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CHANGE OF HOMOGENIZED ABSORPTION TERM IN DIFFUSION PROCESSES WITH REACTION ON THE BOUNDARY OF PERIODICALLY DISTRIBUTED ASYMMETRIC PARTICLES OF CRITICAL SIZE

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Abstract. The main objective of this article is to get a complete characterization of the homogenized global absorption term, and to give a rigorous proof of the convergence, in a class of diffusion processes with a reaction on the boundary of periodically "microscopic" distributed particles (or holes) given through a nonlinear microscopic reaction (i.e. under nonlinear Robin microscopic boundary conditions). We introduce new techniques to deal with the case of non necessarily symmetric particles (or holes) of critical size which leads to important changes in the qualitative global homogenized reaction (such as it happens in many problems of the Nanotechnology). Here we shall merely assume that the particles (or holes) G_ε^j, in the n-dimensional space, are diffeomorphic to a ball (of diameter $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$ for some $C_0 > 0$). To define the corresponding "new strange term" we introduce a one-parametric family of auxiliary external problems associated to canonical cellular problem associated to the prescribed asymmetric geometry G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$ (which is assumed to be merely a Hölder continuous function). We construct the limit homogenized problem and prove that it is a well-posed global problem, showing also the rigorous convergence of solutions, as $\varepsilon \to 0$, in suitable functional spaces. This improves many previous papers in the literature dealing with symmetric particles of critical size.

1. Introduction

It is well-known that the asymptotic behaviour of the solution of many relevant diffusion processes with reaction on the boundary of periodically "microscopic" distributed particles (or holes) is described through the solution of a global reaction-diffusion problem in which the global reaction term (usually an absorption term if the microscopic reactions are given by monotone non-decreasing functions) maintains the same structural properties as the microscopic reaction (see, for instance, [3]) and its many references to previous results in the literature).

A certain critical size of the "microscopic particles" may be responsible of a change in the nature of the homogenized global absorption term, with respect to

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the structural assumptions on the microscopic boundary reaction kinetic. It seems that the first result in that direction was presented in the pioneering paper by V. Marchenko and E. Hruslov [15] dealing with microscopic non-homogeneous Neumann boundary condition (see also the study made by E. Hruslov concerning linear microscopic Robin boundary conditions in [12, 11]). A –perhaps– more popular presentation of the appearance of some "strange term" was due to D. Cioranescu and F.Murat [2] dealing with microscopic Dirichlet boundary conditions (see also [13]).

This change of behavior from the microscopic reaction to the global homogeneized reaction term is one of the characteristics of the nanotechnological effects (see, e.g., [20]) and it does not appear for particles of bigger size (relative to their repetition) than the critical scale (see, e.g., [6] and the references therein). The total identification of the new or "strange" reaction term is an important task which was considered by many authors under different technical assumptions. In the case of nonlinear microscopic boundary reactions the first result in the literature was the 1997 paper by Goncharenko [10] (see also the precedent paper [13]). The identification (and the rigorous proof of the convergence in the homogenization process) requires to assume that the particles (or holes) are symmetric balls of diameter $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$, for some $C_0 > 0$. Many other researches were developed for different problems concerning critical sized balls (see [22, 18, 5] and the references therein). Recently, a unifying study concerning the homogenization for particles (or holes) given by symmetric balls of critical order was presented in [7]: the treatment was extended to a microscopic reaction given by a general maximal monotone graph which allows to include, as special problems, the cases of Dirichlet or nonlinear Robin microscopic boundary conditions. The case of particles of general shape when $n = 2$ was studied in [19], with the limit behaviour being similar to the case of spherical inclusions and $n \geq 2$.

The main task of this paper is to get a complete characterization of the homogenized global absorption term in the class of problems given through a nonlinear microscopic reaction (i.e. under nonlinear Robin microscopic boundary conditions) and for non necessarily symmetric particles (or holes). Here we will merely assume that the particles (or holes) G_ε^j are a rescaled version of a set G_0, diffeomorphic to a ball (where the scaling factor is $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$ for some $C_0 > 0$). To define the corresponding new "strange term" we introduce a one-parametric family of auxiliary external problems associated to canonic cellular problem, which play the role of a "nonlinear capacity" of G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$ (which is assumed to be merely a Hölder continuous function). We construct the limit homogenized problem and prove that it is well-posed global problem, showing also the rigorous convergence of solutions, as $\varepsilon \to 0$, in suitable functional spaces.

2. Statement of main results

Let Ω be a bounded domain in $\mathbb{R}^n$ $n \geq 3$ with a piecewise smooth boundary $\partial\Omega$. The case $n = 2$ requires some technical modifications which will not be presented here. Let G_0 be a domain in $Y = (-\frac{1}{2}, \frac{1}{2})^n$, and $\overline{G_0}$ be a compact set diffeomorphic to a ball. Let $C_0, \varepsilon > 0$ and set

$$a_\varepsilon = C_0\varepsilon^\alpha \quad \text{for } \alpha = \frac{n}{n-2}. \tag{2.1}$$

For $\delta > 0$ and B a set let $\delta B = \{x \,|\, \delta^{-1}x \in B\}$. Assume that ε is small enough so that $a_\varepsilon G_0 \subset \varepsilon Y$. We define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$. For $j \in \mathbb{Z}^n$ we define

$$P_\varepsilon^j = \varepsilon j, \quad Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y, \quad G_\varepsilon^j = P_\varepsilon^j + a_\varepsilon G_0.$$

We define the set of admissible indexes:

$$\Upsilon_\varepsilon = \big\{ j \in \mathbb{Z}^n : G_\varepsilon^j \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset \big\}.$$

Notice that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ where $d > 0$ is a constant. Our problem will be set in the following domain:

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j, \quad \Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon$$

Finally, let

$$\partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega, \quad S_\varepsilon = \partial G_\varepsilon.$$

We consider the following boundary value problem in the domain Ω_ε

$$\begin{gathered} -\Delta u_\varepsilon = f, \quad x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = 0, \quad x \in S_\varepsilon, \\ u_\varepsilon = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.2}$$

where $\gamma = \alpha = \frac{n}{n-2}$, $f \in L^2(\Omega)$, ν is the unit outward normal vector to the boundary S_ε, $\partial_\nu u$ is the normal derivative of u. Furthermore, we suppose that the function $\sigma : \mathbb{R} \to \mathbb{R}$, describing the microscopic nonlinear Neumann boundary condition, is nondecreasing, $\sigma(0) = 0$, and there exist constants k_1, k_2 such that

$$|\sigma(s) - \sigma(t)| \le k_1|s-t|^\alpha + k_2|s-t| \quad \forall s,t \in \mathbb{R}, \quad \text{for some } 0 < \alpha \le 1. \tag{2.3}$$

Remark 2.1. Condition (2.3) means that σ is locally Hölder continuous, but it is only sublinear towards infinity. This condition is weaker than $u \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ or σ Lipschitz, that correspond, respectively, to $k_2 = 0$ and $k_1 = 0$.

Remark 2.2. Condition (2.3) on σ is a purely technical requirement. This kind of regularity can probably be improved. In particular, as shown in [4, 7] the kind of homogenization techniques and result that will be presented later can be expect for any maximal monotone graph σ.

For any prescribed set G_0, as before, and for any given $u \in \mathbb{R}$, we define $\widehat{w}(y; G_0,u)$, for $y \in \mathbb{R}^n \setminus G_0$, as the solution of the following one-parametric family of auxiliary external problems associated to the prescribed asymmetric geometry G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$:

$$\begin{gathered} -\Delta_y \widehat{w} = 0 \quad \text{if } y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_y}\widehat{w} - C_0\sigma(u - \widehat{w}) = 0, \quad \text{if } y \in \partial G_0, \\ \widehat{w} \to 0 \quad \text{as } |y| \to \infty. \end{gathered} \tag{2.4}$$

We will prove in Section 4 that the above auxiliary external problems are well defined and, in particular, there exists a unique solution $\widehat{w}(y; G_0,u) \in H^1(\mathbb{R}^n \setminus \overline{G_0})$, for any $u \in \mathbb{R}$. Concerning the corresponding "new strange term", for any prescribed asymmetric set G_0, as before, and for any given $u \in \mathbb{R}$ we introduce the following definition.

Definition 2.3. Given G_0 we define $H_{G_0} : \mathbb{R} \to \mathbb{R}$ by means of the identity

$$\begin{aligned} H_{G_0}(u) &:= \int_{\partial G_0} \partial_{\nu_y} \widehat{w}(y; G_0,u)\, \mathrm{d}S_y \\ &= C_0 \int_{\partial G_0} \sigma(u - \widehat{w}(y; G_0,u))\, \mathrm{d}S_y, \quad \text{for any } u \in \mathbb{R}. \end{aligned} \tag{2.5}$$

Remark 2.4. Let $G_0 = B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$. We can find the solution of problem (2.4) in the form $\widehat{w}(y; G_0,u) = \frac{\mathcal{H}(u)}{|y|^{n-2}}$, where, in this case, $\mathcal{H}(u)$ is proportional to $H_{B_1(0)}(u)$. We can compute that

$$\begin{aligned} H_{G_0}(u) &= \int_{\partial G_0} \partial_\nu \widehat{w}(u, y)\, \mathrm{d}S_y \\ &= \int_{\partial G_0} (n-2) H_{G_0}(u)\, \mathrm{d}S_y \\ &= (n-2)\mathcal{H}(u)\omega(n), \end{aligned}$$

where $\omega(n)$ is the area of the unit sphere. Hence, due to (2.5), $\mathcal{H}(u)$ is the unique solution of the following functional equation

$$(n-2)\mathcal{H}(u) = C_0 \sigma(u - \mathcal{H}(u)). \tag{2.6}$$

In this case, it is easy to prove that H is nonexpansive (Lipschitz continuous with constant 1). This equation has been considered in many papers (see [7] and the references therein).

We shall prove several results on the regularity and monotonicity of the homogenized reaction $H_{G_0}(u)$ in the next section. Concerning the convergence as $\varepsilon \to 0$ the following statement collects some of the more relevant aspects of this process:

Theorem 2.5. *Let $n \geq 3$, $a_\varepsilon = C_0 \varepsilon^{-\gamma}$, $\gamma = \frac{n}{n-2}$, σ a nondecreasing function such that $\sigma(0) = 0$ and that satisfies (2.3). Let u_ε be the weak solution of (2.2). Then there exists an extension to $H_0^1(\Omega)$, still denoted by u_ε, such that $u_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$ as $\varepsilon \to 0$, where $u_0 \in H_0^1(\Omega)$ is the unique weak solution of*

$$\begin{gathered} -\Delta u_0 + C_0^{n-2} H_{G_0}(u_0) = f \quad \Omega, \\ u_0 = 0 \quad \partial\Omega. \end{gathered} \tag{2.7}$$

Remark 2.6. Since $|H_{G_0}(u)| \leq C(1 + |u|)$ it is clear that $H_{G_0}(u_0) \in L^2(\Omega)$.

