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The extinction versus the blow-up: Global and non-global existence of solutions of source types of degenerate parabolic equations with a singular absorption

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Abstract

We consider nonnegative solutions of degenerate parabolic equations with a singular absorption term and a source nonlinear term:

$$\partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} = f(u, x, t), \quad \text{in } I \times (0, T),$$

with the homogeneous zero boundary condition on $I = (x_1, x_2)$, an open bounded interval in $\mathbb{R}$. Through this paper, we assume that $p > 2$ and $\beta \in (0, 1)$. To show the local existence result, we prove first a sharp pointwise estimate for $|u_x|$. One of our main goals is to analyze conditions on which local solutions can be extended to the whole time interval $t \in (0, \infty)$, the so called global solutions, or by the contrary a finite time blow-up $\tau_0 > 0$ arises such that $\lim_{t\to\tau_0} \|u(t)\|_{L^\infty(I)} = +\infty$. Moreover, we prove that any global solution must vanish identically after a finite time if provided that either the initial data or the source term is small enough. Finally, we show that the condition $f(0, x, t) = 0, \forall (x, t) \in I \times (0, \infty)$ is a necessary and sufficient condition for the existence of solution of equations of this type.

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1. Introduction

In this paper, we are interested in nonnegative solutions of the following equation:

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} = f(u,x,t) & \text{in } I \times (0,T), \\ u(x_1,t) = u(x_2,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{1}$$

where $I = (x_1, x_2)$ is an open bounded interval in $\mathbb{R}$, $\beta \in (0,1)$, $p > 2$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, i.e.:

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \le 0. \end{cases}$$

Note that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose tactically $u^{-\beta}\chi_{\{u>0\}} = 0$ whenever $u = 0$. Through this paper, we always assume that $f : [0,\infty) \times \overline{I} \times [0,\infty) \longrightarrow \mathbb{R}$ is a nonnegative function satisfying the following hypothesis:

$$(H) \begin{cases} f \in \mathcal{C}^1\big([0,\infty) \times \overline{I} \times [0,\infty)\big), \text{ and } f(0,x,t) = 0, \ \forall (x,t) \in I \times (0,\infty), \text{ and} \\ \\ f(u,x,t) \le h(u), \ \forall (x,t) \in I \times (0,\infty), \text{ for some } h \in \mathcal{C}^1([0,\infty)). \end{cases}$$

Nevertheless, in some occasions we shall relax the regularity $f \in \mathcal{C}^1\big([0,\infty) \times \overline{I} \times [0,\infty)\big)$, see Lemma 13, and Theorem 17 below. Our main interest is to consider problem (1) for the case $p > 2$, although several of our results are also valid for the case $p = 2$. This case will be considered in our forthcoming paper.

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In the case N-dimension and $p=2$, equation (1) becomes

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} = f(u,x,t) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$. Problem (2) can be considered as a limit of mathematical models describing enzymatic kinetics (see [1]), or the Langmuir–Hinshelwood model of the heterogeneous chemical catalyst (see, e.g. [27] p. 68, [9], [24] and references therein). This case was studied by the authors in [24], [17], [21], [8], [6], [29], and references therein. These authors focused on studying the existence of solution, and the behaviors of solutions. It is of course that the delicate point is to get the integrability of the singular term $u^{-\beta}\chi_{\{u>0\}}$. In [24], D. Phillips proved the existence of solution for the Cauchy problem associating (2) in the case $f=0$. The case in that $f(u)$ is sub-linear, i.e.: $f(u) \leq C(u+1)$, for $u \geq 0$, was considered by J. Davila and M. Montenegro, [8]. They proved the existence of solution. Moreover, they also showed that the measure of the set $\{(x,t) \in \Omega \times (0,\infty) : u(x,t) = 0\}$ is positive (see also a more general statement in [10]). In other words, the solution may exhibit the quenching (or the extinction) behavior. Moreover, M. Winkler [29] showed that equation (2) with $f=0$ has no uniqueness solution in general.

Recently, problem (1) was considered by Giacomoni et al., [16], with the source term $f(u,x)$ satisfying $f(0,x)=0$, and the natural growth condition, i.e.:

$$0 \leq f(u,x) \leq \lambda u^{q-1} + \nu, \tag{3}$$

with $\lambda, \nu \geq 0$, and $1 < q \leq p$. These authors proved first a local existence result. Unfortunately, their proof of the integrability of the singular term contains a technical point, which was not correctly justified. Then, our first purpose is to prove a local existence of solution of equation (1), even for a more general class of functions $f(u,x,t)$ satisfying (H). As far as we know, our analysis of a general source term $f(u,x,t)$ of the equations of this type has not been studied yet in the literature. Moreover, if f is independent of x, t, then we only assume f a local Lipschitz function on $[0,\infty)$, instead of $f \in C^2([0,\infty))$ required in the previous works (see e.g. [8], [23]). For example, our results can take into account the function $f(u,x,t) = \frac{x^2}{t+1}(e^u - 1)$, which does not satisfy (3). Or, the function $f(u) = (u-1)^+ u$ is a local Lipschitz function on $[0,\infty)$, but it does not belong to $C^1([0,\infty))$.

As in [6] (but now with the additional difficulty of the presence of the source term), to show a local existence result, we first prove a priori pointwise estimate for $|u_x|$ involving a certain power of u, say briefly as follows:

$$|u_x(x,t)|^p \leq C u^{1-\beta}(x,t), \quad \text{for } (x,t) \in I \times (0,T), \tag{4}$$

for some positive constant $C > 0$. It is well known that such an estimate (4) plays an important role in proving the existence of solution for equations of this type. For instance, in the case $p=2$ and $f=0$, estimate (4) was obtained by the authors in [24], [8], [29] (see also [18] for the porous medium of this type).

The second purpose of this article is to study the global existence of solutions. In particular, we are interested in the extinction phenomenon that any solution vanishes identically after a finite time under some circumstances. To illustrate the global existence result, we first consider equation (1) with the simplest model $\lambda f(u) = \lambda u^{q-1}$. In some of our considerations, a crucial role is played by the first eigenvalue λ_I of the Dirichlet problem:

$$\begin{cases} -\partial_x(|\partial_x \phi_I|^{p-2}\partial_x \phi_I) = \lambda_I \phi_I^{p-1} & \text{in } I, \\ \phi_I(x_1) = \phi_I(x_2) = 0, \end{cases} \tag{5}$$

where ϕ_I is the first normalized eigenfunction ($\int_I \phi(x)dx = 1$). It is well known that the value of λ_I is computed as follows:

$$\lambda_I = (p-1)\Big(\frac{\pi_p}{x_2 - x_1}\Big)^p, \text{ with } \pi_p = 2\frac{\pi/p}{\sin(\pi/p)}, \tag{6}$$

see more details in [2], and references therein. Then λ_I decreases when the measure of the spatial domain I increases, and conversely.

For our purpose later, let us remind some classical results on the global and non-global existence of solutions of equation (1) without the singular absorption:

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x = \lambda u^{q-1} & \text{in } I \times (0,T), \\ u(x_1,t) = u(x_2,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } I. \end{cases} \tag{7}$$

In [28], M. Tsutsumi proved that if $q < p$, then problem (7) has global nonnegative solutions whenever initial data u_0 belongs to some Sobolev space. The case $q \geq p$ is quite delicate that there are both nonnegative global solutions, and solutions which blow up in a finite time. Indeed, J.N. Zhao [31] showed that when $q \geq p$, equation (7) has a global solution if the measure of I is small enough, and it has no global solution if the measure of I is large enough. The fact that the first eigenvalue λ_I decreases with increasing domain can be also used as an alternative explanation for Zhao's result. For example, in the critical case $q = p$, Y. Li and C. Xie [22] showed that if $\lambda_I > \lambda$, equation (7) has then a unique globally bounded solution. While, the unique solution blows up in a finite time if $\lambda_I < \lambda$, see Theorem 3.5, [22]. We also note that this one is globally bounded if provided that $\lambda_I = \lambda$ and initial data $u_0(x) \leq \kappa\phi_I(x)$, for some $\kappa > 0$. We would like to refer to the results of H.A. Levine [20], V.A. Galaktionov [13], V.A. Galaktionov and J.L. Vazquez [14], and references therein for a rich source of the blowing-up topic.

Roughly speaking, any weak solution of equation (1) is a sub-solution of equation (7). Thus, the comparison theorem implies that the global existence result holds for equation (1) if provided that either $q < p$, or $q \geq p$ and u_0 (resp. λ, the measure of I) is small enough. Here, we shall show that the global existence of solutions of (1) holds for a general source term $f(u,x,t)$ under some suitable conditions, see Theorem 4 below.

Concerning the quenching phenomenon, let us first mention the semi-linear case $p = 2$. Any weak nonnegative solution of equation (2) is extinct after a finite time if $f \equiv 0$, even beginning with a positive bounded initial data, see e.g. [24], [8], and references therein. The case of a nonnegative initial datum satisfying merely $u_0 \in L^1(\Omega)$ was considered in [7]. Still in the semi-linear case, M. Montenegro [23] considered equation (2) with the source term $\lambda f(u)$, for some $\lambda > 0$, and $f(u)$ is sub-linear. He showed that there exists a positive real number λ_0 so that if $\lambda \in (0, \lambda_0)$, then any solution must vanish identically after a finite time, that he called the complete quenching phenomenon.

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For the quasilinear equation (1), with $p > 2$ and $f \equiv 0$, N. A. Dao, and J. I. Diaz [6] showed that the extinction result also holds for any solution, even beginning with a positive initial data. It is known that the presence of the singular absorption term $u^{-\beta}\chi_{u>0}$ causes the extinction phenomenon. Furthermore, Giacomoni et al., [16] considered equation (1) with the source term λu^{q-1}. These authors showed that the extinction of solution occurs if provided $q \leq p$, and $\lambda_I > \lambda$, see Theorem 2.2, [16]. Their argument is based on the observation that the diffusion term dominates the source term λu^{q-1} in this case (see also M. Montenegro, [23] for the case $p = 2$, and Giacomoni et al., [15] for a quasilinear problem). However, this argument is no longer applicable to other cases, such as: $q \geq p$, or the critical case, $q = p$ and $\lambda = \lambda_I$. Thus, it is natural to address the question of the extinction phenomenon to the general source $f(u, x, t)$. We recall that the presence of supercritical sources terms can be the main reason of the existence of blowing-up solutions for reaction–diffusion parabolic equations. Thus, we shall analyze the interaction between the nonlinear diffusion, the singular absorption, and the nonlinear source term. In fact, we shall show that the singular absorption plays a role not only in preventing the blow-up, but also in driving solution to the extinction under some circumstances. This refers to the title of our paper: "the extinction versus the blow-up".

To illustrate the influence of $u^{-\beta}\chi_{\{u>0\}}$ in the extinction phenomenon, we consider equation (1) with the source $\lambda f(u, x, t) = \lambda u^{p-1}$, and initial data $u_0(x) = \phi_I(x)$. Then, our complete quenching result of this case is as follows: any nonnegative solution of equation (1) is extinct after a finite time if $0 \leq (\lambda - \lambda_I)$ is sufficiently small, see Theorem 21 below. It is interesting to compare with the unique solution of equation (7) which blows up whenever $\lambda > \lambda_I$, see [22]. This result can be explained as follows: when $\lambda > \lambda_I$, the diffusion term $-(|u_x|^{p-2}u_x)_x$ is not strong enough to prevent the blow-up caused by the source λu^{p-1}. In the point of view of inequality (4), the absorption term $u^{-\beta}\chi_{\{u>0\}}$ strengthen the diffusion term, in order to control the influence of the source term. Therefore, we can imagine that an amount of $u^{-\beta}\chi_{\{u>0\}}$ is used to prevent the blow-up, and the remaining part of $u^{-\beta}\chi_{\{u>0\}}$ forces solutions to the extinction. This is a reason why the complete quenching phenomenon of solutions of equation (1) can be extended to the case, where $0 \leq (\lambda - \lambda_I)$ is small enough. At the end of this paper, we will provide some numerical experiences in order to illustrate the difference between the behavior of solutions of both equations (1) and (7). Specifically, the numerical results show that with the same data, the maximal solution of equation (1) vanishes identically after a finite time, while the unique solution of equation (7) is blowing-up, see Section 7.

A different purpose of this paper is to study the non-global existence of solutions of equation (1). To state our results, suppose for simplicity, that $f(u, x, t) = f(u)$ and define

$$F(u) = \int_0^u f(s)ds.$$

It is convenient to introduce the total energy at time $t \geq 0$, associated to the equation (1), by means of the expression

$$E(t) = \int_I \left(\frac{1}{p}|u_x(t)|^p + \frac{1}{1-\beta}u^{1-\beta}(t) - F(u(t))\right)dx. \tag{8}$$

In the supercritical case: $f(u) = u^{q-1}$, for $q > p$, Giacomoni et al. proved that the maximal solution of equation (1) is blowing-up if provided $E(0) < 0$, see Proposition 5.1, [16]. It is

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interesting to ask whether the critical case $p = q$ is belong to the blow-up range or not. Moreover, the other cases of nonlinear source $f(u)$ have not been considered yet for the equations of this type in the literature. For example, the case $f(u) = u^{p-1}\ln(u+1)$ is neither the critical case, nor the supercritical case. Thus, we would like to extend the blowing-up result to more general source $f(u)$. In fact, we will prove that the blowing-up result holds if provided that $\frac{F(u)}{u^p}$ is nondecreasing on $(0,\infty)$, and some additional conditions on initial data u_0, and the measure of I, see Theorem 8 below. Obviously, our result includes the supercritical case and the critical case $f(u) = u^{p-1}$, and the functions like $f(u) = u^{p-1}\ln(u+1)$ above. Remind that any solution of equation (1) exists globally in the sub-critical case. Thus, our blow-up results are sharp in the context of the blow-up range.

Finally, we show that the condition $f(0, x, t) = 0$, $\forall(x, t) \in I \times (0,\infty)$ in (H) cannot be eliminated. If this one is violated then equation (1) have no solution for any small initial data. Thus, this one is a necessary and sufficient condition for the existence of solution of problem (1).

