial_{\nu_p}w_\varepsilon^j|_{\partial G_\varepsilon^j}=\frac{(n-p)\varepsilon^{-\frac{n}{n-p}}}{(p-1)C_0(1-\kappa_\varepsilon)},\tag{17}$$

$$\partial_{\nu_p}w_\varepsilon^j|_{\partial T_\varepsilon^j}=\frac{(n-p)2^{2(n-1)/(p-1)}C_0^{(n-p)/(p-1)}\varepsilon^{1/(p-1)}}{(p-1)(1-\kappa_\varepsilon)},\tag{18}$$

where $\kappa_\varepsilon\to 0$ as $\varepsilon\to 0$ and that $\gamma=\alpha(p-1)$ we obtain, taking into account (9) that

$$K_\varepsilon^1+I_\varepsilon^2=\varepsilon^{-\gamma}\int_{S_\varepsilon}[\sigma(v-H(v))-B_0|H(v)|^{p-2}H(v)]\times(v-H(v)-u_\varepsilon)ds+\beta_\varepsilon$$

$$=\varepsilon^{-\gamma}\int_{S_\varepsilon}[|v_-|^{p-2}v_-](v_+-H(v_+)-u_\varepsilon)ds+\beta_\varepsilon$$

$$=\varepsilon^{-\gamma}\int_{S_\varepsilon}[|v_-|^{p-2}v_-](-u_\varepsilon)ds+\beta_\varepsilon\le\beta_\varepsilon,$$

where $\beta_\varepsilon\to 0$ as $\varepsilon\to 0$ since $u_\varepsilon\ge 0$ on S_ε.

We will use the next lemma to pass to the limit in K_ε^1 (see [10]).

Lemma 2. *Let* $p>1$, $h_\varepsilon\in H_0^1(\Omega)$ *and* $h_\varepsilon\rightharpoonup u_0$ *as* $\varepsilon\to 0$ *in* $H_0^1(\Omega)$, *then*

$$\left|2^{2(n-1)}\varepsilon\sum_{j=1}^{N_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}h_\varepsilon dS-\omega_n\int_\Omega h_0dx\right|\to 0,\quad\varepsilon\to 0,$$

where ω_n *is the surface area of the unit sphere in* $\mathbb{R}^n$.

Due Lemma 2 we deduce that

$$K_\varepsilon^1\to\mathcal{A}(n,p)\int_\Omega|H(v)|^{p-2}H(v)(v-u)dx,\quad\text{as}\quad\varepsilon\to 0.\tag{19}$$

where $\mathcal{A}(n,p)=\left(\frac{n-p}{p-1}\right)^{p-1}C_0^{n-p}\omega_n$. From (15)–(19) we derive that u satisfies following inequality

$$\int_\Omega|\nabla u|^{p-2}\nabla v\nabla(v-u)dx+\mathcal{A}(n,p)\int_\Omega|H(v)|^{p-2}H(v)|^{p-2}(v-u)dx\ge\int_\Omega f(v-u)dx.\tag{20}$$

This inequality implies that u is a weak solution of the problem (13).

In the next theorem we will prove the convergence in the norm of space $W_0^{1,p}(\Omega_\varepsilon)$ of the solution of the problem (1) with a corrector to the solution of the homogenized problem.

Theorem 2. *Let* $\alpha=\frac{n}{n-p}$, $\gamma=\alpha(p-1)$, $p\in(1,2)$, $n\ge 3$. *Suppose that* $u_\varepsilon\in W^{1,p}(\Omega_\varepsilon)$ *is a weak solution of the problem* (1) *and u is a weak solution of the problem* (13) *possessing the additional smoothness* $u\in W^{1,\infty}(\Omega)$. *Then*

$$\|\nabla(u_\varepsilon+W_\varepsilon H(u)-u)\|_{L^p(\Omega^\varepsilon)}\to 0,\quad as\quad\varepsilon\to 0\tag{21}$$

In particular, since $W_\varepsilon\to 0$ *in* $W^{1,q}(\Omega)$ *for* $q<p$, *we have, for all* $q<p$

$$\|\nabla(u_\varepsilon-u)\|_{L^q(\Omega_\varepsilon)}\to 0,\quad\varepsilon\to 0$$

Remark 2. *Under some smoothness hypothesis of* σ_0 *and* f, $u\in W^{1,\infty}(\Omega)$ *is often achieve. See* [1, 5, 7, 9].