3. On the ε-global problem

Some comments on the well-posednes and some a priori estimates concerning the ε-global problem (2.2), when the nondecreasing function $\sigma \in \mathcal{C}(\mathbb{R})$, $\sigma(0) = 0$ satisfies (2.3), are collected in this section. We start by introducing some notations:

Definition 3.1. Let U be an open set and $\Gamma \subset \partial\Omega$. We define the functional space

$$H^1(U, \Gamma) = \overline{\{f \in \mathcal{C}^\infty(U) : f|_\Gamma = 0\}}^{H^1(U)}.$$

Thanks to well-known results (see, e.g. the references given in [7]) there exists a unique weak solution of problem (2.2): i.e. $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is the unique function such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi\, \mathrm{d}x + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(u_\varepsilon)\varphi\, \mathrm{d}S = \int_{\Omega_\varepsilon} f\varphi\, \mathrm{d}x, \tag{3.1}$$

for every $\varphi \in H^1(\Omega_\varepsilon, \partial\Omega)$. As a matter of fact, in order to get a proof of the convergence of u_ε as $\varepsilon \to 0$, under the general assumption (2.3), it is useful to recall that, thanks to the monotonicity of $\sigma(u)$, we can write the weak formulation of (2.2) in the following equivalent way (for details see [6]):

$$\int_{\Omega_\varepsilon} \nabla\varphi \cdot \nabla(\varphi - u_\varepsilon)\,\mathrm{d}x + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(\varphi)(\varphi - u_\varepsilon) ds \geq \int_{\Omega_\varepsilon} f(\varphi - u_\varepsilon)\,\mathrm{d}x, \tag{3.2}$$

for every $\varphi \in H_0^1(\Omega)$.

Concerning some initial apriori estimates, we recall that we can work with $\widetilde{u}_\varepsilon \in H_0^1(\Omega)$ given as an extension of u_ε to Ω such that

$$\|\widetilde{u}_\varepsilon\|_{H^1(\Omega)} \leq K\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}, \quad \|\nabla\widetilde{u}_\varepsilon\|_{L^2(\Omega)} \leq K\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{3.3}$$

where K does not depend on ε. The construction of such an extension is given, e.g., in [17] (the $W^{1,p}$ equivalent, for $p \neq 2$, can be found in [18]).

Now, considering in the weak formulation (3.1) the test function $\varphi = u_\varepsilon$, and using the monotonicity of σ, we obtain

$$\|\nabla u_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2 \leq K. \tag{3.4}$$

where K does not depend on ε. From (3.4) we derive that there are a subsequence of $\widetilde{u}_\varepsilon$ (still denote by $\widetilde{u}_\varepsilon$) and $u_0 \in H_0^1(\Omega)$ such that, as $\varepsilon \to 0$, we have

$$\widetilde{u}_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega), \tag{3.5}$$

$$\widetilde{u}_\varepsilon \to u_0 \quad \text{strongly in } L^2(\Omega). \tag{3.6}$$

In Section 4 we characterize the limit function $u_0 \in H_0^1(\Omega)$.

4. On the regularity of the strange term

4.1. **Auxiliary function $\widehat{w}$.** The existence and regularity of solution in domains

$$\mathcal{O} = \mathbb{R}^n \setminus \overline{G_0} \tag{4.1}$$

which are commonly known as exterior domains, has been extensively studied (see, e.g., [9] and the references therein).

Based on the rate of convergence to 0 as $|y| \to +\infty$ we consider the space

$$\mathbb{X} = \big\{ w \in L^1_{loc}(\mathcal{O}) : \nabla w \in L^2(\mathcal{O}),\ w|_{\partial G_0} \in L^2(\partial G_0),\ |w| \leq \frac{K}{|y|^{n-2}} \big\} \tag{4.2}$$

It is a standard result, known as Weyl's lemma, that any harmonic function is smooth (of class $\mathcal{C}^\infty$) in the interior of the domain. It was first proved for the whole space by Hermann Weyl [21], and later extended by others to any open set in $\mathbb{R}^n$ (see, e.g., [14]).

Remark 4.1. Notice that $\widetilde{w}(y; G_0, u) = -\widehat{w}(y; G_0, -u)$ is a solution of (2.4) that corresponds to $\widetilde{\sigma}(s) = -\sigma(-s)$. Hence, any comparison we prove for $u \geq 0$ we automatically prove for $u \leq 0$.

4.1.1. *A priori estimates.*

Lemma 4.2 (Weak maximum principle in exterior domains)**.** *Assume that $w \in \mathbb{X}$ is such that*

$$\begin{gathered}
-\Delta w \le 0 \quad \mathcal{D}'(\mathcal{O}), \\
w \le 0 \quad \partial G_0.
\end{gathered}$$

Then $w \le 0$ in $\overline{\mathcal{O}}$.

Proof. Let $R > 0$. Consider $\mathcal{O}_R = \mathcal{O} \cap B_R$. Since $w \in \mathbb{X}$ then $w \le \frac{K}{|y|^{n-2}}$. Using the hypothesis $w \le 0$ on ∂G_0 and this fact, $\frac{K}{R^{n-2}}$ on $\partial \mathcal{O}_R$. We can apply the standard weak maximum principle for weak solutions in $\mathcal{O}_R$ to show that $w \le \frac{K}{R^{n-2}}$ on $\overline{\mathcal{O}}_R$. As $R \to +\infty$ we prove the result. □

Analogously, we have the strong maximum principle.

Lemma 4.3. *Let σ nondecreasing, $u \in \mathbb{R}$, $\widehat{w} \in \mathbb{X}$ be a weak solution of* (2.4)*. Then*

$$\min\{0, u\} \le \widehat{w} \le \max\{0, u\} \tag{4.3}$$

Proof. For $u = 0$, $w = 0$ follows from a monotonicity argument. Assume $u > 0$. Let $\psi \in W^{1,\infty}(\mathbb{R})$ non-increasing such that

$$\psi(s) = \begin{cases} 1 & s < \frac{1}{2} \\ 0 & s > 1 \end{cases}$$

and consider the test function $\varphi = (w-u)_+ \psi\big(\frac{d(\cdot, \partial G_0)}{R}\big)$. Then

$$\begin{aligned}
&\int_{\mathcal{O}} |\nabla (w-u)_+|^2 \psi\Big(\frac{d(x,\partial G_0)}{R}\Big)\,\mathrm{d}x + \int_{\mathcal{O}} (w-u)_+ \frac{\psi'\big(\frac{d(x,\partial G_0)}{R}\big)}{R} \nabla w \cdot \nabla d \,\mathrm{d}x \\
&= C_0 \int_{\partial G_0} \sigma(u-w)(w-u)_+ \,\mathrm{d}S \le 0
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \int_{\mathcal{O}} (w-u)_+ \frac{\psi'\big(\frac{d(x,\partial G_0)}{R}\big)}{R} \nabla w \cdot \nabla d \,\mathrm{d}x \Big| \,\mathrm{d}x \\
&\le C \int_{\{\frac{R}{2} < d < R\}} \frac{w}{R} |\nabla w| \,\mathrm{d}x \\
&\le C \Big(\int_{\{\frac{R}{2} < d < R\}} \frac{|w|^2}{R^2} \,\mathrm{d}x \Big)^{1/2} \Big(\int_{\mathcal{O}} |\nabla w|^2 \,\mathrm{d}x \Big)^{1/2} \\
&\le \frac{C}{R^{\frac{n-2}{2}}} \Big(\int_{\mathcal{O}} |\nabla w|^2 \,\mathrm{d}x \Big)^{1/2} \to 0,
\end{aligned}$$

as $R \to \infty$. Therefore,

$$\begin{aligned}
0 &\le \int_{0 \le d < \frac{R}{2}} |\nabla (w-u)_+|^2 \,\mathrm{d}x \le \int_{\mathcal{O}} |\nabla (w-u)_+|^2 \psi\Big(\frac{d(x,\partial G_0)}{R}\Big)\,\mathrm{d}x \\
&\le -\int_{\mathcal{O}} (w-u)_+ \frac{\psi'\Big(\frac{d(x,\partial G_0)}{R}\Big)}{R} \nabla w \cdot \nabla d \,\mathrm{d}x.
\end{aligned}$$

As $R \to +\infty$ we obtain that
$$\int_{\mathcal{O}} |\nabla (w-u)_+|^2 \, \mathrm{d}x = 0. \tag{4.4}$$
In particular $(w-u)_+ \geq 0$ is a constant. Since, as $|y| \to +\infty$ we show that the constant must be $(-u)_+ = 0$ we deduce that $w - u \leq 0$.

If $u < 0$ we apply the previous argument with $\tilde{\sigma}(s) = -\sigma(-s)$. □

Lemma 4.4. *Let $u \in \mathbb{R}$, $w \in \mathbb{X}$ such that $w \leq u$ in ∂G_0 and $-\Delta w \leq 0$ and*
$$K_0 = \max_{z \in \partial G_0} |z|^{n-2}.$$
Then
$$w \leq \frac{K_0 u}{|y|^{n-2}} \quad \forall y \in \overline{\mathcal{O}}.$$

Proof. Notice that
$$\max_{z \in \partial G_0} w(z)|z|^{n-2} \leq u \max_{z \in \partial G_0} |z|^{n-2} = K_0 u.$$
Then
$$w \leq \frac{K_0 u}{|y|^{n-2}} \quad y \in \partial G_0.$$
Since $w - \frac{K_0 u}{|y|^{n-2}}$ is subharmonic and tends to 0 as $|y| \to +\infty$, we can apply the weak maximum principle to deduce that
$$w(y) \leq \frac{K_0 u}{|y|^{n-2}} \quad y \in \mathbb{R}^n \setminus G_0.$$
This proves the result. □

By the same argument it is easy to prove that any classical solution $\widehat{w} \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\overline{\mathcal{O}})$ is, in fact, in $\mathbb{X}$. Furthermore, we have an explicit expression of the K in the definition of $\mathbb{X}$ for the solutions of (2.4):

Lemma 4.5. *Let $\widehat{w} \in \mathbb{X}$ be a solution of* (2.4). *Then*
$$|\widehat{w}(y; G_0, u)| \leq \frac{K_0 |u|}{|y|^{n-2}} \quad \forall y \in \overline{\mathcal{O}}. \tag{4.5}$$

Lemma 4.6. *Let $R_0 = \max_{\partial G_0} |y|$, $\widehat{w} \in \mathbb{X}$ be a weak solution of* (2.4). *Then*
$$\max_{|y|=R} |\nabla \widehat{w}(y; G_0, u)| \leq \frac{K|u|}{(R-R_0)^{n-1}} \quad \forall R > R_0 \tag{4.6}$$
where K does not depend on u or R.