The paper is organized as follows: In the next section, we will give the preliminary and main results, and some definitions. Section 3 is devoted to prove a sharp gradient estimate, which is the key of proving the existence of solution. In Section 4, we shall prove the local existence of solution. In Section 5, we study the global existence of solutions, and the extinction phenomenon of solutions. The non-global existence of solutions is considered in Section 6. Finally, we point out some numerical experiences in the last Section.

2. Preliminary and main results

In the sequel, we always assume $u_0 \geq 0$, and f satisfies (H). At the beginning, let us introduce the notion of a weak solution of equation (1).

Definition 1. Let $u_0 \in L^\infty(I)$. A function $u \geq 0$ is called a weak solution of equation (1) if $u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0,T))$, and $u \in L^p(0,T; W_0^{1,p}(I)) \cap L^\infty(I \times (0,T)) \cap \mathcal{C}([0,T); L^1(I))$ satisfies equation (1) in the sense of distributions $\mathcal{D}'(I \times (0,T))$, i.e.:

$$\int_0^T \int_I \Big(-u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi - f(u,x,t)\phi\Big)\,dxdt = 0, \quad \forall \phi \in \mathcal{C}_c^\infty(I \times (0,T)). \tag{9}$$

Let us call $\Gamma(t)$ is the solution of the equation:

$$\begin{cases} \partial_t \Gamma = h(\Gamma), & \text{in } (0,T), \\ \Gamma(0) = 2\|u_0\|_\infty, \end{cases} \tag{10}$$

where h is the function in (H) above, and T is the maximal existence time of $\Gamma(t)$. Note that T only depends on $\|u_0\|_{L^\infty}$, see Chapter 1, [5].

Then, we have a local existence theorem.

Theorem 2. *Let $u_0 \in L^\infty(I)$. Then, there exists a time $T_0 > 0$ such that equation (1) has a maximal weak solution u in $I \times (0,T_0)$, i.e.: for any weak solution v, we have $v \leq u$ in $I \times (0,T_0)$.*

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Moreover, there is a positive constant $C = C(\beta, p)$ such that

$$|u_x(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\Big(\tau^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)\Theta(D_u f, \Gamma(2T_0)) + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f, \Gamma(2T_0)) + 1\Big), \tag{11}$$

for a.e. $(x,\tau) \in I \times (0, T_0)$*, with* $\Theta(G,r) = \max\limits_{0\le u\le r,(x,t)\in \overline{I}\times[0,2T_0]} \{|G(u,x,t)|\}$.

Besides, if $(u_0^{\frac{1}{\gamma}})_x \in L^\infty(I)$*, then there is a positive constant* $C = C(\beta, p, \|(u_0^{\frac{1}{\gamma}})_x\|_\infty)$ *such that*

$$|u_x(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\left(\Gamma^{1+\beta}(2T_0)\Theta(D_u f, \Gamma(2T_0)) + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f, \Gamma(2T_0)) + 1\right), \tag{12}$$

for a.e. $(x,t) \in I \times (0, T_0)$.

Remark 3. As a consequence of (12), the above solution is continuous at $t = 0$, see Proposition 14 below.

Next, we have a global existence result for the source $\lambda f(u,x,t)$.

Theorem 4. *Let* $u_0 \in L^\infty(I)$*, and* $\lambda > 0$*. Assume that there are an open bounded interval* I_0*, and a positive real number* κ_0 *such that* $I \subset\subset I_0$*, and*

$$\begin{cases} u_0(x) \le \kappa_0\phi_{I_0}(x), & \text{for a.e. } x \in I, \\ \lambda_{I_0}\kappa_0^{p-1}\phi_{I_0}^{p-1}(x) + \kappa_0^{-\beta}\phi_{I_0}^{-\beta}(x) \ge \lambda f(\kappa_0\phi_{I_0}(x), x, t), & \forall (x,t) \in I \times (0,\infty). \end{cases} \tag{13}$$

Recall that λ_{I_0} *and* ϕ_{I_0} *are the first eigenvalue and the first eigenfunction of problem* (5) *in* I_0*. We observe that* $\inf\limits_{x\in I}\{\phi_{I_0}\} > 0$*, so* $\phi_{I_0}^{-\beta}(x)$ *is well defined for any* $x \in I$*. Then, any solution* v *of equation* (1) *exists globally and*

$$v(x,t) \le \kappa_0\phi_{I_0}(x), \quad \text{in } I \times (0,\infty). \tag{14}$$

Remark 5. It is clear that (13) holds if either λ or $\|u_0\|_\infty$ is sufficiently small.

Still consider equation (1) with the source $\lambda f(u,x,t)$, we have then a complete quenching result.

Theorem 6. *Let* $u_0 \in L^\infty(I)$*, and* $h(0) = 0$ *in* (H)*. Then, every weak solution of equation* (1) *vanishes identically after a finite time if provided that either* $\|u_0\|_\infty$ *or* λ *is small enough.*

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As a consequence of the complete quenching result, we will show that the condition $f(0, x, t) = 0$, $\forall (x, t) \in I \times (0, \infty)$ is the necessary and the sufficient condition for the existence of solution of equation (1).

Theorem 7. *Equation (1) has a nonnegative solution for any bounded initial data if and only if* $f(0, x, t) = 0$, $\forall (x, t) \in I \times (0, \infty)$.

Concerning the non-global existence of solutions of equation (1), we have the following theorem.

Theorem 8. *Let* $u_0 \in W_0^{1,p}(I)$, *and* $T > 0$. *Assume that* $f(u, x, t) = f(u)$, *and* $\frac{F(u)}{u^p}$ *is nondecreasing on* $(0, \infty)$. *Then, the maximal solution u in Theorem 2 blows up at a time* $T_0 \in (0, T]$ *if provided*

$$pE(0) + \frac{4(3p-1)}{T(p-2)^2} \int_I u_0^2 dx \leq 0. \tag{15}$$

As a consequence of Theorem 8, we have

Corollary 9. *Let* $u_0 \in W_0^{1,p}(I)$. *Suppose that* $f(u, x, t) = f(u)$, *and* $\frac{F(u)}{u^p}$ *is nondecreasing on* $(0, \infty)$. *Then, the maximal solution u blows up in a finite time if provided* $E(0) < 0$. *Moreover, the blow-up time*

$$T_0 \in \left(0, \frac{4(3p-1)}{-pE(0)(p-2)^2} \int_I u_0^2 dx\right].$$

Proof. Indeed, let

$$T = \frac{4(6p-1)}{-pE(0)(p-2)^2} \int_I u_0^2 dx > 0.$$

Thus, (15) holds, thereby proves the above corollary. □

Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(p, \beta, \tau)$ means that C only depends on p, β, τ. We also denote by $\partial_x u$ (resp. $\partial_t u$) means the partial derivative with respect to x (resp. t). We also write $\partial_x u = u_x$.

3. Pointwise estimates for $|u_x|$

In this part, we shall modify Bernstein's technique to obtain an estimate for $|u_x|$ like (4), the so called gradient estimate in N-dimension. As mentioned in the Introduction, such a gradient estimate of (4) plays a crucial role in proving the existence of solution. The degeneracy of the diffusion operator as $p > 2$ leads obviously to a considerable amount of additional technical

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difficulties. In the case $f = 0$, it is not difficult to show that estimate (4) becomes an equality for a suitable constant C, when considering the stationary equation of (1). That is the reason why such a gradient estimate of this type is called a sharp gradient estimate (since the power of u in (4) cannot bigger or smaller than $1 - 1/\gamma$). By the appearance of the nonlinear diffusion, we shall establish previously the gradient estimates for the solutions of the following regularizing problem.

For any $\varepsilon > 0$, let us set

$$g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s), \text{ with } \psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon}),$$

and $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \psi \le 1$ is a non-decreasing function such that $\psi(s) = 0$, if $s \le 1$; and $\psi(s) = 1$, if $s \ge 2$.

Now fix $\varepsilon > 0$, we consider the following problem:

$$(P_{\varepsilon,\eta}) \begin{cases} \partial_t u - (a(u_x)u_x)_x + g_\varepsilon(u) = f(u), & \text{in } I \times (0,\infty), \\ u(x_1,t) = u(x_2,t) = \eta, & t \in (0,\infty), \\ u(x,0) = u_0(x) + \eta, & x \in I, \end{cases}$$

with $a(u) = b(u)^{\frac{p-2}{2}}$, $b(u) = |u|^2 + \eta^2$; and $\eta \to 0^+$. Note that $a(u_x)$ is a regularization of $|u_x|^{p-2}$. The gradient estimate, presented in this framework is as follows:

Lemma 10. *Let $u_0 \in \mathcal{C}_c^\infty(I)$, $u_0 \ne 0$. Then, there is a time $T_0 \in (0,\infty)$ such that problem $(P_{\varepsilon,\eta})$ admits a unique classical solution $u_{\varepsilon,\eta}$ in $I \times (0, T_0)$.*

i) Moreover, there is a positive constant $C = C(\beta, p)$ such that

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\Big(\tau^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)\Theta(D_u f, \Gamma(2T_0)) + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f, \Gamma(2T_0)) + 1\Big), \tag{16}$$

for $(x,\tau) \in I \times (0, T_0)$.

ii) If $(u_0^{\frac{1}{\gamma}})_x \in L^\infty(I)$, then there is a positive constant $C = C(\beta, p, \|(u^{\frac{1}{\gamma}})_x\|_\infty)$ such that

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\left(\Gamma^{1+\beta}(2T_0)\Theta(D_u f, \Gamma(2T_0)) + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f, \Gamma(2T_0)) + 1\right), \tag{17}$$

for any $(x,\tau) \in I \times (0, T_0)$.

Remark 11. In the case $p = 2$ and $f = 0$, estimate (16) was obtained by the authors in [8], [6], [24]. Note that inequality (17) implies that the solution obtained by passing to the limit as $\eta \to 0$ is continuous at $t = 0$, see Proposition 14 below.

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Proof. We prove i).

Note that equation $(P_{\varepsilon,\eta})$ is non-degenerated. Thanks to the classical result (see [19], [31], [30]), equation $(P_{\varepsilon,\eta})$ possesses a unique classical solution, $u_{\varepsilon,\eta} \in \mathcal{C}^\infty(\overline{I} \times [0,T))$. Moreover, the strong comparison principle yields

$$u_{\varepsilon,\eta}(x,t) \leq \Gamma(t), \quad \text{for } (x,t) \in I \times (0,T). \tag{18}$$

Let us put $T_0 = T/3$. For the sake of brevity, we remove the dependence on ε, η in the notation of $u_{\varepsilon,\eta}$, and put $u = u_{\varepsilon,\eta}$.

Next, we observe that η is a sub-solution of equation $(P_{\varepsilon,\eta})$, so the comparison principle yields

$$\eta \leq u(x,t), \quad \text{in } I \times (0, T_1). \tag{19}$$

For any $0 < \tau < T_0$, let us consider a cut-off function $\xi(t) \in \mathcal{C}_c^\infty(0,\infty)$, $0 \leq \xi(t) \leq 1$ such that

$$\xi(t) = \begin{cases} 1, & \text{on } [\tau, T_0], \\ 0, & \text{outside } (\tau/2, T_0 + \tau/2), \end{cases}$$

and $|\xi_t| \leq \frac{c_0}{\tau}$, for some constant $c_0 > 0$, and put

$$u = \varphi(v) = v^\gamma, \quad w(x,t) = \xi(t) v_x^2.$$

Then, we have

$$w_t - a w_{xx} = \xi_t v_x^2 + 2\xi v_x (v_t - a v_{xx})_x - 2\xi a v_{xx}^2 + 2\xi a_x v_{xx}. \tag{20}$$

From the equation satisfied by u, we get

$$v_t - a v_{xx} = a_x v_x + a v_x^2 \frac{\varphi''}{\varphi'} - \frac{g_\varepsilon(\varphi)}{\varphi'} + \frac{f(\varphi,x,t)}{\varphi'}. \tag{21}$$

Combining (20) and (21) provides us

$$\begin{aligned} w_t - a w_{xx} = {} & \xi_t v_x^2 + 2\xi v_x \left([a(u_x)]_x v_x + a v_x^2 \frac{\varphi''}{\varphi'} - \frac{g_\varepsilon(\varphi)}{\varphi'} + \frac{f(\varphi,x,t)}{\varphi'} \right)_x \\ & - 2\xi a v_{xx}^2 + 2\xi [a(u_x)]_x v_{xx}. \end{aligned} \tag{22}$$

Now, we define $L = \max\limits_{\overline{I} \times [0,2T_0]} \{w(x,t)\}$.

