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LINEAR DIFFUSION WITH SINGULAR ABSORPTION POTENTIAL AND/OR UNBOUNDED CONVECTIVE FLOW: THE WEIGHTED SPACE APPROACH

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Abstract. In this paper we prove the existence and uniqueness of very weak solutions to linear diffusion equations involving a singular absorption potential and/or an unbounded convective flow on a bounded open set of $\mathbb{R}^N$. In most of the paper we consider homogeneous Dirichlet boundary conditions but we prove that when the potential function grows faster than the distance to the boundary to the power -2 then no boundary condition is required to get the uniqueness of very weak solutions. This result is new in the literature and must be distinguished from other previous results in which such uniqueness of solutions without any boundary condition was proved for degenerate diffusion operators (which is not our case). Our approach, based on the treatment on some distance to the boundary weighted spaces, uses a suitable regularity of the solution of the associated dual problem which is here established. We also consider the delicate question of the differentiability of the very weak solution and prove that some suitable additional hypothesis on the data is required since otherwise the gradient of the solution may not be integrable on the domain.

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1. Introduction.

In this paper we want to develop a weighted space approach to study the existence, uniqueness and regularity of linear diffusion equations involving singular and unbounded coefficients of the type

$$-\Delta\omega + \vec{u}\cdot\nabla\omega + V\omega = f \text{ on } \Omega, \tag{1}$$

where V is a very singular potential being in general non negative and locally integrable. To fix ideas, we shall consider mainly the case of Dirichlet boundary conditions

$$\omega = 0 \text{ on } \partial\Omega, \tag{2}$$

but our weighted space approach can also be adapted to the case of Neumann boundary conditions and, what is more remarkable, to the case of **no boundary conditions on** $\partial\Omega$ (but still getting the uniqueness of solutions) for some specially singular potentials (see the subsection 4.2 in section 4). Here Ω is an open bounded smooth (for instance with $\partial\Omega$ of class $C^{2,1}$) of $\mathbb{R}^N$, $N \geqslant 2$, (the case $N = 1$ and u =constant is considerably simpler) . The external forcing term $f(x)$ will be assumed such that

$$f \in L^1(\Omega; \delta) \tag{3}$$

where the weight in this space is given by

$$\delta(x) = d(x, \partial\Omega) \tag{4}$$

(sharper results will require some slight restrictions to (3) (see for instance section 4.3). We recall that (3) is optimal in the cases $V \equiv 0$ and $\vec{u} = \vec{0}$ as it can be shown by explicitly computing the Green kernel for special domains.
Although we shall indicate later the detailed assumptions on the data, we anticipate now that we shall always assume that the convective flow vector $\vec{u}$ satisfies

$$\begin{cases} \vec{u} \in L^N(\Omega)^N, \ \operatorname{div}\vec{u} = 0 & \text{in} \quad \mathcal{D}'(\Omega) \text{ and} \\ \vec{u}\cdot\vec{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

where $\vec{n}$ denotes the unit exterior normal vector to $\partial\Omega$. Notice that, due to (5), the weak solution notion adapted to equation (1) is equivalent to the one defined for the treatment of the equation in divergent form that is

$$-\Delta\omega + \operatorname{div}(\vec{u}\,\omega) + V\omega = f \quad \text{in} \ \ \Omega. \tag{6}$$

It is well-known that the mathematical treatment of diffusion equations such as (1) $\big($or (6)$\big)$ leads to quite satisfactory results (in view of some applications) when the data f, $\vec{u}$ and V are assumed to be bounded. Nevertheless, the main interest of this work concerns the limit cases in which $V(x)$ is assumed to be a singular function (mainly with its singularity located on $\partial\Omega$) and/or when $\vec{u}$ is an unbounded vector (satisfying (5)). Let us indicate some relevant applications leading to the consideration of such limit cases :

1. *The vorticity equation in fluid mechanics.* Equation (1) can be derived from the stationary Navier-Stokes in 2D

$$-\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{F} \tag{7}$$

taking the curl of the equation and setting

$$f = \vec{F}\cdot\vec{k}, \qquad \omega = \operatorname{curl}\vec{u}\cdot\vec{k}, \tag{8}$$

where $\vec{k}$ is the last element of the canonical basis in $\mathbb{R}^3$ (see e.g. [46]). Nevertheless, as far as we know no satisfactory theory is available in the literature under the general condition that $\vec{F}\cdot\vec{k}\in L^1(\Omega;\delta)$.

2. *Schrödinger equation with singular potentials.* It is well-known that the consideration of the bound states $\psi(x,t)=e^{-iEt}\omega(x)$ leads to the stationary Schrödinger equation
$$-\Delta\omega+V(x)\omega=E\omega \qquad \text{in } \ \mathbb{R}^N. \tag{9}$$
The Heisenberg uncertainty principle makes specially interesting the consideration of potentials which are critically singular on $\partial\Omega$ more precisely, such that
$$V(x)\geqslant\frac{c}{\delta(x)^2}, \qquad a.e\ x\in\Omega, \tag{10}$$
for some $c>0$, which implies that $\omega=\dfrac{\partial\omega}{\partial\vec{n}}=0$ on $\partial\Omega$, so that we can assume that $\omega\equiv 0$ on $\mathbb{R}^N-\Omega$ (see [15, 16]). Here we shall not consider any eigenvalue problem like (9) but the study of (1) for potentials $V(x)$ satisfying (10) will be very useful for later works in this direction.

3. *Linearization of singular and/or degenerate nonlinear equations.* For many different purposes, it is very convenient to "approximate" the solutions of quasilinear diffusion equations of the type
$$-\Delta\varphi(w)+\operatorname{div}\big(\vec{\phi}(w)\big)+g(w)=f(x) \qquad \text{in } \ \Omega \tag{11}$$
by the solutions of the associated linearized equation. This is what appears, for instance, in the study of the stability of the associated parabolic or hyperbolic equations and also in some control problems associated with (11). Usually, it is assumed that φ is a strictly increasing function. So by considering $\theta:=\varphi(w)$ we get
$$-\Delta\theta+\operatorname{div}\big(\vec{\psi}(\theta)\big)+h(\theta)=f(x) \qquad \text{in } \ \Omega, \tag{12}$$
with
$$\begin{cases}\vec{\psi}:\mathbb{R}\to\mathbb{R}^N, \quad \vec{\psi}=\vec{\phi}\circ\varphi^{-1},\\ h=g\circ\varphi^{-1}.\end{cases} \tag{13}$$
Now, assume that $\theta_\infty(x)$ is a given solution of (12), satisfying, for instance, $\theta_\infty=0$ on $\partial\Omega$. Then the "formal linearization" of equation (13) around the solution $\theta_\infty(x)$ coincides with equation (1) when we take
$$\vec{u}(x):=\vec{\psi}\big(\theta_\infty(x)\big)$$
and
$$V(x)=h'\big(\theta_\infty(x)\big).$$
What makes difficult the study of the corresponding problem (1) is the fact that in many cases relevant in the reaction-diffusion theory (see e.g. [26]) functions $\vec{\psi}'(r)$ and $h'(r)$ present a singularity at $r=0$ and so, at least on $\partial\Omega$, the coefficients $\vec{u}$ and $\vec{V}$ are singular. A qualitative information on the behavior of $\theta_\infty(x)$ near $\partial\Omega$ allows us to get the precise information about the singularities of $\vec{u}$ and/or V near $\partial\Omega$ $\big($which, for instance, is of the type (10)$\big)$.

4. *Shape optimization in Chemical Engineering.* When dealing with the problem of shape optimization for chemical reactors and applying technics of shape

differentiation, it was shown that if $g \in W^{2,\infty}(\mathbb{R})$, then the solutions u_0 of the problem

$$\begin{cases} -\Delta u + g(u) = f, & \Omega, \\ u = 1, & \partial\Omega, \end{cases} \tag{14}$$

are differentiable with respect to the domain in the sense of Hadamard [25] and after developed in Murat and Simon [32, 43] and the derivative u' in the direction of a deformation $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ is the solution of the problem

$$\begin{cases} -\Delta u' + g'(u_0)u' = 0, \\ u' + \theta \cdot \nabla u \in H_0^1(\Omega). \end{cases} \tag{15}$$

Applying the theory developed for the general case (1), we can give a meaning to the shape derivative if the domain is not smooth as, for example, for root type kinetics (see [17, 24]). These nonlinear terms $g(u)$ are known in chemistry as Freundlich kinetics and have signifiant importance. Once again, taking $V(x) \equiv g'\big(u_0(x)\big)$ we arrive to problem (1).

Some previous papers dealing with data in $L^1(\Omega;\delta)$ and/or singular potentials (with usually $\vec{u} = \vec{0}$) are [20, 18, 37, 1, 29, 40, 6] (see also the references therein).

We also mention that sometimes it is possible to get conclusions for the stationary problem (1) (with $\vec{u} = \vec{0}$) through the consideration of the associated evolution equations (see e.g. [7], [8] and its references).

In this paper we shall work with the notion of "very weak solutions" (v.w.s.) of problem (1).

Definition 1.1. (Very weak solutions of problem (1)). Let f be in $L^1(\Omega;\delta)$ and $\vec{u} \in L^{N,1}(\Omega)^N$ with $\operatorname{div}(\vec{u}) = 0$ in $\mathcal{D}'(\Omega)$, $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega$, V measurable and non negative function. A very weak solution ω of (1) is a function $\omega \in L^{N',\infty}(\Omega)$ satisfying

$$V\omega \in L^1(\Omega;\delta) \text{ and } \int_\Omega \omega\big[-\Delta\phi - \vec{u}\cdot\nabla\phi + V\phi\big]dx = \int_\Omega f\phi\,dx, \tag{16}$$

for all $\phi \in C^2(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$, if $V \in L^1(\Omega;\delta)$, or for all $\phi \in C_c^2(\Omega)$ if $V \in L^1_{loc}(\Omega)$.

Notice that we look for a function in the space $L^{N',\infty}(\Omega)$ where $N' = \frac{N}{N-1}$ instead of $\omega \in L^1(\Omega)$ as usual, in order to get more general assumptions on $\vec{u}$ and V.

We also also point out that our study will be concentrated in the case of "absorption" potentials $V(x) \geqslant 0$ a.e. $x \in \Omega$. In fact, as we shall see later, the study is also applicable to some general potentials such that e.g. $V(x) \geqslant -\lambda$ with $0 < \lambda < \lambda_1$ (λ_1 being the first eigenvalue of the Laplacian on Ω with zero Dirichlet boundary condition). As we shall show, this does not induce a restriction on the growth of the singularity of such absorption potentials near $\partial\Omega$ (in contrast with the well-known results for **negative** potentials, see e.g. [7]).

The detailed definition of the Lorentz spaces $L^{p,q}(\Omega)$ and some other spaces which we shall use in our study will be the object of Section 2 of this paper. Other preliminary results and the statement of some of our main conclusions will be also presented there.

The proof of the existence and uniqueness of a very weak solution (v.w.s.) for (1) needs a deep study of the dual problem associated with (1)

$$\begin{cases} -\Delta\phi - \vec{u}\cdot\nabla\phi + V\phi = T & \text{in} \quad \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Notice the change of sign in the convection term. We anticipate that in some cases no boundary condition will be assumed on ϕ.

In Section 3, we discuss, depending on V and $\vec{u}$, the existence and the regularity of the solution of the dual problem. After this, we shall be concerned with the existence of the very weak solution in $L^{N',\infty}(\Omega)\cap L^1(\Omega; V\delta)$, when $V \geqslant 0$ is locally integrable. We will show that the very weak solution ω of equation (1) under zero Dirichlet boundary condition has its gradient in the Sobolev-Lorentz weighted space $W^1L^{1+\frac{1}{N},\infty}(\Omega;\delta)$ in particular we shall get the estimate

$$\int_{\{x:|\nabla\omega|(x)<\lambda\}} \delta(x)dx \leqslant \frac{constant}{\lambda^{1+\frac{1}{N}}} \text{ for all } \lambda > 0, \tag{18}$$

under the mere assumption $\vec{u} \in L^{N,1}(\Omega)^N$. Thus, we can conclude that $\nabla\omega \in L^1_{loc}(\Omega)$.

The question of uniqueness of v.w.s. given by (16), when V is only in $L^1_{loc}(\Omega)$ is one of the major difficulties in this general framework. When V is sufficiently integrable, say $V \in L^{N,1}(\Omega)$, then we derive the uniqueness thanks to the regularity of the dual problem. If V is only locally integrable, but V is bounded from below by $c\delta^{-r}$, $r > 2$ near the boundary, then the v.w.s. is unique even when no boundary condition is specified on $\partial\Omega$ (but we additionally know that $V\omega \in L^1(\Omega;\delta)$).