Proof. Let $|y_0| = R$. Let B be a ball centered at y_0 of radius $\frac{R-R_0}{2}$. In B we have $|y| \geq \frac{R-R_0}{2}$. Since $\frac{\partial \widehat{w}}{\partial x_i}$ is a harmonic function, and applying Lemma 4.5, we have
$$\frac{\partial \widehat{w}}{\partial x_i}(y_0) = \frac{1}{|B|} \int_B \frac{\partial \widehat{w}}{\partial x_i} \, \mathrm{d}y = \frac{1}{|B|} \int_{\partial B} \widehat{w} \nu_i \, \mathrm{d}S,$$
$$\Big| \frac{\partial \widehat{w}}{\partial x_i}(y_0) \Big| \leq \frac{|\partial B|}{|B|} \frac{K|u|}{(R-R_0)^{n-2}} \leq \frac{K|u|}{(R-R_0)^{n-1}}.$$
This completes the proof. □

4.1.2. *Uniqueness, comparison and approximation of solutions.*

Lemma 4.7. *Let $u \in \mathbb{R}$, σ_1, σ_2 be two nondecreasing functions such that $\sigma_1 \le \sigma_2$ in $[0,+\infty)$ and let $w_1, w_2 \in \mathbb{X}$ satisfy (2.4). Then $w_1 \le w_2$.*

Proof. We subtract the two weak formulations, and consider $\varphi = (w_1 - w_2)_+$ as a test function. We obtain that

$$\int_{\mathbb{R}^n \setminus G_0} |\nabla (w_1 - w_2)_+|^2 \, \mathrm{d}x = \int_{\partial G_0} (\sigma_1(u - w_1) - \sigma_2(u - w_2))(w_1 - w_2)_+ \, \mathrm{d}S$$

Thus, in the set $\{w_2 \le w_1\}$ we have that $u - w_2 \ge u - w_1$ and, hence,

$$\sigma_2(u - w_2) \ge \sigma_2(u - w_1) \ge \sigma_1(u - w_1),$$

so

$$\sigma_1(u - w_1) - \sigma_2(u - w_2) \le 0.$$

Thus, since $(w_1 - w_2)_+ \ge 0$ a.e. in ∂G_0, we have that

$$\int_{\mathbb{R}^n \setminus G_0} |\nabla (w_1 - w_2)_+|^2 \, \mathrm{d}x \le 0. \tag{4.7}$$

Hence $(w_1 - w_2)_+ = c$ constant. Since $(w_1 - w_2)_+ \to 0$ as $|y| \to +\infty$, we have that $c = 0$ and thus $w_1 \le w_2$. □

Corollary 4.8. *There exists, at most, one solution $w \in \mathbb{X}$ of (2.4).*

Lemma 4.9. *Let $\sigma_1, \sigma_2 \in \mathcal{C}(\mathbb{R})$ be two nondecreasing function. Let $\widehat{w}_i(\cdot; G_0, u) \in \mathbb{X}$ be a solution of*

$$\begin{gathered} -\Delta_y \widehat{w}_i = 0 \quad \text{if } y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_y} \widehat{w}_i - C_0 \sigma_i(u - \widehat{w}_i) = 0, \quad \text{if } y \in \partial G_0, \\ \widehat{w_i} \to 0 \quad \text{as } |y| \to \infty. \end{gathered} \tag{4.8}$$

Then

$$\|\nabla(\widehat{w}_1 - \widehat{w}_2)\|^2_{L^2(\mathcal{O})} \le C|u| \|\sigma_1 - \sigma_2\|_{L^\infty(I)}, \tag{4.9}$$

where

$$I = \{u - \widehat{w}_1(y; G_0, u) : y \in \mathbb{R}^n \setminus \overline{G}_0\} \subset \mathbb{R}, \tag{4.10}$$

and C is independent of u.

Proof. By taking as test function $\varphi = \widehat{w}_1 - \widehat{w}_2$ in the weak formulation of these equations we have that

$$\begin{aligned} \|\nabla(\widehat{w}_1 - \widehat{w}_2)\|^2_{L^2(\mathcal{O})} &\le \int_{\mathcal{O}} |\nabla(\widehat{w}_1 - \widehat{w}_2)|^2 \, \mathrm{d}x \\ &\quad + \int_{\partial G_0} (\sigma_2(u - \widehat{w}_2) - \sigma_2(u - \widehat{w}_1))(\widehat{w}_1 - \widehat{w}_2) \, \mathrm{d}S \\ &= \int_{\partial G_0} (\sigma_1(u - \widehat{w}_1) - \sigma_2(u - \widehat{w}_1))(\widehat{w}_1 - \widehat{w}_2) \, \mathrm{d}S \\ &\le \|\sigma_1 - \sigma_2\|_\infty \int_{\partial G_0} |\widehat{w}_1 - \widehat{w}_2| \, \mathrm{d}S \\ &\le C|u| \|\sigma_1 - \sigma_2\|_\infty. \end{aligned}$$

This completes the proof. □

4.1.3. *Existence and regularity.*

Lemma 4.10. *Let $u \in \mathbb{R}$ and σ uniformly Lipschitz. Then, there exists $\widehat{w} \in \mathbb{X}$ a weak solution of* (2.4)*. Furthermore, $\widehat{w}$ satisfies* (4.5)*.*

Proof. Let us assume that $u > 0$. Let $\lambda > 0$, and consider $\mu > 0$ such that

$$F : z \mapsto C_0 \sigma(u - z) + \mu z$$

is nondecreasing. Let $w_0 = 0$. We define the sequence $w_k \in H^1(\mathcal{O})$ as the solutions of

$$\begin{gathered}
-\Delta w_{k+1} + \lambda w_{k+1} = \lambda w_k \quad \mathcal{O}, \\
\partial_\nu w_{k+1} + \mu w_{k+1} = F(w_k) \quad \partial G_0, \\
w_{k+1} \to 0 \quad |y| \to +\infty.
\end{gathered}$$

This sequence is well defined, since $\lambda > 0$ applying the Lax-Milgram theorem. Indeed, if $w_k \in H^1(\mathcal{O})$ then $F(w_k) \in H^{1/2}(\partial G_0)$ so that $w_{k+1} \in H^1(\mathcal{O})$.

Let us show that $0 \le w_k \le w_{k+1} \le u$ a.e. in $\mathcal{O}$ and ∂G_0 for every $n \ge 1$. We start by showing that $0 \le w_1$. This is immediate because $F(0) = C_0 \sigma(u) \ge 0$. Let us now show that, if $w_{k-1} \le w_k$ then $w_k \le w_{k+1}$. Considering the weak formulations:

$$\begin{aligned}
&\int_{\mathcal{O}} \nabla w_{k+1} \nabla v \,\mathrm{d}x + \lambda \int_{\mathcal{O}} w_{k+1} v \,\mathrm{d}x + \mu \int_{\partial G_0} w_k v \,\mathrm{d}S \\
&= \lambda \int_{\mathcal{O}} w_{k+1} v \,\mathrm{d}x + \int_{\partial G_0} F(w_k) v \,\mathrm{d}x
\end{aligned} \tag{4.11}$$

we have that

$$\begin{aligned}
&\int_{\mathcal{O}} \nabla (w_k - w_{k+1}) \nabla v \,\mathrm{d}x + \lambda \int_{\mathcal{O}} (w_k - w_{k+1}) \varphi \,\mathrm{d}x + \mu \int_{\partial G_0} (w_k - w_{k+1}) v \,\mathrm{d}S \\
&= \lambda \int_{\mathcal{O}} (w_{k-1} - w_k) v \,\mathrm{d}x + \int_{\partial G_0} (F(w_{k-1}) - F(w_k)) v \,\mathrm{d}S
\end{aligned}$$

Consider $v = (w_k - w_{k+1})_+ \ge 0$. We have that $w_{k-1} \le w_k$ therefore $w_{k-1} - w_k, F(w_{k-1}) - F(w_k) \le 0$. Hence

$$\begin{aligned}
&\int_{\mathcal{O}} |\nabla (w_k - w_{k+1})_+|^2 \,\mathrm{d}x + \lambda \int_{\mathcal{O}} |(w_k - w_{k+1})_+|^2 \,\mathrm{d}x + \mu \int_{\partial G_0} |(w_k - w_{k+1})_+|^2 \,\mathrm{d}S \\
&= \lambda \int_{\mathcal{O}} (w_k - w_{k+1}) v \,\mathrm{d}x + \int_{\partial G_0} (F(w_{k-1}) - F(w_k)) v \,\mathrm{d}S \le 0
\end{aligned}$$

so that $(w_k - w_{k+1})_+ = 0$. Hence $w_k \le w_{k+1}$ a.e. in $\mathcal{O}$ and in ∂G_0. With an argument similar to the one in Lemma 4.3, one proves that $w_{k+1} \le u$ a.e. in $\mathcal{O}$ and ∂G_0.

The sequence w_k is pointwise increasing a.e. in $\mathcal{O}$. Therefore, there exists a function w such that

$$w_k(y) \nearrow w(y) \quad \text{a.e. } \mathcal{O}. \tag{4.12}$$

Taking traces, the same happens in ∂G_0. Hence

$$w_k(y) \nearrow w(y) \quad \text{a.e. } \partial G_0. \tag{4.13}$$

Thus $F(w_k) \nearrow F(w)$ a.e. in $L^2(\partial G_0)$. Since $F(w) \le F(u)$ and ∂G_0 has bounded measure, we have that

$$F(w_k) \to F(w) \quad \text{in } L^2(\partial G_0). \tag{4.14}$$

We have that

$$-\Delta w_{k+1} = \lambda(w_k - w_{k+1}) \le 0 \quad \mathcal{O}.$$

Hence, w_k are all subharmonic. Then, since $w_k \to 0$ as $|y| \to 0$ and $w_k \in \mathbb{X}$ and $w_k \le u$ on ∂G_0, by Lemma 4.4 we have that

$$0 \le w_k \le \frac{K_0 u}{|y|^{n-2}}.$$

in particular $w_n \in \mathbb{X}$. Passing to the limit we deduce that

$$0 \le w \le \frac{K_0 u}{|y|^{n-2}} \quad y \in \mathcal{O}$$

Hence $w \to 0$ as $|y| \to +\infty$. Applying an equivalent argument to the one in Lemma 4.9 we have that ∇w_k is a Cauchy sequence in $L^2(\mathcal{O})^n$. In particular, there exists $\xi \in L^2(\mathcal{O})^n$ such that

$$\nabla w_k \to \xi \quad \text{in } L^2(\mathcal{O})^n.$$

Consider $\mathcal{O}' \subset \mathcal{O}$ open and bounded. Then we have that

$$\int_{\mathcal{O}'} |\nabla w_k|^2 \, \mathrm{d}y \le \int_{\mathcal{O}} |\nabla w_n|^2 \, \mathrm{d}y$$

is bounded, and

$$\int_{\mathcal{O}'} |w_k|^2 \, \mathrm{d}y \le |u|^2 |\mathcal{O}'|.$$

Hence, there is convergent subsequence in $H^1(\mathcal{O}')$. Any convergent subsequence must have the same limit, so $w_k \rightharpoonup w$ $H^1(\mathcal{O}')$. In particular

$$\xi = \nabla w \quad \text{a.e. } \mathcal{O}'.$$

Since this works for every $\mathcal{O}'$ bounded we have that $\nabla w \in L^2(\mathcal{O})^n$, hence $w \in \mathbb{X}$, and

$$\nabla w_n \to \nabla w \quad \text{in } L^2(\mathcal{O})^n.$$

Using this fact and (4.14), we can pass to the limit in the weak formulation to deduce that

$$\begin{gathered} -\Delta w = 0 \quad \mathcal{O}, \\ \frac{\partial w}{\partial n} = C_0 \sigma(u - w) \quad \partial G_0. \end{gathered}$$

In particular, a solution of (2.4). The same reasoning applies to case $u < 0$. $\square$

Corollary 4.11. *Let $\sigma \in \mathcal{C}(\mathbb{R})$ be nondecreasing be such that*

$$|\sigma(u)| \le C(1 + |u|). \tag{4.15}$$

Then, there exists a unique solution of (2.4).