If $L = 0$, then the conclusion (16) is trivial, and $|u_x(x,\tau)| = 0$, $\forall (x,\tau) \in I \times (0,T_0)$. If not we have $L > 0$, which implies that w attains its maximum at a point $(x_0,t_0) \in I \times (0, 2T_0)$ since $w(x,t) = 0$ on $\partial I \times (0,T_1)$, $w(x,t)|_{t=0} = 0$, and $w(x,t)|_{t \geq 2T_0} = 0$. Note that $\xi(t_0) > 0$, and $|v_x(x_0,t_0)| \neq 0$. Thus, we obtain

$$w_t(x_0,t_0) = w_x(x_0,t_0) = 0,$$

and

$$0 \geq w_{xx}(x_0,t_0) = 2\xi(t_0)v_{xx}^2(x_0,t_0) + 2\xi(t_0)v_x(x_0,t_0)v_{xxx}(x_0,t_0). \tag{23}$$

Since $v_x(x_0,t_0) \neq 0$, we get

$$w_x(x_0,t_0) = 0 \text{ if and only if } v_{xx}(x_0,t_0) = 0. \tag{24}$$

By (24) and (23), we get

$$v_x(x_0,t_0)v_{xxx}(x_0,t_0) \leq 0. \tag{25}$$

At the point (x_0,t_0), a combination of (22) and (24) provides us

$$0 \leq w_t - aw_{xx} = \xi_t v_x^2 + 2\xi v_x\left([a(u_x)]_{xx}v_x + [a(u_x)]_x v_x^2 \frac{\varphi''}{\varphi'} + av_x^2\left(\frac{\varphi''}{\varphi'}\right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x + \left(\frac{f(\varphi,x_0,t_0)}{\varphi'}\right)_x\right),$$

or

$$\begin{aligned} -av_x^3\left(\frac{\varphi''}{\varphi'}\right)_x &\leq \frac{1}{2}\xi_t\xi^{-1}v_x^2 + [a(u_x)]_{xx}v_x^2 + [a(u_x)]_x v_x^3\frac{\varphi''}{\varphi'} \\ &\quad - \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x v_x + \left(\frac{f(\varphi,x_0,t_0)}{\varphi'}\right)_x v_x. \end{aligned} \tag{26}$$

Now, we compute the terms of (26) in detail. Using (24) yields

$$[a(u_x)]_x(x_0,t_0) = (p-2)b^{\frac{p-4}{2}}(u_x)\varphi'\varphi''v_x^3, \tag{27}$$

and

$$[a(u_x)]_{xx}(x_0,t_0) = (p-2)(p-4)b^{\frac{p-6}{2}}(u_x)(\varphi'\varphi'')^2v_x^6 + (p-2)b^{\frac{p-4}{2}}(u_x)(\varphi''^2 + \varphi'\varphi''')v_x^4 + (p-2)b^{\frac{p-4}{2}}(u_x)\varphi'^2 v_x v_{xxx}.$$

Thanks to (25), we get from the last equation

$$[a(u_x)]_{xx}(x_0,t_0) \leq (p-2)(p-4)b^{\frac{p-6}{2}}(u_x)(\varphi'\varphi'')^2v_x^6 + (p-2)b^{\frac{p-4}{2}}(u_x)(\varphi''^2 + \varphi'\varphi''')v_x^4. \tag{28}$$

Next, we have

$$\begin{aligned} v_x\left(\frac{f(\varphi,x_0,t_0)}{\varphi'}\right)_x &= \frac{D_x f(\varphi,x_0,t_0)}{\varphi'}v_x + D_u f(\varphi,x_0,t_0)v_x^2 - f(\varphi,x_0,t_0)\frac{\varphi''}{\varphi'^2}v_x^2 \\ &= \frac{1}{\gamma}D_x f(\varphi,x_0,t_0)v^{1-\gamma}v_x + D_u f(\varphi,x_0,t_0)v_x^2 - (\frac{\gamma-1}{\gamma})f(\varphi,x_0,t_0)v^{-\gamma}v_x^2. \end{aligned}$$

Since $f \geq 0$ and $\gamma > 1$, we get

$$v_x\left(\frac{f(\varphi,x_0,t_0)}{\varphi'}\right)_x \leq \frac{1}{\gamma}D_x f(\varphi,x_0,t_0)v^{1-\gamma}v_x + D_u f(\varphi,x_0,t_0)v_x^2. \tag{29}$$

After that, we handle the following term:

$$v_x\left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x = (g'_\varepsilon - g_\varepsilon\frac{\varphi''}{\varphi'^2})v_x^2 = \left(\psi'_\varepsilon(\varphi)v^{-\beta} - (\beta+\frac{\gamma-1}{\gamma})\psi_\varepsilon(\varphi)v^{-(1+\beta)\gamma}\right)v_x^2.$$

Since $\psi'_\varepsilon \geq 0$, and $0 \leq \psi_\varepsilon \leq 1$, we obtain

$$-v_x\left(\frac{g(\varphi)}{\varphi'}\right)_x \leq (\beta+\frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2. \tag{30}$$

Inserting (27), (28), (29), and (30) into (26) yields

$$\begin{gathered}(\gamma-1)v^{-2}a(u_x)v_x^4 \leq \frac{1}{2}\xi_t\xi^{-1}v_x^2 \\ +\underbrace{(p-2)(p-4)b^{\frac{p-6}{2}}(u_x)(\varphi'\varphi'')^2v_x^8 + (p-2)b^{\frac{p-4}{2}}(u_x)(2\varphi''^2+\varphi'\varphi''')v_x^6}_{\mathcal{B}} + \\ (\beta+\frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \frac{1}{\gamma}D_x f(\varphi,x_0,t_0)v^{1-\gamma}v_x + D_u f(\varphi,x_0,t_0)v_x^2.\end{gathered} \tag{31}$$

By computation, we have

$$\begin{gathered}\mathcal{B} = (p-2)b^{\frac{p-6}{2}}(u_x)v_x^6\Big((p-4)(\varphi'\varphi'')^2v_x^2 + (2\varphi''^2+\varphi'\varphi''')b(u_x)\Big) = \\ (p-2)\varphi'^2b^{\frac{p-6}{2}}(u_x)v_x^8\Big((p-2)\varphi''^2+\varphi'\varphi'''\Big) + \eta^2(p-2)(2\varphi''^2+\varphi'\varphi''')b^{\frac{p-6}{2}}(u_x)v_x^6 = \\ \underbrace{(p-2)(p(\gamma-1)-\gamma)\gamma^2(\gamma-1)v^{2(\gamma-2)}\varphi'^2b^{\frac{p-6}{2}}(u_x)v_x^8}_{\mathcal{B}_1} \\ +\underbrace{\eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)v^{2(\gamma-2)}b^{\frac{p-6}{2}}(u_x)v_x^6}_{\mathcal{B}_2}\end{gathered}$$

Since $p(\gamma-1)-\gamma<0$, we have $\mathcal{B}_1 \leq 0$, thereby proves

$$\mathcal{B} \leq \mathcal{B}_2. \tag{32}$$

By (31) and (32), we get

$$\begin{gathered}(\gamma-1)v^{-2}a(u_x)v_x^4 \leq \frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta+\frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \frac{1}{\gamma}D_x f(\varphi,x_0,t_0)v^{1-\gamma}v_x + \\ D_u f(\varphi,x_0,t_0)v_x^2 + \mathcal{B}_2.\end{gathered} \tag{33}$$

The fact that $b^{\frac{p-2}{2}}(.)$ is an increasing function since $p > 2$ leads to

$$a(u_x) = b^{\frac{p-2}{2}}(u_x) \geq (v_x^2 \varphi'^2)^{\frac{p-2}{2}} = |v_x|^{p-2}\gamma^{p-2}v^{(\gamma-1)(p-2)}.$$

Inserting the last inequality into (33) deduces

$$(\gamma-1)\gamma^{p-2}v^{(\gamma-1)(p-2)-2}|v_x|^{p+2} \leq \frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \frac{1}{\gamma}D_x f(\varphi, x_0, t_0)v^{1-\gamma}v_x + D_u f(\varphi, x_0, t_0)v_x^2 + \mathcal{B}_2.$$

Note that $2-(\gamma-1)(p-2) = (1+\beta)\gamma$, thereby

$$(\gamma-1)\gamma^{p-2}v^{-(1+\beta)\gamma}|v_x|^{p+2} \leq \frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \frac{1}{\gamma}D_x f(\varphi, x_0, t_0)v^{1-\gamma}v_x + D_u f(\varphi, x_0, t_0)v_x^2 + \mathcal{B}_2.$$

Multiplying both sides of the above inequality by $v^{(1+\beta)\gamma}$ yields

$$(\gamma-1)\gamma^{p-2}|v_x|^{p+2} \leq \frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v_x^2 + \frac{1}{\gamma}D_x f(\varphi, x_0, t_0)v^{1+\beta\gamma}v_x + D_u f(\varphi, x_0, t_0)v^{(1+\beta)\gamma}v_x^2 + v^{(1+\beta)\gamma}\mathcal{B}_2. \tag{34}$$

At the moment, if $|v_x(x_0, t_0)| \leq 1$, then we have $w(x_0, t_0) = \xi(t_0)|v_x(x_0, t_0)|^2 \leq 1$. This implies $w(x, t) \leq 1$ since $w(x_0, t_0) = \max_{(x,t)\in\overline{I}\times[0,2T_0]}\{w(x, t)\}$. In particular, we have $w(x, \tau) = v_x^2(x, \tau) \leq 1$, thereby proves

$$|u(x, \tau)|^p \leq \gamma^p u^{1-\beta}(x, \tau),$$

and we get estimate (16).

If not, we have $|v_x(x_0, t_0)| > 1$. Then, it follows from inequality (34) that there is a positive constant $C = C(\beta, p)$ such that

$$|v_x|^{p+2} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + 1\right)v_x^2 + v^{(1+\beta)\gamma}\mathcal{B}_2. \tag{35}$$

We now divide the studying the term $\mathcal{B}_2$ in inequality (35) into two cases.

a) If $\mathcal{B}_2 \leq 0$, then we have from (35)

$$|v_x|^{p+2} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + 1\right)v_x^2. \tag{36}$$

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Since $u(x,t) = v^{\gamma}(x,t) \leq \Gamma(2T_0)$, $\forall (x,t) \in I \times (0, 2T_0)$, inequality (36) deduces

$$|v_x|^p \leq C\left(|\xi_t|\xi^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)|D_u f(\varphi, x_0, t_0)| + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)|D_x f(\varphi, x_0, t_0)| + 1\right). \tag{37}$$

By noting that $0 < \xi(t_0) \leq 1$, we multiply both sides of (37) with $\xi(t_0)^{\frac{p}{2}}$ to get

$$\begin{aligned}|w(x_0,t_0)|^{\frac{p}{2}} &= (\xi(t_0)|v_x|^2)^{\frac{p}{2}} \\ &\leq C\Big(|\xi_t|\xi^{\frac{p}{2}-1}(t_0)\Gamma^{1+\beta}(2T_0) + \xi^{\frac{p}{2}}(t_0)\Gamma^{1+\beta}(2T_0)|D_u f(\varphi, x_0, t_0)| + \\ &\qquad \xi^{\frac{p}{2}}(t_0)\Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)|D_x f(\varphi, x_0, t_0)| + \xi^{\frac{p}{2}}(t_0)\Big).\end{aligned}$$

Since $|\xi_t(t)| \leq \dfrac{c_0}{\tau}$, we obtain

$$|w(x_0,t_0)|^{\frac{p}{2}} \leq C\left(\tau^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)|D_u f(\varphi, x_0, t_0)| + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)|D_x f(\varphi, x_0, t_0)| + 1\right).$$

Remind that $w(x_0,t_0) = \max_{(x,t)\in \overline{I}\times[0,2T_0]} \{w(x,t)\}$, thereby

$$|w(x,\tau)|^{\frac{p}{2}} \leq C\left(\tau^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)|D_u f(\varphi, x_0, t_0)| + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)|D_x f(\varphi, x_0, t_0)| + 1\right),$$

for any $x \in \overline{I}$. In addition, we have $w(x,\tau) = v_x^2(x,\tau)$, thereby proves

$$|v_x(x,\tau)|^{p} \leq C\left(\tau^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)|D_u f(\varphi, x_0, t_0)| + \Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)|D_x f(\varphi, x_0, t_0)| + 1\right),$$

for any $x \in \overline{I}$. Then, we obtain

$$\begin{aligned}|u_x(x,\tau)|^p \leq C_1 u^{1-\beta}(x,\tau)\Big(&\tau^{-1}\Gamma^{1+\beta}(2T_0) + \Gamma^{1+\beta}(2T_0)|D_u f(\varphi, x_0, t_0)| + \\ &\Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)|D_x f(\varphi, x_0, t_0)| + 1\Big),\end{aligned}$$

with $C_1 = C_1(\beta, p) > 0$. The last inequality holds for any $\tau \in (0, T_0)$, so we get estimate (16).

b) If $\mathcal{B}_2 > 0$, we have from the expression of $\mathcal{B}_2$ that $3\gamma - 4 > 0 \Leftrightarrow p < 4(1-\beta)$. Therefore, $b^{\frac{p-6}{2}}(.)$ is a decreasing function, so

$$b^{\frac{p-6}{2}}(u_x) \leq |u_x|^{p-6} = |\varphi'(v)v_x|^{p-6}.$$

Then,

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \leq \eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{2(\gamma-2)+(1+\beta)\gamma+(\gamma-1)(p-6)}|v_x|^p$$

$$\leq \eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{-2(\gamma-1)}|v_x|^p.$$

The last inequality and (35) deduce that there is a positive constant $C = C(\beta, p)$ such that

$$|v_x|^{p+2} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + 1\right)v_x^2 + C\eta^2 v^{-2(\gamma-1)}|v_x|^p.$$

Note that $|v_x(x_0, t_0)| > 1$, thereby proves $|v_x(x_0, t_0)|^p \leq |v_x(x_0, t_0)|^{p+2}$. Thus, it follows from the last inequality that

$$\left(1 - C\eta^2 v^{-2(\gamma-1)}\right)|v_x|^{p+2} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + 1\right)v_x^2.$$

Simplifying v_x^2 in both sides of the last inequality yields

$$\left(1 - C\eta^2 v^{-2(\gamma-1)}\right)|v_x|^{p} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + 1\right).$$

By (19), we have $v = u^{\frac{1}{\gamma}} \geq \eta^{\frac{1}{\gamma}} \Leftrightarrow v^{-2(\gamma-1)} \leq \eta^{-\frac{2(\gamma-1)}{\gamma}}$. Inserting this fact into the indicated inequality above yields

$$\left(1 - C\eta^{2-\frac{2(\gamma-1)}{\gamma}}\right)|v_x|^{p} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + 1\right).$$

Or

$$\left(1 - C\eta^{\frac{2}{\gamma}}\right)|v_x|^{p} \leq C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)|v^{1+\beta\gamma} + |D_u f(\varphi, x_0, t_0)|v^{(1+\beta)\gamma} + 1\right).$$

Since $\eta \to 0^+$, we have $(1 - C\eta^{\frac{2}{\gamma}}) > 0$. Therefore, we obtain

$$|v_x|^p \le C_2 \left(|\xi_t|\xi^{-1} v^{(1+\beta)\gamma} + |D_u f(\varphi, x_0, t_0)| v^{(1+\beta)\gamma} + |D_x f(\varphi, x_0, t_0)| v^{1+\beta\gamma} + 1 \right),$$

with $C_2 = \dfrac{C}{1 - C\eta^{\frac{2}{\gamma}}}$. This inequality is just a version of (37). By the same analysis as in **a)**, we also get estimate (16).