Proof of Theorem 2. Inequality (6) implies that

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla(\phi-u_\varepsilon)dx+\int_{S_\varepsilon}\sigma_0(\phi)(\phi-u_\varepsilon)ds\ge\int_{\Omega_\varepsilon}f(\phi-u_\varepsilon)dx.\tag{22}$$

In inequality (22) we substitute $\phi=u-W_\varepsilon H(u)$ and in the weak formulation of problem (13), namely,

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla vdx+\mathcal{A}(n,p)\int_\Omega|H(u)|^{p-2}H(u)vdx=\int_\Omega fvdx$$

we take, as a test function, $v = -\Psi_\varepsilon$, where $\Psi_\varepsilon = u - W_\varepsilon H(u) - \tilde u_\varepsilon$ and $\tilde u_\varepsilon$ is a $W^{1,p}$-extension u_ε on Ω. Let us define,

$$\xi_1^\varepsilon = \Phi_p(\nabla u_\varepsilon), \quad \xi_2 = \Phi_p(\nabla u).$$

By adding (22) and the integral identity for u, we obtain $I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon \geq I_4^\varepsilon$, where

$$I_1^\varepsilon = \int_{\Omega_\varepsilon} (\xi_1^\varepsilon - \xi_2)\cdot\nabla\Psi_\varepsilon dx, \quad I_2^\varepsilon \int_{G_\varepsilon} \xi_2\cdot\nabla\Psi_\varepsilon dx,$$

$$I_3^\varepsilon = \varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(u - H(u))\Psi_\varepsilon dx$$

$$-\mathcal{A}(n,p)\int_\Omega |H(u)|^{p-2}H(u)\Psi_\varepsilon dx,$$

$$I_4^\varepsilon = \int_{G_\varepsilon} f\Psi_\varepsilon dx.$$

It is clear that $I_2^\varepsilon, I_4^\varepsilon \to 0$ as $\varepsilon \to 0$ due to weak convergence and the fact that $|G_\varepsilon| \to 0$. We define

$$\xi_3^\varepsilon = \Phi_p(\nabla(W_\varepsilon H(u))), \quad \xi_4^\varepsilon = \Phi_p(\nabla(u - W_\varepsilon H(u))).$$

We decompose $I_1^\varepsilon = J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon$, where

$$J_1^\varepsilon = \int_{\Omega_\varepsilon}(\xi_1^\varepsilon - \xi_4^\varepsilon)\cdot\nabla\Psi_\varepsilon dx,$$

$$J_2^\varepsilon = \int_{\Omega_\varepsilon}(\xi_4^\varepsilon - \xi_2 + \xi_3^\varepsilon)\cdot\nabla\Psi_\varepsilon dx, \quad J_3^\varepsilon = -\int_{\Omega_\varepsilon}\xi_3^\varepsilon\cdot\nabla\Psi_\varepsilon dx.$$

Applying Lemma 3 we have that

$$|J_2^\varepsilon| \leq C\int_{\Omega_\varepsilon}|\nabla u|^{\frac{p-1}{2}}|\nabla(W_\varepsilon H(u))|^{\frac{p-1}{2}}$$

$$\times|\nabla(u - W_\varepsilon H(u) - u_\varepsilon)|dx \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, we can write

$$J_3^\varepsilon = -\int_{\Omega_\varepsilon}|\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon\nabla(|H(u)|^{p-2}H(u)\Psi_\varepsilon)\,dx + \delta_\varepsilon$$

$$= -\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}\partial_{\nu_p}w_\varepsilon^j|H(u)|^{p-2}H(u)\Psi_\varepsilon ds$$

$$-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}\partial_{\nu_p}w_\varepsilon^j|H(u)|^{p-2}H(u)\Psi_\varepsilon ds + \delta_\varepsilon,$$

where $\delta_\varepsilon \to 0$. Therefore, $J_3^\varepsilon + I_3^\varepsilon \to 0$ as $\varepsilon \to 0$, due to the explicit expression of $\partial_{\nu_p}w_\varepsilon^j$ and H. So, finally, $J_1^\varepsilon \to 0$. We will use the following inequality (see [2]). For all $1 < p < 2$ and $\xi, \eta \in \mathbb{R}^n$

$$C\frac{|\xi-\eta|^2}{|\xi|^{2-p}+|\eta|^{2-p}} \leq (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta). \tag{23}$$