Proof. Let us assume first that $u > 0$. Let $\sigma_m \in C^1([0, |u|])$ be a pointwise increasing sequence that approximates σ uniformly in $[0, |u|]$. Since σ_m is Lipschitz, then $\widehat{w}_m$ exists by the previous part. Because of Lemma 4.7, the sequence $\widehat{w}_m$ of solutions of (2.4) is pointwise increasing. Since we know that, we have that $\widehat{w}_m \le u$ then, for a.e. $y \in \mathcal{O}$, $\widehat{w}_m(y)$ is a bounded and increasing sequence

$$\widehat{w}_m(y) \nearrow w(y).$$

for some $w(y)$. In particular

$$0 \le w(y) \le \frac{K_0 u}{|y|^{n-2}} \quad y \in \mathcal{O}.$$

Applying as in the proof of Lemma 4.10 we deduce that $w \in \mathbb{X}$ and it is a solution of (2.4). The proof for $u < 0$ follows in the same way, by taking a pointwise decreasing sequence σ_m. □

With the same techniques we can prove the following (applying that $u - w \ge 0$ for $u \ge 0$ and $u - w \le 0$ for $u \le 0$):

Lemma 4.12. *Let $u \in \mathbb{R}$, $\mathcal{O}' \subset \mathcal{O}$ bounded, σ, σ_m be nondecreasing continuous functions such that $\sigma(0) = \sigma_m(0) = 0$ and $|\sigma_m| \le |\sigma|$ and $\sigma_m \to \sigma$ in $\mathcal{C}([-2|u|, 2|u|])$. Then:*

$$\widehat{w}_m(\cdot; G_0, u) \to \widehat{w}(\cdot; G_0, u) \quad \text{strongly in } H^1(\mathcal{O}'). \tag{4.16}$$

Furthermore,

1. *If $u \ge 0$ then $\widehat{w}_m \nearrow \widehat{w}$ a.e. $y \in \mathcal{O}$ and $y \in \partial G_0$.*
2. *If $u \le 0$ then $\widehat{w}_m \searrow \widehat{w}$ a.e. $y \in \mathcal{O}$ and $y \in \partial G_0$.*

4.1.4. *Lipschitz continuity with respect to u.*

Lemma 4.13. *For every $y \in \mathbb{R}^n \setminus G_0$, $\widehat{w}(y; G_0,u)$ is a nondecreasing Lipschitz-continuous function with respect to u. In fact,*

$$|\widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2)| \le |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R},\ \forall y \in \mathbb{R}^n \setminus G_0. \tag{4.17}$$

Furthermore, for every $y \in \partial G_0$, $\partial_\nu \widehat{w}(y; G_0, u)$ is also nondecreasing in u.

Proof. Let us consider first that $\sigma \in \mathcal{C}^1(\mathbb{R})$. We have that $\widehat{w}(\cdot; G_0, u) \in \mathcal{C}(\overline{\mathcal{O}}) \cap \mathcal{C}^2(\mathcal{O})$ for every $u \in \mathbb{R}$ and the equation is satisfied pointwise (see [14]).

Let us first consider $u_1 > u_2$. We want to prove the following

$$0 \le \widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2) \le u_1 - u_2 \tag{4.18}$$

$$\partial_\nu \widehat{w}(u_1; G_0, y) \ge \partial_\nu \widehat{w}(y; G_0, u_2). \tag{4.19}$$

That

$$\widehat{w}(u_1; G_0, y) \ge \widehat{w}(y; G_0, u_2). \tag{4.20}$$

follows from the comparison principle. Indeed, let us plug $\widehat{w}(u_1; G_0, y)$ in the equation for $\widehat{w}(y; G_0, u_2)$:

$$\begin{gathered} -\Delta \widehat{w}(u_1; G_0, y) = 0 \quad \mathbb{R}^n \setminus G_0, \\ \begin{aligned} &\partial_{\nu_y} \widehat{w}(u_1; G_0, y) - C_0 \sigma(u_2 - \widehat{w}(u_1; G_0, y)) \\ &= C_0 \left(\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2)) \right) \ge 0 \quad \partial G_0, \end{aligned} \\ \widehat{w}(u_1; G_0, y) \to 0 \quad |y| \to +\infty. \end{gathered} \tag{4.21}$$

Therefore, $\widehat{w}(u_1; G_0, y)$ is a supersolution of the problem for $\widehat{w}(y; G_0, u_2)$. Applying the comparison principle we deduce (4.20).

We define

$$g(u_1, u_2, y) = \widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2) \ge 0. \tag{4.22}$$

The function g is the solution of the following elliptic problem:

$$\begin{gathered}\Delta_y g = 0 \quad \text{if } y \in \mathbb{R}^n \setminus \overline{G_0},\\ \partial_{\nu_y} g - C_0\Big(\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2))\Big) = 0 \quad \text{if } y \in \partial G_0,\\ g \to 0 \quad \text{as } |y| \to \infty.\end{gathered} \tag{4.23}$$

Let us consider the boundary condition for $y \in \partial G_0$:

$$\partial_{\nu_y} g(y) = C_0(\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2)))$$

multiplying by $u_1 - u_2 - g(u_1, u_2, y)$, and applying the monotonicity of σ, we have

$$(\partial_\nu g(y))(u_1 - u_2 - g(y)) \geq 0 \quad \forall y \in \partial G_0. \tag{4.24}$$

Let $g(y_0) = \max_{\partial G_0} g$ for some $y_0 \in \partial G_0$. By the strong maximum principle $g(y_0) = \max_{\mathbb{R}^n \setminus G_0} g$. Hence $g(y) \leq g(y_0)$ for $y \in \mathbb{R}^n \setminus G_0$ and we have

$$\partial_{\nu_y} g(y_0) \geq 0.$$

Assume, first, that σ is strictly increasing. We study two cases. If $\partial_{\nu_y} g(y_0) > 0$ then, by (4.24),

$$u_1 - u_2 \geq g(y_0) \geq g(y) \quad \forall y \in \mathbb{R}^n \setminus G_0.$$

If $\partial_\nu g(y_0) = 0$ then, by (4.23),

$$\begin{gathered}\sigma(u_1 - \widehat{w}(y_0; G_0, u_1)) = \sigma(u_2 - \widehat{w}(y_0; G_0, u_2))\\ u_1 - \widehat{w}(y_0; G_0, u_1) = u_2 - \widehat{w}(y_0; G_0, u_2)\\ u_1 - u_2 = g(y_0) \geq g(y) \quad \forall y \in \mathbb{R}^n \setminus G_0.\end{gathered}$$

Either way, we deduce that (4.18) holds. Hence,

$$\sigma(u_1 - \widehat{w}(u_1; G_0, y)) \geq \sigma(u_2 - \widehat{w}(y; G_0, u_2)) \quad \forall y \in \partial G_0$$

so (4.19) holds. This concludes the proof when σ is strictly increasing.

Let σ be a nondecreasing function and $U = \max\{|u_1|, |u_2|\}$. We consider an approximation sequence σ_m of σ in $[-2U, 2U]$ by strictly increasing smooth functions such that $|\sigma_m| \leq |\sigma|$. Consider $\widehat{w}_m$ as defined in Lemma 5.7. We have that

$$u_i - \widehat{w}(y; G_0, u_i) \in [-2U, 2U] \quad \forall i = 1, 2, \forall y \in \mathbb{R}^n \setminus \overline{G}_0.$$

By the previous part $\widehat{w}_m$ satisfies (4.18) and (4.19). Applying Lemma 4.12 we have a.e.-pointwise convergence $\widehat{w}_m(u_i, y) \to \widehat{w}(u_i, y)$ for $i = 1, 2$, up to a subsequence, as $m \to +\infty$. Therefore (4.18) and (4.19) hold almost everywhere in y. Since $\widehat{w}$ is continuous, (4.18) and (4.19) hold everywhere. This concludes the proof in the case $u_1 > u_2$.

If $u_1 < u_2$ we can exchange the roles of u_1 and u_2 in (4.18) to deduce (4.17). This concludes the proof. □

4.1.5. *Auxiliary function $\widehat{w}_\varepsilon^j$.* We conclude this section by introducing the following function:

Definition 4.14. Let $u \in \mathbb{R}$, $j \in \Upsilon_\varepsilon$ and $\varepsilon > 0$. We define

$$\widehat{w}_\varepsilon^j(x; G_0, u) = \widehat{w}\Big(\frac{x - P_\varepsilon^j}{a_\varepsilon}; G_0, u\Big). \tag{4.25}$$

It is clear that this function is the solution of the problem

$$\begin{gathered}
-\Delta \widehat{w}_\varepsilon^j = 0 \quad \mathbb{R}^n \setminus G_\varepsilon^j, \\
\partial_n \widehat{w}_\varepsilon^j - \varepsilon^{-\gamma}\sigma(u - \widehat{w}_\varepsilon^j) = 0 \quad \partial G_\varepsilon^j \\
\widehat{w}_\varepsilon^j \to 0 \quad |x| \to +\infty.
\end{gathered} \tag{4.26}$$

We have the following estimates:

Lemma 4.15. *Let $\varepsilon, r > 0$ and $x \in \partial T_{r\varepsilon}^j$. Then*

$$|\widehat{w}_\varepsilon^j(x; G_0, u)| \le \frac{K|u|}{\left|\frac{x-P_\varepsilon^j}{a_\varepsilon}\right|^{n-2}} \le \frac{K|u| a_\varepsilon^{n-2}}{r^{n-2}\varepsilon^{n-2}} \le \frac{K|u|}{r^{n-2}}\varepsilon^2 \tag{4.27}$$

where K does not depend on $r, |u|$ or ε.

