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On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via singular potentials: the multi-dimensional case

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Abstract We start by pointing out an important ambiguity in the mathematical treatment of the study of bound state solutions of the Schrödinger equation for infinite well type potentials (studied for the first time in a pioneering article of 1928 by G. Gamow). An alternative to get a "localizing effect" for the wave packet solution of time dependent Schrödinger equation with potentials becoming singular on the boundary of a compact region $\overline{\Omega}$ is here offered in terms of " Hardy type potentials" in which the potential behaves like the distance to the boundary to the power $\alpha = -2$. We show that in this case the probability to find the particle outside Ω is zero once we assume that at $t = 0$ the particle is located in Ω. The paper extends to the N-dimensional and evolution cases some previous results by the author.

Keywords Schrödinger equation singular potentials · Infinite well potential · Flat solutions

Mathematics Subject Classification 35B60 · 35J60 · 81Q05

1 Introduction

This paper is a companion of a previous paper by the author concerning the one-dimensional case [22] (already quoted in a textbook on Quantum Mechanics [54]).

We consider the Schrödinger equation with potentials $V(x)$ becoming singular on the boundary of a regular open bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$ after identifying (for simplicity in our proposes) the usual parameters $\hbar$ (the renormalized Planck constant) and $2m$ (m being the mass of the particle) with 1. So, if $\mathbf{i} = \sqrt{-1}$, our problem becomes

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0, \infty) \times \mathbb{R}^N, \\ \psi(0, x) = \psi_0(x) & \text{on } \mathbb{R}^N. \end{cases} \tag{1}$$

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Let us recall that in his 1928 pioneering article Gamow [40] proved, for the first time, the *tunneling effect* which, among many other applications, lead to the construction of the electronic microscope and the correct study of the alpha radioactivity. Most of his study was concerning with the *bound states* $\psi(x,t) = e^{-\mathbf{i}Et}u(x)$ [E denotes the energy and in the following we shall denote it also by λ], i.e. with $u(x)$ solving the stationary equation

$$-\Delta u + V(x)u = \lambda u \quad \text{in} \quad \mathbb{R}^N, \tag{2}$$

for a given potential $V(x)$. He was specially interested in the Coulomb potential but he offered some reasons to truncate such a potential when $0 < |x| < r'$ for some $r' > 0$. Then he proposed to replace the resulting potential by a simple potential which keeps the main properties of the original one: in this way he proposed, it seems that for the first time in the literature, what today is usually called as the *finite well potential*

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^N - \Omega. \end{cases} \tag{3}$$

In his paper Ω was an one-dimensional interval, as for instance $(-R, R)$, and $V(x) \equiv V_0$ for some $V_0 > 0$ and $q > 0$, but more general situations were considered also later in the literature. It seems that the first reference dealing with the limit case, the so called *infinite well potential,*

$$V_\infty(x : R, V_0) = \begin{cases} V_0 & \text{if } x \in \Omega, \\ +\infty & \text{if } x \notin \Omega, \end{cases} \tag{4}$$

for some $V_0 \in \mathbb{R}$ (without loss of generality we can assume $V_0 \geq 0$) was the book by the 1977 Nobel Prix Mott [48]. Since 1930 to our-days, the *infinite well potential problem* was selected as one of the best pedagogical, mathematical and physical models in Quantum Mechanics and was considered as a basic example in any text-book in the field (see, e.g. the survey [10] which includes 248 references on the subject). In many textbooks this case is presented as a limit case of the associate *finite well potential* (3). In fact, there is an abuse of the notation in the above terminology. What is really true is that we can introduce as a definition of solution u of the *infinite well potential problem* (i.e. problem (2) with V given by (4) the function $u = \lim_{q\to\infty} u_q$ with u_q solution of (2) associated to the potential $V_{q,\Omega}(x)$ given by (3) (see Lemma 2.1 of [22] and [26]). It is usually claimed that $u = \lim_{q\to\infty} u_q$ satisfies Eq. (2) for the *infinite well potential* but, as we shall explain now, this is not correct since some other terms appear in the limit equation (which, in fact must be understood in the distributional sense).

In contrast with the case of the finite well potential (3) the usual study of the *infinite well potential,* such as it is presented in most of the textbooks, contains an ambiguity which, curiously enough, it seems unseen before: it is said in many textbooks that to solve the equation in $\mathbb{R}^N$ outside Ω it is necessary to impose that the solution $u(x)$ of (2) let $u(x) \equiv 0$ if $x \notin \Omega$ (a better justification of this fact can be given through the approximation of such potential by a sequence of truncated potentials V_q and passing to the limit on the associated solutions u_q as $q \to +\infty$: see [22]). Thus the study of problem (2) leads to solve the associated Dirichlet problem on Ω

$$DP(V, \lambda, \Omega) \begin{cases} -\Delta u + V(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This Dirichlet problem can be almost explicitly solved in many cases. For instance, for the radially symmetric case of the N-dimensional *infinite well potential* over a ball $\Omega = B_R(0)$, for some $R > 0$ and with $V(x) = V(|x|)$, the differential equation and the conservation of the orbital angular momentum leads to the equation

$$-\frac{d^2u_{n,l}}{dr^2}(r) - \frac{N-1}{r}\frac{du_{n,l}}{dr}(r) + \frac{l(l+1)}{r^2}u_{n,l} + V(r)u_{n,l} = \lambda_{n,l}u_{n,l} \tag{5}$$

with $r = |x|$ and $l \geq 0$. In the case $N = 3$ and (4) with $V_0 = 0$ the solution is given by

$$u_{n,l}(r) = Cj_l(\alpha_n r)$$

with j_l the *spherical Bessel function of the first kind* and α_n such that

$$j_l(\alpha_n R) = 0. \tag{6}$$

If $\mu_{n,l}$ is the nth positive zero of the function $j_l(s)$, then the energies of the bound state are

$$\lambda_{n,l} := \frac{\mu_{n,l}^2}{R^2},$$

(see, e.g. [39,54]). For $l = 0$ equation (6) reduces to $\sin \alpha R = 0$, $\mu_{n,0} = n\pi$ and the problem is equivalent to the one-dimensional case (see [22,39,54]). In terms of the original value of the parameters m and $\hbar$, and denoting again the energy by E we get the countable set of energies

$$E_{n,l} := \frac{\hbar^2}{2m}\lambda_{n,l}$$

(see also some PDE's textbook as, e.g., Strauss [55]).

The ambiguity in this mathematical treatment arises because the derivatives of such $u_{n,l}$ are discontinuous over $\partial\Omega$ (i.e. on $r = R$), and thus such $u_{n,l}$ are not solutions of the equation (5) in the whole domain $\mathbb{R}^N$ (i.e. $r \in [0, +\infty)$) in the sense of distributions but of the different equation

$$-\frac{d^2u_{n,l}}{dr^2}(r) - \frac{2}{r}\frac{du_{n,l}}{dr}(r) + \frac{l(l+1)}{r^2}u_{n,l}(r) = \lambda_{n,l}u_n(r) + k_{n,l}(R)\delta_{\{R\}}(r), \quad \text{in} \quad (0, +\infty) \tag{7}$$

with $k_{n,l}(R) \neq 0$, since the second derivative develops a Dirac delta $\delta_{\{R\}}$ (see also [22]). The presence of such discontinuities was noticed previously in the literature (see, e.g. [39, page 140] and the survey [10]) but, as far as we know, it seems that a careful analysis of this ambiguity, and the study of some alternative potential $V(x)$ preventing this ambiguity, was not considered before.

Besides pointing out such ambiguity, the main goal of this paper is to present a set of results offering some kind of alternative and extending to the $N-$ dimensional case the results of the author [22] dealing only with nonnegative solutions $u \geq 0$ of the stationary problem $DP(V, \lambda, \Omega)$ in the one-dimensional case, $\Omega = (-R, R)$.