The uniqueness proof relies on the $L^1(\Omega;\delta)$-accretiveness property of the operator (see [36]) $T\varpi = -\Delta\varpi + \operatorname{div}(\vec{u}\varpi)$ when $\varpi \in L^1(\Omega;\delta^{-r}) \cap W^{1,1}_{loc}(\Omega)$. This is given through the following local version of the Kato's type inequality

$$\int_\Omega \varpi_+ T^*\psi\, dx \leqslant \int_\Omega \psi \operatorname{sign}_+(\varpi)T\varpi\, dx, \text{ whenever } T\varpi \in L^1_{loc}(\Omega),\ \psi \in \mathcal{D}(\Omega), \tag{19}$$

and a special approximation of test function φ in $C^2(\overline{\Omega})$ by a sequence of functions of the type $\varphi_n(x) = \delta(x)^r h_n(x)$ with $h \in C^2_c(\Omega)$ and $r > 0$ (see Lemma 4.4). We point out that, besides the concrete interest of (19) in itself; such an inequality has many consequences since it allows to apply the semigroup operators theory on suitable functional spaces.

Concerning very weak solutions (where no differentiability is asked to the function ω), a natural question (originally set by H. Brézis in 1972 when $\vec{u} = 0$) is then: when should we have $|\nabla\omega|$ in $L^1(\Omega)$? The answer to this question will require some suitable additional integrability conditions on f and $\vec{u}$.

Note that to get some additional integrability for the very weak solutions ω is a delicate task. Indeed, we shall show that for some special cases of $\vec{u} \in C^{0,\alpha}(\overline{\Omega})$, $\alpha > 0$, there exists $f \in L^1_+(\Omega;\delta)$ such that $||\omega||_{L^{N'}} = +\infty$ when $N \geqslant 3$.This leads to an additional question: under what conditions could we improve the integrability of ω, to say $\omega \in L^{N'}(\Omega)$? The answer to this question is also one of the main results of this paper.

Before stating the study of the main equation (1), we shall recall some notations and functional spaces that we shall use.

2. **Notations, preliminary definitions and results.** Before stating our main results concerning equation (1) we need to recall some notations and some functional spaces which are relevant for the study of the "dual problem" (17) under very general regularity assumptions on the coefficients $\vec{u}$ and T.

Definition 2.1. ($\mathrm{bmo}(\mathbb{R}^N)$) [23]. A locally integrable function f on $\mathbb{R}^N$ is said to be in $\mathrm{bmo}(\mathbb{R}^N)$ if

$$\sup_{0<\mathrm{diam}\,(Q)<1} \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \sup_{\mathrm{diam}\,(Q)\geqslant 1} \frac{1}{|Q|}\int_Q |f(x)|dx$$
$$\equiv ||f||_{\mathrm{bmo}(\mathbb{R}^N)} < +\infty,$$

where the supremum is taken over all cube $Q\subset\mathbb{R}^N$ the sides of which are parallel to the coordinates axes.

Here $f_Q = \dfrac{1}{|Q|}\displaystyle\int_Q f(y)dy.$

Definition 2.2. ($\mathrm{bmo}_r(\Omega)$) [11, 12]. A locally integrable function f on a Lipschitz bounded domain Ω is said to be in $\mathrm{bmo}_r(\Omega)$ (r stands for restriction) if

$$\sup_{0<\mathrm{diam}\,(Q)<1} \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \int_\Omega |f(x)|dx \equiv ||f||_{\mathrm{bmo}_r(\Omega)} < +\infty, \tag{20}$$

where the supremum is taken over all cube $Q\subset\Omega$ the sides of which are parallel to the coordinates axes.

In this case, there exists a function $\widetilde{f}\in\mathrm{bmo}(\mathbb{R}^N)$ such that

$$\widetilde{f}\Big|_\Omega = f \text{ and } ||\widetilde{f}||_{\mathrm{bmo}(\mathbb{R}^N)} \leqslant c_\Omega\cdot||f||_{\mathrm{bmo}_r(\Omega)}. \tag{21}$$

Remark 1. The above definition adapted to the case where the domain Ω is bounded, is equivalent to the definition given in [12, 11]. The main property (21) is due to P.W Jones [27].

This extension result implies that $\mathrm{bmo}_r(\Omega)$ embeds continuously into $L_{exp}(\Omega)$ (a space which we shall introduce below in Definition 2.5.)

Definition 2.3. **(Campanato space $\mathcal{L}^{2,N}(\Omega)$)** A function $u\in\mathcal{L}^{2,N}(\Omega)$ if

$$||u||_{L^2(\Omega)} + \sup_{x_0\in\Omega, r>0}\left[r^{-N}\int_{Q(x_0,r)\cap\Omega}|u-u_r|^2\,dx\right]^{\frac{1}{2}} := ||u||_{\mathcal{L}^{2,N}(\Omega)} < +\infty.$$

Here

$$u_r := \frac{1}{|Q(x_0;r)\cap\Omega|}\int_{Q(x_0;r)\cap\Omega} u(x)\,dx.$$

In fact the two above definitions are equivalent:

Theorem 2.1. [40] *For a Lipschitz bounded domain Ω one has*

$$\mathcal{L}^{2,N}(\Omega) = \mathrm{bmo}_r(\Omega), \textit{ with equivalent norms.}$$

We set

$$L^0(\Omega) = \Big\{v:\Omega\to\mathbb{R} \text{ Lebesgue measurable}\Big\}$$

and we denote by $L^p(\Omega)$ the usual Lebesgue space $1\leqslant p\leqslant +\infty$. Although it is not too standard, we shall use the notation $W^{1,p}(\Omega) = W^1L^p(\Omega)$ for the associate Sobolev space. We shall need the following definitions:

Definition 2.4. (of the distribution function and monotone rearrangement) Let $u \in L^0(\Omega)$. The distribution function of u is the decreasing function

$$m = m_u : \mathbb{R} \mapsto [0, |\Omega|]$$

$$m_u = m_u(t) = \text{measure}\{x : u(x) > t\} = |\{u > t\}|.$$

The generalized inverse u_* of m is defined by

$$u_*(s) = \inf\Big\{t : |\{u > t\}| \leqslant s\Big\}, \quad s \in [0, |\Omega|[$$

it is called the decreasing rearrangement of u. We shall set $\Omega_* =]0, |\Omega|\,[$.

We recall now the following definitions:

Definition 2.5. Let $1 \leqslant p \leqslant +\infty,\ 0 < q \leqslant +\infty$:

- If $q < +\infty$, one defines the following norm for $u \in L^0(\Omega)$

$$||u||_{p,q} = ||u||_{L^{p,q}} := \left[\int_{\Omega_*} \left[t^{\frac{1}{p}} |u|_{**}(t)\right]^q \frac{dt}{t}\right]^{\frac{1}{q}} \text{ where } |u|_{**}(t) = \frac{1}{t}\int_0^t |u|_*(\sigma)d\sigma.$$

- If $q = +\infty$,

$$||u||_{p,\infty} = \sup_{0<t\leqslant|\Omega|} t^{\frac{1}{p}} |u|_{**}(t).$$

The space $L^{p,q}(\Omega) = \Big\{u \in L^0(\Omega) : ||u||_{p,q} < +\infty\Big\}$ is called a **Lorentz space**.

- If $p = q = +\infty$, $\qquad L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$.

The dual of $L^{1,1}(\Omega)$ is called $L_{exp}(\Omega)$

Remark 2. We recall that $L^{p,q}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega)$ for any $p > 1,\ q \geqslant 1$.

For $\alpha > 0$, we define

$$L^\alpha_{exp}(\Omega) = \left\{ v : \Omega \to \mathbb{R}, \ \sup_{0<s<|\Omega|} \frac{|v|_*(s)}{\left(1 - \operatorname{Log}\dfrac{s}{|\Omega|}\right)^\alpha} < +\infty \right\},$$

$$L^p(\operatorname{Log} L)^\alpha = \left\{ f : \Omega \to \mathbb{R}, \ \int_{\Omega_*} \left[\left(1 - \operatorname{Log}\frac{s}{|\Omega|}\right)^\alpha |f|_*(s)\right]^p ds < +\infty \right\}.$$

When there is no possible confusion, we denote by the same symbol the space product V^N and V.

We recall also that if $v,\ u \in L^1(\Omega)$, then

$$v_{*u} \doteq \lim_{\lambda \searrow 0} \frac{(u + \lambda v)_* - u_*}{\lambda}$$

exists in a weak sense and it is called the relative rearrangement of v with respect to u. More precisely, we have the following result (see [31, 35]).

Theorem 2.2. *Let Ω be a bounded measurable set in $\mathbb{R}^N$, u and v two functions in $L^1(\Omega)$ and let $w : \overline{\Omega}_* \to \mathbb{R}$ be defined by:*

$$w(s) = \int_{\{u>u_*(s)\}} v(x)dx + \int_0^{s-|u>u_*(s)|} \left(v\Big|_{\{u=u_*(s)\}}\right)_*(\sigma)d\sigma,$$

where $v\Big|_{\{u=u_*(s)\}}$ is the restriction of v to $\{u = u_*(s)\}$.

Then

$$\frac{(u+\lambda v)_* - u_*}{\lambda} \underset{\lambda\to 0}{\rightharpoonup} \frac{dw}{ds} \ \textit{in} \ \begin{cases} L^p(\Omega_*)\textit{-weak} & \textit{if } v \in L^p(\Omega),\ 1 \leqslant p < +\infty \\ L^\infty(\Omega_*)\textit{-weak-star} & \textit{if } v \in L^\infty(\Omega) \end{cases}.$$

Moreover, $\left|\dfrac{dw}{ds}\right|_{L^p(\Omega_*)} \leqslant |v|_{L^p(\Omega)}$.

One property that we shall use for the relative rearrangement is the following one:

Proposition 1. *Let $v \geqslant 0$, and u be two functions in $L^1(\Omega)$. Then*

$$(v_{*u})_{**} \leqslant v_{**}.$$

There is a link between the derivative of u_* and the relative rearrangement of the gradient of u as it was proved in [35, 41]. We will use only the following result (see [35])

Theorem 2.3. *(a) Let $u \in W_0^{1,1}(\Omega), u \geqslant 0$. Then*

$$-u_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} |\nabla u|_{*u}(s) \quad \textit{a.e in } \Omega_*,$$

and

$$-u_{**}'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} (|\nabla u|_{*u})_{**}(s) \ \textit{a.e. in} \ \ \Omega_*.$$

(b) Let $u \in W^{1,1}(\Omega)$. Then if Ω is a Lipschitz connected open set of $\mathbb{R}^n$

$$-u_*'(s) \leqslant \frac{\min(s, |\Omega| - s)^{\frac{1}{N}-1}}{Q(\Omega)} |\nabla u|_{*u}(s),$$

where $Q(\Omega)$ is a suitable constant depending only on Ω.

Note that u_* is in $W_{loc}^{1,1}(\Omega_*)$ under statements (a) and (b) (see [35, 41]).

Let V be a Banach space contained in $L^1_{loc}(\Omega)$.The norm on V is denoted by $||\cdot||_V$ (or simply $||\cdot||$). We define the Sobolev space over V, for $m \in \mathbb{N}$ by

$$W^m V = \Big\{ v \in L^1_{loc}(\Omega) : D^\alpha v \in V \text{ for any } |\alpha| = \alpha_1 + \ldots + \alpha_N \leqslant m \Big\}.$$

In particular, $W_0^1 V = W^1 V \cap W_0^{1,1}(\Omega)$.

The following density result can be found in [22, 38, 40]:

Theorem 2.4. (Density) *Let Ω be a bounded set of class $C^{1,1}$. Then, the set $\Big\{\varphi \in C^2(\overline{\Omega}) : \varphi = 0$ on $\partial\Omega\Big\}$ is dense in $\Big\{\varphi \in W^2 L^{p,q}(\Omega) : \varphi = 0$ on $\partial\Omega\Big\}$, $1 < p < +\infty,\ 1 \leqslant q \leqslant +\infty$.*

Remark 3. Here and along the paper $\vec{u}$ is at least in $L^N(\Omega)^N$, div $(\vec{u}) = 0$ in $\mathcal{D}'(\Omega)$ and $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$, if $N \geqslant 3$ and $\vec{u} \in L^{2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$ if $N = 2$. The value of $\vec{u}\cdot\vec{n}$ on $\partial\Omega$ is defined through the Green's formula (see [46]).