Finally, we prove ii).

The proof of estimate (17) is most likely to the one of estimate (16), so we just make a slight change. Let us consider a cut-off function $\overline{\xi}(t) \in \mathcal{C}^\infty(\mathbb{R})$ instead of $\xi(t)$ above, $0 \le \overline{\xi}(t) \le 1$ such that

$$\overline{\xi}(t) = \begin{cases} 1, & \text{if } t < T_0, \\ 0, & \text{if } t > 2T_0, \end{cases}$$

and $\overline{\xi}_t(t) \le 0$. Then, we observe that

Either $w(x,t)$ attains its maximum at the initial data

$$\max_{(x,t) \in I \times [0,2T_0]} w(x,t) = w(x_0, 0) = \overline{\xi}(0) v_x^2(x_0, 0) \le \|(u_0^{\frac{1}{\gamma}})_x\|_\infty^2, \quad \text{for some } x_0 \in I,$$

which implies

$$|u_x(x,t)|^p \le \gamma^p \|(u_0^{\frac{1}{\gamma}})_x\|_\infty^p u^{1-\beta}(x,t), \quad \text{for any } (x,t) \in I \times (0, 2T_0). \tag{38}$$

Thus, we get estimate (17) immediately.

Or there is a point $(x_0, t_0) \in I \times (0, 2T_0)$ such that $\max\limits_{(x,t) \in I \times [0,2T_0]} w(x,t) = w(x_0, t_0)$, since $w(.,t) = 0$ for $t \ge 2T_0$.

Then, we repeat the proof of i) until (34). It is convenient for us to rewrite inequality (34) here.

$$(\gamma - 1)\gamma^{p-2}|v_x|^{p+2} \le \frac{1}{2}\overline{\xi}_t \overline{\xi}^{-1} v^{(1+\beta)\gamma} v_x^2 + (\beta + \frac{\gamma - 1}{\gamma}) v_x^2 + \frac{1}{\gamma} D_x f(\varphi, x_0, t_0) v^{1+\beta\gamma} v_x + D_u f(\varphi, x_0, t_0) v^{(1+\beta)\gamma} v_x^2 + v^{(1+\beta)\gamma} \mathcal{B}_2.$$

Since $\overline{\xi}_t(t) \le 0$, we get from the indicated inequality

$$\begin{aligned} (\gamma - 1)\gamma^{p-2}|v_x|^{p+2} &\le (\beta + \frac{\gamma - 1}{\gamma}) v_x^2 + \frac{1}{\gamma} D_x f(\varphi, x_0, t_0) v^{1+\beta\gamma} v_x \\ &\quad + D_u f(\varphi, x_0, t_0) v^{(1+\beta)\gamma} v_x^2 + v^{(1+\beta)\gamma} \mathcal{B}_2. \end{aligned}$$

By repeating the proof of i) after this inequality, we obtain

$$|u_x(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\left(\Gamma^{1+\beta}(2T_0)\Theta(D_u f,\Gamma(2T_0))+\Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f,\Gamma(2T_0))+1\right), \quad (39)$$

with $C=C(\beta,p)$. A combination of (38) and (39) yields estimate (17). This puts an end to the proof of Lemma 10. □

Remark 12. If $f(u,x,t)$ is independent of x-variable, then the term $\Theta(D_x f,.)$ in both estimates (16) and (17) can be eliminated. Thus, (16) and (17) are relaxed respectively as follows:

$$|u_x(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\left(\tau^{-1}\Gamma^{1+\beta}(2T_0)+\Gamma^{1+\beta}(2T_0)\Theta(D_u f,\Gamma(2T_0))+1\right), \quad (40)$$

and

$$|u_x(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\left(\Gamma^{1+\beta}(2T_0)\Theta(D_u f,\Gamma(2T_0))+1\right), \quad (41)$$

for $(x,\tau)\in I\times(0,T_0)$.

If $f(u,x,t)=f(u)$, and f is merely a local Lipschitz function on $[0,\infty)$, then we have the following result.

Lemma 13. *Suppose that f is merely a local Lipschitz nonnegative function on $[0,\infty)$, and $f(0)=0$. Then equation $(P_{\varepsilon,\eta})$ has a unique solution (denoted by u for short), which satisfies*

$$|\partial_x u(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\left(\tau^{-1}\Gamma^{1+\beta}(2T_0)+\Gamma^{1+\beta}(2T_0)Lip(f,\Gamma(2T_0)+1\right), \quad (42)$$

for $(x,\tau)\in I\times(0,T_0)$, where $Lip(f,\Gamma(2T_0))$ is the local Lipschitz constant of f on the closed interval $[0,\Gamma(2T_0)]$.

Moreover, if $(u_0^{\frac{1}{\gamma}})_x\in L^\infty(I)$, then we have

$$|\partial_x u(x,\tau)|^p \le Cu^{1-\beta}(x,\tau)\left(\Gamma^{1+\beta}(2T_0)Lip(f,\Gamma(2T_0)+1\right), \quad (43)$$

with $C=C(\beta,p,\|(u_0^{\frac{1}{\gamma}})_x\|_\infty)>0$.

Proof. At the beginning, we regularize f on $[0,\infty)$. To do it, we extend f by 0 in $(-\infty,0)$ (still denoted by f). Let f_n be the standard regularization of f on $\mathbb{R}$. Then, we consider equation $(P_{\varepsilon,\eta})$ with the source $f_n(u)$ instead of $f(u)$. Thanks to Lemma 10 and Remark 12, equation $(P_{\varepsilon,\eta})$ possesses a unique classical solution, denoted by u_n, satisfying

$$|\partial_x u_n(x,\tau)|^p \le Cu_n^{1-\beta}(x,\tau)\left(\tau^{-1}\Gamma^{1+\beta}(2T_0)+\Gamma^{1+\beta}(2T_0)\Theta(D_u f,\Gamma(2T_0))+1\right), \quad (44)$$

for any $(x,t)\in I\times(0,T_0)$.

On the other hand, Rademacher's theorem (see [11]) ensures that

$$\Theta(f_n', \Gamma(2T_0)) \le Lip(f, \Gamma(2T_0) + \frac{1}{n}) \le Lip(f, 2\Gamma(2T_0)). \tag{45}$$

By (44) and (45), we observe that $|\partial_x u_n(x,t)|$ is bounded by a constant not depending on n. Then, the classical argument allows us to pass to the limit as $n \to \infty$ to get

$$u_n \to u, \quad \partial_x u_n \to \partial_x u, \quad \text{pointwise in } I \times (0, T_0).$$

Thus, gradient estimate (42) follows. Similarly, we also obtain estimate (43). □

Next, we shall show that $u_{\varepsilon,\eta}$ is a Lipschitz function on $I \times (\tau, T_0)$ with a Lipschitz constant C being independent of ε, η.

Proposition 14. *Let $u_{\varepsilon,\eta}$ be the solution of problem $(P_{\varepsilon,\eta})$ above. Then, for any $\tau \in (0, T_0)$ there is a positive constant $C(\beta, p, |I|, \tau, T_0, \|u_0\|_\infty)$ such that*

$$|u_{\varepsilon,\eta}(x,t) - u_{\varepsilon,\eta}(y,s)| \le C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall x, y \in \overline{I}, \quad \forall s, t \in (\tau, T_0). \tag{46}$$

Moreover, if $(u_0^{\frac{1}{\gamma}})_x \in L^\infty(I)$, then there is a constant $C = C(\beta, p, T_0, |I|, \|u_0\|_\infty, \|(u_0^{\frac{1}{\gamma}})_x\|_\infty) > 0$ such that inequality (46) holds for $x, y \in \overline{I}$, and for $s, t \in [0, T_0)$.

Proof. For the sake of brevity, we keep the notation $u = u_{\varepsilon,\eta}$.

Let us first extend u by η outside I, still denoted as u. Multiplying equation $(P_{\varepsilon,\eta})$ by $\partial_t u$, and using integration by parts yield

$$\int_s^t \int_I |\partial_t u|^2 + a(u_x)u_x \partial_t u_x + g_\varepsilon(u)\partial_t u \, dx d\sigma = \int_s^t \int_I f(u, x, \sigma)\partial_t u \, dx d\sigma,$$
$$\text{for } t > s \ge \tau. \tag{47}$$

Next, we observe that

$$a(u_x)u_x\partial_t u_x = \left(|u_x|^2 + \eta^2\right)^{\frac{p-2}{2}} \frac{1}{2}\partial_t(|u_x|^2) = \frac{1}{p}\partial_t(|u_x|^2 + \eta^2)^{\frac{p}{2}}.$$

By this fact, we deduce from the above equation

$$\int_s^t \int_I |\partial_t u(x,\sigma)|^2 dx d\sigma$$
$$\le \int_I \frac{1}{p}\left(|u_x(x,s)|^2 + \eta^2\right)^{\frac{p}{2}} dx + \int_I G_\varepsilon(u(x,s))dx + \int_s^t \int_I f(u, x, \sigma)\partial_t u dx d\sigma$$

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$$\leq \int_I \frac{1}{p}\left(|u_x(x,s)|^2+\eta^2\right)^{\frac{p}{2}}dx + \frac{1}{1-\beta}\int_I u(x,s)^{1-\beta}dx + \int_s^t\int_I f(u,x,\sigma)\partial_t u dx d\sigma$$

with

$$G_\varepsilon(r) = \int_0^r g_\varepsilon(s)ds \leq \int_0^r s^{-\beta}ds = \frac{r^{1-\beta}}{1-\beta}.$$

It follows from Holder's inequality that

$$\int_s^t\int_I |\partial_t u|^2 dx d\sigma \leq \int_I \frac{1}{p}\left(|u_x(x,s)|^2+\eta^2\right)^{\frac{p}{2}}dx + \frac{1}{1-\beta}\int_I u(x,s)^{1-\beta}dx + \left(\int_s^t\int_I f^2(u,x,\sigma)dx d\sigma\right)^{\frac{1}{2}}\left(\int_s^t\int_I |\partial_t u|^2 dx d\sigma\right)^{\frac{1}{2}}.$$

By Young's inequality, we obtain

$$\frac{1}{2}\int_s^t\int_I |\partial_t u|^2 dx d\sigma$$

$$\leq \frac{1}{p}\int_I |u_x(s)|^p dx + \frac{1}{1-\beta}\int_I u^{1-\beta}(s)dx + \frac{1}{2}\int_s^t\int_I f^2(u,x,\sigma)dx d\sigma + O(\eta),$$

where $\lim\limits_{\eta\to 0} O(\eta) = 0$. Since $f(u,x,t) \leq h(u)$, we obtain

$$\frac{1}{2}\int_s^t\int_I |\partial_t u|^2 dx d\sigma \leq \frac{1}{p}\int_I |u_x(s)|^p dx + \frac{1}{1-\beta}\int_I u^{1-\beta}(s)dx + \frac{1}{2}\int_s^t\int_I h^2(u)dx d\sigma + 1.$$

Or

$$\frac{1}{2}\int_s^t\int_I |\partial_t u|^2 dx d\sigma \leq \frac{1}{p}\int_I \|u_x(s)\|_\infty^p dx + \frac{1}{1-\beta}\int_I \Gamma^{1-\beta}(T_0)dx + \frac{1}{2}\int_s^t\int_I \bar{h}^2 dx d\sigma + 1,$$

for any $\tau < s < t < T_0$, and $\bar{h} = \max\limits_{0\leq s\leq\Gamma(T_0)}\{h(s)\}$. Thus, there is a constant $C = C(\beta, p, |I|, T_0)$ such that

$$\int_s^t\int_I |\partial_t u|^2 dx d\sigma \leq C\left(\|u_x(s)\|_\infty^p + \Gamma^{1-\beta}(T_0) + \bar{h}^2 + 1\right). \tag{48}$$

Thanks to (16), we have

$$|u_x(x,s)|^p \le Cu^{1-\beta}(x,s)\Big(s^{-1}\Gamma^{1+\beta}(2T_0)+\Gamma^{1+\beta}(2T_0)\Theta(D_u f,\Gamma(2T_0))$$
$$\Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f,\Gamma(2T_0))+1\Big),$$

for any $s>\tau$. Then,

$$|u_x(x,s)|^p \le C\Gamma^{1-\beta}(2T_0)\Big(\tau^{-1}\Gamma^{1+\beta}(2T_0)+\Gamma^{1+\beta}(2T_0)\Theta(D_u f,\Gamma(2T_0))+$$
$$\Gamma^{\frac{1+\beta\gamma}{\gamma}}(2T_0)\Theta(D_x f,\Gamma(2T_0))+1\Big),$$

for any $s>\tau$.

The last inequality and (48) imply that $\|\partial_t u\|_{L^2(I\times(\tau,T_0))}$ is bounded by a constant depending only on $\beta, p, \tau, |I|, T_0, \|u_0\|_\infty$.

Next, for any $x, y\in I$ and for $T_0>t>s>\tau$, we set

$$r=|x-y|+|t-s|^{\frac{1}{3}}.$$

According to the Mean Value Theorem, there is a real number $\bar{x}\in I_r(y)$ such that

$$|\partial_t u(\bar{x},\sigma)|^2=\frac{1}{|I_r(y)|}\int_{I_r(y)}|\partial_t u(l,\sigma)|^2dl=\frac{1}{2r}\int_{I_r(y)\cap I}|\partial_t u(l,\sigma)|^2dl\le\frac{1}{2r}\int_I|\partial_t u(l,\sigma)|^2dl \tag{49}$$

Note that $\partial_t u(.,t)=0$ outside I.