In some sense, our main aim can be stated in terms of the following *inverse free boundary problem*: find a class of potentials $V(x)$ such that the solution of the Schrödinger Eq. (1) let *localized* for any $t > 0$, in the sense that if we start with a localized initial wave packet $\psi_0 \in H^1(\mathbb{R} : \mathbb{C})$, i.e. such that

$$\text{support } \psi_0 \subset \overline{\Omega},$$

then the particle still remains permanently confined in Ω in the sense that

$$\text{support}\psi(t, .) \subset \overline{\Omega} \quad \text{for any} \quad t > 0.$$

We recall that, in contrast with Classical Mechanics, in Quantum Mechanics the incertitude appears (the Heisenberg principle). For instance for a free particle (i.e. with $V(x) \equiv 0$), in nonrelativistic Quantum Mechanics, if the wave function $\psi(t, .)$ at time $t = 0$ vanishes outside some compact region $\overline{\Omega}$ then at an arbitrarily short time later the wave function is nonzero arbitrarily far away from the original region $\overline{\Omega}$. This is an easy consequence of the free propagator (see, e.g., [51]). Thus the wave function instantaneously spreads to infinity and the probability of finding the particle arbitrarily far away from the initial region is nonzero for any $t > 0$. Recall that this concerns a nonrelativistic theory and so this superluminal propagation is not a philosophical contradiction. See also [18,38,39,42–44]. Nevertheless, there are many relevant applications for which it is very important to have some kind of partial localization of the particle and so in many textbooks the *infinite well potential* is presented as an example of simple potential for which such partial localization occurs. Unfortunately, this is no coherent since the solution obtained trough the associated Dirichlet problem is not solution of the Schrödinger Eq. (1) but only of a variation of it. In some sense the modified Eq. (7) can be understood as the equation corresponding to an *effective potential* which presents a singularity on $\partial\Omega$ of the type $C/d(x, \partial\Omega)$. Indeed, formally we can write

$$k_{n,l}(R)\delta_{\{R\}}(r) = \frac{k_{n,l}(R)\delta_{\{R\}}(r)}{u(r)}u(r)$$

and since we know that $\underline{K}_{n,E}\,|R-r| \le u_n(r) \le \overline{K}_{n,E}\,|R-r|$ for $r \in [0, R)$ (see, e.g. [50] and [11]) we get that Eq. (7) can be understood as the equation associated to the effective potential

$$W(|x|) = \frac{l(l+1)}{|x|^2} + \frac{(-k_{n,l}(R))\delta_{\{R\}}(|x|)}{|R-|x||}.$$

In the more general class of potentials $V(x)$ with a singularity on $\partial\Omega$ of the type

$$\frac{\underline{C}}{d(x,\partial\Omega)^{\alpha}} \le V(x) \le \frac{\overline{C}}{d(x,\partial\Omega)^{\alpha}} \quad \text{a.e.} \quad x \in \Omega, \tag{8}$$

for some $\alpha > 0$ and some $\overline{C} > \underline{C} \ge 0$ the answer to the confinement question depends strongly of the value of α. For $\alpha \in (0, 2)$ it can be proved (see, e.g., [22] and part iv) of Theorem 4.1 below) that there is a *tunneling effect* since (support $\psi(t, .)) \cap (\mathbb{R}^N - \overline{\Omega}) \ne \phi$ for $t > 0$. The main goal of this paper is to show that this changes drastically if $\alpha = 2$ (case in which $V(x)$ can be called as *Hardy type absorption potentials* for many different reasons which will be presented in this paper). The case $\alpha > 2$ requires some special notion of solution (see [26]) and will be treated separately in a different paper. So, our main result (see Theorem 4.1 below) proves that if $\alpha = 2$ the probability to find the particle outside Ω is zero assumed that at $t = 0$ it is located in Ω.

Notice that in the radial case the singularity at the origin is always present (even for bounded potentials) due to the term $\frac{l(l+1)}{r^2}u_{n,l}(r)$, nevertheless there is a large class of singular potentials presenting singularities for other values of $r > 0$. This is the case, for instance of the so called Pösch-Teller potential (see [49])

$$V(x) = V(|x|) = \frac{1}{2}V_0\left\{\frac{k(k-1)}{\sin^2\alpha\,|x|} + \frac{\mu(\mu-1)}{\cos^2\alpha\,|x|}\right\}, \tag{9}$$

for some $V_0, \alpha > 0, k, \mu \ge 0$, intensively studied since 1933 (see, e.g. the monograph [41]). The special case of $V_0 = 2, \alpha = 1$ and $\mu = 0$ was studied in [17] as an important examples of the so-called *supersymmetric potentials* (SUSY).

The content of this paper is organized in the following way. In Sect. 2 we show the localization of the associated eigenfunctions by using an energy method which allow to show that the eigenfunctions of the corresponding eigenvalue problem on Ω are " flat solutions" (in the sense that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$). We also extend, in Sect. 2, the study of flat solutions for a class of problems in which the singularity over $\partial\Omega$ is not presented by any absorption potential but for a diffusion coefficient. This remarkable property of some *linear problems* seems to be not considered before in the previous literature except in a recent paper dealing with some eigenvalue problem under symmetry assumptions (see [35]).

The application of the super and subsolution method to the study of the flat eigenfunctions of the associated Dirichlet problem is presented in Sect. 3. In some sense this can be understood as the natural extension to the N- dimensional case of the technique of proof developed in [22] for the one-dimensional case. Unfortunately the situation becomes much harder for $N > 1$ (for instance, there is a lack of information on the classification of the set of nodal solutions of the auxiliary semilinear problem, in contrast to the very rich answers given in [28,31] for $N = 1$). In fact we improve the results of [22] since here the study is not limited to the first eigenfunction (which is positive in Ω) but to any eigenfunction corresponding to the set of countable eigenvalues. The estimates, near $\partial\Omega$, found in this section are sharper than the ones obtained in Sect. 2.

Finally, Sect. 4 is devoted to the consideration of the evolution problem (1). Besides to state and prove the main result of this paper (Theorem 4.1) we present several commentaries on possible generalizations. In particular it shown that the wave function $\psi(t, x)$ corresponding to the Pösch–Teller potential can exhibit " holes" for finite-time intervals, in contrast with many other singular potential satisfying (44) with $\alpha \in [0, 2)$ (see [18,42–44]).

2 An energy method for the study of flat eigenfunctions

2.1 The Schrödinger equation with singular potentials

As mentioned before, given a potential $V \in L^1_{loc}(\Omega)$, $V > 0$ on Ω, an open regular bounded set of $\mathbb{R}^N$, $N \geq 1$ (the value of V on $\mathbb{R}-\Omega$ being irrelevant for our purposes: see Remarks 2.3 and 4.2 below), the study of problem (2) leads to the consideration of the associated Dirichlet problem on Ω

$$DP(V, \lambda, \Omega) \begin{cases} -\Delta u + V(x)u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

In which follows we shall consider *Hardy type absorption potentials,* i.e. such that

$$\frac{\underline{C}}{d(x, \partial\Omega)^2} \leq V(x) \leq \frac{\overline{C}}{d(x, \partial\Omega)^2} \quad \text{a.e.} \quad x \in \Omega, \tag{10}$$

for some $\overline{C} > \underline{C} > 0$. It is useful to introduce the following notation:

Definition 2.1 We say that a function $u \in H^1_0(\Omega)$ is a " *flat solution*" of problem $DP(V, \lambda, \Omega)$ if u satisfies $DP(V, \lambda, \Omega)$, $u(x) \neq 0$ for a.e. $x \in \Omega$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

A very detailed analysis of the eigenvalues and eigenfuctions of the linear problem $DP(V, \lambda, \Omega)$, under condition (10) for the special case of $\overline{C} = \underline{C} > 0$, was carried out in [16] (see also some classical approach in some PDE's or Quantum Mechanics textbooks, as, e.g. [39,51,54,55]) but they do not consider the possibility to get flat solutions as eigenfunctions.

As a general first result we have:

Proposition 2.1 *Assume* (10)*, then there exists a sequence of eigenvalues* $\lambda_n \to +\infty$*,* $\lambda_1 > \lambda_{1,\Omega}$ *(the first eigenvalue for the Dirichlet problem for the* $-\Delta$ *operator on* Ω*),* λ_1 *is isolated and* $u_1 > 0$ *on* Ω*.*

Proof We start by arguing as in the proof of Theorem 3.2 of [30]. For any $h \in L^2(\Omega)$ we define the operator $Th = z \in H_0^1(\Omega)$ solution of the linear problem

$$\begin{cases} -\Delta z + V(x)z = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

This operator is well defined since problem (11) has a unique (weak) solution $z \in H_0^1(\Omega)$. This follows from applying the Lax-Milgram Lemma to the associated bilinear form in $H_0^1(\Omega)$

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega V(x)uv\, dx$$

which is well-defined, continuous and coercive. Indeed, taking into account that

$$V(x) \leq \frac{\overline{C}}{d(x, \partial\Omega)^2} \quad \text{a.e.} \quad x \in \Omega,$$

(thanks to assumption (10)) Hardy's inequality implies that

$$\frac{1}{\overline{C}} \int_\Omega V(x)u^2 dx \leq \int_\Omega \frac{u^2}{d(x)^2}\, dx \leq k \int_\Omega |\nabla u|^2\, dx$$

for some suitable constant $k = k(\Omega)$ and then

$$a(u, u) \leq C\, \|u\|^2_{H_0^1(\Omega)}$$

for some $C > 0$, which implies that a is continuous (the coerciveness of a is obvious since $V(x) \geq \underline{C}/\max_\Omega d(x, \partial\Omega)^2$). Thus, for any $h \in L^2(\Omega)$, there exists a unique $Th \in H_0^1(\Omega)$ solution of the above equation and it is easy to see that the composition with the (compact) embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is a selfadjoint compact linear operator $\widetilde{T} = i \circ T : L^2(\Omega) \to L^2(\Omega)$ for which we obtain in the usual way a sequence of eigenvalues $\lambda_n \to +\infty$. By well-known results (see e.g. [12,51]) we know that $\lambda_1 > 0$. In fact, since $V(x) \geq 0$, we know that $\lambda_1 > \lambda_{1,\Omega}$. Moreover we know that λ_1 is isolated and that $u_1 > 0$. □

Remark 2.1 As mentioned in the Introduction, we recall that in the one-dimensional case $\Omega = (-R, R)$ we have $\lambda_{1,\Omega} = \left(\frac{\pi}{2R}\right)^2$ and that by well known results, if $\Omega = B_R(0) \subset \mathbb{R}^N$ and $V(x) = V(|x|)$ with $N > 1$ then $\lambda_{1,\Omega} > \left(\frac{\pi}{2R}\right)^2$.