The following density result can be proved using the same argument as for the L^p-case (see [46, 13])

Proposition 2. (Density of smooth functions). *Let $1 < p < +\infty$ and $1 \leqslant q \leqslant \infty$. Then the closure of the set*

$$\mathcal{V} = \Big\{ \vec{u} \in C_c^\infty(\Omega)^N : \operatorname{div}(\vec{u}) = 0 \text{ in } \Omega \Big\}$$

in $L^{p,q}(\Omega)^N$ (resp. $(L^N(\operatorname{Log} L)^\alpha)^N$, $\alpha > 0$) is the space

$$\overline{\mathcal{V}} := \Big\{ \vec{u} \in L^{p,q}(\Omega)^N \text{ (resp. } (L^N(\operatorname{Log}\, L)^\alpha)^N,\ \alpha > 0 \text{) } : \operatorname{div}(\vec{u}) = 0, \vec{u} \cdot \vec{n} = 0 \text{ on } \partial\Omega \Big\}.$$

Due to Proposition 2, a standard approximation argument leads to :

Lemma 2.6. *For all Lipschitz mappings $G : \mathbb{R} \to \mathbb{R}$, and for all $\phi \in W_0^1 L^{N'}(\Omega)$ with $N' = \dfrac{N}{N-1}$, one has*

$$\int_\Omega (\vec{u} \cdot \nabla \phi)\, G(\phi)\, dx = 0.$$

Lemma 2.7. *For all $\varpi \in H_0^1(\Omega)$, and for all $\phi \in H_0^1(\Omega)$*

$$\int_\Omega (\vec{u} \cdot \nabla \varpi)\, \phi\, dx = -\int_\Omega \vec{u} \cdot \nabla \phi\, \varpi\, dx.$$

Let us remark that,

- if $N \geqslant 3$

$$\left| \int_\Omega \vec{u} \cdot \nabla \varpi\, \phi\, dx \right| \leqslant ||\vec{u}||_{L^N} ||\nabla \varpi||_{L^2} ||\phi||_{L^{2^*}} \text{ where } \frac{1}{2^*} + \frac{1}{2} + \frac{1}{N} = 1, \tag{22}$$

- if $N = 2$ the above inequality holds true after replacing N by $2 + \varepsilon$ and 2^* by $\dfrac{2(2+\varepsilon)}{\varepsilon}$.

We shall need the following classical result (see [28]) :

Lemma 2.8. *Let $X \hookrightarrow_c Y \hookrightarrow Z$ be three Banach spaces each continuously embedded in the next one, the first inclusion is supposed to be compact. Then, for all $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $\forall\, \phi \in X$*

$$||\phi||_Y \leqslant \varepsilon ||\phi||_X + c_\varepsilon ||\phi||_Z.$$

3. **Existence, uniqueness, regularity and results for the dual problem.**

3.1. **Case where the potential V is only measurable and bounded from below.**

We first study the solvability of the dual problem (17) (equivalent to (23) below and the regularity of its solutions.

The following result, consequence of the Lax-Milgram theorem, is a remarkable fact due to the low regularity assumed on the data $\vec{u}$ and V:

Proposition 3. *Let $T \in H^{-1}(\Omega)$ (dual space of $H_0^1(\Omega)$), $\vec{u}$ satisfying (5) and let $V \in L^0(\Omega)$ satisfying $V \geqslant -\lambda$ for some $\lambda \in [0, \lambda_1)$ where λ_1 is the first eigenvalue of $-\Delta$ under the zero Dirichlet boundary condition. Define $W = \Big\{ \varphi \in H_0^1(\Omega) : (V + \lambda)\varphi^2 \in L^1(\Omega) \Big\}$, and let W' denotes its dual.*

Then, there exists a unique $\phi \in H_0^1(\Omega)$, with $(V + \lambda)\phi^2 \in L^1(\Omega)$, such that

$$(\mathcal{P})_{V,T} \qquad -\Delta \phi - \vec{u} \cdot \nabla\, \phi + V \phi = T \text{ in } W'. \tag{23}$$

Moreover,

$$||\phi||_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \leqslant \frac{\lambda_1}{\lambda_1 - \lambda} ||T||_{H^{-1}(\Omega)},$$

$$\left(\int_\Omega (V+\lambda)\phi^2 dx \right)^{\frac{1}{2}} \leqslant \left(\frac{\lambda_1}{\lambda_1 - \lambda} \right)^{\frac{1}{2}} ||T||_{H^{-1}(\Omega)},$$

$$V\phi \in L^1_{loc}(\Omega).$$

If furthermore $V \in L^1_{loc}(\Omega)$, *then the equation* (23) *holds in the sense of distributions in* $\mathcal{D}'(\Omega)$

Proof. We endow W with the following norm

$$[\varphi]_W^2 = ||\varphi||^2_{H_0^1(\Omega)} + \int_\Omega (V+\lambda)\varphi^2 dx.$$

Let us consider the bilinear form on W given by

$$\begin{aligned} a(\psi,\varphi) &= \int_\Omega \nabla\psi \cdot \nabla\varphi dx - \int_\Omega \vec{u} \cdot \nabla\psi\varphi dx + \int_\Omega (V+\lambda)\psi\varphi dx \\ &\quad -\lambda \int_\Omega \psi\varphi dx, \quad (\psi,\varphi) \in W^2. \end{aligned}$$

Then, by Lemmas 2.6 and 2.7

$$a(\psi,\psi) = \int_\Omega |\nabla\psi|^2 - \lambda \int_\Omega \psi^2 dx + \int_\Omega (V+\lambda)\psi^2 dx \geqslant \alpha_0 \left[\int_\Omega (V+\lambda)\psi^2 + \int_\Omega |\nabla\psi|^2 \right], \tag{24}$$

with $\alpha_0 > 0$.

According to the above remark (22), since $\vec{u} \in L^N(\Omega)^N$, the bilinear form is continuous on W and we have

$$|a(\psi,\varphi)| \leqslant M[\psi]_W[\varphi]_W \ ,$$

with $M = 3(1+||\vec{u}||_{L^N})$. Moreover, since $W \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow W'$ we have

$$\langle T,\psi\rangle_{H^{-1}H_0^1} \leqslant ||T||_{H^{-1}}[\psi]_W, \ \forall\,\psi \in W.$$

Thus we may apply the Lax-Milgram theorem to derive the existence of a unique $\phi \in W$, such $a(\phi,\psi) = \langle T,\psi\rangle_{H^{-1}H_0^1}$ $\forall\,\psi \in W$. The estimate on ϕ follows from (24).

If $V \in L^1_{loc}(\Omega)$ then one has

$$\mathcal{D}(\Omega) \subset W.$$

Moreover, since $\int_\Omega (V+\lambda)\phi^2 dx$ is finite, the Cauchy-Schwarz inequality yields

$$0 \leqslant \int_{\Omega'} (V+\lambda)|\phi| dx \leqslant \left(\int_\Omega (V+\lambda)\phi^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega'} (V+\lambda) dx \right)^{\frac{1}{2}} < +\infty \tag{25}$$

for any open set Ω' relatively compact in Ω.

Writing

$$\int_{\Omega'} |V\phi| dx \leqslant \int_{\Omega'} (V+\lambda)|\phi| dx + \lambda \int_\Omega |\phi| dx,$$

the right hand is finite taking into account (25) and the fact that $\phi \in L^2(\Omega)$. Thus, we have $\forall\,\Omega' \subset\subset \Omega,\ V\phi \in L^1(\Omega')$. We conclude that $V\phi \in L^1_{loc}(\Omega)$. □

As usual in some problems of Quantum Mechanics (see e.g. Lemma 2.1 of [15]) it is very useful to approximate the solution $\phi \in H_0^1(\Omega)$ of the dual problem (23) found in Proposition 3 by a sequence of solutions ϕ_k corresponding to a sequence of bounded potentials V_k approximating V. Let us define V_k by

$$V_k = \min(V, k).$$

Proposition 4. (**Approximation by bounded potentials**). *Let $T \in H^{-1}(\Omega)$, $\vec{u}$ and V as in Proposition 3. Then, the sequence $\phi_k \in H_0^1(\Omega)$ of solutions of the problems*

$$(\mathcal{P})_{V_k,T} : \int_\Omega \nabla\phi_k \cdot \nabla\psi dx - \int_\Omega \vec{u}\nabla\phi_k\phi dx + \int_\Omega V_k\phi_k\psi dx = \langle T, \psi\rangle, \quad \forall\, \psi \in H_0^1(\Omega),$$

converges to ϕ strongly in $H_0^1(\Omega)$, where ϕ is the unique solution of $(\mathcal{P})_{V,T}$ found in Proposition 3.

Sketch of the proof of Proposition 4. One has, following the arguments of the Proposition 3, that

$$||\phi_k||_{H_0^1} + \left(\int_\Omega (V_k+\lambda)\phi_k^2 dx\right)^{\frac{1}{2}} \leqslant 2\left(\frac{\lambda_1}{\lambda_1-\lambda}\right)||T||_{H^{-1}(\Omega)}. \tag{26}$$

Thus, ϕ_k remains in a bounded set of $H_0^1(\Omega)$. So we may assume that it converges to a function φ weakly in $H_0^1(\Omega)$ and a.e. in Ω. The above relation (26) implies that:

$$\left(\int_\Omega (V+\lambda)\varphi^2 dx\right)^{\frac{1}{2}} + ||\varphi||_{H_0^1} \leqslant 2\left(\frac{\lambda_1}{\lambda_1-\lambda}\right)||T||_{H^{-1}(\Omega)}. \tag{27}$$

This shows that $\varphi \in W$ (where W is the space defined in the proof of Proposition 3). Moreover, since for all $\psi \in W$ we have $\vec{u}\psi \in L^{2^{*\prime}}(\Omega)$ (see the above remark), we deduce

$$\lim_{k\to+\infty}\int_\Omega \vec{u}\cdot\nabla\phi_k\psi dx = \int_\Omega \vec{u}\cdot\nabla\varphi\psi dx. \tag{28}$$

The sequence $(V_k+\lambda)\phi_k\psi$ satisfies Vitali's condition, since for any measurable subset $B \subset \Omega$, we have

$$\left|\int_B (V_k+\lambda)\phi_k\psi dx\right| \leqslant 2\left(\frac{\lambda_1}{\lambda_1-\lambda}\right)||T||_{H^{-1}(\Omega)}\left(\int_B (V+\lambda)\psi^2 dx\right)^{\frac{1}{2}} \tag{29}$$

and

$$\lim_{k\to+\infty}(V_k+\lambda)(x)\phi_k(x)\psi(x) = (V+\lambda)(x)\varphi(x)\psi(x). \tag{30}$$

Thus

$$\lim_{k\to+\infty}\int_\Omega (V_k+\lambda)\phi_k\psi dx = \int_\Omega (V+\lambda)\varphi\,\psi dx. \tag{31}$$

We then deduce that φ is solution of the problem $(\mathcal{P})_{V,T}$ and by uniqueness $\varphi = \phi$. Therefore, the whole sequence ϕ_k converges to ϕ weakly in W and strongly in $L^2(\Omega)$.

To prove the strong convergence in $H_0^1(\Omega)$, let us note, using the equations $(\mathcal{P})_{V_k,T}$ and $(\mathcal{P})_{V,T}$, that

$$\begin{aligned}\lim_{k\to+\infty}\int_\Omega |\nabla\phi_k|^2 dx + \int_\Omega (V_k+\lambda)\phi_k^2 dx &= \lambda\int_\Omega \phi^2 dx + \langle T, \phi\rangle \\ &= \int_\Omega (V+\lambda)\phi^2 + \int_\Omega |\nabla\phi|^2 dx.\end{aligned}$$

Therefore, if we introduce $U_k = (\nabla\phi_k; \phi_k\sqrt{V_k+\lambda}) \in L^2(\Omega)^{N+1}$, $U_\infty = (\nabla\phi; \phi\sqrt{V+\lambda})$ we have

- $\lim\limits_{k\to+\infty} |U_k|^2_{L^2(\Omega)^{N+1}} = |U_\infty|^2_{L^2(\Omega)^{N+1}}$,
- U_k converges to U_∞ weakly in $L^2(\Omega)^{N+1}$.

Thus U_k converges to U_∞ strongly in $L^2(\Omega)^{N+1}$. □

Remark 4. Let us notice that for $\phi \in L^2(\Omega)$ the conditions $(V+\lambda)\phi^2 \in L^1(\Omega)$ and $|V|\phi^2 \in L^1(\Omega)$, $\phi \in L^2(\Omega)$ are equivalent. Indeed, since $V+\lambda = |V+\lambda|$,

$$\int_\Omega |V|\phi^2 dx \leqslant \int_\Omega (V+\lambda)\phi^2 dx + \lambda\int_\Omega \phi^2 \leqslant \int_\Omega |V|\phi^2 dx + 2\lambda\int_\Omega \phi^2 dx.$$

For this reason, from now, **we will assume that** $\lambda = 0$.