Now, we have from Holder's inequality and (49)

$$|u(\bar{x},t)-u(\bar{x},s)|^2\le(t-s)\int_s^t|\partial_t u(\bar{x},\sigma)|^2d\sigma\le\frac{(t-s)}{2r}\int_s^t\int_I|\partial_t u(l,\sigma)|^2dld\sigma,$$

or

$$|u(\bar{x},t)-u(\bar{x},s)|^2\le\frac{1}{2}(t-s)^{\frac{2}{3}}\int_s^t\int_I|\partial_t u(l,\sigma)|^2dld\sigma. \tag{50}$$

From (48) and (50), there is a constant $C=C(\beta,p,|I|,\tau,T_0)>0$ such that

$$|u(\bar{x},t)-u(\bar{x},s)|\le C(t-s)^{\frac{1}{3}},\quad \forall\tau<s<t<T_0. \tag{51}$$

Now, it is sufficient to show (46). Indeed, we have the triangle inequality

$$|u(x,t)-u(y,s)| \le |u(x,t)-u(y,t)| + |u(y,t)-u(y,s)| \le |u(x,t)-u(y,t)|+$$
$$|u(y,t)-u(\bar{x},t)| + |u(\bar{x},t)-u(\bar{x},s)| + +|u(\bar{x},s)-u(y,s)|,$$

where $\bar{x} \in I_r(y)$ is above. Then, the conclusion (46) just follows from (51), estimate (16), and the Mean Value Theorem.

Finally, if $(u_0^{\frac{1}{\gamma}})_x \in L^\infty(I)$ then the constant C in (46) does not depend on τ by using estimate (17) instead of using estimate (16). Thus, we get the above proposition. □

Note that the estimates in the proof of Lemma 10 and Proposition 14 are independent of η, ε. This observation allows us to pass to the limit as $\eta \to 0$ in order to get estimate (16) (resp. (17)) for problem (P_ε) below.

$$(P_\varepsilon)\begin{cases} \partial_t u - \partial_x\left(|\partial_x u|^{p-2}\partial_x u\right) + g_\varepsilon(u) = f(u,x,t) & \text{in } I\times(0,T_0), \\ u(x_1,t)=u(x_2,t)=0 & t\in(0,T_0), \\ u(x,0)=u_0(x) & \text{on } I. \end{cases} \tag{52}$$

Then, we have the following result

Theorem 15. *Let $u_0 \in L^\infty(I)$. Then, there exists a finite time $T_0>0$ so that equation $(P\varepsilon)$ possesses a unique bounded weak solution u_ε in $I\times(0,T_0)$. Furthermore, u_ε satisfies estimate (16) for a.e. $(x,t)\in I\times(0,T_0)$, and the regularity result in Proposition 14.*

Moreover, if $(u_0^{\frac{1}{\gamma}})_x \in L^\infty(I)$, then u_ε satisfies estimate (17).

Proof. Let us first assume that $u_0 \in \mathcal{C}_c^\infty(I)$. It is well known that problem (P_ε) possesses a unique solution u_ε in $I\times(0,T_0)$, which is the limit of solution $u_{\varepsilon,\eta}$ of problem $(P_{\varepsilon,\eta})$ as $\eta\to 0$, see details in Theorem 2.1, [31]. As a result, u_ε fulfills estimate (16) for a.e. $(x,t)\in I\times(0,T_0)$, and the Lipschitz property (46).

If $u_0\in L^\infty(I)$, then we make a regularization to u_0 by considering a sequence $\{u_{0,n}\}_{n\ge1}\subset \mathcal{C}_c^\infty(I)$ such that $u_{0,n}\overset{n\to\infty}{\longrightarrow} u_0$ in $L^r(I)$, for any $r\ge 1$, and $\|u_{0,n}\|_{L^\infty}\le\|u_0\|_{L^\infty}$.

For any $\varepsilon>0$ fixed, there exists a unique solution $u_{\varepsilon,n}$ of problem (P_ε) corresponding to initial data $u_{0,n}$.

Since $u_{\varepsilon,n}$ satisfies (46), it follows from the Ascoli–Azela Theorem that there is a subsequence (still denoted as $\{u_{\varepsilon,n}\}$) such that $u_{\varepsilon,n}$ converges to u_ε, uniformly on any compact of $\overline{I}\times(0,T_0)$ as $n\to\infty$. Using the diagonal argument deduces that $u_{\varepsilon,n}$ converges to u_ε, pointwise in $\overline{I}\times(0,T_0)$ up to a subsequence.

Then, we obtain from the boundedness of $u_{\varepsilon,n}$ and the Dominated Convergence Theorem

$$u_{\varepsilon,n}\overset{n\to\infty}{\longrightarrow} u_\varepsilon, \quad \text{in } L^r(I\times(0,T_0)),\ \forall r\ge 1. \tag{53}$$

At the moment, we derive some priori estimates for $u_{\varepsilon,n}$.

Using $u_{\varepsilon,n}$ as a test function in equation satisfied by $u_{\varepsilon,n}$ and integrating on I yield

$$\frac{1}{2}\frac{d}{dt}\int_I u_{\varepsilon,n}^2(t)dx + \int_I \left(|\partial_x u_{\varepsilon,n}|^p + g_\varepsilon(u_{\varepsilon,n})u_{\varepsilon,n}\right)dx = \int_I f(u_{\varepsilon,n},x,s)u_{\varepsilon,n}dx. \tag{54}$$

After integrating both sides of (54) on $(0, t)$, we get

$$\frac{1}{2}\int_I u_{\varepsilon,n}^2(t)dx + \int_0^t\int_I \left(|\partial_x u_{\varepsilon,n}|^p + g_\varepsilon(u_{\varepsilon,n})u_{\varepsilon,n}\right)dxds$$
$$= \int_0^t\int_I f(u_{\varepsilon,n}, x, s)u_{\varepsilon,n}dxds + \frac{1}{2}\int_I u_{\varepsilon,n}^2(0)dx$$
$$\leq \int_0^t\int_I h(u_{\varepsilon,n})u_{\varepsilon,n}dxds + |I|\|u_0\|^2_{L^\infty(I)}.$$

This leads to

$$\frac{1}{2}\int_I u_{\varepsilon,n}^2(t)dx + \int_0^t\int_I \left(|\partial_x u_{\varepsilon,n}|^p + g_\varepsilon(u_{\varepsilon,n})u_{\varepsilon,n}\right)dxds$$
$$\leq T_0|I|\bar{h}\|u_{\varepsilon,n}\|_{L^\infty(I\times(0,T_0))} + |I|\|u_0\|^2_{L^\infty(I)}. \tag{55}$$

Estimate (55) and the boundedness of $u_{\varepsilon,n}$ by $\Gamma(T_0)$ on $I \times (0, T_0)$ imply that $u_{\varepsilon,n}$ is bounded in $W^{1,p}(I \times (0, T_0))$ by a constant being independent of ε and n.

Next, we show that $g_\varepsilon(u_\varepsilon)$ is bounded in $L^1(I \times (0, T_0))$ by a constant not depending on ε. In fact, we have L^1-*estimate*

$$\int_I u_{\varepsilon,n}(T_0)dx + \int_0^{T_0}\int_I g_\varepsilon(u_{\varepsilon,n})dxds \leq \int_0^{T_0}\int_I f(u_{\varepsilon,n}, x, s)dxds + \int_I u_{\varepsilon,n}(0)dx.$$

Thus, we obtain

$$\int_0^{T_0}\int_I g_\varepsilon(u_{\varepsilon,n})dxds \leq T_0|I|\bar{h} + \|u_{0,n}\|_{L^1(I)}. \tag{56}$$

Thanks to the Dominated Convergence Theorem, we can pass to the limit in (56) to get

$$\int_0^{T_0}\int_I g_\varepsilon(u_\varepsilon)dxds \leq T_0|I|\bar{h} + \|u_0\|_{L^1(I)}. \tag{57}$$

Next, from equations satisfied by $u_{\varepsilon,n}$ and $u_{\varepsilon,m}$, we have

$$\partial_t(u_{\varepsilon,n} - u_{\varepsilon,m}) - (|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m})_x + g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})$$
$$= f(u_{\varepsilon,n}, x, t) - f(u_{\varepsilon,m}, x, t).$$

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Multiplying the above equation with $w_{n,m} = u_{\varepsilon,n} - u_{\varepsilon,m}$ and integrating on $I \times (0,t)$ yield

$$\frac{1}{2}\int_I w_{n,m}^2(t)dx + \int_0^t\int_I (|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m})\partial_x w_{n,m}dxds+ \\ \int_0^t\int_I \big(g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})\big)w_{n,m}dxds = \\ \int_0^t\int_I \big(f(u_{\varepsilon,n},x,t) - f(u_{\varepsilon,m},x,t)\big)w_{n,m}dxds + \frac{1}{2}\int_I w_{n,m}^2(0)dx, \tag{58}$$

for any $t \in (0,T_0)$.

By the strong monotonicity of p-Laplace operator, and the global Lipschitz property of g_ε, we get

$$\begin{cases} c\int_0^t\int_I |\partial_x w_{n,m}|^p dxds \le \int_0^t\int_I (|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m})\partial_x w_{n,m}dxds, \\ \int_0^t\int_I |g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})||w_{n,m}|dxds \le C_\varepsilon \int_0^t\int_I |w_{n,m}|^2 dxds, \\ \int_0^t\int_I |f(u_{\varepsilon,n},x,t) - f(u_{\varepsilon,m},x,t)||w_{n,m}|dxds \le \int_0^t\int_I 2\bar{h}|w_{n,m}|dxds, \end{cases} \tag{59}$$

where $c > 0$, and C_ε is the global Lipschitz constant of g_ε.

From (59) and (58), we get

$$\frac{1}{2}\int_I w_{n,m}^2(t)dx + c\int_0^t\int_I |\partial_x w_{n,m}|^p dxds \le 2\bar{h}\int_0^t\int_I |w_{n,m}|dxds + \\ C_\varepsilon\int_0^t\int_I |w_{n,m}|^2 dxds + \frac{1}{2}\int_I |u_{0,n} - u_{0,m}|^2 dx. \tag{60}$$

Since $\{u_{0,n}\}_{n\ge 1}$ is a Cauchy sequence in $L^2(I)$, and by (53), it follows from (60) that $\{\partial_x u_{\varepsilon,n}\}_{n\ge 1}$ is a Cauchy sequence in $L^p(I \times (0,T_0))$. Thus, we get

$$\partial_x u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon, \quad \text{in } L^p(I \times (0,T_0)). \tag{61}$$

As a result, there is a subsequence of $\{\partial_x u_{\varepsilon,n}\}_{n\ge 1}$ (still denoted as $\{\partial_x u_{\varepsilon,n}\}_{n\ge 1}$) such that

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$$\partial_x u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon, \quad \text{for a.e. } (x,t) \in I \times (0,T_0).$$

This implies that u_ε also fulfills estimate (16) for a.e. $(x,t) \in I \times (0,T_0)$.

On the other hand, by (60) we observe that $u_{\varepsilon,n}$ is a Cauchy sequence in $\mathcal{C}([0,T_0];L^2(I))$, thereby proves

$$u_\varepsilon \in \mathcal{C}([0,T_0];L^2(I)). \tag{62}$$

Now, we will show that u_ε satisfies equation (52) in $\mathcal{D}'(I\times(0,T_0))$. In fact, we write equation satisfied by $u_{\varepsilon,n}$ under variational form as follows

$$\int_0^{T_0}\int_I \Big(-u_{\varepsilon,n}\phi_t + |\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n}\phi_x + g_\varepsilon(u_{\varepsilon,n})\phi - f(u_{\varepsilon,n},x,s)\phi\Big)\,dxds = 0,$$

for any $\phi \in \mathcal{C}_c^\infty(I\times(0,T_0))$. Thanks to the Dominated Convergence Theorem, (53), and (61), we obtain after letting $n\to\infty$

$$\int_0^{T_0}\int_I \Big(-u_\varepsilon\phi_t + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon\phi_x + g_\varepsilon(u_\varepsilon)\phi - f(u_\varepsilon,x,s)\phi\Big)\,dxds = 0,$$
$$\forall\phi \in \mathcal{C}_c^\infty(I\times(0,T_0)).$$

In brief, u_ε is a weak solution of equation (52).

Finally, the uniqueness result follows from the standard argument due to the local Lipschitz property of $f(.,x,t)$. □

4. Local existence

In this section, we give the proof of Theorem 2.

Proof. For any $\varepsilon > 0$, by the result of Theorem 15, there is a unique weak solution u_ε of equation (52) in $I\times(0,T_0)$. We first claim that $\{u_\varepsilon\}_{\varepsilon>0}$ is an increasing sequence. In fact, we observe that

$$g_\varepsilon(s) \le g_{\varepsilon'}(s), \quad \text{for any } \varepsilon > \varepsilon' > 0,\ \forall s \ge 0.$$

This implies that u_ε is a super-solution of equation satisfied by $u_{\varepsilon'}$. Therefore, the comparison principle yields

$$u_\varepsilon \ge u_{\varepsilon'}, \quad \text{in } I\times(0,T_0),$$

or we get the above claim. As a result, there is a nonnegative function u such that $u_\varepsilon \downarrow u$ as $\varepsilon\to 0$. We would like to emphasize that the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ will be intensively used in what follows, although one can utilize Ascoli–Azela Theorem to show that $u_\varepsilon \to u$.

According to (56), the boundedness of f in $I\times(0,T_0)$, and the fact that $u_\varepsilon \downarrow u$ in $I\times(0,T_0)$, we can use a result of gradient convergence of Boccardo et al., [4], [3] in order to obtain

$$\partial_x u_\varepsilon \xrightarrow{\varepsilon\to 0} \partial_x u, \quad \text{for a.e. } (x,t) \in I \times (0, T_0). \tag{63}$$

The reader who is interested the proof of (63) in detail can find in [6]. As a result, u_x fulfills estimate (16) for a.e. $(x,t) \in I \times (0, T_0)$, and

$$\partial_x u_\varepsilon \xrightarrow{\varepsilon\to 0} \partial_x u, \quad \text{in } L^r(I \times (t_1, t_2)), \quad \forall r \geq 1, \quad \text{for } 0 < t_1 < t_2 < T_0. \tag{64}$$

Next, let us show that

$$u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0, T_0)). \tag{65}$$

From (57), applying Fatou's lemma deduces that there is a function $\Phi \in L^1(I \times (0,\infty))$ such that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = \Phi, \quad \text{in } L^1(I \times (0, T_0)). \tag{66}$$

The monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ ensures that

$$g_\varepsilon(u_\varepsilon)(x,t) \geq g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t),$$

so

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)(x,t) = \Phi \geq u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for a.e. } (x,t) \in I \times (0, T_0). \tag{67}$$

Thus, conclusion (65) just follows from (66) and (67). Actually, we will show at the end that

$$\Phi = u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(I \times (0,\infty)). \tag{68}$$

Now, we demonstrate that u must satisfy equation (1) in the sense of distribution.