As usual in Quantum Mechanics we shall pay attention to the associate eigenfunctions with normalized L^2-norm, i.e. such that

$$\|u_n\|_{L^2(\Omega)} = 1. \tag{12}$$

The following result shows that the only assumption (10) suffices to ensure that any eigenfunction u_n is a flat solution.

Theorem 2.1 *Let u_n be an eigenfunction associated to the eigenvalue λ_n. Then u_n is a flat solution of $DP(V, \lambda_n, \Omega)$. In fact, there exists $\overline{K}_n > 0$ such that*

$$|u_n(x)| \leq \overline{K}_n d(x, \partial\Omega)^2 \quad a.e. \quad x \in \Omega. \tag{13}$$

The main idea of the proof will consist in the use of an appropriate set of test functions and a Moser-type iterative argument (see [34,36]) leading to a quantitative estimate of $\|u_n\|_{L^\infty(\Omega)}$ in terms of λ_n. Notice that taking $\varphi = u_n$ as test function, and using (10) we conclude that

$$\int_\Omega |\nabla u_n|^2 \, dx + \int_\Omega \frac{C}{\delta(x)^2} |u_n|^2 \, dx \leq \lambda_n \int_\Omega |u_n|^2 \, dx = \lambda_n. \tag{14}$$

This provides an estimate of $\|u\|_{H_0^1(\Omega)}$ in terms of λ_n. In the one-dimensional case, since $H_0^1(\Omega) \subset L^\infty(\Omega)$ this implies an L^∞-estimate but this is not so if $N \geq 2$. Moreover, we shall get a sharper L^∞-estimate valid for any dimension N.

Given $n \in \mathbb{N}$ and $M, \kappa > 0$, we consider the set of truncate test functions of the form

$$\varphi(x) = v_{n,M}^{2\kappa+1}(x), \quad \text{with} \quad v_{n,M}(x) := \min\{|u_n(x)|, M\} sign(u_n(x)). \tag{15}$$

Since $\varphi \in H_0^1(\Omega)$ is an appropriate test function

$$\begin{aligned}
&(2\kappa + 1) \int_\Omega \left|v_M^{2\kappa}(x)\right| |\nabla u_n|^2 \, dx + \int_\Omega \frac{C}{\delta(x)^2} \left|v_M^{2\kappa+1}(x)\right| |u_n| \, dx \\
&\leq (2\kappa + 1) \int_\Omega \left|v_M^{2\kappa}(x)\right| |\nabla u_n|^2 \, dx + \int_\Omega V(x) \left|v_M^{2\kappa+1}(x)\right| |u_n| \, dx \\
&= \lambda_n \int_\Omega \left|v_M^{2\kappa+1}(x)\right| |u_n| \, dx
\end{aligned} \tag{16}$$

where from now and in what follows we use the simplified notation $v_M = v_{n,M}$ and

$$\delta(x) = d(x, \partial\Omega).$$

The following lemma was proved in [34] for the case $n = 1$ (and some other additional conditions) but it can be easily adapted to the case of changing sign eigenfunctions.

Lemma 2.1 [34] *Let $n \in \mathbb{N}$ and $M, \kappa > 0$. Then*

$$\begin{aligned}
\lim_{M \to +\infty} \int_\Omega \tfrac{C}{\delta(x)^2} \left|v_M^{2(\kappa+1)}(x)\right| dx &= \lim_{M \to +\infty} \int_\Omega \tfrac{C}{\delta(x)^2} \left|v_M^{2\kappa+1}(x)\right| |u_n| \, dx \\
&= \int_\Omega \tfrac{C}{\delta(x)^2} \left|u_n^{2(\kappa+1)}(x)\right| dx \leq \lambda_n \int_\Omega \left|u_n^{2(\kappa+1)}(x)\right| dx.
\end{aligned}$$

Proof Since $|v_M| \nearrow |u_n|$ monotonically as $M \nearrow +\infty$, for a.e. $x \in \Omega$, we get the conclusion from the monotone convergence theorem and the inequality

$$\begin{aligned}
\int_\Omega \frac{C}{\delta(x)^2} \left|v_M^{2(\kappa+1)}(x)\right| dx &\leq \int_\Omega \frac{C}{\delta(x)^2} \left|v_M^{2\kappa+1}(x)\right| |u_n| \, dx \\
&\leq \lambda_n \int_\Omega \left|v_M^{2\kappa+1}(x)\right| |u_n| \, dx \leq \lambda_n \int_\Omega \left|u_n^{2(\kappa+1)}(x)\right| dx,
\end{aligned}$$

which is a consequence of the energy estimate (16) (remember that we already know the uniform in M estimate $\int_\Omega \left|u_n^{2(\kappa+1)}(x)\right| dx \leq 2R\lambda_n^{2(\kappa+1)}$). □

A Moser-type iterative argument leads to an inequality which can be understood as an *inverse Hölder type inequality*:

Lemma 2.2 [34] *Let $n \in \mathbb{N}$ and let $q \in (1, +\infty)$ if $N \leq 2$ and $q = 2^* := 2N/(N-2)$ if $N \geq 3$. Then there exists a sequence $\kappa_m \to +\infty$ if $m \to +\infty$ and a constant $C(q, \underline{C}, \lambda_n, \Omega)$ such that*

$$\|u_n\|_{2(\kappa_m+1)q} \leq C(q, \underline{C}, \lambda_n, \Omega)\, \|u_n\|_q \text{ for any } m \in \mathbb{N}. \tag{17}$$

In particular,

$$\|u_n\|_\infty \leq C(q, \underline{C}, \lambda_n, \Omega)\, \|u_n\|_q \,.$$

Remark 2.2 The proof of this lemma is identical to the one given in [34] although there are some slight differences in the framework considered there. The dependence with respect Ω appears not only because, obviously, λ_n depend on Ω but also because in the proof it is used that $C_\Omega \leq 1/\delta^2(x)$ for any $x \in \Omega$ and for some $C_\Omega > 0$.

Proof of Theorem 2.1 From Lemma 2.1

$$\left[\int_\Omega \frac{\underline{C}}{\delta(x)^2} \left|u_n^{2(\kappa+1)}(x)\right| dx\right]^{\frac{1}{2(\kappa+1)}} \leq \lambda_n^{\frac{1}{2(\kappa+1)}}\, \|u_n\|_{2(\kappa+1)} \,.$$

Then, by Lemma 2.2, for any fixed $q \in (1, +\infty)$ and for each $m \in \mathbb{N}$,

$$\left[\int_\Omega \frac{\underline{C}}{\delta(x)^2} \left|u_n^{2(\kappa_m+1)}(x)\right| dx\right]^{\frac{1}{2(\kappa_m+1)}} \leq \lambda_n^{\frac{1}{2(\kappa_m+1)}}\, \|u_n\|_{2(\kappa_m+1)q} \leq C(q, \underline{C}, \lambda_n, R)\, \|u_n\|_q \,.$$

Making $m \to +\infty$ we get that there exists $\overline{K}_n = \overline{K}_n(q, \underline{C}, \lambda_n, R)$ such that

$$\frac{1}{\delta(x)^2}\, |u_n(x)| \leq \sup_{y \in \Omega} ess \frac{1}{\delta(y)^2}\, |u_n(y)| \leq \overline{K}_n$$

which proves (13). □

As a particular consequence of Theorem 2.1 it is possible to offer a correct alternative to the "localizing" process suggested by Gamow in his paper [40].

Corollary 2.1 *Let Ω be an open regular bounded set of $\mathbb{R}^N$, $N \geq 1$. For any $q \in [0, +\infty)$ consider the potential*

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^N - \Omega. \end{cases}$$

Assume (10). Then there exists a countable set of eigenvalues λ_n and eigenfunctions $\widetilde{u}_{n,q}$ of the Schrö dinger equation

$$-\Delta u + V_{q,\Omega}(x) u = \lambda_n u \ \text{ in } \mathbb{R}^N, \tag{18}$$

such that

$$\widetilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N - \Omega, \end{cases}$$

where λ_n and $u_n(x)$ are the eigenvalues and eigenfunctions of the Dirichlet problem $DP(V, \lambda, \Omega)$. Moreover the same conclusion holds for $q = +\infty$ if we define the corresponding solution as $\widetilde{u}_{n,\infty}(x) = \lim_{q \nearrow +\infty} \widetilde{u}_{n,q}(x)$.