Proposition 5. *Under the same assumptions as for Proposition 3 (with $\lambda = 0$), if $T \geqslant 0$, $T \in L^1(\Omega)\cap H^{-1}(\Omega)$ then $\phi \geqslant 0$.*

Proof. We have $\phi_- \in W$ and

$$0 \geqslant -\int_\Omega |\nabla\phi_-|dx - \int_\Omega V\phi_- dx = \int_\Omega T\phi_- dx \geqslant 0.$$

Thus

$$\phi_- = 0.$$

□

For the treatment of (1) we shall need some additional regularity for the solutions of the dual problem (23) independent of $\vec{u}$ or V. We start by proving the boundedness of ϕ by means of some rearrangement technics ([35] p.126 of Th 5.5.1, see also [45]).

We point out that $L^{\frac{N}{2},1}(\Omega) \hookrightarrow H^{-1}(\Omega)$.

Proposition 6. **(L^∞-estimates).** *Let ϕ be the solution of (23) when $T \in L^{\frac{N}{2},1}(\Omega)$, $V \geqslant 0$. Then $\phi \in L^\infty(\Omega)$ and there exists a constant $K_N(\Omega)$ independent of $\vec{u}$ and V such that*

$$||\phi||_{L^\infty(\Omega)} \leqslant K_N(\Omega)||T||_{L^{\frac{N}{2},1}(\Omega)}.$$

Proof. We shall argue in a way similar to the proof of Theorem 5.3.1 in [35]. According to Proposition 4 , it is enough to prove the proposition for $V \in L^\infty_+(\Omega)$, and and for $T \geqslant 0$, since the equation (23) is linear. Thus $\phi \geqslant 0$, therefore, in this proof $v = |\phi| = \phi$, but we shall keep the notation v because in the general case we cannot use anymore this maximum principle. Let $v = |\phi|$, $G_s(\sigma) = (\sigma - v_*(s))_+ \operatorname{sign}(\sigma)$, $\sigma \in \mathbb{R}$, $s \in \Omega_*$. The mapping $\sigma \mapsto G_s(\sigma)$ is Lipschitz. Then following Lemma 2.6

$$\int_\Omega (\vec{u}\cdot\nabla\phi)G_s(\phi)\,dx = 0.$$

Therefore, we derive

$$\int_\Omega \nabla\phi\cdot\nabla G_s(\phi) = \int_{v>v_*(s)} |\nabla\phi|^2 dx = \int_\Omega T(x)G_s(\phi)(x)dx - \int_\Omega V(x)G_s(\phi)dx.$$

Differentiating this relation with respect to s, we find

$$\frac{d}{ds}\int_{v>v_*(s)} |\nabla\phi|^2 dx = -v_*'(s)\int_{v>v_*(s)} \big(T(x)-V(x)\big)dx \leqslant -v_*'(s)\int_0^s T_*(\sigma)d\sigma$$

where T_* is the monotone rearrangement of T (we use the fact that $V \geqslant 0$). Therefore, we arrive at

$$\big[\,|\nabla\phi|^2\big]_{*v}(s) \leqslant -v'_*(s)\int_0^s T_*(\sigma)d\sigma. \tag{32}$$

Since

$$|\nabla\phi| = |\nabla v|, \text{ and } \quad -v'_*(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}|\nabla v|_{*s}(s)$$

$\big($the PSR property (see Theorem 3 of [35])$\big)$ and $|\nabla v|_{*v} \leqslant \big[\,|\nabla v|^2\big]^{\frac12}_{*v}$, we infer from (32)

$$-v'_*(s) \leqslant \frac{s^{\frac{2}{N}-2}}{(N\alpha_N^{\frac{1}{N}})^2}\int_0^s T_*(\sigma)d\sigma. \tag{33}$$

Thus, integrating (33) between 0 to $|\Omega|$, we find

$$||\phi||_{L^\infty} \leqslant c_N\int_0^{|\Omega|} s^{\frac{2}{N}}T_{**}(s)\frac{ds}{s} \equiv c_N||T||_{L^{\frac{N}{2},1}(\Omega)}.$$

□

An analogous result can be obtained when $T = -\mathrm{div}\,(\vec{F})$, with $\vec{F} \in L^{N,1}(\Omega)^N$.

Proposition 7. *Let* $N \geqslant 2$, *and let* ϕ *be a solution of* (23) *when* $T = -\mathrm{div}\,(\vec{F})$, $\vec{F} \in L^{N,1}(\Omega)^N$ *if* $N \geqslant 3$, $\vec{F} \in L^{2+\varepsilon}(\Omega)^2$ *if* $N = 2$. *Then* $\phi \in L^\infty(\Omega)$ *and there exists a constant* $K_N(\Omega) > 0$ *independent of* $\vec{u}$ *and* V *such that*

$$||\phi||_{L^\infty(\Omega)} \leqslant K_N(\Omega)||\vec{F}||_{L_V} \text{ with } L_V = L^{N,1}(\Omega)^N \text{ if } N \geqslant 3,\ L^{2+\varepsilon}(\Omega)^2 \text{ if } N = 2.$$

Proof. For convenience, we write F for $\vec{F}$. Thanks to Proposition 4, we can use the same test function $G_s(\phi)$ as in the proof of Proposition 6. Then

$$\int_\Omega \nabla\phi\cdot\nabla G_s(\phi)dx + \int_\Omega V(x)G_s(\phi)dx = \int_\Omega F\cdot\nabla G_s(\phi)dx.$$

We differentiate this equation with respect to s as before, for a.e. $s \in \Omega_*$, and find

$$\big[\,|\nabla v|^2\big]_{*v}(s) - v'_*(s)\int_{v>v_*(s)} V(x)dx = \big[\,F\cdot\nabla\phi\,\big]_{*v}(s). \tag{34}$$

Since, $V \geqslant 0$ and $v'_*(s) \leqslant 0$, we obtain

$$\big[\,|\nabla v|^2\big]_{*v}(s) \leqslant \big[\,|\,F\,|^2\big]^{\frac12}_{*v}\big[\,|\nabla v|^2_{*v}\big]^{\frac12}(s), \tag{35}$$

$$\big[\,|\nabla v|^2\big]^{\frac12}_{*v}(s) \leqslant \big[\,|\,F\,|^2\big]^{\frac12}_{*v}(s). \tag{36}$$

We have as before:

$$-v'_*(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}\big[\,|\nabla v\,|^2\big]^{\frac12}_{*v}(s). \tag{37}$$

We infer that for a.e. s

$$-v'_*(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}\big[\,|\,F\,|^2\big]^{\frac12}_{*v}. \tag{38}$$

Integrating this relation between 0 and $|\Omega|$ and using the Hardy-Littlewood inequality (see [35] p.118-121) we obtain

$$||\phi||_{L^\infty} \leqslant \begin{cases} c_N \displaystyle\int_{\Omega_*} \sigma^{\frac{1}{N}-1} \big(\,|\,F\,|^2\big)^{\frac{1}{2}}_{**}(\sigma) d\sigma, & \text{if } N \geqslant 3, \\ c_{2,\varepsilon} ||F\,||_{L^{2+\varepsilon}(\Omega)^2}, & \text{if } N = 2. \end{cases}$$

We conclude as in [35] p. 118-120, Proposition 5.2.2. □

Remark 5. The problem considered in this Section 3.1 was previously considered by other authors in the special case of $\overrightarrow{u} \equiv \overrightarrow{0}$ (see, e.g. [14] and its references), nevertheless we emphasize that the results of this section must be understood as preliminary results with respect the study we shall present in the following sections of this paper. In particular, what is specially important for us is to obtain a continuous dependence estimate with respect to the data (namely the velocity $\overrightarrow{u}$, the potential V, and the right hand side f) since we need to carry out several perturbations of those data in the next sections. As far as we know, such estimates are new in the literature (and, of course, they were not given in the above mentioned reference).

3.2. **Some regularity results with an integrable potential V and bounded from below.** As a first consequence of Proposition 3 and Proposition 7 we can deduce Meyer's type regularity giving a better information on the gradient of the solution of (23).

Proposition 8. **($W^1L^{p,q}$-estimate)** *Let $N \geqslant 2$. Assume that there exists $p > N$ and $q \in [1, +\infty]$, such that*

$$\begin{cases} \vec{u} \in L^{p,q}(\Omega)^N \quad V \geqslant 0,\ V \in L^{r,q}(\Omega),\ r = \dfrac{Np}{N+p}, \\ T = -\text{div}\,(\vec{F}) \quad \textit{with } \vec{F} \in L^{p,q}(\Omega)^N. \end{cases}$$

Then, the unique solution ϕ of the equation (23) belongs to $W^1L^{p,q}(\Omega)$. Moreover, there exists a constant $K_{pq} > 0$ independent of $\vec{u}$ such that :

$$||\nabla\phi||_{L^{p,q}(\Omega)} \leqslant K_{pq}\,(1 + ||\vec{u}||_{L^{p,q}} + ||V||_{L^{r,q}})\,||F||_{L^{p,q}(\Omega)^N}.$$

Proof. (We shall simply write $F,\ F_0,\ F_1$ for $\vec{F},\ \vec{F_0},\ \vec{F_1}$). We first assume that $\vec{u} \in \mathcal{V}$. We know from Proposition 7 that $\phi \in L^\infty(\Omega)$ and that there exists a constant independent of $\vec{u}$, V and $\vec{F}$ and V such that

$$||\phi||_\infty \leqslant K_N(\Omega)||F||_{L^{p,q}(\Omega)}. \tag{39}$$

Therefore, there exists a vector field $F_0 \in L^{p,q}(\Omega)^N$ such that

$$V\phi = -\text{div}\,(F_0) \text{ and } ||F_0||_{L^{p,q}} \leqslant K_{1,N}(\Omega)||V||_{L^{r,q}}||\phi||_\infty,$$

that is

$$||F_0||_{L^{p,q}} \leqslant K_{1N}(\Omega)||V||_{L^{r,q}}||F||_{L^{p,q}(\Omega)}.$$

Setting $F_1 = F - F_0$, we can write (23) as

$$-\Delta\phi = -\text{div}\,(F_1 - \vec{u}\phi). \tag{40}$$

But, we have $\vec{u}\phi \in L^{p,q}(\Omega)^N$ since $\phi \in L^\infty(\Omega)$ according to the above Proposition 7. Hence

$$||\vec{u}\phi||_{L^{p,q}(\Omega)^N} \leqslant ||\vec{u}||_{L^{p,q}}||\phi||_{L^\infty} \leqslant K_N||F||_{L^{p,q}}||\vec{u}||_{L^{p,q}}.$$

We may apply the $W^1L^{p,q}$ result to (40) (see [42, 9, 2, 36]) to deduce that

$$||\nabla\phi||_{L^{p,q}} \leqslant K_p||F_1-\vec{u}\phi||_{L^{p,q}} \leqslant K_{pNq}(1+||\vec{u}||_{L^{p,q}}+||V||_{L^{r,q}})||F||_{L^{p,q}}. \tag{41}$$

For the general case, we consider $u_k \in \mathcal{V}$ such that $u_k \to u$ strongly in $L^{p,q}(\Omega)^N$. Let ϕ_k be the solution of equation (23) where ϕ is replaced by ϕ_k

$$-\Delta\phi_k - \vec{u}_k\nabla\phi_k + V\phi_k = T = -\mathrm{div}\,(F).$$

The sequence $(\phi_k)_k$ satisfies

$$||\phi_k||_{L^\infty} \leqslant K_N||F||_{L^{r,q}} \text{ and } ||\phi_k||_{H_0^1} \leqslant ||T||_{H^{-1}},$$

and then $(\phi_k)_k$ converges weakly in $H_0^1(\Omega)$ to ϕ the solution of (23). Since ϕ_k satisfies (41), we deduce that ϕ also satisfies (41) and (23). ☐

As an immediate consequence of the above result.

Proposition 9. *Let $\vec{u}$ and $\vec{F}$ be in $L^{p,\infty}(\Omega)^N$ for some $p > N$. Then, the solution of (23) satisfies*

$$\phi \in C^{0,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{p}.$$

Proof. According to the Sobolev embedding (see [35]), we have

$$W^1L^{p,\infty}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \text{ with } \alpha = 1-\frac{N}{p}.$$

☐

Now we shall consider the case of more general data $\vec{u}$ and V.

Proposition 10. *Assume that $\vec{u}$ and $\vec{F}$ are in $\mathrm{bmo}_r(\Omega)^N$ and V is in $\mathrm{bmo}_r(V)$. Then the solution ϕ of the equation (23) satisfies*

1. $\vec{u}\phi \in \mathrm{bmo}_r(\Omega)^N$
2. $\nabla\phi \in \mathrm{bmo}_r(\Omega)^N$.