Lemma 4.16. *For $\varepsilon, r > 0$ be such that $a_\varepsilon < \frac{r\varepsilon}{2R_0}$. Let $x \in \partial T_{r\varepsilon}^j$. Then*

$$|\nabla \widehat{w}_\varepsilon^j(x; G_0, u)| \le \frac{K|u|}{r^{n-1}}\varepsilon, \tag{4.28}$$

where K does not depend on r, ε or j.

Proof. By the definition of $\widehat{w}_\varepsilon^j$ we have

$$\nabla \widehat{w}_\varepsilon^j(x; G_0, u) = a_\varepsilon^{-1}(\nabla \widehat{w})\Big(\frac{x - P_\varepsilon^j}{a_\varepsilon}; G_0, u\Big)$$

Therefore, for $x \in \partial T_{r\varepsilon}^j$,

$$\begin{aligned}
|\nabla \widehat{w}_\varepsilon^j| = a_\varepsilon^{-1}\big|\nabla \widehat{w}\Big(\frac{x-P_\varepsilon^j}{a_\varepsilon}\Big)\big| &\le \frac{K|u|a_\varepsilon^{-1}}{\Big(|\frac{x-P_\varepsilon^j}{a_\varepsilon}| - R_0\Big)^{n-1}} \\
&\le \frac{K|u|a_\varepsilon^{n-2}}{(r\varepsilon - a_\varepsilon R_0)^{n-1}} \le \frac{K|u|a_\varepsilon^{n-2}}{\big(\frac{r\varepsilon}{2}\big)^{n-1}} \\
&\le \frac{K|u|}{r^{n-1}}\varepsilon.
\end{aligned}$$

This completes the proof. □

4.2. **Properties of H_{G_0}.**

Lemma 4.17. *H_{G_0} is a nondecreasing function. Furthermore:*

1. *If σ satisfies (2.3), then so does H_{G_0}.*
2. *If $\sigma \in \mathcal{C}^{0,\alpha}(\mathbb{R})$, then so is H_{G_0}.*
3. *If $\sigma \in \mathcal{C}^1(\mathbb{R})$, then H_{G_0} is locally Lipschitz continuous.*
4. *If $\sigma \in W^{1,\infty}(\mathbb{R})$, then so is H_{G_0}.*

Proof. Let us prove the monotonicity of $H_{G_0}(u)$ given by (2.5). Let $u_1 > u_2$. By applying (4.19) we deduce that $H_{G_0}(u_1) \ge H_{G_0}(u_2)$.

Assume (2.3). Indeed, taking into account (4.17) we deduce

$$\begin{aligned} &|H_{G_0}(u) - H_{G_0}(v)| \\ &\le C_0 \int_{\partial G_0} |\sigma(u - \widehat{w}(y; G_0,u)) - \sigma(v - \widehat{w}(y; G_0, v))| \, \mathrm{d}S_y \\ &\le C_0 k_1 \int_{\partial G_0} \Big(|u - v| + |\widehat{w}(y; G_0,u) - \widehat{w}(y; G_0, v)| \Big)^\alpha \, \mathrm{d}S_y \\ &\quad + C_0 k_2 \int_{\partial G_0} \Big(|u - v| + |\widehat{w}(y; G_0,u) - \widehat{w}(y; G_0, v)| \Big) \, \mathrm{d}S_y \\ &\le K_1 |u - v|^\alpha + K_2 |u - v| \end{aligned} \tag{4.29}$$

In particular, if $u \in \mathcal{C}^{0,\alpha}(\mathbb{R})$, then $k_2 = 0$ and $K_2 = 0$.

Assume now that $\sigma \in \mathcal{C}^1(\mathbb{R})$. Let $u_1, u_2 \in \mathbb{R}$. We have that, for $y \in \partial G_0$

$$\begin{aligned} &|\partial_{\nu_y} \widehat{w}(u_1; G_0, y) - \partial_{\nu_y} \widehat{w}(u_2, y)| \\ &= C_0 |\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2))| \\ &\le C |\sigma'(\xi)| \Big(|u_1 - u_2| + |\widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2)| \Big) \\ &\le C |\sigma'(\xi)| |u_1 - u_2|. \end{aligned}$$

for some ξ between $u_1 - \widehat{w}(y; G_0, u_1)$ and $u_2 - \widehat{w}(y; G_0, u_2)$. Since $|\widehat{w}(u, y)| \le |u|$, for every $K \subset \mathbb{R}$ compact there exists a constant C_K such that

$$|\partial_{\nu_y} \widehat{w}(y; G_0, u_1) - \partial_{\nu_y} \widehat{w}(y; G_0, u_2)| \le C_K |u_1 - u_2| \quad \forall u_1, u_2 \in K.$$

Therefore,

$$|H_{G_0}(u) - H_{G_0}(v)| \le \widetilde{C}_K |u_1 - u_2| \quad \forall u_1, u_2 \in K.$$

Let $\sigma \in W^{1,\infty}(\mathbb{R})$. By approximation by nondecreasing functions $\sigma_n \in W^{1,\infty} \cap \mathcal{C}^1$, we obtain that

$$|\partial_{\nu_y} \widehat{w}(y; G_0, u_1) - \partial_{\nu_y} \widehat{w}(y; G_0, u_2)| \le 2 \|\sigma'\|_\infty |u_1 - u_2|. \tag{4.30}$$

Therefore,

$$|H_{G_0}(u) - H_{G_0}(v)| \le 2 \|\sigma'\|_\infty |\partial G_0| |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}. \tag{4.31}$$

This completes the proof. □

Lemma 4.18. *Let $u \in \mathbb{R}$. Let σ, σ_m be nondecreasing continuous functions such that $\sigma(0) = \sigma_m(0) = 0$ satisfy* (2.3) *with the same constants k_1, k_2 and α, $|\sigma_m| \le |\sigma|$ and $\sigma_m \to \sigma$ in $\mathcal{C}([-2U, 2U])$ for some $U > 0$. Then*

$$H_{G_0,m} \to H_{G_0} \quad \text{in } \mathcal{C}([-U, U]). \tag{4.32}$$

Proof. Let $u \in [0, U]$. By Lemma 4.12 we know that

$$u - \widehat{w}_m(y; G_0, u) \searrow u - \widehat{w}(y; G_0, u) \text{ for a.e. } y \in \partial G_0.$$

In particular, due to the dominated convergence theorem, $H_{G_0,m}(u) \to H_{G_0}(u)$. An equivalent argument applies to $u \in [-U, 0]$. Hence

$$H_{G_0,m} \to H_{G_0} \quad \text{pointwise in } [-U, U].$$

Since all σ_m satisfy (2.3) with the same k_1, k_2, α, we know that H_m satisfies (4.29) with the same K_1, K_2 and α. Hence, $H_{G_0,m}$ is an equicontinuous sequence. Applying the Ascoli-Arzela theorem we know that the sequence is relatively compact in $\mathcal{C}([-U, U])$ with the supremum norm. It has, at least, a uniformly convergent subsequence. Since every convergent subsequence has to converge to H_{G_0}, we know that the the whole sequence $H_{G_0,m}$ converges to H_{G_0} uniformly in $[-U,U]$. $\square$

Remark 4.19. When ∂G_0 is assumed C^2 it is possible to develop other type of techniques (which we shall not present in detail here) showing the existence and uniqueness of solution $\widehat{w}(y;G_0,u)$. Indeed, the existence of $\widehat{w}(y;G_0,u)$ can be built through passing to the limit after a truncation of the domain process (with the artificial boundary condition $\widehat{w}(y;G_0,u) = 0$ on the new truncated boundary). The maximum principle for classical solutions (see, e.g. [14], [8, p.206] or [1]) allows to get universal a priori estimates which justify the weak convergence and thanks to the monotonicity of the nonlinear term in the interior boundary condition the passing to the limit can well-justified. In addition, it can be proved (see the indicated references) that the limit is also a classical solution on the whole exterior domain. Moreover, the same technique (i.e. the maximum principle for classical solutions) implies the comparison, uniqueness and continuous dependence of the solution $\widehat{w}(y;G_0,u)$.

5. Proof in the smooth case $\sigma \in \mathcal{C}^1(\mathbb{R})$

5.1. **Auxiliary function w_ε^j.** To pass to the limit as $\varepsilon \to 0$ in (3.2) we need some auxiliary functions.

Definition 5.1. Let $u \in \mathbb{R}$, $\varepsilon > 0$ and $j \in \Upsilon_\varepsilon$. We define the function $w_\varepsilon^j(\cdot;G_0,u)$ as the solution of the problem

$$\begin{gathered} \Delta w_\varepsilon^j = 0 \quad \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G_\varepsilon^j}, \\ \partial_{\nu_x} w_\varepsilon^j - \varepsilon^{-\gamma}\sigma(u - w_\varepsilon^j) = 0 \quad \text{if } x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0 \quad \text{if } x \in \partial T^j_{\varepsilon/4}, \end{gathered} \tag{5.1}$$

where

$$T_r^j = \{x \in \mathbb{R}^n : |x - P_\varepsilon^j| \le r\}, \tag{5.2}$$

P_ε^j is the center of Y_ε^j. Finally, we define

$$W_\varepsilon(x;G_0,u) = \begin{cases} w_\varepsilon^j(x;G_0,u) & \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G_\varepsilon^j}, j \in \Upsilon_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \cup_{j\in\Upsilon_\varepsilon} \overline{T^j_{\varepsilon/4}}. \end{cases} \tag{5.3}$$

Applying the comparison principle we obtain the following result.

Lemma 5.2. *Let $u \ge 0$. Then $0 \le w_\varepsilon^j(\cdot;G_0,u) \le \widehat{w}_\varepsilon^j(\cdot;G_0,u)$. If $u \le 0$ then $\widehat{w}_\varepsilon^j(\cdot;G_0,u) \le w_\varepsilon^j(\cdot;G_0,u) \le 0$.*

Remark 5.3. Note that, if $u = 0$, then $w_\varepsilon^j(\cdot;G_0,0) \equiv 0$.

Let us prove some properties of $W_\varepsilon(x;G_0,u)$. First, we introduce the following lemma.