For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, $\phi \in \mathcal{C}_c^\infty(I \times (0, T_0))$, in the equation satisfied by u_ε. Then, using integration by parts yields

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u_\varepsilon)\phi_t + \frac{1}{\eta}|\partial_x u_\varepsilon|^p \psi'(\frac{u_\varepsilon}{\eta})\phi + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon \phi_x \psi_\eta(u_\varepsilon) +$$
$$g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi + f(u_\varepsilon, x, s)\psi_\eta(u_\varepsilon)\phi \Big) dxds = 0,$$

with $\Psi_\eta(u) = \int_0^u \psi_\eta(s)ds$. Note that we use the function $\psi_\eta(.)$ in order to avoid the singularity of the term $u^{-\beta}\chi_{\{u>0\}}$, as u is near 0. Thus, there is no problem of going to the limit as $\varepsilon \to 0$ in the indicated equation:

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$$\int_{Supp(\phi)} \Big(- \Psi_\eta(u)\phi_t + \frac{1}{\eta}|u_x|^p \psi'(\frac{u}{\eta})\phi$$
$$+ |u_x|^{p-2}u_x\phi_x\psi_\eta(u) + u^{-\beta}\psi_\eta(u)\phi + f(u,x,s)\psi_\eta(u)\phi \Big) dxds = 0.$$

Next, we go to the limit as $\eta \to 0$ in the above equation. By (63), (64), (65), and the boundedness of f and u in $I \times (0, T_0)$, it is not difficult to verify

$$\begin{cases} \lim_{\eta\to 0} \int_{Supp(\phi)} \Psi_\eta(u)\phi_t dxds = \int_{Supp(\phi)} u\phi_t dxds, \\ \lim_{\eta\to 0} \int_{Supp(\phi)} |u_x|^{p-2}u_x\phi_x\psi_\eta(u)dxds = \int_{Supp(\phi)} |u_x|^{p-2}u_x\phi_x dxds, \\ \lim_{\eta\to 0} \int_{Supp(\phi)} u^{-\beta}\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi dxds, \\ \lim_{\eta\to 0} \int_{Supp(\phi)} f(u,x,s)\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} f(u,x,s)\phi dxds. \end{cases} \tag{69}$$

(Note that the assumption $f(0, x, t) = 0$ is used in the final limit of (69)).

On the other hand, we show that

$$\lim_{\eta\to 0} \int_{Supp(\phi)} \frac{1}{\eta}|\partial_x u|^p \psi'(\frac{u}{\eta})\phi dxds = 0. \tag{70}$$

In fact, since u satisfies estimate (16) for a.e. $(x, t) \in I \times (0, T_0)$, we have

$$\frac{1}{\eta}\int_{Supp(\phi)} |\partial_x u|^p|\psi'(\frac{u}{\eta})\phi|dxds \leq C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{1-\beta}dxds$$
$$\leq 2C \int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}dxds,$$

where the constant $C > 0$ is independent of η. Since $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $I \times (0, T_0)$ (see (65)), we obtain

$$\lim_{\eta\to 0} \int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}dxds = 0,$$

which implies the conclusion (70). A combination of (69) and (70) deduces

$$\int\limits_{Supp(\phi)} \big(-u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u,x,s)\phi\big)dxds = 0. \tag{71}$$

In other words, u satisfies equation (1) in $\mathcal{D}'(I\times(0,T_0))$.

As mentioned above, we prove (68) now. The fact that u_ε is a weak solution of (52) gives us

$$\int\limits_{Supp(\phi)} \big(-u_\varepsilon\phi_t + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon\partial_x\phi + g_\varepsilon(u_\varepsilon)\phi + f(u_\varepsilon,x,s)\phi\big)dxds = 0,$$

for $\phi\in\mathcal{C}_c^\infty(I\times(0,T_0))$, $\phi\geq 0$. Then, letting $\varepsilon\to 0$ deduces

$$\int\limits_{Supp(\phi)} \left(-u\phi_t + |u_x|^{p-2}u_x\phi_x\right)dxds + \lim_{\varepsilon\to 0}\int\limits_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi\, dxds + \int\limits_{Supp(\phi)} f(u,x,t)\phi\, dxds = 0. \tag{72}$$

By (71) and (72), we get

$$\lim_{\varepsilon\to 0}\int\limits_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi\, dxds = \int\limits_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi\, dxds. \tag{73}$$

According to (66), (73) and Fatou's lemma, we obtain

$$\int\limits_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi dxds \geq \int\limits_{Supp(\phi)} \Phi\phi\, dxds, \quad \forall\phi\in\mathcal{C}_c^\infty(I\times(0,T_0)), \phi\geq 0.$$

The last inequality and (67) yield conclusion (68).

Finally, the conclusion $u\in\mathcal{C}([0,T_0];L^1(I))$ is well known, so we skip its proof and refer to the compactness result in Theorem 1.1, [25], (see also [6], [26]).

In conclusion, u is a weak solution of equation (1).

We complete this Section by proving that u is the maximal solution of equation (1).

Proposition 16. *Let v be any weak solution of equation (1) on $I\times(0,T_0)$. Then, we have*

$$v(x,t)\leq u(x,t), \quad \textit{for a.e. } (x,t)\in I\times(0,T_0).$$

In fact, for any $\varepsilon>0$, we observe that

$$g_\varepsilon(v)\leq v^{-\beta}\chi_{\{v>0\}}.$$

Thus,

$$\partial_t v - \left(|v_x|^{p-2}v_x\right)_x + g_\varepsilon(v)\leq f(v,x,t), \quad \text{in } \mathcal{D}'(I\times(0,T_0)),$$

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which implies that v is a sub-solution of equation (P_ε). By the comparison principle, we get

$$v(x,t) \leq u_\varepsilon(x,t), \quad \text{for a.e. } (x,t) \in I \times (0,T_0).$$

Letting $\varepsilon \to 0$ yields the result. Thus, we complete the proof of Theorem 2. □

If $f(u,x,t) = f(u)$, we have then a local existence of solution.

Theorem 17. *Assume that $f(u,x,t) = f(u)$, and f is a local Lipschitz function on $[0,\infty)$. Then, equation (1) has a solution in $I \times (0,T_0)$ satisfying gradient estimate (42).*

Moreover, if $(u_0^{\frac{1}{\gamma}})_x \in L^\infty(I)$, then the above solution fulfills estimate (43)

Proof. The result is proved by a combination of the proof of Theorem 2 and Lemma 13. We leave it to the reader. □

5. Global existence of solution, and the extinction phenomenon

5.1. Global existence and the extinction of solution

It suffices to prove Theorem 4, and Theorem 6 for the maximal solution u.

Proof of Theorem 4. We first note that the local existence of solution u in $I \times (0,T_0)$ is established by Theorem 2. To prove that u is a global solution of equation (1), it is sufficient to show that u is bounded by a constant not depending on t.

In fact, let us put

$$\Phi(x) = \kappa_0 \phi_{I_0}(x). \tag{74}$$

We have $\inf_{x\in I}\{\phi_{I_0}(x)\} > 0$ since $I \subset\subset I_0$. Then, for any $\varepsilon \in \left(0, \frac{1}{2}\inf_{x\in I}\{\Phi(x)\}\right)$, we get $g_\varepsilon(\Phi) = \Phi^{-\beta}$, likewise

$$\begin{aligned}\mathcal{L}_\varepsilon(\Phi) &:= \Phi_t - (|\Phi_x|^{p-2}\Phi_x)_x + g_\varepsilon(\Phi) - \lambda f(\Phi,x,t) \\ &= -(|\Phi_x|^{p-2}\Phi_x)_x + \Phi^{-\beta} - \lambda f(\Phi,x,t).\end{aligned} \tag{75}$$

By (13), we observe that $\mathcal{L}_\varepsilon(\Phi) \geq 0$, in $\mathcal{D}'(I \times (0,T_0))$, so Φ is a super-solution of equation (P_ε). Thus, the strong comparison theorem yields

$$u_\varepsilon(x,t) \leq \Phi(x), \quad \forall (x,t) \in I \times (0,T_0).$$

The standard argument deduces the global existence of solution u_ε. Then, u exists globally and the conclusion (14) follows immediately by the monotonicity of u_ε. □

Next, we will show that for a given $\lambda > 0$, the maximal solution u must vanish identically after a finite time if $\|u_0\|_\infty$ is small enough.

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Proof of Theorem 6. One hand, the assumption $h(0) = 0$ implies that $f(s, x, t) \to 0$ as $s \to 0$, uniformly for any $(x, t) \in I \times (0, \infty)$. Other hand, since $\|u_0\|_\infty$ is sufficiently small, we can then choose an open bounded interval I_0 containing I, and $\kappa_0 > 0$ small as well such that (13) holds. Thanks to Theorem 4, the maximal solution u exists globally, and it is bounded by $M = \sup_{x\in I}\{\Phi(x)\}$, where Φ is the function in (74). Note that M is as small as $\|u_0\|_\infty$.

Using the test function u to equation (1) gives us

$$\frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + \int_I \left(|u_x(t)|^p + u^{1-\beta}(t)\right)dx = \lambda \int_I f(u, x, t)u dx$$
$$\leq \lambda M^\beta \max_{0\leq u\leq M}\{h(u)\}\int_I u^{1-\beta}(t)dx.$$

Or

$$\frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + \int_I |u_x(t)|^p dx + (1 - c_M)\int_I u^{1-\beta}(t)dx = 0, \tag{76}$$

where $c_M = \lambda M^\beta \max_{0\leq u\leq M}\{h(u)\}$ tends to 0 as $M \to 0$. Thus, $(1 - c_M) > c_0 > 0$, when $\|u_0\|_\infty$ is small enough. It follows from (76) that

$$\frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + c_0\int_I \left(|u_x(t)|^p dx + u^{1-\beta}(t)\right)dx \leq 0. \tag{77}$$

Now, using Garliardo–Nirenberg's inequality yields

$$\|u(t)\|_{L^2(I)} \leq c\|u_x(t)\|^\theta_{L^p(I)}\|u(t)\|^{1-\theta}_{L^1(I)} = c\left(\int_I |u_x(t)|^p dx\right)^{\frac{\theta}{p}}\left(\int_I u(t)dx\right)^{1-\theta},$$

with $\theta = \dfrac{1}{4 - 2p^{-1}}$. Thus,

$$\|u(t)\|_{L^2(I)} \leq c\left(\int_I \left(|u_x(t)|^p + u(t)\right)dx\right)^{\frac{\theta}{p}+1-\theta} \leq c\left(\int_I \left(|u_x(t)|^p + M^\beta u^{1-\beta}(t)\right)dx\right)^{\frac{\theta}{p}+1-\theta}$$
$$\leq c_1\left(\int_I \left(|u_x(t)|^p + u^{1-\beta}(t)\right)dx\right)^{\frac{\theta}{p}+1-\theta},$$

with $c = c(p)$, $c_1 = c_1(\beta, p, M) > 0$.

Then, we obtain

$$\left(\int_I u^2(t)dx\right)^\sigma \le c_2 \int_I \left(|u_x(t)|^p + u^{1-\beta}(t)\right) dx, \tag{78}$$

with $\sigma = \dfrac{1}{2(\frac{\theta}{p} + 1 - \theta)} \in (0, 1)$, and $c_2 = c_2(\beta, p, M) > 0$.

By (77) and (78), there is a positive constant $c_3 = c_3(\beta, p, M) > 0$ such that

$$z'(t) + c_3 z^\sigma(t) \le 0, \quad \text{for } t > 0, \tag{79}$$

with $z(t) = \|u(t)\|^2_{L^2(I)}$.

If we can show that there is a time $t_0 \in [0, \infty)$ such that $z(t_0) = 0$. It follows then from (79) that $z(t) = 0$, for any $t > t_0$, thereby proves Theorem 6.

In fact, we argue by a contradiction. Assume that $z(t) > 0$, for any $t > 0$. Solving the ordinary differential inequality (79) yields

$$z^{1-\sigma}(t) + c_3(1-\sigma)t \le \|u(0)\|^{2(1-\sigma)}_{L^2(I)}, \quad \forall t > 0, \tag{80}$$

which leads to a contradiction as t is sufficiently large.

In other word, we complete the proof of Theorem 6. $\square$

Remark 18. Inequality (80) implies that the extinction time of u, denoted by $T^\star \le \dfrac{\|u(0)\|^{2(1-\sigma)}_{L^2(I)}}{c_3(1-\sigma)}$.

Remark 19. Similarly, we also obtain the complete quenching result for the case λ small. As a result, Theorem 6 follows.

Now, we will show that $f(0, x, t) = 0$, for any $(x, t) \in I \times (0, \infty)$ is the necessary and the sufficient condition for the existence of solution of equation (1) for any small initial data.

Theorem 20. *Let $0 \le f \in \mathcal{C}^1\big([0, \infty) \times \overline{I} \times [0, \infty)\big)$. Assume that there exists a function $h \in \mathcal{C}^1([0, \infty))$ such that $f(u, x, t) \le h(u)$, for any $(x, t) \in I \times (0, \infty)$. Assume that equation (1) has a solution for any initial data. Then, we have $f(0, x, t) = 0$, for any $(x, t) \in I \times (0, \infty)$.*

Proof. We argue by a contradiction that there exists $(x_0, t_0) \in I \times (0, \infty)$, such that

$$f(0, x_0, t_0) > 0. \tag{81}$$

Let $\|u_0\|_\infty$ be sufficiently small, and v be a weak solution of equation (1).