Proof. Since $\mathrm{bmo}_r(\Omega) \hookrightarrow L^{p,q}(\Omega)$ for all $p > N$ and $q \in [1,+\infty]$, we deduce from Proposition 8 and Proposition 9 that :

$$\phi \in C^{0,\alpha}(\overline{\Omega}) \quad \forall\,\alpha \in \ [0,1\,[\text{ and } -\Delta\phi = -\mathrm{div}\,(\vec{F}_1 - \vec{u}\phi),$$

where $\vec{F}_1$ was defined in the proof of Proposition 8 (see equation (40)). From Stegenga multiplier's result, $\vec{u}\phi \in \mathrm{bmo}_r(\Omega)^N$ whenever $\vec{u}$ is in $\mathrm{bmo}_r(\Omega)^N$ [44, 47]. Therefore $\vec{F}_1 - \vec{u}\phi \in \mathrm{bmo}_r(\Omega)^N$. We may appeal to Campanato's result [10] to derive then that $\nabla\phi \in \mathrm{bmo}_r(\Omega)^N$ and

$$||\nabla\phi||_{\mathrm{bmo}_r} \leqslant K\Big(||F||_{\mathrm{bmo}_r} + ||\vec{u}\,\phi||_{\mathrm{bmo}_r} + ||F_0||_{\mathrm{bmo}_r}\Big).$$

☐

We shall end this paragraph by proving a $W^2L^{p,q}(\Omega)$-regularity result for the solutions of the dual problem (23) which will lead to interesting conclusions for the direct problem (1).

For this, we shall use the following ADN constant

$$K^s_{pq} = \sup_{v\in H_0^1(\Omega)\cap W^2L^{p,q}(\Omega)} \frac{||v||_{W^2L^{p,q}(\Omega)}}{||v||_{L^{p,q}(\Omega)} + ||\Delta v||_{L^{p,q}(\Omega)}}, \tag{42}$$

which is finite due to the well-known Agmon-Douglis-Nirenberg's regularity result combined with the Marcinkiewicz interpolation Theorem.

We shall improve now the regularity obtained in Proposition 10. We consider $\varepsilon_0 > 0$ (fixed) so that $K^s_{pq}\varepsilon_0||\vec{u}||_{L^{p,q}(\Omega)} \leqslant \frac{1}{2}$.

Proposition 11. **($W^2L^{p,q}(\Omega)$ regularity for $p > N$)** *Let ϕ be the solution of (23) when $T \in L^{p,q}(\Omega)$, $p > N$, $q \in [1, +\infty]$. Assume, furthermore, that $\vec{u} \in L^{p,q}(\Omega)^N$ and $V \in L^{p,q}(\Omega)$. Then*

$$\phi \in W^2L^{p,q}(\Omega).$$

Moreover, there exist constants c_{ε_0}, $K_{pqN} > 0$ such that

$$||\phi||_{W^2L^{p,q}(\Omega)} \leqslant \frac{K_{pqN}c_{\varepsilon_0}(1 + ||V||_{L^{p,q}} + ||\vec{u}||_{L^{p,q}(\Omega)})}{1 - K^s_{pq}\varepsilon_0||\vec{u}||_{L^{p,q}(\Omega)}}||T||_{L^{p,q}(\Omega)}.$$

Proof. We assume first that $\vec{u} \in \mathcal{V}$. Arguing as in Proposition 8, since we can assume that $T = \operatorname{div} \vec{F}$ for suitable $\vec{F}$ we get that the solution ϕ of (23) is in $W^1L^{p,q}(\Omega)$ and then

$$-\Delta\phi = \vec{u}\,\nabla\phi + T - V\phi \in L^{p,q}(\Omega).$$

By the Agmon-Douglis-Nirenberg regularity results and the Marcinkiewicz interpolation theorem we deduce that $\phi \in W^2L^{p,q}(\Omega)$. Moreover, since $p > N$ and $q \in [\,1, +\infty]$, we have the following continuous embeddings :

$$W^2L^{p,q}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \hookrightarrow L^{p,q}(\Omega).$$

The first inclusion is compact so we may appeal to Lemma 2.8 to derive that $\forall\,\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$||\nabla\,\phi||_\infty \leqslant \varepsilon||\phi||_{W^2L^{p,q}(\Omega)} + c_\varepsilon||\phi||_{L^{p,q}(\Omega)}. \tag{43}$$

From the equation satisfied by ϕ, we have

$$||\Delta\phi||_{L^{p,q}(\Omega)} \leqslant ||\vec{u}||_{L^{p,q}(\Omega)}||\nabla\phi||_\infty + ||T||_{L^{p,q}(\Omega)} + ||V||_{L^{p,q}}||\phi||_\infty, \tag{44}$$

and using the ADN constant

$$||\phi||_{W^2L^{p,q}(\Omega)} \leqslant K^s_{pq}\Big(||\phi||_{L^{p,q}(\Omega)} + ||\Delta\phi||_{L^{p,q}(\Omega)}\Big). \tag{45}$$

We combine those last three equations and derive that for any $\varepsilon > 0$

$$\begin{aligned} ||\phi||_{W^2L^{p,q}(\Omega)}(1 - \varepsilon K^s_{pq}||\vec{u}||_{L^{p,q}(\Omega)}) \quad \leqslant \quad & K^s_{pq}||\phi||_{L^{p,q}(\Omega)}\Big(1 + c_\varepsilon||\vec{u}||_{L^{p,q}(\Omega)}\Big) \\ & + K^s_{pq}||T||_{L^{p,q}(\Omega)}(1 + ||V||_{L^{p,q}})K_{2N}. \end{aligned} \tag{46}$$

Next, we consider $\vec{u}_k \in \mathcal{V}$ such that $\vec{u_k} \to \vec{u} \in \overline{\mathcal{V}}$. Then, choosing $\varepsilon = \varepsilon_0 > 0$ such that $\varepsilon_0 K^s_{pq}\sup\limits_k ||\vec{u_k}||_{L^{p,q}(\Omega)} \leqslant \dfrac{1}{2}$, we deduce from relation (46) that ϕ_k corresponding to the solution of (23) , that is $-\Delta\phi_k - \vec{u_k}\cdot\nabla\phi_k + V\phi_k = T \in L^{p,q}(\Omega)$, belongs to a bounded set of $W^2L^{p,q}(\Omega)$ when k varies. Therefore, the strong limit ϕ in $C^1(\overline{\Omega})$ is the solution of (23) and it satisfies also the relation (46) for all $\varepsilon \in]0, \varepsilon_0\,]$. From Proposition 6, we have

$$||\phi||_{L^{p,q}(\Omega)} \leqslant K_N(\Omega)||T||_{L^{p,q}(\Omega)}. \tag{47}$$

Combining relations (46) and (47) with $\varepsilon = \varepsilon_0$, we derive the result. ☐

The case where $p = N$ can also be treated in the same way provided that the norm of $\vec{u}$ in $L^{N,1}(\Omega)$ is small enough in the sense that

$$||\vec{u}||_{L^{N,1}(\Omega)} \quad \leqslant \quad \theta K^{s0}_{N1} \text{ for some } \theta \in [\,0, 1\,[, \tag{48}$$

$$K^{s0}_{N1} \quad = \quad K^s_{N1} \sup_{\phi \in H^1_0(\Omega) \cap W^2L^{N,1}(\Omega)} \frac{||\nabla\phi||_\infty}{||\phi||_{W^2L^{N,1}}}. \tag{49}$$

Proposition 12. (Regularity in $W^2L^{N,1}(\Omega)$). *Let ϕ be the solution of (23) when $T \in L^{N,1}(\Omega)$, $V \in L^{N,1}(\Omega)$. Assume that $\vec{u}$ satisfies relation (48). Then $\phi \in W^2L^{N,1}(\Omega)$. Moreover, there exists a constant $K'_N(\Omega)$ (independent of $\vec{u}$) such that*

$$||\phi||_{W^2L^{N,1}(\Omega)} \leqslant \frac{K'_N(\Omega)(1+||V||_{L^{N,1}})}{1-K^{s0}_{N1}||\vec{u}||_{L^{N,1}}}||T||_{L^{N,1}(\Omega)}.$$

Proof. The proof follows the same argument as for the proof of Proposition 11. Nevertheless, the embedding $W^1L^{N,1} \subset C(\overline{\Omega})$ is not compact and this explains the condition (48). □

There are many other spaces between the space $L^{p,1}(\Omega)$ and $L^{N,1}(\Omega)$ for which we can obtain a regularity result for the second derivatives of ϕ.

Here we want only to consider the space $\Lambda = (L^N(\operatorname{Log} L)^{\frac{\beta}{N}})^N$ for $\beta > N-1$.

Indeed this space is included in $L^{N,1}(\Omega)$ and contains $L^p(\Omega)$ for all $p > N$.

Theorem 3.1. (Regularity in $W^2L^N(\Omega)$). *Let T and V be in $L^N(\Omega)$, $\vec{u} \in \Lambda$, $\operatorname{div}(\vec{u}) = 0$ and $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Then the unique solution ϕ of (23) belongs to $W^2L^N(\Omega)$ and choosing $\varepsilon > 0$ such that $\varepsilon||\vec{u}||_\Lambda \leqslant \frac{1}{2}$, there exists a constant $K_\varepsilon > 0$ such that*

$$||\phi||_{W^2L^N(\Omega)} \leqslant \frac{K_\varepsilon(1+||\vec{u}||_\Lambda+||V||_{L^N})}{1-\varepsilon||\vec{u}||_\Lambda}||T||_{L^N(\Omega)}.$$

The proof firstly depends on the following Trudinger's type embedding :

Lemma 3.1. (Trudinger's embedding) *We have*

$$W^1_0L^N(\Omega) \hookrightarrow L^{\frac{1}{N'}}_{exp}(\Omega).$$

Moreover, for all $v \in W^1_0L^N(\Omega)$

$$\sup_{t\leqslant|\Omega|}\frac{|v|_*(t)}{\left(1+\operatorname{Log}\frac{|\Omega|}{t}\right)^{\frac{1}{N'}}} \leqslant K_0||\nabla v||_{L^N(\Omega)}, \text{ with } K_0 = \frac{1}{N\alpha_N^{\frac{1}{N}}}.$$

Proof. According to the pointwise Sobolev inequality for the relative rearrangement, we have for $u = |v|$ (see Theorem 2.3)

$$-u'_*(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}|\nabla u|_{*u}(s). \tag{50}$$

We integrate this formula from t to $|\Omega|$ knowing that $u_*(|\Omega|) = 0$, and using the Hölder inequality, we get

$$u_*(t) \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}\int_t^{|\Omega|} s^{\frac{1}{N}-1}|\nabla u|_{*u}(s)ds \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}\left(\operatorname{Log}\frac{|\Omega|}{t}\right)^{\frac{1}{N'}}||\,|\nabla u|_{*u}||_{L^N}. \tag{51}$$

Therefore from (51), implies using Theorem 2.2

$$\sup_{t\leqslant|\Omega|}\frac{u_*(t)}{\left(1+\operatorname{Log}\frac{|\Omega|}{t}\right)^{\frac{1}{N'}}} \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}||\,|\nabla u|_{*u}||_{L^N} \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}||\nabla u||_{L^N}.$$

□

The key result for the proof of Theorem 3.1 is the following compactness inclusion:

Theorem 3.2. (Compact inclusion for $W_0^1L^N(\Omega)$). *$W_0^1L^N(\Omega)$ is compactly embedded in $L^\alpha_{exp}(\Omega)$ for $\alpha > \dfrac{1}{N'}$.*

Proof. Let $(u_n)_n$ be a bounded sequence in $W_0^1L^N(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $W_0^1L^N(\Omega)$-weakly and almost everywhere in Ω. Let $c = \underset{n}{\mathrm{Max}}\, ||u_n - u||_{L_{exp}^{\frac{1}{N'}}} < +\infty$.