Lemma 5.4 (Uniform trace constant)**.** *There exists a constant $C_T > 0$ such that, for all $\varepsilon > 0$*

$$\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |f|^2 \,\mathrm{d}S \le C_T \int_{T^j_{\varepsilon/4} \setminus G_\varepsilon^j} |\nabla f|^2 \,\mathrm{d}x \quad \forall f \in H^1\big(T^j_{\varepsilon/4} \setminus \overline{G_\varepsilon^j}, \partial T^j_{\frac{\varepsilon}{4}}\big) \tag{5.4}$$

Proof. First, we extend f to $H_0^1(Y_\varepsilon^j)$ where $Y_\varepsilon^j = \varepsilon j + \varepsilon Y$. In [17] we find that

$$\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |f|^2 \le C\Big(\int_{Y_\varepsilon^j} |f|^2 + \int_{Y_\varepsilon^j} |\nabla f|^2 \Big). \tag{5.5}$$

Since $f = 0$ on ∂Y_ε^j, taking $\tilde f(y) = f(P_\varepsilon^j + \varepsilon y)$, we have

$$\int_Y |\tilde f|^2 \, \mathrm{d}y \le C \int_Y |\nabla \tilde f|^2 \, \mathrm{d}y. \tag{5.6}$$

Since $\nabla_x f = \varepsilon \nabla_y \tilde f$ we have

$$\int_{Y_\varepsilon^j} |f|^2 \le \varepsilon^2 \int_{Y_\varepsilon^j} |\nabla f|^2. \tag{5.7}$$

Hence, the result is proved. $\square$

We have some precise estimates on the norm of W_ε:

Lemma 5.5. *For all $u \in \mathbb{R}$, we have*

$$\|\nabla W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K(|u| + |u|^2), \tag{5.8}$$

$$\|W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^2. \tag{5.9}$$

Proof. Let $u \in \mathbb{R}$ be fixed. If we take w_ε^j as a test function in weak formulation of problem (5.1) we obtain

$$\int_{T^j_{\varepsilon/4} \setminus G_\varepsilon^j} |\nabla w_\varepsilon^j|^2 \, \mathrm{d}x - \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} \sigma(u - w_\varepsilon^j) w_\varepsilon^j \, \mathrm{d}S = 0.$$

We rewrite this as follows:

$$\int_{T^j_{\varepsilon/4} \setminus G_\varepsilon^j} |\nabla w_\varepsilon^j|^2 \, \mathrm{d}x + \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} \sigma(u - w_\varepsilon^j)(u - w_\varepsilon^j) \, \mathrm{d}S = \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} \sigma(u - w_\varepsilon^j) u \, \mathrm{d}S.$$

Since σ is nondecreasing we have that

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4} \setminus G_\varepsilon^j)} \le \varepsilon^{-\gamma} |u| \int_{\partial G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)| \, \mathrm{d}S.$$

Because of (2.3) and that $|s|^\alpha \le 1 + |s|$ for every $s \in \mathbb{R}$, we have

$$\begin{aligned}
\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)| \, \mathrm{d}S &\le k_1 \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j|^\alpha \, \mathrm{d}S + k_2 \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j| \, \mathrm{d}S \\
&\le k_1 \varepsilon^{-\gamma} |\partial G_\varepsilon^j| + (k_1 + k_2) \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j| \, \mathrm{d}S.
\end{aligned}$$

Applying Lemma 5.4 and that, for every $a, b, C \in \mathbb{R}$ it holds that $ab \le \frac{C^2}{2} a^2 + \frac{1}{2C^2} b^2$, we obtain

$$\begin{aligned}
(k_1 + k_2)\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j| \, \mathrm{d}S &\le \varepsilon^{-\gamma} C |\partial G_\varepsilon^j| + \frac{1}{2C_T |u|} \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j|^2 \, \mathrm{d}S \\
&\le C|u| \varepsilon^{-\gamma} |\partial G_\varepsilon^j| + \frac{1}{2C_T|u|} \|u - w_\varepsilon^j\|^2_{L^2(\partial G_\varepsilon^j)} \\
&\le C|u| \varepsilon^n + \frac{1}{2|u|} \|\nabla(u - w_\varepsilon^j)\|^2_{L^2(T^j_{\varepsilon/4} \setminus G_\varepsilon^j)} \\
&= C|u| \varepsilon^n + \frac{1}{2|u|} \|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4} \setminus G_\varepsilon^j)}.
\end{aligned}$$

Therefore,

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^n + \frac{1}{2}\|\nabla w_\varepsilon^j\|^2_{L^2(\partial G^j_\varepsilon)}.$$

Thus, we have

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^n.$$

Adding over $j \in \Upsilon_\varepsilon$, and taking into account that $\#\Upsilon_\varepsilon \le d\varepsilon^{-n}$, we deduce that (5.8) holds. Using Friedrich's inequality we obtain

$$\|w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le \varepsilon^2 K\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)},$$

so (5.9) holds. This completes the proof. □

5.2. **Auxiliary function $v_\varepsilon^j = w_\varepsilon^j - \widehat{w}_\varepsilon^j$.** Let us define:

$$v_\varepsilon^j = w_\varepsilon^j - \widehat{w}_\varepsilon^j. \tag{5.10}$$

This functions is the solution of the problem

$$\begin{gathered}\Delta v_\varepsilon^j = 0 \quad \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\\ \partial_\nu v_\varepsilon^j - \varepsilon^{-\gamma}(\sigma(u-w_\varepsilon^j) - \sigma(u-\widehat{w}_\varepsilon^j)) = 0, \quad \text{if } x \in \partial G^j_\varepsilon,\\ v_\varepsilon^j = -\widehat{w}_\varepsilon^j(x;G_0,u), \quad \text{if } x \in \partial T^j_{\varepsilon/4}.\end{gathered} \tag{5.11}$$

Lemma 5.6. *The following estimates hold*

$$\sum_{j\in\Upsilon_\varepsilon} \|\nabla(w_\varepsilon^j(x;G_0,u) - \widehat{w}_\varepsilon^j(x;G_0,u))\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^2, \tag{5.12}$$

$$\sum_{j\in\Upsilon_\varepsilon} \|w_\varepsilon^j(x;G_0,u) - \widehat{w}_\varepsilon^j(x;G_0,u)\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^4. \tag{5.13}$$

Proof. From Lemma 5.2 it is clear that

$$|v_\varepsilon^j(x;G_0,u)| \le |\widehat{w}_\varepsilon^j(x;G_0,u)| \quad \forall x \in \overline{T^j_{\varepsilon/4}} \setminus G^j_\varepsilon. \tag{5.14}$$

Integrating by parts $v_\varepsilon^j(\Delta v_\varepsilon^j)$ and using (5.11) we deduce that

$$\begin{aligned}&\int_{T^j_{\varepsilon/4}\setminus G^j_\varepsilon} |\nabla v_\varepsilon^j|^2\,\mathrm{d}x - \varepsilon^{-\gamma}\int_{\partial G^j_\varepsilon}(\sigma(u-w_\varepsilon^j) - \sigma(u-\widehat{w}))v_\varepsilon^j\,\mathrm{d}S\\ &= -\int_{\partial T^j_{\varepsilon/4}} (\partial_\nu v_\varepsilon^j)\widehat{w}_\varepsilon^j(x;G_0,u)\,\mathrm{d}S\end{aligned}$$

By the monotonicity of σ and applying Green's first identity, we have

$$\begin{aligned}\|\nabla v_\varepsilon^j\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} &\le -\int_{\partial T^j_{\varepsilon/4}} (\partial_\nu v_\varepsilon^j)\widehat{w}_\varepsilon^j(x;G_0,u)\,\mathrm{d}S\\ &= -\int_{T^j_{\varepsilon/4}\setminus T^j_{\varepsilon/8}} \nabla v_\varepsilon^j \nabla\widehat{w}_\varepsilon^j\,\mathrm{d}x + \int_{\partial T^j_{\varepsilon/8}} (\partial_\nu v_\varepsilon^j)\widehat{w}_\varepsilon^j\,\mathrm{d}S.\end{aligned}$$

Applying Lemmas 4.15 and 4.16 we have

$$\begin{gathered}|v_\varepsilon^j(x;G_0,u)| \le |\widehat{w}_\varepsilon^j(x;G_0,u)| \le K|u|\varepsilon^2.\\ |\nabla\widehat{w}_\varepsilon^j(x;G_0,u)| \le K|u|\varepsilon\end{gathered}$$

for all $x \in T^j_{\varepsilon/8}$, where K does not depend on ε. Since v^j_ε is harmonic, denoting $T^x_r = \{z \in \mathbb{R}^n : |x - z| < r\}$ we have

$$|\frac{\partial v^j_\varepsilon}{\partial x_i}(x)| = \frac{1}{|T^x_{\varepsilon/16}|}\Big|\int_{T^x_{\varepsilon/16}} \frac{\partial v^j_\varepsilon}{\partial x_i}\,\mathrm{d}x\Big| = \frac{K}{\varepsilon^n}\Big|\int_{\partial T^x_{\varepsilon/16}} v^j_\varepsilon \nu_i \,\mathrm{d}S\Big| \le K|u|\varepsilon.$$

for all $x \in T^j_{\varepsilon/4} \setminus T^j_{\frac{\varepsilon}{8}}$, since $T^x_{\varepsilon/16} \subset T^j_{\varepsilon/4} \setminus T^j_{\varepsilon/16}$. Hence, we have

$$\Big|\int_{T^j_{\varepsilon/4}\setminus T^j_{\varepsilon/8}} \nabla v^j_\varepsilon \nabla \widehat{w}^j_\varepsilon \,\mathrm{d}x\Big| \le K(|u| + |u|^2)\varepsilon^{n+2},$$
$$\Big|\int_{\partial T^j_{\frac{\varepsilon}{8}}} (\partial_\nu v^j_\varepsilon)\widehat{w}^j_\varepsilon \,\mathrm{d}S\Big| \le K(|u| + |u|^2)\varepsilon^{n+2}.$$

From this we deduce that

$$\|\nabla v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^{n+2}.$$

From Friedrich's inequality,

$$\|v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^{n+4}.$$

Then, adding over $j \in \Upsilon_\varepsilon$ we obtain

$$\sum_{j\in\Upsilon_\varepsilon} \|\nabla v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^2,$$
$$\sum_{j\in\Upsilon_\varepsilon} \|v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^4.$$

This estimates completes the proof. □

5.3. **Convergence of integrals over** $\cup_{j\in\Upsilon_\varepsilon}\partial T^j_{\varepsilon/4}$.

Lemma 5.7. *Let $H_{G_0}(u)$ be defined by formula (2.5), $\phi \in C^\infty_0(\Omega)$ and $h_\varepsilon, h \in H^1_0(\Omega)$ be such that $h_\varepsilon \rightharpoonup h$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Then, we have that*

$$-\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\frac{\varepsilon}{4}}} \Big(\partial_\nu \widehat{w}^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon))\Big) h_\varepsilon(x)\,\mathrm{d}S = C_0^{n-2} \int_\Omega H_{G_0}(\phi(x))\, h(x)\,\mathrm{d}x \tag{5.15}$$

where ν is an unit outward normal vector to $T^j_{\varepsilon/4}$.