Proof of Corollary 2.1 Thanks to Theorem 2.1 we have $\widetilde{u}_n \in H^1(\mathbb{R}^N)$. Moreover on $\mathbb{R}^N - \Omega$ we trivially have $-\Delta u + V_{a,\Omega}(x)u = \lambda_n u$, so the conclusion follows. The convergence $\widetilde{u}_{n,\infty}(x) = \lim_{q \nearrow +\infty} \widetilde{u}_{n,q}(x)$ follows the same arguments as in the one-dimensional case (see Lemma 2.1 of [22] and Proposition 4 of [26]). □

Remark 2.3 Notice that no Dirac delta is generated on the boundary $\partial\Omega$ once we assume (10). Moreover, by construction of the extension $\widetilde{u}_n(x)$ over $\mathbb{R}^N - \Omega$ the value of of the extension of $V(x)$ over $\mathbb{R}^N - \Omega$ is irrelevant. Notice that this is peculiar to the special construction of our solution $\widetilde{u}_n$ since otherwise some conditions on the behaviour of $V(x)$ for $|x|$ large enough must be assumed for the existence of weak solutions (see, e.g. [15,37,51] and their references).

Remark 2.4 It is possible to consider some unbounded domains of Ω with the help of the modification of the eigenvalue problem with an auxiliary weight function (so that the right hand side reads as $\lambda_n \sigma(x) u$ for some weight function $\sigma(x)$). Some related results can be found in [32,35,36].

2.2 Flat solution of a problem with singular diffusion coefficient (without absorption term)

In fact, the same type of localizing conclusions also holds for other types of linear eigenvalue problems in which the singularity appears merely in the diffusion operator. Consider for instance the singular diffusion problem

$$SD(a, \Omega, f) \begin{cases} -\mathrm{div}(a(|x|)\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where now we assume

$$\Omega = B_R(0), \quad \text{for some} \quad R > 0,$$

and the crucial assumption

$$a(|x|) = \delta(x)^{-\gamma} = (R - |x|)^{-\gamma} \text{ for some } \gamma \in (0, 1). \tag{20}$$

By defining the weighted Sobolev space $W^{1,p}(\Omega, \delta^{-\gamma})$ as usual (see, e.g. [36,47]), equipped with the norm

$$\|u\|_{W^{1,p}(\Omega,\delta^{-\gamma})} := \left(\sum_{|\alpha| \le 1} \int_\Omega \left| D^\alpha u \right|^p \delta(x)^{-\gamma} dx \right)^{1/p}$$

the existence (comparison and uniqueness) of a weak solution $u \in H_0^1(\Omega, \delta^{-\gamma})$ of the linear singular problem is a well-known result if $f \in L^2(\Omega, \delta^{-\gamma})$ and, in fact, the existence and comparison of the so-called " entropy solutions" holds if $f \in L^1(\Omega, \delta^{-\gamma})$ (see, e.g. [19] and its references). As a matter of fact, by applying the Hardy inequality it is possible to show that

$$\|u\|_{W_0^{1,p}(\Omega,\delta^{-\gamma})} := \left(\sum_{|\alpha| = 1} \int_\Omega \left| D^\alpha u \right|^p \delta(x)^{-\gamma} dx \right)^{1/p}$$

is an equivalent norm for the space $H_0^1(\Omega, \delta^{-\gamma})$ and that the existence, comparison and uniqueness, of weak solutions $u \in H_0^1(\Omega, \delta^{-\gamma})$ can be extended to more general right hand side data $f \in L^2(\Omega,\sigma)$ for some weight function $\sigma(|x|)$ once that the embedding $H_0^1(\Omega,\delta^{-\gamma}) \subset L^2(\Omega,\sigma)$ is compact. In the radial case, by taking the weights

$$\rho(r) := r^{N-1}(R-r)^{-\gamma} \quad \text{and} \quad \sigma(r) := r^{N-1}(R-r)^{\beta},$$

the result holds if

$$\left[\gamma > -1 \text{ and } \beta \geq -1\right] \quad \text{or} \quad \left[\beta < -1 \text{ and } \gamma > -2-\beta\right]. \tag{21}$$

(see [35]).

Theorem 2.2 *Assume $f \in L^2(\Omega,(R-|x|)^{\beta})$ such that*

$$|f(x)| \leq C\delta(x)^{1+\gamma+\beta} \quad a.e. \quad x \in \Omega. \tag{22}$$

Then u is a flat solution and

$$|u(x)| \leq K\delta(x)^{1+\gamma} \quad a.e. \quad x \in \Omega.$$

Proof Consider the auxiliary radially symmetric problem with singular diffusion

$$SD(a,\lambda,\beta,R) = \begin{cases} -\mathrm{div}(a(|x|)\nabla U) = \lambda(R-|x|)^{\beta}U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases}$$

under conditions (21). It was shown in ([35]) that there is a first eigenvalue λ_1 of $SD(a,\lambda,\beta,R)$ which is simple and that, modulo a multiplicative constant, the associated eigenfunction U_1 satisfies

$$0 < U(|x|) \leq K(R-|x|)^{1+\gamma} \quad \text{a.e.} \quad x \in \Omega.$$

Then, by taking

$$\overline{u}(x) = \frac{C}{\lambda_1}(R-|x|)^{\beta}U(|x|)$$

we have that $-\mathrm{div}(a(|x|)\nabla u) \leq -\mathrm{div}(a(|x|)\nabla\overline{u})$ on Ω and since $u \leq \overline{u}$ on $\partial\Omega$ we get that $u(x) \leq K\delta(x)^{1+\gamma}$ a.e. $x \in \Omega$ thanks to the comparison principle. The proof of $u(x) \geq -K\delta(x)^{1+\gamma}$ a.e. $x \in \Omega$ is similar. □

Remark 2.5 Notice that the conclusion holds even if $f(x)$ is singular near the boundary $\partial\Omega$. For instance functions of the type $f(x) = C\delta(x)^{-\theta}$satisfy all the requirements to generate a flat solution in $u \in H_0^1(\Omega,\delta^{-\gamma})$ if we take $\theta \in (0,1/2)$. In the case of L^1-solutions it is enough to assume $\theta \in (0,1)$.

Remark 2.6 Many variations of the above theorem can be proved. For instance, it is possible to consider weighted p-Laplacian quasilinear diffusion operators, the equation can be generalized to equations containing transport and absorption terms and the result holds for transmission solutions for non necessarily symmetric domains Ω such that $\Omega \supset B_R(0)$, for some $R > 0$ and we assume

$$f(x) = \begin{cases} f_R(x) & \text{if } x \in B_R(0), \\ 0 & \text{if } x \in \Omega - B_R(0), \end{cases}$$

with $f_R \in L^2(\Omega,(R-|x|)^{\beta})$ satisfying (22). In that case the (unique) solution of $SD(a,\Omega,f)$ is given by

$$u(x) = \begin{cases} u_R(x) & \text{if } x \in B_R(0) \\ 0 & \text{if } x \in \Omega - B_R(0). \end{cases}$$

Remark 2.7 In contrast to the large amount of papers on elliptic free boundary problems (see, e.g. the monographs [2,21]) the occurrence of flat solutions when the only " mechanism" justifying such a phenomenon is the singularity of the diffusion coefficient seems to have being unexplored in the previous literature. This property could justify, perhaps, the appearance of some free boundaries for some quasilinear equations under anomalous criteria to the usual ones and which involve (in different ways) the 1-Laplacian operator (see, e.g. [1,25]).

3 Application of the super and subsolution method to the study of flat solutions

3.1 On the radially symmetric semilinear auxiliary problem

In this subsection we shall pay attention to the radially symmetric problem $DP(V, \lambda, \Omega)$ when we assume that

$$\Omega = B_R(0), \quad \text{for some} \quad R > 0 \quad \text{and} \quad V(x) = V(|x|) \quad \text{for a.e.} \quad x \in \Omega.$$

As in the one-dimensional case ([22]) it will be useful to start by considering the auxiliary semilinear eigenvalue type problem

$$P(R, m, V_0, \lambda) \equiv \begin{cases} -\Delta v + V_0 |v|^{m-1} v = \lambda v, & v \geq 0 \quad \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

for a given $V_0 > 0$ and $m \in (0, 1)$. We shall prove:

Proposition 3.1 *(i) Let*

$$\lambda^{\#} = R^{2N^2} \tag{23}$$

with $\omega_N := |B_1(0)|$. Then, for any $\lambda > \lambda^{\#}$, there exists a radially symmetric weak solution u_λ of problem $P(R, m, V_0, \lambda)$ such that

$$\|v_\lambda\|_{L^\infty(\Omega)} \leq \frac{C}{\lambda^{\frac{1}{1-m}}} \tag{24}$$

for some $C > 0$ depending only on R. In addition the above weak solutions v_λ have compact support in Ω and satisfy

$$v_m(x) \leq \overline{K} d(x, \partial(\mathit{support}\ v_m))^{2/(1-m)} \quad \mathit{for\ any}\ x \in \mathit{support}\ v_m \tag{25}$$

for some constant $\overline{K}$.

(ii) For $\lambda > \lambda^{\#}$, there exists $\underline{K} > 0$

$$\underline{K} d(x, \partial(\mathit{support}\ v_m))^{2/(1-m)} \leq v_m(x) \quad \mathit{for\ any}\ x \in \mathit{support}\ v_m \tag{26}$$

for some constant $\underline{K}$.