For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{c}{\left(1+\mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\alpha-\frac{1}{N'}}} \leqslant \varepsilon \text{ for all } t \leqslant \delta.$$

Therefore, we have : if $t \leqslant \delta$

$$\frac{|u_n - u|_*(t)}{\left(1+\mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\alpha}} \leqslant \frac{c}{\left(1+\mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\alpha-\frac{1}{N'}}} \leqslant \varepsilon;$$

if $t > \delta$ then, since $|u_n - u|_*$ is nonincreasing

$$|u_n - u|_*(t) \leqslant \frac{1}{\delta}\int_0^\delta |u_n - u|_*(s)ds,$$

so that

$$\sup_{t\geqslant\delta} \frac{|u_n - u|_*(t)}{\left(1+\mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\alpha}} \leqslant \frac{1}{\delta}\int_0^\delta |u_n - u|_*(s)ds.$$

The right hand side of this inequality tends to zero as n goes to infinity. Hence, for $n \geqslant n_\varepsilon$ with n_ε large enough

$$\sup_{0<t<|\Omega|} \frac{|u_n - u|_*(t)}{\left(1+\mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\alpha}} \leqslant \varepsilon.$$

□

As a corollary of the above theorem, since $W^2L^N \cap W_0^1L^N \hookrightarrow W_0^1L^\alpha_{exp} \hookrightarrow L^N$, we have:

Corollary 1. (of Theorem 3.2) *Let $\alpha > \frac{1}{N'}$. Then, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that $\forall\, v \in W^2L^N(\Omega) \cap H_0^1(\Omega)$*

$$||\nabla v||_{L^\alpha_{exp}} \leqslant \varepsilon ||\Delta v||_{L^N} + c_\varepsilon ||v||_{L^N}.$$

Proof. We use the equivalence of norms $||v||_{W^2L^N(\Omega)\cap H_0^1} \equiv ||\Delta v||_{L^N} + ||v||_{L^N}$ and apply Lemma 2.8 with

$$Y = W_0^1L^\alpha_{exp}(\Omega), \quad X = W^2L^N(\Omega) \cap H_0^1(\Omega), \quad Z = L^N(\Omega).$$

□

Proof of Theorem 3.1. We first assume that $\vec{u} \in \mathcal{V}$, and $T \in L^\infty(\Omega)$. Then, the unique solution ϕ of (23) satisfies

$$\begin{aligned} ||\Delta\phi||_{L^N} &\leqslant ||T||_{L^N} + ||\vec{u}\cdot\nabla\phi||_{L^N} + ||V||_{L^N}||\phi||_\infty \\ &\leqslant K_N(1 + ||V||_N)||T||_N + ||\vec{u}\cdot\nabla\phi||_{L^N}. \end{aligned} \tag{52}$$

We have

$$||\vec{u}\cdot\nabla\phi||_{L^N}^N \leqslant \int_{\Omega_*} |\vec{u}|_*^N |\nabla\phi|_*^N\,dt$$
$$\leqslant \sup_{t\in\Omega_*} \frac{|\nabla\phi|_*^N(t)}{\left(1+\mathrm{Log}\,\frac{|\Omega|}{t}\right)^\beta} \int_{\Omega_*} |\vec{u}|_*^N(t)\left(1+\mathrm{Log}\,\frac{|\Omega|}{t}\right)^\beta dt,$$

which implies

$$||\vec{u}\nabla\phi||_{L^N} \leqslant ||\nabla\phi||_{L^\alpha_{exp}}||\vec{u}||_\Lambda \text{ with } \alpha = \frac{\beta}{N} > \frac{1}{N'}. \tag{53}$$

Let $\varepsilon > 0$ be fixed. There exists $c_\varepsilon > 0$ such that

$$||\vec{u}\cdot\nabla\phi||_{L^N} \leqslant (\varepsilon||\Delta\phi||_{L^N} + c_\varepsilon||\phi||_{L^N})||\vec{u}||_\Lambda$$

(see Corollary 1 of Theorem 3.2). Combining this with relation (52), we have $\forall\,\varepsilon > 0,\ \exists\, c_\varepsilon^1 > 0$

$$||\Delta\phi||_{L^N}(1-\varepsilon||\vec{u}||_\Lambda) \leqslant c_\varepsilon^1(1+||\vec{u}||_\Lambda + ||V||_{L^N})||T||_{L^N}. \tag{54}$$

Secondly, we consider $T \in L^N(\Omega)$ and $\vec{u} \in \overline{\mathcal{V}}$. There exist $\vec{u_k} \in \mathcal{V}$ such that $\vec{u}_k \to \vec{u}$ strongly in Λ and $T_k \in L^\infty(\Omega)$ with

$$||T_k||_{L^N} \leqslant ||T||_{L^N}.$$

Then from relation (54), the solution ϕ_k of (23) satisfies

$$||\Delta\phi_k||_{L^N}(1-\varepsilon||\vec{u_k}||_\Lambda) \leqslant c_\varepsilon^1(1+||\vec{u}_k||_\Lambda + ||V||_{L^N})||T||_{L^N}. \tag{55}$$

We choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0 \sup_k ||u_k||_\Lambda \leqslant \frac{1}{2}.$$

Then ϕ_k remains in a bounded set of $W^2L^N(\Omega)\cap H_0^1(\Omega)$. So it converges to ϕ weakly in $W^2L^N(\Omega)\cap H_0^1(\Omega)$ and we have

$$||\Delta\phi||_{L^N}(1-\varepsilon_0||\vec{u}||_\Lambda) \leqslant c_{\varepsilon_0}^1(1+||\vec{u}||_\Lambda + ||V||_{L^N})||T||_{L^N}, \tag{56}$$

and

$$||\phi||_{L^N} \leqslant |\Omega|^{\frac{1}{N}}||\phi||_\infty \leqslant K_N(\Omega)||T||_{L^N(\Omega)}$$

(according to Proposition 6). This gives the results. □

4. Very weak solutions of problem (1) with and without the Dirichlet boundary condition.

We now want to apply all those regularity results to the study of equation (1). We first start with some definitions of the weak solution associated with (1).

4.1. Existence and regularity of the very weak solution for a locally integrable potential $V \geqslant 0$.

We start by considering the existence of very weak solutions of equation (1) with the Dirichlet boundary condition (23) when the potential V is a nonnegative locally integrable function.

We can use the definition of very weak solution (see Definition 1.1).

Theorem 4.1. *Let $f \in L^1(\Omega;\delta)$. Let $\vec{u}$ be in $L^{p,1}(\Omega)^N$ with* $\mathrm{div}\,(\vec{u}) = 0$ *in $\mathcal{D}'(\Omega)$, $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Furthermore, assume that either $p > N$ or $p = N$ and $||\vec{u}||_{L^{N,1}} < K_{N1}^{s0}$ (see (48)). Then, there exists a very weak solution ω in the sense of (16), which is unique, if $V \in L^{p,1}(\Omega)$.*

Remark 6. In section 4.2, we shall discuss the uniqueness of the v.w.s when $V \notin L^{N,1}(\Omega)$.

Proof. First, we assume that $f \geqslant 0$. Let $u_j \in \mathcal{V}$ be such that $\vec{u}_j \to \vec{u}$ strongly in $L^{p,1}(\Omega)^N$ and $f_j \in L^\infty(\Omega)$ such that $0 \leqslant f_j(x) \leqslant f(x)$ a.e and $f_j(x) \to f(x)$ a.e. According to Proposition 4, Proposition 11 or Proposition 12, there exists a unique function $\omega_j \geqslant 0$ such that

$$\begin{cases} -\Delta\omega_j + \vec{u}_j \cdot \nabla\omega_j + V_j\omega_j = f_j, \\ \omega_j \in H_0^1(\Omega) \cap W^2L^{p,1}(\Omega), \end{cases} \tag{57}$$

which is equivalent to saying that

$$\begin{cases} \displaystyle\int_\Omega \omega_j\Big[-\Delta\phi - \vec{u}_j \cdot \nabla\phi\Big]dx = \int_\Omega f_j\phi\, dx - \int_\Omega V_j\omega_j\phi dx, \\ \forall\, \phi \in W^2L^{p,1}(\Omega) \cap H_0^1(\Omega). \end{cases} \tag{58}$$

We argue as in [20, 18, 36]. Let E be a measurable subset of Ω and χ_E its characteristic function. Then, there exists a non negative function $\phi_j \in W^2L^m(\Omega)$, $\forall\, m < +\infty$, satisfying

$$\begin{cases} -\Delta\phi_j - \vec{u}_j\nabla\phi_j = \chi_E \text{ in } \Omega, \\ \phi_j = 0 \text{ on } \partial\Omega. \end{cases} \tag{59}$$

We consider a small number $\varepsilon > 0$ such $\varepsilon \sup\limits_j ||\vec{u}_j||_{L^{N,1}} \leqslant \dfrac{1}{2}$. Therefore, we have

$$||\phi_j||_{W^2L^{N,1}} \leqslant K_0||\chi_E||_{L^{N,1}} \leqslant K_1|E|^{\frac{1}{N}}.$$

Thus

$$\begin{aligned} \int_E \omega_j dx = \int_\Omega \omega_j\big[-\Delta\phi_j - \vec{u_j}\nabla\phi_j\big]dx \quad &\leqslant \quad \int_\Omega f_j\phi_j \leqslant K_1\left(\int_\Omega |f_j|\delta\right)||\phi_j||_{W^2L^{N,1}} \\ &\leqslant \quad K_0|E|^{\frac{1}{N}}\int_\Omega |f_j|\delta dx. \end{aligned} \tag{60}$$

By the Hardy-Littlewood property we conclude that

$$\sup_{t\leqslant|\Omega|} t^{\frac{1}{N'}}|\omega_j|_{**}(t) \leqslant K_0\int_\Omega |f_j|\delta dx \leqslant K_0\int_\Omega |f|\delta dx. \tag{61}$$

Moreover, choosing $\phi = \varphi_1$ as the test function with $-\Delta\varphi_1 = \lambda_1\varphi_1$, and $\varphi_1 = 0$ on $\partial\Omega$, we have

$$\begin{aligned} \lambda_1\int_\Omega \omega_j\varphi_1 dx + \int_\Omega V_j\omega_j\varphi_1 dx \quad &\leqslant \quad ||\nabla\varphi_1||_\infty\, ||\omega_j||_{L^{N',\infty}}||\vec{u}_j||_{L^{N,1}} + c\int_\Omega |f_j|\delta dx \\ &\leqslant \quad c\big(1 + ||\vec{u}_j||_{L^{N,1}}\big)\int_\Omega |f_j|\delta dx, \end{aligned}$$

for a suitable constant $c > 0$. Thus $V_j\omega_j$ remains in a bounded set of $L^1(\Omega;\delta)$ and

$$\int_\Omega V_j\omega_j\delta dx \leqslant c\big(1 + ||\vec{u}_j||_{L^{N,1}}\big)\int_\Omega |f_j|\delta dx. \tag{62}$$

If f has a constant sign, we write $f_j = f_{j+} - f_{j-}$ with $f_{j+} = \max(f_j, 0) \geqslant 0$.

Denoting by ω_j^+ the v.w.s. associated to f_{j+} and by ω_j^- the one associated to f_{j-},we see that $\omega_j = \omega_j^+ - \omega_j^-$ satisfies (58) and we have also the estimates (61) and (62).

In particular, since $|\omega_j| \leqslant \omega_j^+ + \omega_j^-$

$$\int_\Omega V_j|\omega_j|\delta dx \leqslant c\big(1+||u_j||_{L^{N,1}}\big)\int_\Omega |f_j|\delta dx. \tag{63}$$

We conclude that $(\omega_j)_j$ converges weak-* to ω in $L^{N',\infty}(\Omega) = \big(L^{N,1}(\Omega)\big)^*$. To obtain a strong convergence, we need a local estimate of the gradient. For that purpose, we shall prove the boundedness of ω_j in the Lorentz-Sobolev weighted space $W^1L^{1+\frac{1}{N},\infty}(\Omega;\delta)$. For this, we shall need the following result due to Philippe Bénilan and co-authors whose proof can be found in [5] Lemma 4.2, with generalization in [40].

Proposition 13. *Let $v \in L^1(\Omega,\delta^\alpha)$, and $\alpha \in [0,1]$. Assume that there exists a constant $c_0 > 0$ such that for all $k > 0$*

$$T_k(v) := \min(|v|;k)\ \mathrm{sign}\,(v) \in W^1L^2(\Omega,\delta^\alpha),$$

and

$$\int_\Omega |\nabla T_k(v)|^2\delta^\alpha dx + \int_\Omega |T_k(v)|^2\delta^\alpha dx \leqslant c_0 k. \tag{64}$$

Then, there exists a constant c, depending continuously on $c_0 > 0$, such that for all $\lambda > 0$

$$\int_{\{x:|\nabla v|(x)>\lambda\}} \delta^\alpha(x)dx \leqslant \frac{c}{\lambda^{1+\frac{1}{N+\alpha-1}}}.$$

In particular, if v_j is a sequence converging weakly in $L^1(\Omega)$ to a function v, satisfying the inequality (64)

$$\int_\Omega |\nabla T_k(v_j)|^2\delta^\alpha dx \leqslant c_0 k \qquad \forall j,\ \forall k,$$

then v_j converges to v weakly in $W^{1,q}(\Omega')$ for all $q \in \left[1, \dfrac{N+\alpha}{N+\alpha-1}\right[$ and all $\Omega' \subset\subset \Omega$, with a subsequence, $v_j(x) \to v(x)$ a.e. in Ω.