Proof. Let us consider the auxiliary problem

$$\begin{gathered}
\Delta\theta^j_\varepsilon = \mu^j_\varepsilon \quad x \in Y^j_\varepsilon \setminus \overline{T}^j_{\varepsilon/4},\ j \in \Upsilon_\varepsilon,\\
-\partial_\nu \theta^j_\varepsilon = \partial_\nu \widehat{w}^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon)) \quad x \in \partial T^j_\varepsilon,\\
-\partial_\nu \theta^j_\varepsilon = 0 \quad x \in \partial Y^j_\varepsilon,\\
\langle \theta^j_\varepsilon \rangle_{Y^j_\varepsilon\setminus \overline{T}^j_{\varepsilon/4}} = 0,
\end{gathered} \tag{5.16}$$

where ν is a unit inwards normal vector of the boundary of $Y^j_\varepsilon \setminus T^j_{\varepsilon/4}$. We choose, against the convention, the inward normal vector so that it coincides with the unit outward normal vector of $T^j_{\varepsilon/4} \setminus G^j_\varepsilon$ in their shared boundary. We changed the sign accordingly. The constant μ_ε^j is given by the compatibility condition of the problem (5.16):

$$\begin{aligned}\mu_\varepsilon^j \varepsilon^n \big|Y \setminus T_{1/4}^0\big| &= \int_{\partial T_{\varepsilon/4}^j} \partial_\nu \widehat{w}_\varepsilon^j(x; G_0, \phi(P_\varepsilon^j))\,\mathrm{d}S \\ &= -\int_{\partial G_\varepsilon^j} \partial_\nu \widehat{w}_\varepsilon^j(x; G_0, \phi(P_\varepsilon^j))\,\mathrm{d}S \\ &= -a_\varepsilon^{n-2} \int_{\partial G_0} \partial_{\nu_y} \widehat{w}(\phi(P_\varepsilon^j), y)\,\mathrm{d}S_y,\end{aligned}$$

Therefore,

$$\mu_\varepsilon^j = \frac{-a_\varepsilon^{n-2} H_{G_0}(\phi(P_\varepsilon^j))}{|Y \setminus T_{1/4}^0|\varepsilon^n} = \frac{-C_0^{n-2} H_{G_0}(\phi(P_\varepsilon^j))}{|Y \setminus T_{1/4}^0|},$$

From the integral identity for the problem (5.16) we obtain

$$-\int_{Y_\varepsilon^j \setminus T_\varepsilon^j} |\nabla \theta_\varepsilon^j|^2\,\mathrm{d}x = \mu_\varepsilon^j \int_{Y_\varepsilon^j \setminus T_{\varepsilon/4}^j} \theta_\varepsilon^j dx - \int_{\partial T_{\varepsilon/4}^j} \Big(\partial_\nu \widehat{w}_\varepsilon^j(x; G_0, \phi(P_\varepsilon^j))\Big)\,\theta_\varepsilon^j\,\mathrm{d}S. \quad (5.17)$$

Applying Lemma 4.16 and using the estimates from [17], we deduce

$$\begin{aligned}&\int_{\partial T_{\varepsilon/4}^j} \big|\big(\partial_{\nu_x} \widehat{w}_\varepsilon^j(x; G_0, \phi(P_\varepsilon^j))\big)\theta_\varepsilon^j\big|\,\mathrm{d}S \\ &\le K|\phi(P_\varepsilon^j)|\varepsilon \int_{\partial T_{\varepsilon/4}^j} |\theta_\varepsilon|\,\mathrm{d}S \\ &\le K|\phi(P_\varepsilon^j)|\varepsilon^{\frac{n-1}{2}+1} \|\theta_\varepsilon^j\|_{L_2(\partial T_{\varepsilon/4}^j)} \\ &\le K|\phi(P_\varepsilon^j)|\varepsilon^{\frac{n+1}{2}} \big\{\varepsilon^{-\frac12} \|\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j \setminus \overline{T}_{\varepsilon/4}^j)} + \sqrt{\varepsilon}\|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j \setminus \overline{T}_{\varepsilon/4}^j)}\big\} \\ &\le K|\phi(P_\varepsilon^j)|\varepsilon^{\frac{n+2}{2}} \|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon \setminus \overline{T}_{\varepsilon/4}^j)}.\end{aligned}$$

In particular, since $|\phi(P_\varepsilon^j)| \le \|\phi\|_\infty$ we can make a uniform bound, independent of j and ε. Thus, we have

$$\|\nabla \theta_\varepsilon^j\|^2_{L_2(Y_\varepsilon^j \setminus \overline{T}_\varepsilon^j)} \le K\varepsilon^{n+2}. \quad (5.18)$$

Adding over $j \in \Upsilon_\varepsilon$ we have

$$\sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j \setminus \overline{T}_{\varepsilon/4}^j} |\nabla \theta_\varepsilon^j|^2\,\mathrm{d}x \le K\varepsilon^2. \quad (5.19)$$

Hence, by the definition of θ_ε^j, we obtain

$$\begin{aligned}&\Big|\sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} \Big(\partial_\nu \widehat{w}_\varepsilon^j(x; G_0, \phi(P_\varepsilon^j))\Big) h_\varepsilon\,\mathrm{d}S - \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j \setminus \overline{T_{\varepsilon/4}^j}} \mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x\Big| \\ &= \Big|\sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j \setminus \overline{T_{\varepsilon/4}^j}} \nabla \theta_\varepsilon^j \nabla h_\varepsilon\,\mathrm{d}x\Big| \le K\varepsilon\|h_\varepsilon\|_{H_1(\Omega, \partial\Omega)}.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} \Big(\partial_\nu \widehat{w}_\varepsilon^j(x; G_0, \phi(P_\varepsilon^j))\Big)\, h_\varepsilon\,\mathrm{d}S = \lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j \setminus \overline{T_{\varepsilon/4}^j}} \mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x.$$

From the definition of μ_ε^j we deduce

$$\begin{aligned}
&\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x+\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}H_{G_0}(\phi(x))\,h_\varepsilon\,\mathrm{d}x\\
&=-\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\Big(H_{G_0}(\phi(P_\varepsilon^j))-H_{G_0}(\phi(x))\Big)h_\varepsilon\,\mathrm{d}x.
\end{aligned}$$

Using (4.17) we obtain

$$\begin{aligned}
&\Big|\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}(H_{G_0}(\phi(P_\varepsilon^j))-H_{G_0}(\phi(x)))h_\varepsilon\,\mathrm{d}x\Big|\\
&\le K\|h_\varepsilon\|_{L_2(\Omega)}\max_j\Big|\int_{\partial G_0}\partial_{\nu_y}\widehat{w}(y;G_0,\phi(P_\varepsilon^j))-\partial_{\nu_y}\widehat{w}(y;G_0,\phi(x))\,\mathrm{d}S_y\Big|\\
&=K\|h_\varepsilon\|_{L_2(\Omega)}\max_j\Big|\int_{\partial G_0}\sigma\Big(\phi(P_\varepsilon^j)-\widehat{w}(y;G_0,\phi(x))\Big)\\
&\quad-\sigma\Big(\phi(P_\varepsilon^j)-\widehat{w}(y;G_0,\phi(P_\varepsilon^j))\Big)\,\mathrm{d}S_y\Big|\\
&\le K\max_j\Big(\Big|\widehat{w}(y;G_0,\phi(P_\varepsilon^j))-\widehat{w}(y;G_0,\phi(x))\Big|\\
&\quad+\Big|\widehat{w}(y;G_0,\phi(P_\varepsilon^j))-\widehat{w}(y;G_0,\phi(x))\Big|^\alpha\Big)\\
&\le K\max_j\Big(|\phi(P_\varepsilon^j)-\phi(x)|+|\phi(P_\varepsilon^j)-\phi(x)|^\alpha\Big)\\
&\le K(a_\varepsilon+a_\varepsilon^\alpha)\to 0\quad\text{as }\varepsilon\to 0.
\end{aligned}$$

Hence

$$\lim_{\varepsilon\to0}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x=-\lim_{\varepsilon\to0}\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}H_{G_0}(\phi(x))h_\varepsilon\,\mathrm{d}x.$$

From [16, Corollary 1.7] we derive

$$\lim_{\varepsilon\to0}\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}H_{G_0}(\phi(x))h_\varepsilon\,\mathrm{d}x=C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))h\,\mathrm{d}x.$$

This completes the proof. □

Lemma 5.8. *Let $H_{G_0}(u)$ be defined by formula (2.5), $\phi\in C_0^\infty(\Omega)$ and $h_\varepsilon,h\in H_0^1(\Omega)$ be such that $h_\varepsilon\rightharpoonup h$ in $H_0^1(\Omega)$ as $\varepsilon\to0$. Then, we have*

$$-\lim_{\varepsilon\to0}\sum_{j\in\Upsilon}\int_{\partial T_{\frac{\varepsilon}{4}}^j}\big(\partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\big)\,h_\varepsilon\,\mathrm{d}S=C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))h\,\mathrm{d}x.\tag{5.20}$$

Proof. Using Lemma 5.6 and applying Green's identity we obtain

$$\begin{aligned}
&\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}\Big(\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))-\partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\Big)h_\varepsilon\,\mathrm{d}S\\
&=-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}\partial_\nu v_\varepsilon^j h_\varepsilon\,\mathrm{d}S\\
&=-\sum_{j\in\Upsilon_\varepsilon}\int_{T_{\varepsilon/4}^j\setminus G_\varepsilon^j}\nabla v_\varepsilon^j\nabla h_\varepsilon\,\mathrm{d}x+\int_{\partial G_\varepsilon^j}\partial_\nu v_\varepsilon^j h_\varepsilon\,\mathrm{d}S
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j\in\Upsilon_\varepsilon}\int_{T^j_{\varepsilon/4}\setminus G^j_\varepsilon} \nabla v^j_\varepsilon \nabla h_\varepsilon \,\mathrm{d}x \\
&\quad + \varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G^j_\varepsilon} \big(\sigma(\phi(P^j_\varepsilon) - w^j_\varepsilon) - \sigma(\phi(P^j_\varepsilon) - \widehat{w})\big)\, h_\varepsilon \,\mathrm{d}S.
\end{aligned}
$$

From Cauchy's inequality and the properties of v^j_ε we have

$$
\begin{aligned}
\Big|\sum_{j\in\Upsilon_\varepsilon}\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla v^j_\varepsilon \nabla h_\varepsilon \,\mathrm{d}x\Big| &\le \varepsilon^{-1}\sum_{j\in\Upsilon_\varepsilon}\|\nabla v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}})} + \varepsilon\|\nabla h_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \\
&\le K\varepsilon.
\end{aligned}
$$