Proof Since $\lambda^{\#} = \left(\frac{|\Omega|}{\omega_N}\right)^{2N}$, the first part of property i) is a particular case of Theorem 1 of [24] (see also [30]). We also recall that by the results of [46] any solution of $P(R, m, V_0, \lambda)$ must be radially symmetric. So, it remains to prove estimate (25) and (26). If we make the change of variables

$$v_\lambda(x) = \left(\frac{V_0}{\lambda}\right)^{\frac{1}{1-m}} U(\sqrt{\lambda} x), \tag{27}$$

with v_λ solution of $P(R, m, V_0, \lambda)$, then U satisfies

$$P(L,m) = \begin{cases} -\Delta U + |U|^{m-1}U = U & \text{in } B_L(0), \\ U = 0 & \text{on } \partial B_L(0), \end{cases} \tag{28}$$

with

$$L = \sqrt{\lambda} R.$$

In order to get the estimates mentioned in i) and ii) we shall use the fact that

$$-U''(r) - \frac{N-1}{r}U'(r) + f(U(r)) = 0$$

where

$$f(x,u) := u^m - u.$$

Then

$$F(r) := \int_0^r f(s)ds = \frac{r^2}{2} - \frac{r^{m+1}}{m+1}.$$

Notice that $f(s) < 0$ if $0 < s < 1 := r_f$ and $f(s) > 0$ if $1 < s$. On the other hand $F(s) < 0$ if $0 < s < r_F = (2/(1+m))^{1/(1-m)}$ and $F(s) > 0$ for $s > r_F$. As a matter of fact, if $U_1 \in C^1([0, L])$ is the solution of the one-dimensional equation

$$\begin{bmatrix} -U_1''(r) + f(U_1(r)) = 0 \; r \in (0,1), \\ U_1'(0) = 0, U_1(L) = 0, \end{bmatrix}$$

it was shown in [22] that the flat solution satisfies that

$$U_1(0) = r_F$$

and that

$$\frac{1}{\sqrt{2}} \int_{U_1(x)}^{r_F} \frac{dr}{(-F(r))^{1/2}} = |x|, \quad \text{for} \quad |x| \leq L,$$

where L is such that

$$L = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/2}}. \tag{29}$$

Notice that $L < +\infty$ due to the assumption $m \in (0, 1)$. Moreover, it was shown in [22] that

$$0 \leq U_1(r) \leq \overline{M} \, |L - r|^{\frac{1-m}{2}} \quad \text{for some} \quad \overline{M} > 0.$$

Notice that $U_1(r)$ is a supersolution to our problem and that it satisfies estimate (25). Coming back to the N-dimensional problem, we know that $U \in C^1([0, L])$ and that $U'(r) \leq 0$ for any $r \in [0, L]$. Moreover, as mentioned before, there exists $L_0 \in (0, L]$ such that

$$U(r) = 0 \text{ for any } r \in [L_0, L].$$

Without loss of generality we can assume that there exists $L_1 \in (0, L_0)$ such that

$$0 < U(L_1) \leq 1.$$

On the ring $r \in (L_1, L_0)$ we know that $0 \leq U(r) \leq 1$, so that $-F(U(.))$ is a monotone non-decreasing function of U over such ring. Let us define $Y(s)$, when $s \in [0, L_0 - L_1]$, by the expression

$$U(r) = Y(L_1 + s).$$

Then $Y'(s) \leq 0$ and

$$Y''(s) + \frac{N-1}{L_1+s} Y'(s) = -f(Y(s)).$$

Since

$$\frac{N-1}{L_0} \leq \frac{N-1}{L_1+s} \leq \frac{N-1}{L_1} \text{ if } s \in [0, L_0 - L_1]$$

we get

$$Y''(s) + \frac{N-1}{L_1} Y'(s) \leq -F'(Y(s)) \leq Y''(s) + \frac{N-1}{L_0} Y'(s).$$

Multiplying by $Y'(s)$ we have

$$e^{-\underline{\Lambda} s}(\frac{1}{2}\left(Y'(s)\right)^2 e^{\underline{\Lambda} s})' \leq -F(Y(s))' \leq e^{-\overline{\Lambda} s}(\frac{1}{2}\left(Y'(s)\right)^2 e^{\overline{\Lambda} s})' \tag{30}$$

where

$$\underline{\Lambda} := \frac{N-1}{L_1} \quad \text{and} \quad \overline{\Lambda} := \frac{N-1}{L_0}.$$

In particular, since $Y'(s) = Y(s) = 0$ if $s = L_0 - L_1$, by integrating in (30) we get

$$-\sqrt{2e^{-\overline{\Lambda} s}}\sqrt{-e^{\overline{\Lambda} s} F(Y(s))} \leq -Y'(s) \leq -\sqrt{2e^{-\underline{\Lambda} s}}\sqrt{-e^{\underline{\Lambda} s} F(Y(s))}. \tag{31}$$

Thus, if for a positive parameter θ we denote by $\eta_\theta(s)$ to the solution of the ordinary differential equation

$$\begin{cases} -\eta_\theta'(s) = \theta\sqrt{-F(\eta(s))} \\ \eta_\theta(L - L_1) = 0, \end{cases}$$

by the comparison of solutions for ordinary differential equations, we get that

$$\eta_{\underline{\theta}}(s) \leq Y(s) \leq \eta_{\overline{\theta}}(s) \quad \text{for any} \quad s \in (\delta, L_0 - L_1]$$

for some $\delta, \underline{\theta}, \overline{\theta} > 0$. Thanks to the estimate (31) we get that there exist two positive constants $\underline{M} < \overline{M}$ such that

$$\underline{M}\tau^{\frac{1-m}{2}} \leq \frac{1}{\sqrt{2}} \int_0^\tau \frac{dr}{\sqrt{-F(r)}} \leq \overline{M}\tau^{\frac{1-m}{2}} \tag{32}$$

for any $\tau \in (0, 1)$ and (25) (26) follows. □

Remark 3.1 We recall that by the results of [45] then there is a unique flat solution of problem $P(L, m)$ for a suitable $L = \sqrt{\lambda}R$, but we do not know if this value of L corresponds exactly to $\sqrt{\lambda^{\#}}R$. In the one-dimensional case this value of L can be determined explicitly: see expression $\gamma(2/(1+m))^{1/(1-m)})$ in formula (2.6) of [22]).

3.2 Non-radially symmetric case

In the case of a general regular open bounded set Ω of $\mathbb{R}^N$ it is possible to improve the estimate given in Theorem 2.1 implying that the eigenfunctions are flat solutions.

Theorem 3.1 *Assume (10). Let u_n be the normalized eigenfunction of $DP(V, \lambda_n, \Omega)$ associated to the eigenvalue λ_n. Then there exists $m \in (0, 1/2)$ and a constant $\overline{K}_{n,E}$ such that*

$$|u_n(x)| \leq \overline{K}_{n,E} d(x, \partial\Omega)^{\frac{2}{1-m}} \quad a.e.\ x \in \Omega. \tag{33}$$

Proof From (10) and Theorem 2.1 we know that

$$\begin{cases} -\Delta u_n + V(x)u_n = \lambda_n u \leq \lambda_n |u| \leq \lambda_n \overline{K}_n d(x, \partial\Omega)^2 & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

So, by the comparison principle, $u_n(x) \leq \widehat{u}_n(x)$ a.e. $x \in \Omega$, where $\widehat{u}_n \geq 0$ is the unique solution of

$$\begin{cases} -\Delta\widehat{u}_n + V(x)\widehat{u}_n = \lambda_n \overline{K}_n d(x, \partial\Omega)^2 & \text{in} \quad \Omega, \\ \widehat{u}_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

In particular, if we denote

$$\delta(x) = d(x, \partial\Omega) \tag{34}$$

then

$$-\Delta\widehat{u}_n + \frac{C}{\delta(x)^2}\widehat{u}_n \leq \lambda_n \overline{K}_n \delta(x)^2.$$

Now, let $x_0 \in \partial\Omega$ and for $m \in (0, 1/2)$ consider the barrier function

$$U(x; x_0) = K\,|x - x_0|^{\frac{2}{1-m}}.$$

It was shown in Lemma 6 of [21] that

$$-\Delta U + \mu |U|^{m-1} U = C(K)\,|x - x_0|^{\frac{2m}{1-m}},$$

with

$$C(K) := (\mu K^m - K^{m-1} \frac{m^{(1-m)}(2m + N(1-m))}{(1-m)^m}).$$

In particular, if $K = K_\mu$

$$K_\mu < \left[\frac{\mu(1-m)^2}{2(2m + N(1-m))}\right]^{\frac{1}{1-m}}, \tag{35}$$

we have that $C(K_\mu) > 0$ (notice that $C(K) = 0$ if in (35) the symbol $<$ is replaced by $=$). Consider the set

$$\Omega_R(x_0) := \Omega \cap B_R(x_0).$$

It is clear that $\delta(x) \leq |x - x_0|$ on $\Omega_R(x_0)$. Then if K satisfies (35) we get

$$\begin{aligned} -\Delta U + \frac{\mu}{K^{(1-m)}\delta(x)^2} U &\geq -\Delta U + \frac{\mu}{K\,|x - x_0|^2} U = -\Delta U + \frac{\mu}{|U|^{1-m}} U \\ &= C(K)\,|x - x_0|^{\frac{2m}{1-m}} \geq C(K)\delta(x)^{\frac{2m}{1-m}}. \end{aligned}$$