We first need to prove the following a priori estimate :

Proposition 14. *Let ω_j be the solution of (57), ω its weak limit in $L^{N',\infty}(\Omega)$. Under the same assumptions as for Theorem 3.1, there exists a constant $c_0 > 0$ such that:*

$$\int_\Omega \big|\nabla T_k(\omega_j)\big|^2\delta dx + \int_\Omega \big|\nabla T_k(\omega)\big|^2\delta dx \leqslant c_0 k \quad \forall\, k > 0,\ \forall\, j.$$

Proof. Let φ_1 be the first eigenvalue of the Dirichlet problem $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$. Then, there exist constants such that $c_1\delta(x) \leqslant \varphi_1(x) \leqslant c_2\delta(x)\quad \forall\, x \in \Omega$. We consider the approximate problem given in equation (57) say

$$\begin{cases} -\Delta\omega_j + \vec{u}_j\cdot\nabla\omega_j + V_j\omega_j = f_j, \\ \omega_j \in W_0^{1,1}(\Omega)\cap W^2L^{p,1}(\Omega), \end{cases}$$

with $|f_j(x)| \leqslant |f(x)|$, $f_j \to f$ a.e, $\vec{u}_j \to \vec{u}$ in $L^{p,1}(\Omega)^N$−strongly and $\omega_j \to \omega$ weakly-* in $L^{N',\infty}(\Omega)$.

For $k > 1$, we choose $T_k(\omega_j)\varphi_1$ as a test function; then $V_j\omega_j T_k(\omega_j)\varphi_1 \geqslant 0$ and we derive after some integrations by parts :

$$\int_\Omega |\nabla T_k(\omega_j)|^2\varphi_1 dx + \lambda_1\int_\Omega \varphi_1\left(\int_0^{\omega_j} T_k(\sigma)d\sigma\right) dx - \int_\Omega \vec{u}_j \cdot \nabla\varphi_1 \int_0^{\omega_j} T_k(\sigma)d\sigma dx \leqslant c_2 k\int_\Omega |f|\delta dx. \tag{65}$$

This relation implies:

$$\int_\Omega |\nabla T_k(\omega_j)|^2\delta(x) \leqslant c_3 k\int_\Omega |\omega_j|\delta dx + c_2 k\int_\Omega |f|\delta dx + c_3 k\int_\Omega |\vec{u}_j|\,|\omega_j| dx. \tag{66}$$

By the Hölder inequality

$$\int_\Omega |\vec{u}_j|\,|\omega_j| dx \leqslant c_4 ||\vec{u}_j||_{L^{N,1}} \cdot ||\omega_j||_{L^{N',\infty}} \leqslant c_4 ||\vec{u}_j|| \int_\Omega |f|\delta dx. \tag{67}$$

From relation (66) and (67), we then have :

$$\int_\Omega |\nabla T_k(\omega_j)|^2\delta(x)dx \leqslant c_5(1 + ||\vec{u}_j||_{L^{N,1}})\left(\int_\Omega |f|\delta dx\right) k. \tag{68}$$

Letting $j \to +\infty$, we deduce from (68) and Proposition 13 :

$$\int_\Omega |\nabla T_k(\omega)|^2\delta(x)dx \leqslant c_0 k \text{ with } c_0 = c_5(1 + ||\vec{u}||_{L^{N,1}})\int_\Omega |f|\delta dx.$$

Then the $L^{N',\infty}$-regularity of ω implies

$$\int_\Omega |T_k(\omega)|^2\delta dx \leqslant c_0 k\int_\Omega |\omega| dx.$$

□

Corollary 2 (of Propositions 13 and 14)**.** *Let ω be as in the proof of the previous proposition. Then, there exists a constant $c_6 > 0$ such that*

$$||\nabla\omega||_{L^{1+\frac{1}{N},\infty}(\Omega;\delta)} \leqslant c_6 \int_\Omega |f(x)|\delta(x)dx.$$

In particular, we have, for all $q < 1 + \dfrac{1}{N}$,

$$\int_\Omega |\nabla\omega|^q\delta(x)dx \leqslant c_q\int_\Omega |f(x)|\delta(x)dx.$$

To pass to the limit in (57), we argue as in [19] p. 1041. We emphasize the main differences due to the additional term $\vec{u}\cdot\nabla\omega$.

Let us note that by the above Proposition 11, we have (for a subsequence still denoted as $(\omega_j)_j$) that

1. $\omega_j(x) \to \omega(x)$ a.e. (and thus $V_j\omega_j \to V\omega$ a.e. in Ω).
2. $\omega_j \rightharpoonup \omega$ weakly in $W^{1,q}(\Omega;\delta)$, $\forall\, q < 1 + \dfrac{1}{N}$.
3. $\omega_j \to \omega$ strongly in $L^r(\Omega)$, for any $r < N'$.

In particular, we deduce from the above statement 1., relation (63) and Fatou's lemma

Lemma 4.1. *Under the assumptions of Theorem 4.1 and Proposition 14 one has*

$$\int_\Omega V|\omega|\delta dx \leqslant c\Big(1 + ||u||_{L^{N,1}}\Big)\int_\Omega |f|\delta dx.$$

Lemma 4.2. *Under the assumptions of Theorem 4.1 and Proposition 14 one has*

$$\lim_{j\to+\infty}\int_\Omega |\vec{u}_j\omega_j - \vec{u}\omega|dx = 0.$$

Proof. Since $\vec{u}_j \to \vec{u}$ in $L^{N,1}(\Omega)$, and a.e. in Ω, we have

$$\lim_{j\to+\infty} \vec{u}_j(x)\omega(x) = \vec{u}(x)\omega(x) \ a.e.$$

It is enough to show that $(\vec{u}_j\omega_j)_j$ satisfies Vitali's condition : $\forall\,\varepsilon > 0\ \exists\eta > 0$ such that if $E \subset \Omega$ is measurable with $|E| \leqslant \eta$ then

$$\limsup_{j\to+\infty}\int_E |\vec{u}_j\omega_j|dx \leqslant \varepsilon.$$

But from Hölder's inequality we have

$$\int_E |\vec{u}_j\omega_j|dx \leqslant ||\vec{u}_j||_{L^{N,1}(E)}||\omega_j||_{L^{N',\infty}(\Omega)} \leqslant c||\vec{u}_j||_{L^{N,1}(E)},$$

so that

$$\limsup_{j\to+\infty}\int_E |\vec{u}_j\omega_j|dx \leqslant c||\vec{u}||_{L^{N,1}(E)}.$$

Since

$$||\vec{u}||_{L^{N,1}(E)} \xrightarrow[|E|\to 0]{} 0,$$

we derive that it satisfies the Vitali condition. Therefore, we have proved the lemma. □

Then we have the following result analogous to Lemma 2.3 of [19].

Lemma 4.3. *We assume that $V \in L^1_{loc}(\Omega)$, and $V \geqslant 0$. Then*

$$V_j\omega_j\delta \rightharpoonup V\omega\delta \quad \text{weakly in } L^1_{loc}(\Omega).$$

Furthermore, if $V \in L^1(\Omega;\delta)$, then

$$V_j\omega_j\delta \rightharpoonup V\omega\delta \text{ weakly in } L^1(\Omega).$$

Proof. Let $t \in \mathbb{R}_+$. Consider a sequence of functions γ_m in $C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ such that

$$\begin{array}{rcl} \gamma_m' \geqslant 0 & & \forall\, s \in \mathbb{R}, \\ \gamma_m(s) & \to & -1 \text{ for } s < -t \text{ as } m \to +\infty, \\ \gamma_m(s) & \to & 1 \text{ for } s > t \text{ as } m \to +\infty, \\ \gamma_m(s) & = & 0 \text{ on } -t \leqslant s \leqslant t, \end{array}$$

and let $\varphi_1 \in C^2(\overline{\Omega})$ with $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, $\lambda_1 > 0$.

Taking $\varphi_1\gamma_m(\omega_j)$ as a test function in relation (57) we get

$$\begin{aligned} \int_\Omega \nabla\omega_j\cdot\nabla\big(\varphi_1\gamma_m(\omega_j)\big) + \int_\Omega V_j\omega_j\varphi_1\big(\gamma_m(\omega_j)\big)dx \ &+ \int_\Omega \vec{u}_j\cdot\nabla\omega_j\gamma_m(\omega_j)\varphi_1 dx \\ &= \int_\Omega f_j\gamma_m(\omega_j)\varphi_1 dx. \qquad (69) \end{aligned}$$

We write $\nabla\omega_j\gamma_m(\omega_j) = \nabla\left[\int_0^{\omega_j}\gamma_m(\sigma)dx\right]$ so that

$$\begin{aligned}\int_\Omega(\vec{u}_j\cdot\nabla\omega_j)\gamma_m(\omega_j)\varphi_1 dx &= -\int_\Omega \operatorname{div}(u_j\varphi_1)\int_0^{\omega_j}\gamma_m(\sigma)d\sigma dx\\ &= -\int_\Omega \vec{u}_j\nabla\varphi_1\left(\int_0^{\omega_j}\gamma_m(\sigma)d\sigma\right)dx.\end{aligned}$$

As $m\to+\infty$, treating the remaining terms in (69) as in [19], we derive

$$\int_{|\omega_j|>t} V_j|\omega_j|\delta dx \leqslant c\left[\int_{|\omega_j|\geqslant t}|f|\delta dx + \int_{|\omega_j|\geqslant t}|\omega_j|\delta dx + \int_{|\omega_j|\geqslant t}|\vec{u}_j|\,|\omega_j|dx\right]. \qquad (70)$$

This relation proves that $V_j\omega_j\delta$ remains in a bounded set of $L^1(\Omega)$ but also that the set $\Big\{V_j|\omega_j|\delta,\ j\in\mathbb{N}\Big\}$ is x compact for the $\sigma(L^1;L^\infty)$-topology, so we may appeal to the Dunford-Pettis to conclude. Indeed, let us set

$$\Gamma_j(t) := \int_{|\omega_j|\geqslant t}|f(x)|\delta(x)dx + \int_{|\omega_j|\geqslant t}|\omega_j|\delta dx + \int_{|\omega_j|\geqslant t}|\vec{u}_j\omega_j|dx.$$

For a.e. $t>0$,

$$\lim_{j\to+\infty}\Gamma_j(t) = \Gamma(t) = \int_{|\omega|>t}|f(x)|\delta(x)dx + \int_{|\omega|>t}|\omega|\delta dx + \int_{|\omega|>t}|\vec{u}\omega|dx,$$

and

$$\Big|\big\{|\omega|>t\big\}\Big| + \sup_j\Big|\big\{|\omega_j|>t\big\}\Big| \leqslant \frac{constant}{t}\xrightarrow[t\to+\infty]{} 0,$$

we deduce that for any $\varepsilon>0$, there exists $t_\varepsilon>0$ such that, for all $j\in\mathbb{N}$,

$$\Gamma_j(t_\varepsilon)\leqslant\varepsilon.$$

Let $\Omega_0\subset\Omega$ such that $V\delta\in L^1(\Omega_0)$ (thus $\Omega_0\neq\Omega$ if V is only locally integrable). Then by the Lebesgue convergence dominate theorem for a.e. t,

$$\lim_{j\to+\infty}\int_{\Omega_0}\Big|\chi_{|\omega_j|\leqslant t}(x)V_j\omega_j(x) - \chi_{\{|\omega|\leqslant t\}}(x)V(x)\omega(x)\Big|\delta(x)dx = 0,$$

since

$$\lim_{|A|\to 0}\int_A V|\omega|\delta dx = 0 \quad \Big(V\omega\delta\in L^1(\Omega)\Big).$$

Therefore there exists $\eta>0$ such that if $A\subset\Omega_0,\ |A|\leqslant\eta$, then for all $j\in\mathbb{N}$,

$$\int_{A\cap\{|\omega_j|\leqslant t_\varepsilon\}}V_j|\omega_j|\delta dx\leqslant\varepsilon.$$

Hence, for all $j\in\mathbb{N}$, all $A\subset\Omega_0$, with $|A|\leqslant\eta$

$$\int_A V_j|\omega_j|\varphi dx\leqslant\Gamma_j(t_\varepsilon) + \int_A V_j|\omega_j|\delta dx\leqslant 2\varepsilon.$$

This conclude the proof of Lemma 7. □

The passage to the limit, we will distinguish two different cases :