Using the estimates from Lemma 5.6 we deduce

$$
\begin{aligned}
&\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\Big|\int_{\partial G^j_\varepsilon} \big(\sigma(\phi(P^j_\varepsilon) - w^j_\varepsilon) - \sigma(\phi(P^j_\varepsilon) - \widehat{w}^j_\varepsilon)\big)\, h_\varepsilon \,\mathrm{d}S\Big| \\
&\le \varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G^j_\varepsilon} \|\sigma'\|_{L^\infty([-2\|\phi\|_\infty, 2\|\phi\|_\infty])} |v^j_\varepsilon||h_\varepsilon| \,\mathrm{d}S \\
&\le K\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G^j_\varepsilon} |v^j_\varepsilon||h_\varepsilon| \,\mathrm{d}S \\
&\le K\varepsilon\varepsilon^{-\gamma/2}\|h_\varepsilon\|_{L_2(S_\varepsilon)} \\
&\le K\varepsilon\|\nabla h_\varepsilon\|_{L^2(\Omega)},
\end{aligned}
$$

where K depends on $\|\phi\|_\infty$. Therefore,

$$
\Big|\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\varepsilon/4}} \big(\partial_\nu \widehat{w}^j_\varepsilon(x;G_0,\phi(P^j_\varepsilon)) - \partial_\nu w^j_\varepsilon(x;G_0,\phi(P^j_\varepsilon))\big)\, h_\varepsilon \,\mathrm{d}S\Big| \le K\varepsilon. \tag{5.21}
$$

From this inequality and Lemma 5.7 we deduce that

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon}\int_{\partial T^j_{\varepsilon/4}} \Big(\partial_\nu w^j_\varepsilon(x;G_0,\phi(P^j_\varepsilon))\Big) h_\varepsilon \,\mathrm{d}S \\
&= -\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\varepsilon/4}} \Big(\partial_\nu \widehat{w}^j_\varepsilon(x;G_0,\phi(P^j_\varepsilon))\Big)\, h_\varepsilon \,\mathrm{d}S \\
&= C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))\, h \,\mathrm{d}x.
\end{aligned}
$$

This completes the proof. □

5.4. **Proof of Theorem 2.5 for $\sigma \in \mathcal{C}^1(\mathbb{R})$.** Let $\phi \in C_0^\infty(\Omega)$. We define

$$
\widetilde{W}_\varepsilon(x;\phi) = \begin{cases} W_\varepsilon(x;G_0,\phi(P^j_\varepsilon)) & Y^j_\varepsilon \setminus \overline{G^j_\varepsilon}, j\in\Upsilon_\varepsilon \\ 0 & \Omega\setminus \cup_{j\in\Upsilon_\varepsilon}\overline{Y}^j_\varepsilon, j\in\Upsilon_\varepsilon. \end{cases} \tag{5.22}
$$

We have that $\widetilde{W}_\varepsilon(\cdot;\phi) \in H_0^1(\Omega)$ and $\widetilde{W}_\varepsilon(\cdot;\phi) \rightharpoonup 0$ in $H^1(\Omega)$ as $\varepsilon \to 0$. Using $\varphi = \phi - \widetilde{W}_\varepsilon(x;\phi)$ as a test function in inequality (3.2) we obtain

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(\phi - \widetilde{W}_\varepsilon(x;\phi))\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi)) - u_\varepsilon)\,\mathrm{d}x \\
&+ \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \sigma(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon)\,\mathrm{d}S \\
&\geq \int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x.
\end{aligned}
\tag{5.23}
$$

Taking into account that $w_\varepsilon^j(x;G_0,u)$ is a solution of the problem (5.1), we can rewrite this in the form

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x \\
&- \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\frac{\varepsilon}{4}}^j} \partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))(\phi - u_\varepsilon)\,\mathrm{d}S \\
&- \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \sigma(\phi(P_\varepsilon^j) - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon)\,\mathrm{d}S
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
&+ \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \sigma(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon)\,\mathrm{d}S \qquad (5.25)\\
&\geq \int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

We choose the boundary condition for w_ε^j so that (5.24) cancels (5.25) out in the limit. We observe that

$$
\begin{aligned}
\rho_\varepsilon &= \Big|\varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \Big(\sigma(\phi(P_\varepsilon^j) - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))) - \sigma(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))\Big) \\
&\qquad \times \Big(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon\Big)\,\mathrm{d}S\Big| \\
&\leq \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \|\sigma'\|_{L^\infty([-U,U])} \|\nabla\phi\|_{L^\infty(\Omega)} a_\varepsilon |\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon|\,\mathrm{d}S \\
&\leq K a_\varepsilon \to 0,
\end{aligned}
$$

where $U = 2\|\phi\|_\infty$ and K depends of $\|\phi\|_\infty$. Taking this into account we have

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x \\
&\qquad - \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\frac{\varepsilon}{4}}^j} \partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))(\phi - u_\varepsilon)\,\mathrm{d}S \\
&\geq \int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x - \rho_\varepsilon.
\end{aligned}
\tag{5.26}
$$

From Lemma 5.5 we have

$$
\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x = \int_\Omega \nabla\phi\nabla(\phi - u_0)\,\mathrm{d}x, \tag{5.27}
$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x = \int_\Omega f(\phi - u_0)\,\mathrm{d}x. \tag{5.28}$$

Applying Lemma 5.8 for $h_\varepsilon = \phi - u_\varepsilon$ we have

$$-\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\frac{\varepsilon}{4}}} \Big(\partial_\nu w^j_\varepsilon(x;G_0,\phi(P^j_\varepsilon))\Big)(\phi-u_\varepsilon)\,\mathrm{d}S = C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))(\phi-u_0)dx.$$

Therefore u_0 satisfies the inequality

$$\int_\Omega \nabla\phi\nabla(\phi-u_0)\,\mathrm{d}x + C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))(\phi-u_0)\,\mathrm{d}x \ge \int_\Omega f(\phi-u_0)\,\mathrm{d}x.$$

for any $\phi \in H^1_0(\Omega)$. Therefore, $u \in H^1_0(\Omega)$ satisfies the identity

$$\int_\Omega \nabla u_0\nabla\phi\,\mathrm{d}x + C_0^{n-2}\int_\Omega H_{G_0}(u_0)\phi\,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x,$$

where $\phi \in H^1_0(\Omega)$. Thus, u is a weak solution of (2.7). This completes the proof of the Theorem 2.5 when σ is $\mathcal{C}^1(\mathbb{R})$. □

6. Proof in the Hölder-continuous case

Let $\sigma \in \mathcal{C}(\Omega)$ be satisfying (2.3). Applying [6, Lemma 2] we deduce there a sequence of nondecreasing functions $\sigma_\delta \in C^1(\mathbb{R})$ such that $\sigma_\delta(0) = 0$, $|\sigma_\delta| \le |\sigma|$ and $\sigma_\delta \to \sigma$ in $\mathcal{C}(\mathbb{R})$. Therefore σ_δ satisfies (2.3). Applying the result in the previous section, we have that

$$P_\varepsilon u_{\varepsilon,\delta} \rightharpoonup u_\delta \quad \text{in } H^1(\Omega). \tag{6.1}$$

where u_δ is the solution of (2.7) with H_δ instead of H_{G_0}.

By the approximation lemmas in [6] we have

$$\|\nabla(u_\varepsilon - u_{\varepsilon,\delta})\|_{L^2(\Omega_\varepsilon)} \le C\|\sigma_\delta - \sigma\|_\infty \tag{6.2}$$

Therefore,

$$\|\nabla(u - u_\delta)\|_{L^2(\Omega)} \le C\|\sigma_\delta - \sigma\|_\infty \tag{6.3}$$

Since, by Lemma 4.18, H_{δ,G_0} converges uniformly over compacts to HG, applying standard methods (see Lemma 7.1) we deduce that $u_\delta \to \widehat{u}_0$, where $\widehat{u}_0$ is the solution of (2.7). Notice that, due to Lemma 4.17, we have that, if $u_0 \in L^2(\Omega)$ then $H_{G_0}(u_0) \in L^2(\Omega)$.

By uniqueness of the limit $u_0 = \widehat{u}$ and it is the solution of (2.7). This completes the proof of Theorem 2.5 in the general case. □

7. Appendix: A convergence lemma

Lemma 7.1. *Let $H_m, H : \mathbb{R} \to \mathbb{R}$ be nondecreasing functions that satisfy* (2.3) *with the same constants k_1, k_2, and such that $H_m \to H$ uniformly over compacts. Let u_m, u be the corresponding solutions of* (2.7) *with H_m and H respectively. Then*

$$u_m \rightharpoonup u \ \textit{in}\ H^1_0(\Omega). \tag{7.1}$$

Proof. We have

$$\int_\Omega |\nabla u_m|^2\,\mathrm{d}x \le C\int_\Omega |f|^2\,\mathrm{d}x \tag{7.2}$$

Therefore, up to a subsequence, there is a weak limit in $H^1_0(\Omega)$, let this be $\widetilde{u}$. A further subsequence guaranties that

$$u_m \to \widetilde{u} \quad \text{in } L^2(\Omega),$$

$$u_m \to \widetilde{u} \quad \text{a.e. } \Omega.$$

Let $x \in \Omega$ such that $u_m(x) \to u(x)$ in $\mathbb{R}$. In particular the sequence is bounded so $H_m(u_m(u(x))) \to H(u(x))$ because of the uniform convergence over compact sets. Hence

$$H_m(u_m) \to H(\widetilde{u}) \quad \text{a.e. in } \Omega. \tag{7.3}$$

On the other hand, we have

$$\begin{gathered}
|H_m(u_m)| \le k_1|u_m|^\alpha + k_2|u_m| \le k_1 + (k_1+k_2)|u_m|, \\
\int_\Omega |H(u_m)|^2 \,\mathrm{d}x \le C\Big(|\Omega| + \int_\Omega |u_m|^2 \,\mathrm{d}x\Big) \\
\le C\Big(|\Omega| + \int_\Omega |f|^2 \,\mathrm{d}x\Big)
\end{gathered}$$

Hence, up to a subsequence, there exists $\widetilde{H} \in L^2(\Omega)$ such that

$$H_m(u_m) \rightharpoonup \widetilde{H} \text{ in } L^2(\Omega).$$

By Egorov's theorem, we have that, for every $\delta > 0$ there exists A_δ measurable such that $|A_\delta| < \delta$ and $H_m(u_m) \to H(\widetilde{u})$ uniformly $\Omega \setminus A_\delta$. Since $H_m(u_m) \rightharpoonup \widetilde{H}$ in $L^2(\Omega \setminus A_\delta)$ we have that $H(\widetilde{u}) = \widetilde{H}$ a.e. in $\Omega \setminus A_\delta$. Hence $H(\widetilde{u}) = \widetilde{H}$ in a.e. Ω, so

$$H_m(u_m) \rightharpoonup H(\widetilde{u}) \text{ in } L^2(\Omega).$$

By passing to the limit in the weak formulation we deduce that $\widetilde{u} = u$. $\square$

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