Thus, if

$$\frac{\mu}{K_\mu^{(1-m)}} \leq \underline{C} \tag{36}$$

and

$$C(K_\mu) R^{\frac{2m}{1-m}} \geq \lambda_n \overline{K}_n R^2 \tag{37}$$

we have that

$$-\Delta \widehat{u}_n + \frac{C}{\delta(x)^2} \widehat{u}_n \leq -\Delta U + \frac{C}{\delta(x)^2} U \text{ in } \Omega_R(x_0)$$

(recall that $\frac{2m}{1-m} < 2$ since $m \in (0, 1/2)$). Notice that conditions (36) and (37) hold if we take μ large enough, then

$$R \leq \left[\frac{C(K_\mu)}{\lambda_n \overline{K}_n} \right]^{\frac{(1-m)}{2(1-2m)}} \tag{38}$$

(notice that $C(K_\mu) \nearrow +\infty$ if $\mu \nearrow +\infty$). On the other hand, by choosing R large enough we trivially have

$$\widehat{u}_n \leq U \text{ on } \partial\Omega_R(x_0),$$

since on $\partial\Omega_R(x_0) - \partial\Omega$ (i.e. if $|x - x_0| = R$)

$$\widehat{u}_n(x) \leq \|\widehat{u}_n\|_\infty \leq U(x) = K_\mu R^{\frac{2}{1-m}}$$

once we assume

$$R \geq \left(\frac{\|\widehat{u}_n\|_\infty}{K_\mu} \right)^{\frac{1-m}{2}}. \tag{39}$$

In consequence, by taking μ large enough we can choose R satisfying (38) and (39). Then, for such a choice of μ and R, by the comparison principle,

$$\widehat{u}_n(x) \leq K_\mu |x - x_0|^{\frac{2}{1-m}} \text{ in } \overline{\Omega_R(x_0)}.$$

Moreover, as $x_0 \in \partial\Omega$ is arbitrarily chosen, by taking the envelop of $U(x; x_0)$ when $x_0 \in \partial\Omega$, we get the conclusion. The estimate from above of u_n is obtained in a similar way. □

Remark 3.2 Notice that $m \in (0, 1/2)$ implies that $\frac{2}{1-m} < 4$ and thus the improved estimate (33) and the standard regularity for linear equations shows that the solution u_n is at least of class $C^3(\overline{\Omega})$ (see also [29] and [33]). This same conclusion can be obtained by using as supersolution $U(x) = C\delta(x)^b$ for some $b < 4$ and playing with the Euler ordinary differential equation in a similar way to Lemma 2.8 of [3] (see also [4]).

We shall end this subsection by showing an estimate from below for the eigenfunctions u_n.

Proposition 3.2 *We have*

$$Kd(x, \partial\Omega)^{\frac{2}{1-m}} \leq |u_n(x)| \quad a.e. \quad x \in \Omega.$$

Proof Let $\omega \subset \Omega$. Assume for simplicity that $u_n(x) > 0$ on ω. Then, for any $\varepsilon > 0$ small, $m \in (0, 1)$ and for any $x_0 \in \partial\omega_\varepsilon$, where

$$\omega_\varepsilon = \{x \in \omega \quad \text{such that } d(x, \omega) \geq \varepsilon\},$$

let $\underline{u}$ be the unique solution of the eigenvalue type problem

$$\begin{cases} -\Delta\underline{u} + \mu\underline{u}^m = \lambda_n(\Omega)g(x)\underline{u} & \text{in } \omega, \\ \underline{u} = 0 & \text{on } \partial\omega, \end{cases} \tag{40}$$

for

$$\frac{\lambda_n(\omega_\varepsilon)}{\lambda_n(\Omega)} \geq g(x) \geq C(K)d(x, \omega_\varepsilon)^{\frac{2m}{1-m}} \text{ a.e. } x \in \omega_\varepsilon,$$
$$\frac{\lambda_n(\omega_\varepsilon)}{\lambda_n(\Omega)} = g(x) \text{ on } \omega - \omega_\varepsilon,$$

(we know that $\lambda_n(\omega_\varepsilon) > \lambda_n(\Omega)$). Then, if ω_ε is a ball, arguing as in the above subsection and using local barrier functions we conclude that

$$\underline{u}(x) \geq Cd(x, \omega_\varepsilon)^{\frac{2}{1-m}} \quad \text{a.e.} \quad x \in \omega_\varepsilon.$$

Since we cannot apply directly the comparison principle, we shall apply the iterative method of super and subsolutions so that if $\underline{u}(x)$ and $\overline{u}(x)$ are respectively a subsolution and a supersolution of $DP(V, \lambda_n, \Omega)$ such that

$$\underline{u}(x) \leq \overline{u}(x) \quad \text{for any} \quad x \in \Omega, \tag{41}$$

then we get the existence of a minimal $\underline{u}^*(x)$ and maximal $\overline{u}^*(x)$ solution of $DP(V, \lambda_n, \Omega)$ such that

$$\underline{u}(x) \leq \underline{u}^*(x) \leq \overline{u}^*(x) \leq \overline{u}(x) \quad \text{for a.e.} \quad x \in \Omega.$$

As a subsolution we take $\underline{u}(x)$ extended by zero on $\Omega - \omega_\varepsilon$ and as a supersolution we take the solution of

$$\begin{cases} -\Delta\overline{u}_n + V(x)\overline{u}_n = \lambda_n \|u_n\|_\infty & \text{in} \quad \Omega, \\ \overline{u}_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Notice that by well known results

$$\overline{u}_n(x) \geq Cd(x, \Omega) \quad \text{a.e.} \quad x \in \Omega.$$

By taking ω small enough we know (see [30]) that $\lambda_n(\omega_\varepsilon)$ is large enough and that $\left\|\underline{u}\right\|_\infty$ can be assumed as small as desired, so that we have (41). Since ε and ω are arbitrarily chosen we get the result. □

Remark 3.3 Notice the absence of contradiction with some papers showing the non-existence of solutions (see, e.g. [13]) in which they consider non-absorption Hardy potentials, i.e. with non positive constants $\overline{C}$ and $\underline{C}$.

Remark 3.4 Some of the ideas of this paper can be adapted to the study the existence of "large solutions" of the same type of linear equation

$$\begin{cases} -\Delta u + V(x)u = f(x) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

when the potential V satisfies (10) (see [23]). Notice that in contrast with [3] it is not required the presence in the equation of any superlinear term of the form u^m with $m > 1$. We recall (Theorem 2.10 of [33]) that given $f \in L^1(\Omega : \delta)$, $f(x) \geq 0$ a.e. $x \in \Omega$, $V_0 > 0$, the existence of a large solution of the semilinear problem

$$\begin{cases} -\Delta v + V_0 v^m = f(x) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega, \end{cases}$$

requires now the crucial assumption $m > 1$.

4 The evolution case

As mentioned in the Introduction we consider the Schrödinger equation with potentials becoming singular on the boundary of a regular open bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$. As before, we identify $\hbar$ and $2m$ with 1. So our problem becomes

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0,\infty) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x) & \text{on } \mathbb{R}^N. \end{cases} \tag{42}$$

We consider the case of potentials with a singularity over $\partial\Omega$, i.e., such that there exists $q \in [0, +\infty)$ such that

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^N - \Omega, \end{cases} \tag{43}$$

and $V \in L^1_{loc}(\Omega)$ satisfies

$$\frac{\underline{C}}{d(x,\partial\Omega)^\alpha} \leq V(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^\alpha} \quad \text{a.e.} \quad x \in \Omega, \tag{44}$$

for some $\alpha > 0$ and some $\overline{C} > \underline{C} \geq 0$. Our interest is the study of the time evolution of localized initial wave packets $\psi_0 \in H^1(\mathbb{R} : \mathbb{C})$, i.e. such that

$$\text{support } \psi_0 \subset \overline{\Omega}.$$

The behaviour of the support of the particle $\psi(t,.)$ depends of the exponent α. Let us study the permanent confinement in Ω question under assumption (44) and more specially for $\alpha = 2$ (condition (10)).

Theorem 4.1 *Assume* (10) *and let* $\psi_0 \in H^1(\mathbb{R}^N : \mathbb{C})$ *such that support* $\psi_0 \subset \overline{\Omega}$*.Then*

(i) For $q > 0$ *consider the the extended potential* $V(x) = V_{q,\Omega}(x)$ *given by* (43)*. Then Problem* (42) *has a unique solution* $\psi \in C([0,+\infty) : L^2(\mathbb{R}^N : \mathbb{C}))$ *with* $\psi \in L^2(0,T : H^1(\mathbb{R}^N : \mathbb{C}))$ *and* $V_{q,\Omega}(x)\psi \in L^2(0,T : L^2(\mathbb{R}^N : \mathbb{C}))$ *for any* $T > 0$.