Hence, for $\xi = \xi_1^\varepsilon$ and $\xi = \xi_4^\varepsilon$, we deduce that

$$C\int_{\Omega_\varepsilon}\frac{|\nabla(u_\varepsilon - u + W_\varepsilon H(u))|^2}{|\nabla u_\varepsilon|^{2-p} + |\nabla(u - W_\varepsilon H(u))|^{2-p}}dx$$

$$\leq \int_{\Omega_\varepsilon}(\xi_1^\varepsilon - \xi_4^\varepsilon)\cdot\nabla\Psi_\varepsilon dx = J_1^\varepsilon \to 0.$$

as $\varepsilon \to 0$. Using Holder's inequality (21), which concludes the proof.

APPENDIX A

AN AUXILIARY LEMMA

Lemma 3. *Let $p \in (1,2)$, $n \geq 2$. Then there exists constant $C = C(n,p)$ such that for all $a, b \in \mathbb{R}^n$ following inequality is valid*

$$||a-b|^{p-2}(a-b) - (|a|^{p-2}a - |b|^{p-2}b)| \leq C(|a||b|)^{\frac{p-1}{2}}.$$

Proof. Without loss of generality we can assume that $|a| \geq |b| > 0$. Let $u = \frac{a}{|a|}$, $v = \frac{b}{|b|}$, $|u| = |v| = 1$, $\xi = u\cdot v$, $\xi \in [-1, 1]$, $k = \frac{a}{|b|} \geq 1$. The desired inequality written in these new variables takes the following form

$$||ku - v|^{p-2}(ku - v) - (k^{p-1}u - v)| \leq Ck^{(p-1)/2}.$$

By squaring this inequality we get

$$\Re(k,\xi) = (k^2 - 2k\xi + 1)^{p-1} + k^{2(p-1)} + 1 - 2k^{p-1}\xi$$
$$-2(k^2 - 2k\xi + 1)^{(p-2)/2}(k^p + 1 - k\xi - k^{p-1}\xi) \leq C^2k^{p-1}.$$

Consider function

$$f(x,\xi) = \frac{\Re(k,\xi)}{k^{p-1}} = k^{p-1}\left(1 - \frac{2\xi}{k} + \frac{1}{k^2}\right)^{p-1}$$
$$+ k^{p-1} + k^{1-p} - 2\xi - 2\left(1 - \frac{2\xi}{k} + \frac{1}{k^2}\right)^{(p-2)/2}$$
$$\times(k^{p-1} - \xi - k^{p-2}\xi + k^{-1}).$$

Decomposing functions $(1 - 2\xi/k + 1/k^2)^\beta$ for $\beta = p-1, (p-2)/2$ in Taylor series as $k \to \infty$, $k > 1 + \sqrt{2}$, and identifying the coefficients of corresponding degrees, we obtain

$$f(k,\xi) = \alpha k^{1-p} + \beta k^{p-2} + o\left(\frac{1}{k}\right),$$

where α and β depend only on p and ξ. Hence, $f(k, \xi) \to 0$ as $k \to \infty$. Thus there exists $k_1 > 1 + \sqrt{2}$ such that $f(k, \xi) < 1$ for all $k > k_1$, $|\xi| \le 1$. It's easy to show that function $f(k, \xi)$ is continuous on the set $D = \{(k, \xi)|1 \le k \le k_1, |\xi| \le 1\}$. So there exists a positive constant M that depends on p such that $\max_{(k,\xi)\in D} |f(k, \xi| \le M$. Hence, function $|f|$ is bounded by $\max(M, 1)$ for all permissible k and ξ.

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