By $f(0, x_0, t_0) > 0$, we can assume without loss of generality that $h(0) = 1$. Consider an open interval I_0 such that $I \subset\subset I_0$.

Put

$$\kappa_0 = \frac{\|u_0\|_\infty}{\inf\limits_{x \in I}\{\phi_{I_0}(x)\}}, \quad \text{and } \Phi_0(x) = \kappa_0 \phi_{I_0}(x),$$

where ϕ_{I_0} is the first eigenfunction of equation (5) in I_0. We note that $\inf_{x\in I}\{\phi_{I_0}(x)\} > 0$. Then for any $\varepsilon \in \left(0, \frac{1}{2}\inf_{x\in I}\{\phi_{I_0}(x)\}\right)$, we consider

$$\begin{aligned}\overline{\mathcal{L}_\varepsilon}(\Phi_0) &= (\Phi_0)_t - (|(\Phi_0)_x|^{p-2}(\Phi_0)_x)_x + g_\varepsilon(\Phi_0) - h(\Phi_0)\\ &= \lambda_{I_0}\kappa_0^{p-1}\phi_{I_0}^{p-1} + \kappa_0^{-\beta}\phi_{I_0}^{-\beta} - h(\kappa_0\phi_{I_0}), \quad \text{in } \mathcal{D}'(I).\end{aligned}$$

Next, we observe

$$\kappa_0^{-\beta}\phi_{I_0}^{-\beta} \geq \|u_0\|_\infty^{-\beta}\left(\frac{\inf_{x\in I}\{\phi_{I_0}(x)\}}{\max_{x\in I}\{\phi_{I_0}(x)\}}\right)^{\beta},$$

which implies that $\kappa_0^{-\beta}\phi_{I_0}^{-\beta}$ is large when $\|u_0\|_\infty$ is small. While, $h(\kappa_0\phi_{I_0})$ is bounded by a constant. Thus, $\overline{\mathcal{L}_\varepsilon}(\Phi_0) > 0$ in $\mathcal{D}'(I)$, thereby

$$\mathcal{L}_\varepsilon(\Phi_0) \geq \overline{\mathcal{L}_\varepsilon}(\Phi_0) > 0.$$

Or, Φ_0 is a super-solution of equation (P_ε).

On the other hand, v is a sub-solution of equation (P_ε), see the proof in Proposition 16. Therefore, the comparison principle deduces

$$v(x) \leq \kappa_0\phi_{I_0}(x), \quad \text{in } I \times (0, \infty).$$

By the same analysis as in the proof of Theorem 6, we obtain

$$v(x,t) = 0, \quad \text{for any } (x,t) \in I \times (T^\star, \infty), \tag{82}$$

where $T^\star \leq c\|u(0)\|_{L^2(I)}^{2(1-\sigma)}$, for some constant $c > 0$, see Remark 18.

By equation (1) and (82), we obtain

$$f(0,x,t) = 0, \quad \text{for any } x \in I, \text{ and for } t > c\|u(0)\|_{L^2(I)}^{2(1-\sigma)}.$$

Since $\|u_0\|_\infty$ is small enough, then $t_0 > c\|u(0)\|_{L^2(I)}^{2(1-\sigma)}$. Thereby, $f(0,x,t_0) = 0$, for any $x \in I$. This contradicts (81). Or, we complete the proof of Theorem 20. □

5.2. *The critical case $\lambda f(u,x,t) = \lambda u^{p-1}$*

As mentioned in the Introduction, it is interesting to study our extinction result for the critical case: $f(u) = \lambda u^{p-1}$. We will show that the maximal solution of equation (1) vanishes after a finite time if provided $0 \leq (\lambda - \lambda_I)$ small enough. We emphasize that the proof of the following theorem will illustrate the role of the singular absorption term $u^{-\beta}\chi_{\{u>0\}}$ in preventing blow-up and in forcing solution to the extinction.

Theorem 21. *Let $u_0(x) = \phi_I(x)$. Assume that $0 \leq (\lambda - \lambda_I)$ is sufficiently small. Then, the maximal solution u vanishes identically after a finite time.*

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Proof. We first claim the global existence of the maximal solution u. The idea is similar to the one of Theorem 4.

Indeed, let us put $I_\delta = (x_1 - \delta, x_2 + \delta)$, where $\delta > 0$ small enough. Let λ_{I_δ} and ϕ_{I_δ} be the first eigenvalue and the first eigenfunction of problem (5) in I_δ. Because λ_{I_δ} is a continuous function with respect to I_δ, then we can choose $\delta > 0$ small such that $|\lambda_{I_\delta} - \lambda|$ is small as well.

Fix $\delta > 0$ small, there is a positive constant $k_0 > 0$ such that

$$k_0\phi_{I_\delta}(x) \geq \phi_0(x), \quad \forall x \in I,$$

and for any $\varepsilon \in \left(0, \frac{1}{2}\inf_{x\in I}\{\phi_\delta(x)\}\right)$, we have

$$\mathcal{L}_\varepsilon(k_0\phi_\delta) = \lambda_{I_\delta}k_0^{p-1}\phi_{I_\delta}^{p-1}(x) + k_0^{-\beta}\phi_{I_\delta}^{-\beta}(x) - \lambda k_0^{p-1}\phi_{I_\delta}^{p-1}(x).$$

Since $\phi_{I_\delta}^{-\beta}(x)$ is large when x is near to x_1 (resp. x_2), and $|\lambda - \lambda_{I_\delta}|$ is small enough, then it is not difficult to verify that $\mathcal{L}_\varepsilon(k_0\phi_\delta) > 0$ in $\mathcal{D}'(I \times (0, T_0))$. Thus, the comparison theorem yields

$$u_\varepsilon(x,t) \leq k_0\phi_\delta(x), \quad \forall (x,t) \in I \times (0, T_0), \tag{83}$$

which implies the global existence of u_ε. Or, we get the above claim.

Next, we prove the extinction result for the maximal solution u.

By multiplying the test function u to equation (1), we have

$$\frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + \int_I \left(|u_x(t)|^p + u^{1-\beta}(t)\right)dx = \int_I \lambda u^p dx. \tag{84}$$

Since $\lambda_I = \inf_{v\in W_0^{1,p}(I)}\{\frac{\int_I |v_x|^p dx}{\int_I |v|^p dx}\}$, we have

$$\int_I u^p dx \leq \frac{1}{\lambda_I}\int_I |u_x|^p dx.$$

A combination of this inequality and (84) deduces

$$\frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + \int_I u^{1-\beta}(t)dx \leq (\frac{\lambda}{\lambda_I} - 1)\int_I |u_x(t)|^p dx. \tag{85}$$

Thanks to estimate (16), and the boundedness of u in (83), we obtain

$$|u_x(x,t)|^p \leq C(\beta,p)u^{1-\beta}(x,t)\left(t^{-1}M^{1+\beta} + (p-1)M^{p+\beta-1} + 1\right),$$
$$\text{for a.e. } (x,t) \in I \times (0,\infty),$$

with $M = \sup_{x\in I}\{k_0\phi_\delta(x)\}$. Therefore, we get for any $t \geq 1$,

$$|u_x(x,t)|^p \leq C(\beta,p)\left(M^{1+\beta}+(p-1)M^{p+\beta-1}+1\right)u^{1-\beta}(x,t). \tag{86}$$

Insert inequality (86) into the right hand side of (85) to get

$$\frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + C_1\int_I u^{1-\beta}(t)dx \leq 0, \quad \text{for } t>1, \tag{87}$$

with $C_1 = 1 - C(\beta,p)\left(M^{1+\beta}+(p-1)M^{p+\beta-1}+1\right)\left(\frac{\lambda}{\lambda_I}-1\right) > 0$. Note that $C_1>0$ since $|\lambda-\lambda_I|$ is small enough.

From (86) and (87), there is a constant $C_2>0$ such that

$$\frac{d}{dt}\int_I u^2(t)dx + C_2\int_I \left(|u_x(t)|^p + u^{1-\beta}(t)\right)dx \leq \frac{1}{2}\frac{d}{dt}\int_I u^2(t)dx + C_1\int_I u^{1-\beta}(t)dx \leq 0, \quad \text{for } t>1. \tag{88}$$

By the same argument as in the proof of Theorem 6 after (77), we obtain

$$z'(t) + C_3 z^\sigma(t) \leq 0, \quad \text{for } t>1, \tag{89}$$

with $C_3>0$, and $z(t)=\|u(t)\|^2_{L^2(I)}$.

Now, if we can show that there is a time $t_0\in[1,\infty)$ such that $z(t_0)=0$. It follows then from (89) that $z(t)=0$, $\forall t>t_0$. Then, we complete the proof of Theorem 21.

By contradiction, we assume that $z(t)>0$ for any $t>1$. Solving the ordinary differential inequality (89) yields

$$z^{1-\sigma}(t) + C_3(1-\sigma)(t-1) \leq \|u(1)\|^{2(1-\sigma)}_{L^2(I)}, \quad \forall t>1,$$

which is ridiculous when t is large enough. In other words, we get the above theorem. □

6. Non-global existence of solution

We give the proof of Theorem 8. The proof follows from the lemmas below.

Lemma 22. *Let $u_0\in W_0^{1,p}(I)$. Suppose that $f(u,x,t)=f(u)$. Then, the maximal solution u of equation (1) in $I\times(0,T)$ satisfies the energy relations: for any $t\in(0,T)$*

$$\frac{1}{2}\int_I u^2(x,t)dx - \frac{1}{2}\int_I u_0^2(x)dx = \int_0^t\int_I \left(uf(u) - |u_x|^p - u^{1-\beta}\right)dxds, \tag{90}$$

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$$\int_0^t \int_I u_t^2 dxds + \int_I \left(\frac{1}{p} |u_x(t)|^p + \frac{1}{1-\beta} u^{1-\beta}(t) - F(u(t)) \right) dx \\ \leq \int_I \left(\frac{1}{p} |(u_0)_x|^p + \frac{1}{1-\beta} u_0^{1-\beta} - F(u_0) \right) dx. \tag{91}$$

Proof. The proof of Lemma 22 is classical, so we skip it, and refer its proof to Theorem 2.1, [16]. □

Lemma 23. *Assume that the hypotheses of Lemma 22 holds. Suppose that $\frac{F(u)}{u^p}$ is nondecreasing on $(0, \infty)$. If $u \in L^\infty(I \times (0, T))$, then we have*

$$pE(0) + \frac{4(3p-1)}{T(p-2)^2} \int_I u_0^2 dx > 0. \tag{92}$$

Proof. Let us put

$$\alpha = \frac{2}{T(p-2)^2} \int_I u_0^2 dx, \quad t_0 = \frac{T}{2}(p-2),$$

and

$$H(t) = \frac{1}{2} \int_0^t \int_I u^2(x,s) dxds + \frac{1}{2} \int_I (T-t) u_0^2(x) dx + \alpha (t + t_0)^2.$$

Obviously, we have from (90)

$$H'(t) = \frac{1}{2} \int_I u^2(x,t) dx - \frac{1}{2} \int_I u_0^2(x) dx + 2\alpha(t+t_0)$$
$$= \int_0^t \int_I \left(uf(u) - |u_x|^p - u^{1-\beta} \right) dxds + 2\alpha(t+t_0),$$
$$H''(t) = \int_I \left(uf(u) - |u_x|^p - u^{1-\beta} \right) dx + 2\alpha. \tag{93}$$

By Holder's inequality

$$\left| \frac{1}{2} \int_I u^2(x,t) dx - \frac{1}{2} \int_I u_0^2(x) dx \right| = \left| \frac{1}{2} \int_0^t \int_I (u^2)_t dxds \right|$$

$$\leq \int_0^t \int_I |uu_t| dxds \leq \left(\int_0^t \int_I u^2 dxds \right)^{\frac{1}{2}} \left(\int_0^t \int_I u_t^2 dxds \right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} |H'(t)|^2 \leq & \left(\int_0^t \int_I u^2 dxds \right) \left(\int_0^t \int_I u_t^2 dxds \right) + 4\alpha^2 (t+t_0)^2 \\ & + 4\alpha(t+t_0) \left(\int_0^t \int_I u^2 dxds \right)^{\frac{1}{2}} \left(\int_0^t \int_I u_t^2 dxds \right)^{\frac{1}{2}}. \end{aligned} \tag{94}$$

By the definition of $H(t)$, we have for any $t \in [0, T]$

$$\begin{cases} 4\alpha^2 (t+t_0)^2 \leq 4\alpha H(t), \\ t + t_0 \leq \sqrt{\frac{H(t)}{\alpha}}, \\ \int_0^t \int_I u^2(x,s) dxds \leq 2H(t). \end{cases} \tag{95}$$

Inserting (95) into the right hand side of (94) yields

$$|H'(t)|^2 \leq H(t) \left(\int_0^t \int_I u_t^2 dxds + 4\sqrt{2\alpha} \left(\int_0^t \int_I u_t^2 dxds \right)^{\frac{1}{2}} + 4\alpha \right)$$

By Schwarz's inequality, we obtain

$$|H'(t)|^2 \leq H(t) \left(2 \int_0^t \int_I u_t^2 dxds + 12\alpha \right) \tag{96}$$

Thus, it follows from (93) and (96) that

$$\begin{aligned} & H(t)H''(t) - \frac{p}{2}|H'(t)|^2 \\ & \geq H(t) \left(\int_I \left(uf(u) - |u_x|^p - u^{1-\beta} \right) dx + 2\alpha - p \int_0^t \int_I u_t^2 dxds - 6p\alpha \right) \end{aligned}$$

Thanks to (91), we get

$$H(t)H''(t) - \frac{p}{2}|H'(t)|^2 \geq H(t)\Big(\int_I \big(uf(u) - pF(u)\big)(t)dx + \frac{p+\beta-1}{1-\beta}\int_I u^{1-\beta}(t)dx + \int_I \big(pF(u_0) - |(u_0)_x|^p - \frac{p}{1-\beta}u_0^{1-\beta}\big)dx - 2\alpha(3p-1)\Big)$$