(ii) The problem

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0,\infty) \times \Omega, \\ \psi = \mathbf{0} & \text{on } (0,\infty) \times \partial\Omega, \\ \psi(0,x) = \psi_0(x) & \text{on } \Omega, \end{cases} \tag{45}$$

has a unique solution $\psi_\Omega \in C([0,+\infty) : H^2(\Omega : \mathbb{C}) \cap H^1_0(\Omega : \mathbb{C}))$*, and we have the Galerkin decomposition*

$$\psi_\Omega(t,x) = \sum_{n=1}^{\infty} \mathbf{a}_n e^{-\mathbf{i}\lambda_n t} u_n(x), \tag{46}$$

with convergence at least in $L^2(\Omega : \mathbb{C})$, where λ_n and u_n are the eigenvalues and eigenfunctions given in Proposition 2.1 (renormalized by (12)) for any n and

$$\mathbf{a}_n = \int_\Omega \psi_0(x) u_n(x) dx.$$

(iii) Assume that

$$\sum_{n=1}^{\infty} |\mathbf{a}_n| \, \overline{K}_n < +\infty \tag{47}$$

where $\overline{K}_n > 0$ was given in Theorem 2.1. Then

$$|\psi_\Omega(t,x)| \leq K d(x, \partial\Omega)^2 \quad \textit{for any } t > 0 \textit{ and a.e. } x \in \Omega, \tag{48}$$

for some $K > 0$. In consequence, the unique solution of (42) for the extended potential $V_{q,\Omega}(x)$ is given by

$$\psi(t,x) = \begin{cases} \psi_\Omega(t,x) & \textit{if } x \in \Omega, \\ \mathbf{0} & \textit{if } x \in \mathbb{R}^N - \Omega, \end{cases} \tag{49}$$

and thus support $\psi(t,.) \subset \overline{\Omega}$ for any $t > 0$.

(iv) (Tunneling effect or instantaneous propagation). If $V(x)$ satisfies (44) with $\alpha \in [0,2)$ then

$$(\textit{support}\psi(t,.)) \cap (\mathbb{R}^N - \overline{\Omega}) \neq \phi \textit{ for } t > 0.$$

Proof To prove i) we rewrite problem (42) in terms of an abstract Cauchy problem over the Banach space $X = L^2(\mathbb{R}^N : \mathbb{C})$ of the form

$$\begin{cases} \frac{dv}{dt}(t) + Av(t) = 0 & \text{in } X \\ v(0) = v_0 \end{cases} \tag{50}$$

with $v(t) = \psi(t,,.)$ and $A : D(A) \to X$ defined by

$$\begin{cases} D(A) = \{\mathbf{w} \in H_0^1(\mathbb{R}^N : \mathbb{C}) & \text{such that } (-\Delta + V(x))\mathbf{w} \in L^2(\mathbb{R}^N : \mathbb{C})\} \\ A\mathbf{w} = \mathbf{i}(-\Delta + V(x))\mathbf{w}, & \text{if } \mathbf{w} \in D(A). \end{cases}$$

Then the operator A is m-accretive in X. Indeed, given $\mathbf{g} \in L^2(\mathbb{R}^N : \mathbb{C})$ to study the existence of solution of the equation $A\mathbf{w} + \mu\mathbf{w} = \mathbf{g}$, for any $\mu > 0$ we observe that if $\mathbf{g} = g_r + \mathbf{i}g_i$ and $\mathbf{w} = w_r + \mathbf{i}w_i$ then we had to solve the uncoupled system

$$\begin{cases} -\Delta w_r + V_{q,\Omega}(x) w_r + \mu w_r = -g_i(x) & \text{in } \mathbb{R}^N, \\ -\Delta w_i + V_{q,\Omega}(x) w_i + \mu w_i = g_r(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{51}$$

The bilinear form

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} (V_{q,\Omega}(x) + \mu) uv \, \mathrm{d}x$$

is clearly coercive in $H^1(\mathbb{R}^N)$ which shows the uniqueness of solutions of (51) once that $V_{q,\Omega}(x)\mathbf{w} \in L^2(\mathbb{R}^N : \mathbb{C})$. Moreover, given $M > 0$, by truncating the potential $V_{q,\Omega}(x)$ by

$$V_{q,\Omega}^M(x) = \begin{cases} V_{q,\Omega}(x) & \text{if } V_{q,\Omega}(x) < M, \\ M & \text{if } V_{q,\Omega}(x) \geq M, \end{cases}$$

the associated bilinear form

$$a^M(u,v)=\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\mathrm{d}x+\int_{\mathbb{R}^N}(V^M_{q,\Omega}(x)+\mu)uv\,\mathrm{d}x$$

is not only coercive but also continuous on $H^1(\mathbb{R}^N)$. Then, by applying the Hille–Yosida theorem (see, e.g. [12,51]) we get the existence and uniqueness of a solution $\mathbf{w}_M=w_{M,r}+\mathbf{i}w_{M,i}$ of the associated uncoupled system (51). Moreover by multiplying by $\mathbf{w}_M$ we get the estimate

$$\|\nabla w_{M,r}\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\left(V^M_{q,\Omega}(x)+\frac{\mu}{2}\right)w^2_{M,r}\mathrm{d}x\le C(\mu)\,\|g_i\|^2_{L^2(\mathbb{R}^N)}.$$

Since by the comparison principle we get that $\left|w_{M,r}(x)\right|\ge\left|w_{M',r}(x)\right|$ for a.e. $x\in\mathbb{R}^N$ if $M'>M$, then the monotone convergence implies the existence of a subsequence such that $w_{M,r}\to w_r$ in $L^2(\mathbb{R}^N)$ (and thus $\nabla w_{M,r}\rightharpoonup\nabla w_M$ in $L^2(\mathbb{R}^N)$). In consequence

$$\|\nabla w_r\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\left(V_{q,\Omega}(x)+\frac{\mu}{2}\right)w^2_r\mathrm{d}x\le C(\mu)\,\|g_i\|^2_{L^2(\mathbb{R}^N)},$$

and so we get the existence of $w_r\in H^1(\mathbb{R}^N)$ such that $V_{q,\Omega}(x)w_r\in L^2(\mathbb{R}^N)$. The existence (and uniqueness) of solutions for the case of w_i is entirely similar. The regularity mentioned on ψ in i) (and some other additional regularity properties) can be obtained in a standard way by multiplying by ψ (see [14]).
The proof of (ii) is similar to the one of part (i) but now we rewrite problem (45) in terms of an abstract Cauchy problem (50) on the Banach space $X=L^2(\Omega:\mathbb{C})$ with $v(t)=\psi(t,,.)$ and $A:D(A)\to X$ defined by

$$\begin{cases}D(A)=\{\mathbf{w}\in H^1_0(\Omega:\mathbb{C})\quad\text{such that }(-\Delta+V(x))\mathbf{w}\in L^2(\Omega:\mathbb{C})\}\\ A\mathbf{w}=\mathbf{i}(-\Delta+V(x))\mathbf{w}\quad\text{if }\mathbf{w}\in D(A).\end{cases}$$

Then the operator A is m-accretive in X. Indeed, given $\mathbf{g}\in L^2(\Omega:\mathbb{C})$ the existence of solution of the equation $A\mathbf{w}+\mu\mathbf{w}=\mathbf{g}$, for any $\mu>0$ is consequence of the application of the Lax-Milgram theorem to the bilinear form

$$a(u,v)=\int_\Omega\nabla u\cdot\nabla v\mathrm{d}x+\int_\Omega(V(x)+\mu)uv\,\mathrm{d}x$$

since now it is coercive in $H^1_0(\Omega)$ [it suffices to apply Hardy's inequality as in Proposition 2.1]. The accretivity is again a consequence of the Hardy's inequality and the fact that $\underline{C}>0$ in (10). Moreover by adapting to our framework some previous results in the literature for free particles $V=0$ (see, e.g. Remark 1.4.36 of [14] and also [51]) we can get a Galerkin decomposition as mentioned in (46). Indeed, since the operator defined in the proof of Proposition 2.1] is compact we know that $(u_n)_{n\ge1}$ is a Hilbert basis of $L^2(\Omega)$. Then given $\psi_0\in L^2(\mathbb{R}^N:\mathbb{C})$, $\psi_0=\psi_{0,r}+\mathbf{i}\psi_{0,i}$ we define $\mathbf{a}_n=a_{n,r}+\mathbf{i}a_{n,i}$ by

$$a_{n,r}:=\int_\Omega\psi_{0,r}(x)u_n(x)dx\quad\text{and}\quad a_{n,i}:=\int_\Omega\psi_{0,i}(x)u_n(x)dx$$

so that

$$\psi_0=\sum_{n=1}^{\infty}\mathbf{a}_nu_n(x).$$

For $k\in\mathbb{N}$ consider let $\psi_{0,k}\in L^2(\Omega:\mathbb{C})$ and $\psi_{\Omega,k}(t,x)$ defined by

$$\psi_{0,k}(x) = \sum_{n=1}^{k} \mathbf{a}_n u_n(x),$$

and

$$\psi_{\Omega,k}(t,x) = \sum_{n=1}^{k} \mathbf{a}_n e^{-\mathbf{i}\lambda_n t} u_n(x). \tag{52}$$