Since $\frac{F(u)}{u^p}$ is non-decreasing on $(0,\infty)$, we have

$$\frac{d}{du}\left(\frac{F(u)}{u^p}\right) \geq 0, \quad \text{for } u > 0,$$

thereby proves

$$uf(u) - pF(u) \geq 0.$$

Then

$$H(t)H''(t) - \frac{p}{2}|H'(t)|^2 \geq H(t)\Big(\int_I \big(pF(u_0) - |(u_0)_x|^p - \frac{p}{1-\beta}u_0^{1-\beta}\big)dx - 2\alpha(3p-1)\Big).$$

Or

$$H(t)H''(t) - \frac{p}{2}|H'(t)|^2 \geq H(t)\Big(-pE(0) - \frac{4(3p-1)}{T(p-2)^2}\int_{I_0} u_0^2 dx\Big).$$

Now, if conclusion (92) fails, then

$$pE(0) + \frac{4(3p-1)}{T(p-2)^2}\int_{I_0} u_0^2 dx \leq 0,$$

thereby

$$H(t)H''(t) - \frac{p}{2}|H'(t)|^2 \geq 0,$$

or

$$\frac{d^2}{dt^2}\big(H^{1-\frac{p}{2}}(t)\big) \leq 0, \quad \text{for } t \in (0,T). \tag{97}$$

Clearly

$$H(0) > 0, \quad \big(H^{1-\frac{p}{2}}\big)'(0) < 0, \quad \frac{-H^{1-\frac{p}{2}}(0)}{\big(H^{1-\frac{p}{2}}\big)'(0)} = T.$$

Solving the ordinary differential inequality (97) yields

$$H(t) \geq \left(\frac{1}{H^{1-\frac{p}{2}}(0) + t\left(H^{1-\frac{p}{2}}\right)'(0)} \right)^{\frac{1}{\frac{p}{2}-1}},$$

which implies that $H(t)$ is blowing-up at a time $\tau_0 \in (0, T]$. This contradicts the boundedness of $H(t)$ in $[0, T]$, so Lemma 23 is proved. □

Lemma 24. *Suppose that the hypotheses of Lemma 23 holds. Let u be the maximal local solution of equation (1). If we assume that*

$$pE(0) + \frac{4(3p-1)}{T(p-2)^2} \int_I u_0^2 dx \leq 0, \tag{98}$$

then there exists a time $\tau_0 \in (0, T]$ such that

$$\lim_{t \to \tau_0} \|u(t)\|_{L^\infty(I)} = +\infty.$$

Proof. Let $\tau = \sup\left\{t \in (0, T) : u \in L^\infty(I \times (0, \tau))\right\}$.

If $\tau < T$, then there is a positive constant $M > 0$ such that

$$\|u(t)\|_\infty \leq M, \quad \text{for } t \in (0, \tau].$$

Put

$$f_M(u) = \begin{cases} f(u), & \text{if } u < M+1, \\ f(M+1), & \text{if } u \geq M+1. \end{cases}$$

It is clear that f_M is a global Lipschitz function on $[0, \infty)$. We consider the following equation:

$$\begin{cases} \partial_t v - (|v_x|^{p-2} v_x)_x + v^{-\beta} \chi_{\{v>0\}} = f_M(u) & \text{in } I \times (0, T), \\ v(x_1, t) = v(x_2, t) = 0 & t \in (0, T), \\ v(x, 0) = u_0(x) & \text{in } I. \end{cases} \tag{99}$$

Thanks to Lemma 13, and the boundedness of f_M, equation (99) has a maximal continuous global solution v. Since u and v are the two maximal solutions on $(0, \tau]$, then we obtain

$$v = u, \quad \text{on } (0, \tau],$$

which implies that $v \leq M$ on $(0, \tau]$. It follows from the continuity of v that

$$v \leq M+1, \quad \text{on } (0, \tau + \varepsilon],$$

for some $\varepsilon > 0$. Note that $f_M(s) = f(s)$, if $s \leq M+1$. This implies that v is also the maximal solution of equation (1) in $I \times (0, \tau + \varepsilon)$. This contradicts the definition of τ.

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If $\tau = T$, then u is bounded in $I \times (0, T)$. This contradicts Lemma 23. Thus, u must blow up at a time $\tau_0 \in (0, T]$. Or, Lemma 24 is proved. □

To complete the proof of Theorem 8, it remains to show that the set

$$\mathcal{E}(T) = \left\{ u_0 \in W_0^{1,p}(I) : pE(0) + \frac{4(3p-1)}{T(p-2)^2} \int_I u_0^2 dx \leq 0 \right\}$$

is not empty. Then, we have

Lemma 25. *Let $u_0 \in W_0^{1,p}(I)$, and $u_0 \neq 0$. Assume that $\lim\limits_{u \to \infty} \dfrac{F(u)}{u^p} = +\infty$. Then, $\kappa u_0 \in \mathcal{E}(T)$ for $\kappa > 1$ large enough.*

In the critical case: $f(u) = u^{p-1}$, then $\kappa u_0 \in \mathcal{E}(T)$ if $|I|$ and κ are sufficiently large.

Proof. By contradiction, if $\kappa u_0 \notin \mathcal{E}(T)$, for any $\kappa \geq 1$, then we have

$$p \int_I F(\kappa u_0) dx < \kappa^p \int_I |(u_0)_x|^p dx + \frac{p\kappa^{1-\beta}}{1-\beta} \int_I u_0^{1-\beta} dx + \frac{4(3p-1)\kappa^2}{T(p-2)^2} \int_I u_0^2 dx.$$

Or

$$\frac{p}{\kappa^p} \int_I F(\kappa u_0) dx < \int_I |(u_0)_x|^p dx + \frac{p\kappa^{1-\beta-p}}{1-\beta} \int_I u_0^{1-\beta} dx + \frac{4(3p-1)\kappa^{2-p}}{T(p-2)^2} \int_I u_0^2 dx. \quad (100)$$

It is clear that the right hand side of (100) is bounded by a constant not depending on the parameter $\kappa \geq 1$.

On the other hand, we have

$$\frac{1}{\kappa^p} \int_I F(\kappa u_0) dx = \int_I u_0^p \frac{F(\kappa u_0)}{(\kappa u_0)^p} dx \to +\infty, \quad \text{as } \kappa \to +\infty.$$

This contradicts the boundedness of $\dfrac{1}{\kappa^p} \displaystyle\int_I F(\kappa u_0) dx$, so we get the conclusion for the case:

$$\lim_{u \to \infty} \frac{F(u)}{u^p} = +\infty.$$

Next, we prove for the critical case.

Without loss of generality, we assume $I = (-l, l)$, with $l > 1$ large enough. We consider an initial data $u_0 \in C_c^\infty(I)$, such that $u_0 = 1$, on $(-l+1, l-1)$, $0 \leq u_0 \leq 1$, $u_0 = 0$ outside $(-l, l)$, and $|(u_0)_x| \leq c_0$.

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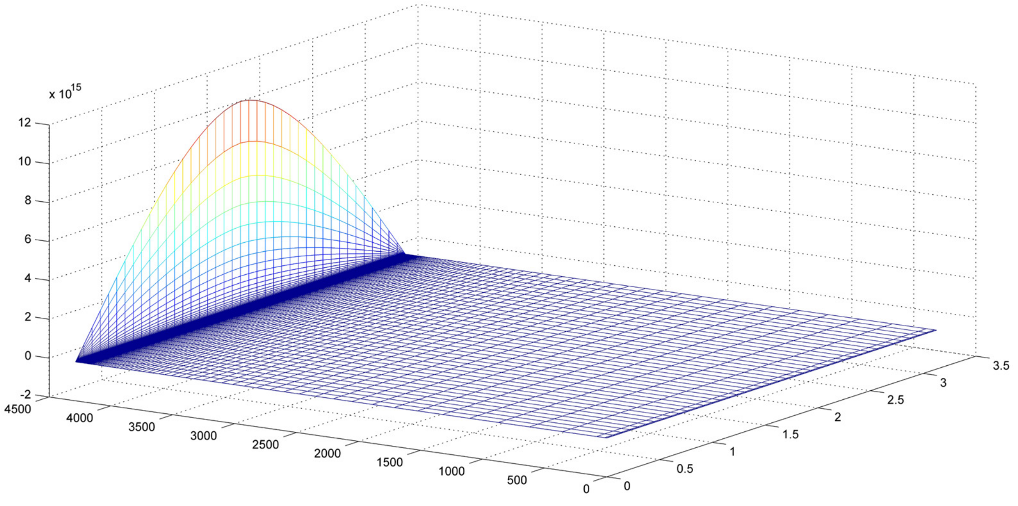

Fig. 1. Evolution of the unique solution of equation (7).

Then

$$\frac{1}{\kappa^p}\int_I F(\kappa u_0)dx = \frac{1}{p}\int_I u_0^p(x)dx \geq \frac{2}{p}(l-1),$$

$$\int_I |(u_0)_x|^p dx + \frac{p\kappa^{1-\beta-p}}{1-\beta}\int_I u_0^{1-\beta}dx$$
$$+\frac{4(3p-1)\kappa^{2-p}}{T(p-2)^2}\int_I u_0^2 dx \leq C(\beta, p, c_0, T)\left((\kappa^{1-\beta-p}+\kappa^{2-p})l+1\right).$$

Thus, $\kappa u_0 \in \mathcal{E}(T)$ if $\kappa, |I|$ are sufficiently large. Or, we get the above lemma.

This puts an end to the proof of Theorem 8. □

7. Numerical experiences

In this part, we illustrate our theoretical results with some numerical experiences. Our numerical scheme mimics the one in the paper of Ferrera et al., [12]. Similarly, we use the linear finite elements with mass lumping in a uniform mess for the space variable to discretize our equations (1) and (7). The reader who is interested in detail can find in [12].

In the sequel, we consider equation (1) and equation (7) for the case: $q = p = 2.3$, $I = (0, L)$, and $u_0(x) = x(L-x)$, and $f(u) = \lambda u^{q-1}$. We fix $\beta = 0.8$, $L = 3.1273$. It follows then from (6) that $\lambda_I = 0.9999$.

With $\lambda = 1 > \lambda_I$ (just a little bit difference), the unique solution of equation (7) blows up after $t = 4286$, see Fig. 1.

With $\lambda = 1.269$, the maximal solution of equation (1) vanishes after $t = 7.6$, see Fig. 2.

With $\lambda = 1.270$, the maximal solution of equation (1) blows up at $t = 23$, see Fig. 3 (compare to the case $\lambda = 1.269$ in Fig. 2).

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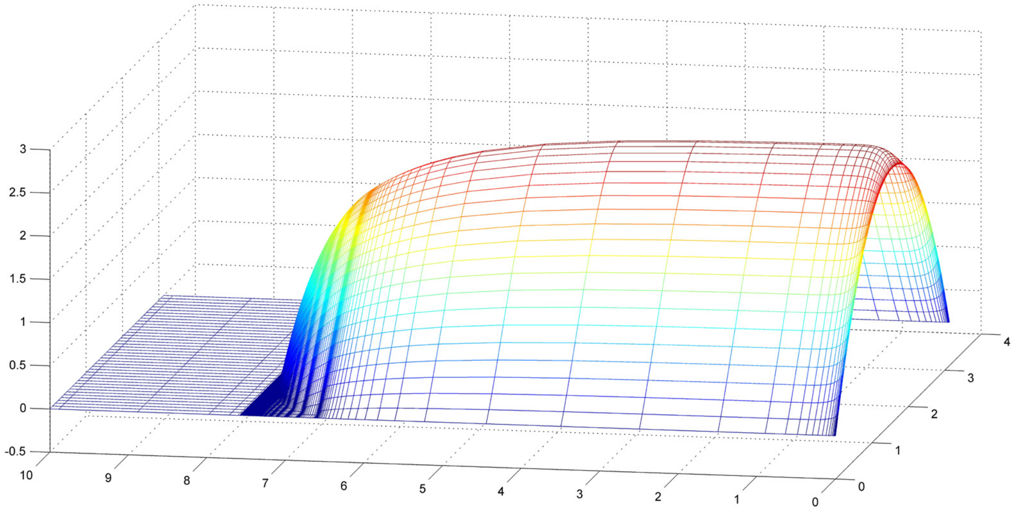

Fig. 2. Evolution of the maximal solution of equation (1).

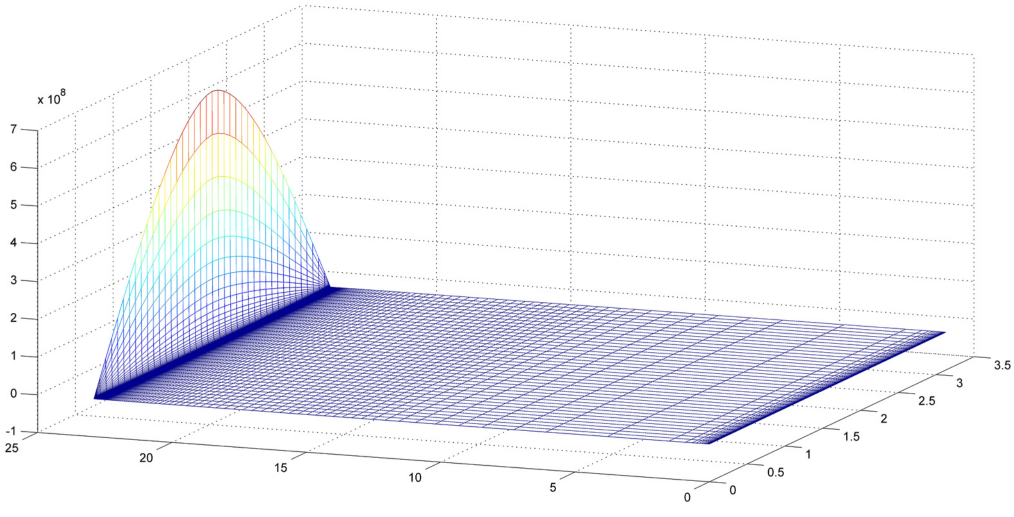

Fig. 3. Evolution of the maximal solution of equation (1).

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