It is clear that $\psi_{\Omega,k} \in C([0,+\infty) : H^2(\Omega : \mathbb{C}) \cap H_0^1(\Omega : \mathbb{C}))$ and that $\psi_{\Omega,k}(0,.) = \psi_{0,k}(.)$. Moreover

$$\mathbf{i}\frac{\partial \psi_{\Omega,k}}{\partial t} = \sum_{n=1}^{k} \mathbf{a}_n e^{-\mathbf{i}\lambda_n t} \lambda_n u_n = \sum_{n=1}^{k} \mathbf{a}_n e^{-\mathbf{i}\lambda_n t}(-\Delta u_n + V(x)u_n) = -\Delta\psi_{\Omega,k} + V(x)\psi_{\Omega,k},$$

so $\psi_{\Omega,k}$ is the solution of problem (45) corresponding to the initial datum $\psi_{0,k}$. Since the set

$$\bigcup_{k\geq 1} \left\{ \sum_{n=1}^{k} \mathbf{a}_n u_n;\ \left(a_{n,r}\right)_{1\leq n\leq k}, \left(a_{n,i}\right)_{1\leq n\leq k} \subset \mathbb{R}^k \right\}$$

is dense in $L^2(\Omega : \mathbb{C})$ we get the Galerkin decomposition (46).
Conclusion (iii) follows from Theorem 2.1 and assumption (47) since

$$\begin{aligned} \left|\psi_{\Omega,k}(t,x)\right| &\leq \sum_{n=1}^{k} |\mathbf{a}_n| \left|e^{-\mathbf{i}\lambda_n t}\right| |u_n(x)| \\ &\leq \sum_{n=1}^{k} |\mathbf{a}_n| \overline{K}_n d(x,\partial\Omega)^2 \quad \text{for any} \quad t > 0 \quad \text{and a.e.} \quad x \in \Omega. \end{aligned} \tag{53}$$

Moreover, the extension $\psi(t,x)$ by zero outside Ω, given by (49) satisfies that $\psi \in C([0,+\infty) : L^2(\mathbb{R}^N : \mathbb{C}))$ with $\psi \in L^2(0,T : H^1(\mathbb{R}^N : \mathbb{C}))$ and $V_{q,\Omega}(x)\psi \in L^2(0,T : L^2(\mathbb{R}^N : \mathbb{C}))$ for any $T > 0$ and solves the problem (1), so it coincides with the uniqueness of solution of it obtained in i).
Property iv) is consequence of the *Unique Continuation Property* obtained under the assumption (44) and $\alpha \in [0,2)$ (see, e.g. Theorem XIII.57 of [20,53]). The conclusion can also be obtained as an application of the Paley-Wiener theorem (see, e.g. [52]). □

Corollary 4.1 *Under the conditions of Theorem 4.1 assumption (47) holds if for instance*

$$a_n \equiv 0 \quad \textit{for any} \quad n \geq n_0, \quad \textit{for some} \quad n_0 \in \mathbb{N}.$$

□

Remark 4.1 I conjecture that conclusion (48) can be obtained trough the use of some energy method similar to the one used in the elliptic case (see Theorem 2.1) and that (47) holds once we know merely that $\psi_0 \in H_0^1(\Omega : \mathbb{C})$ but at this moment their proofs are open questions.

Remark 4.2 The case $q = +\infty$ can be also considered (see Corollary 2.1). In fact, for the existence of a solution satisfying that support $\psi(t,x) \subset \Omega$ for any $t > 0$ the value of V on $\mathbb{R}^N - \Omega$ is irrelevant (see Remark 2.3).

Since no assumption on the connectness of the domain Ω was made in Theorem 4.1 the conclusion applies to domains with " holes":

Corollary 4.2 *(i) Assume* (10) *and*

$\Omega = \Omega_0 - \cup_{k=1}^{r} \overline{D}_k$, *for some regular open bounded sets,* Ω_0, D_k *of* $\mathbb{R}^N$ *with* $D_k \subset\subset \Omega_0$.

Then, if $\psi_0 \in H^1(\mathbb{R}^N : \mathbb{C})$ *and* $\psi_0(x) = \mathbf{0}$ *for a.e.* $x \in \cup_{k=1}^{r} \overline{D}$ *for some and assumption* (47) *holds then the same happens for* $\psi(t, x)$, *for any* $t > 0$.

(ii) Consider the Pösch–Teller potential (9) *and let* $\psi_0 \in H^1(\mathbb{R}^N : \mathbb{C})$ *such that* $\psi_0(x) = \mathbf{0}$ *for a.e.* $|x| \in [0, +\infty) - \left([\frac{j\pi}{\alpha}, \frac{(j+1)\pi}{\alpha}] \cup [\frac{m\pi}{\alpha}, \frac{(m+1)\pi}{\alpha}]\right)$ *with* $0 \leq j < j+1 < m$. *Then support* $\psi(t, x) \subset \{x \in \mathbb{R}^N$ *such that* $|x| \in [\frac{j\pi}{\alpha}, \frac{(j+1)\pi}{\alpha}] \cup [\frac{m\pi}{\alpha}, \frac{(m+1)\pi}{\alpha}]\}$ *for any* $t > 0$.

Proof i) is a direct consequence of Theorem 4.1. *For the proof of ii) it is enough to observe that*$V(|x|)$ is $\frac{\pi}{\alpha}$ periodic and that if we take $\Omega = \{x \in \mathbb{R}^N$ such that $|x| \in (\frac{j\pi}{\alpha}, \frac{(j+1)\pi}{\alpha}) \cup (\frac{m\pi}{\alpha}, \frac{(m+1)\pi}{\alpha})\}$ then the Pösch–Teller potential (9) satisfies assumption (10). □

Remark 4.3 The conclusion of the above Corollary can be contrasted with the study of the cases in which the potential V grows as $d(x, \partial\Omega)^{-\alpha}$ with $\alpha \in [0, 2)$ considered, for instance, in [18,42–44], where it was shown that the wave function $\psi(t, x)$ cannot exhibit " holes" for finite-time intervals. Although the study of the Pösch-Teller potential was initiated with the important paper [49], as far as we know, no rigorous proof of the statement ii) was given in the previous literature.

Remark 4.4 As mentioned in the case of the associate eigenvalue problem (Remark 2.4) the case of Ω unbounded can be also considered under suitable assumptions on $V(x)$ for $|x|$ large. For instance, if $V(x) = V(|x|)$ we can assume

$$\frac{\underline{C}}{r^2} \leq V(r) \leq \frac{\overline{C}}{r^2} \quad \text{for} \quad r \in (0, \varepsilon) \quad \text{for some} \quad \varepsilon > 0 \tag{54}$$

and

$$\underline{C} \leq \lim \inf_{r \to +\infty} V(r) r^2 \leq \lim \sup_{r \to +\infty} V(r) r^2 \leq \overline{C}.$$

Notice that under the above condition the spectrum is still countable (see, e.g., [39]). This is the case, for instance, of the " effective potential" associated to the Yukawa potential: also called " screened Coulomb potential")

$$W(r) = \frac{L_0}{\mu r^2} + \frac{k}{r} e^{-\frac{r}{a}}$$

with L_0 the angular momentum, μ the reduced mass and $k, a > 0$ some given parameters. In that case, the conclusion is the existence of solutions (for suitable initial data) such that

$$\begin{cases} |\psi(t, |x|)| \leq K\,|x|^2 & \text{for any } t > 0 \text{ and } |x| \in (0, \varepsilon) \text{ for some } \varepsilon > 0, \\ |\psi(t, |x|)| \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

Remark 4.5 As a variant of Theorem 4.1 (see also Remark 3.4), when the potential V satisfies (10) and under suitable assumptions on the initial data, it is possible to show the existence of "large solutions" of the linear problem

$$\begin{cases} \mathbf{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0, \infty) \times \Omega, \\ \psi = +\infty & \text{on } (0, \infty) \times \partial\Omega, \\ \psi(0, x) = \psi_0(x) & \text{on } \Omega, \end{cases} \tag{55}$$

(see [23]).

Remark 4.6 The case $\alpha > 2$ remains similar to the case $\alpha = 2$ but the notion of solution must be understood in a suitable way (see the examples given in [37] and the mathematical study of the associated stationary problem made in [26]).

Remark 4.7 A different type of localizing results concerning the non-linear Schrödinger equation, arising in nonlinear optics,

$$\mathbf{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \boldsymbol{a}|\psi|^{\sigma}\psi, \text{ in } (0,\infty)\times\mathbb{R}^N,$$

can be established if $\sigma \in (-1, 0)$. We recall that in most of the papers in the literature it is assumed $\sigma = 2$, nevertheless there are many applications in which $\sigma \in (-1, 0)$. In a series of papers in collaboration with Bégout [5–9] we prove precise estimates on the location of the support of $\psi(x, t)$, whose boundary gives rise to a free boundary associated to the problem. The techniques of proof are some extensions of the ones of [2] and are entirely different to the ones used in the present paper.

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