1. Case $V \in L^1(\Omega;\delta)$ For all $\phi \in C^2(\overline{\Omega}),\ \phi = 0$, we have
$$\lim_j \int_\Omega V_j\omega_j\phi dx = \int_\Omega V\omega\phi dx \tag{71}$$
(since $\dfrac{\phi}{\delta} \in L^\infty(\Omega)$ and $V_j\omega_j\delta$ converges to $V\omega\delta$ for $\sigma(L^1;L^\infty)$ topology). Therefore, since
$$-\int_\Omega \omega_j\Delta\phi dx - \int_\Omega \vec{u}_j\omega_j\nabla\phi dx + \int_\Omega V_j\omega_j\phi dx = \int_\Omega f_j\phi dx, \tag{72}$$
we let $j \to +\infty$ to deduce that ω is a v.w.s. using Lemma 4.2 and the convergences of ω_j.
2. Case $V \in L^1_{loc}(\Omega)$ We consider $\phi \in W^2L^{N,1}(\Omega)$ with support ϕ be a compact in Ω. Then the same argument holds since $V_j\omega_j\delta$ tends to $V\omega\delta$ weakly in $L^1_{loc}(\Omega)$. Then (71) and (72) hold true
$$\begin{cases} \displaystyle\int_\Omega \omega\big[-\Delta\phi - \vec{u}\nabla\phi + V\phi\big]dx = \int_\Omega f\phi dx, \\ \forall\,\phi \in W^2L^{N,1}(\Omega),\ \text{support}(\phi) \text{ compact in } \Omega. \end{cases} \tag{73}$$

If $V \in L^{p,1}(\Omega)$, the solution is unique. Indeed, if we denote by ω the difference of two solutions then
$$\int_\Omega \Big[-\Delta\varphi - \vec{u}\nabla\varphi + V\varphi\Big]\omega dx = 0 \quad \forall\,\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \delta\Omega.$$

Let us consider the function ϕ solution of
$$\begin{cases} -\Delta\phi - \vec{u}\nabla\phi + V\phi = \text{sign}\,(\omega), \\ \phi \in H^1_0(\Omega). \end{cases} \tag{74}$$

Then $\phi \in W^2L^{N,1}\Omega \hookrightarrow C^1(\overline{\Omega})$ for $V \in L^{N,1}(\Omega)$. Thus
$$\int_\Omega \omega\big[-\Delta\phi - \vec{u}\nabla\phi + V\phi\big]dx = 0, \tag{75}$$
since $\Big\{\varphi \in C^2(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega\Big\}$ is dense in $W^2L^{N,1}(\Omega) \cap H^1_0(\Omega)$. Combining the relations (74) and (75) we find :
$$\int_\Omega |\omega|dx = 0 \quad i.e.\ \omega \equiv 0.$$

$\square$

4.2. A result of uniqueness of solution when the potential is bounded from below by $c\,\delta^{-r}, r > 2$.

The purpose of this section is to show the following uniqueness result.

Theorem 4.2. *Assume that V is locally integrable $V \geqslant 0$, and such that*
$$\exists\, c > 0,\ V(x) \geqslant c\delta(x)^{-r},\ \textit{in a neighborhood } U \textit{ of the boundary, with } \ r > 2.$$
Then, the v.w.s. ω found in Theorem 4.1 is unique.

This theorem relies on the following general result which does not require any information about the boundary condition, since the required additional information is written in another way :

Theorem 4.3. (Comparison principle) *Let ϖ be in $L^1(\Omega;\delta^{-r})\cap W^{1,1}_{\mathrm{loc}}(\Omega)$, $r>1$. Let $\varpi \in L^{N',\infty}(\Omega)$ and $\vec{u}\in L^{p,1}(\Omega)$ with $p>N$ or $p=N$ with a small norm. Assume that*

$$\mathrm{L}\varpi \doteq -\Delta\varpi + \mathrm{div}\,(\vec{u}\varpi) \leqslant 0 \ \text{ in } \mathcal{D}'(\Omega).$$

Then

$$\varpi \leqslant 0 \ \text{ in } \Omega.$$

As an immediate corollary of the above theorem we have

Corollary 3. of Theorem 4.3 *Assume the hypotheses of Theorem 4.3 hold and let $f\in L^1_{loc}(\Omega)$. Then there exists at most one function $\varpi\in L^1(\Omega;\delta^{-r})\cap W^{1,1}_{\mathrm{loc}}(\Omega)$, $r>1$ solution of $\mathrm{L}\varpi = f$ in $\mathcal{D}'(\Omega)$.*

For the proof of Theorem 4.3, we need the following extension of the Kato's inequality whose proof is similar to the one given in [30] :

Theorem 4.4. (Local Kato's inequality) *Let $\varpi\in W^{1,1}_{loc}(\Omega)$ with $\vec{u}\,\varpi\in L^1_{loc}(\Omega)$. Assume that $\mathrm{L}\varpi = -\,\Delta\varpi + \mathrm{div}\,(\vec{u}\,\varpi)$ belongs to $L^1_{loc}(\Omega)$. Then*

1. $\forall\,\psi\in\mathcal{D}(\Omega),\ \psi\geqslant 0,\ \displaystyle\int_\Omega \varpi_+ \mathrm{L}^*\psi dx \leqslant \int_\Omega \psi \mathrm{sign}_+(\varpi)\mathrm{L}(\varpi)dx,$

$$i.e.\ \mathrm{L}(\varpi_+)\leqslant \mathrm{sign}_+(\varpi)\mathrm{L}(\varpi) \quad in \quad \mathcal{D}'(\Omega).$$

2. $\mathrm{L}(|\varpi|)\leqslant \mathrm{sign}\,(\varpi)\mathrm{L}(\varpi)$ *in* $\mathcal{D}'(\Omega)$.
   Here

$$\mathrm{sign}_+(\sigma) = \begin{cases} 1 & if\ \sigma>0,\\ 0 & if\ \sigma\leqslant 0,\end{cases} \qquad \mathrm{sign}\,(\sigma) = \begin{cases} 1 & if \sigma>0,\\ -1 & if\ \sigma<0,\end{cases}$$

$$\mathrm{L}^*\psi = -\Delta\psi - \vec{u}\cdot\nabla\psi,\ \ for\ \psi\in C^\infty_c(\Omega).$$

Proof. Following [30], we first remark that for any $\alpha\in C^\infty_c(\Omega)$, $\mathrm{L}(\alpha\varpi)\in L^1(\Omega)$ since, one has, in $\mathcal{D}'(\Omega)$,

$$\mathrm{L}(\alpha\varpi) = \alpha\mathrm{L}\varpi - \varpi\Delta\alpha - 2\nabla\varpi\cdot\nabla\alpha + (\vec{u}\varpi)\cdot\nabla\alpha\in L^1(\Omega).$$

Thus, the conclusion 1 will be proved if we show that

$$\mathrm{L}(\alpha\varpi)_+ \leqslant \mathrm{sign}_+(\alpha\omega)\mathrm{L}(\alpha\varpi) \ \text{ in } \ \mathcal{D}'(\Omega).$$

For this purpose, we may assume that $\varpi\in W^{1,1}(\Omega)$ with compact support and $\mathrm{L}\varpi\in L^1(\Omega)$. Moreover, if $\rho_j\in C^\infty_c(\mathbb{R}^N)$ is a sequence of mollifiers, and $\varpi\star\rho_j\in C^\infty_c(\Omega)$ we have

$$\mathrm{L}(\varpi\star\rho_j) = \mathrm{L}\varpi\star\rho_j \to \mathrm{L}\varpi \ \text{ in } \ L^1(\Omega).$$

So, it is sufficient to show the inequality number for $\varpi\in C^\infty_c(\Omega)$. From here, we argue as for the case where L is replaced by the Laplacian operator (see Proposition 1.5.4 p.21 in [30] for more details). We approximate the functions sign_+ by a sequence of convex, non-decreasing functions h_ε such that

$$\lim_{\varepsilon\to 0} h'_\varepsilon(t) = \mathrm{sign}_+(t);\quad \lim_{\varepsilon\to 0} h_\varepsilon(t) = t_+$$

$$\sup_{\varepsilon>0}|h'_\varepsilon|(t) \text{ is independent of } \varepsilon.$$

Thus, for all $\psi\in C^\infty_c(\Omega)$, $\psi\geqslant 0$, we have

$$\int_\Omega h_\varepsilon(\varpi)\mathrm{L}^*\psi dx \leqslant \int_\Omega \psi h'_\varepsilon(\varpi)\mathrm{L}\varpi dx, \tag{76}$$

where $\mathrm{L}^*\psi = -\Delta\psi - \vec{u}\cdot\nabla\psi$.

Indeed, $\psi h'_\varepsilon(\overline{\omega})$ is in $C^\infty_c(\Omega)$ and then the convexity of h_ε implies

$$\int_\Omega \psi h'_\varepsilon(\overline{\omega})\mathrm{L}\overline{\omega}dx \geqslant -\int_\Omega h_\varepsilon(\overline{\omega})\Delta\psi dx + \int_\Omega \vec{u}\psi h'_\varepsilon(\overline{\omega})\cdot\nabla\overline{\omega}dx.$$

Since $\mathrm{div}\,(\vec{u}) = 0$, and $h'_\varepsilon(\overline{\omega})\nabla\overline{\omega} = \nabla h_\varepsilon(\overline{\omega})$ we have

$$\int_\Omega \vec{u}\psi h'_\varepsilon(\overline{\omega})\cdot\nabla\overline{\omega}dx = \int_\Omega \vec{u}\psi\cdot\nabla h_\varepsilon(\overline{\omega})dx = -\int_\Omega \vec{u}\cdot\nabla\psi h_\varepsilon(\overline{\omega})dx.$$

Thus we get (76).

As in [30], letting $\varepsilon \to 0$, we have

$$\int_\Omega \overline{\omega}_+\mathrm{L}^*\psi dx \leqslant \int_\Omega \psi\mathrm{sign}_+(\overline{\omega})\mathrm{L}\overline{\omega}dx \quad \forall\,\psi \in \mathcal{D}(\Omega),\ \psi \geqslant 0.$$

We derive conclusion 1, as in [30], for $\overline{\omega} \in W^{1,1}_c(\Omega)$ and the same for conclusion 2. □

To extend the set of test functions from $\mathcal{D}(\Omega)$ to other sets of functions we need the following approximation result.

Lemma 4.4. (Approximation of functions in $W^{m,\infty}(\Omega)$ by a sequence in $W^{m,\infty}_c(\Omega)$) *Let $W^{m,\infty}_c(\Omega) = \Big\{\varphi \in W^{m,\infty}(\Omega)$ with compact support$\Big\}$, $1 < m < +\infty$ and assume that $\partial\Omega$ is of class C^m, $r > 0$. Then, for $\varphi \in W^{m,\infty}(\Omega)$ there exists a sequence $(\varphi_n)_n$, $\varphi_n \in W^{m,\infty}_c(\Omega)$, such that*

1. *$\delta^r(D^\alpha\varphi_n) \to \delta^r(D^\alpha\varphi)$ strongly in $L^\infty(\Omega)$, for all α such that $|\alpha| < r$.*
2. *Moreover, if $\varphi \in W^{1,\infty}_0(\Omega)$ then*
$$\sup_n ||\nabla\varphi_n||_\infty \leqslant c_\Omega||\nabla\varphi||_\infty,\ (c_\Omega \text{ with independent of } \varphi),$$
$$\delta^r(D^\alpha\varphi_n) \to \delta^r(D^\alpha\varphi) \text{ strongly in } L^\infty(\Omega) \text{ for } |\alpha| < r+1.$$
3. *If $\varphi \geqslant 0$ then one can take $\varphi_n \geqslant 0$.*
4. *If $\varphi \in C^m(\overline{\Omega})$ then $\varphi_n \in C^m_c(\Omega)$. By the density of $C^\infty_c(\Omega)$ in $C^m_c(\Omega)$, φ_n in this case can be taken in $C^\infty_c(\Omega)$.*

Proof. Let $h \in C^\infty(\mathbb{R})$ be such that $0 \leqslant h \leqslant 1$, $h(\sigma) = \begin{cases} 1 & \text{if } \sigma \geqslant 1, \\ 0 & \text{if } \sigma \leqslant 0 \end{cases}$.

Since $\partial\Omega \in C^m$, δ is of class C^m in a neighborhood U of $\partial\Omega$ (see [22]). Let $0 < \varepsilon < 1$ be such that

$$\Big\{x \in \Omega : \delta(x) \leqslant \varepsilon\Big\} \subset U$$

and define, for $x \in \Omega$,

$$h_\varepsilon(x) = h\left(\frac{2\delta(x)-\varepsilon}{\varepsilon}\right), \tag{77}$$

so that $h_\varepsilon(x) = 1$ if $\delta(x) > \varepsilon$, $h_\varepsilon(x) \to 1$ as $\varepsilon \to 0$, and $h_\varepsilon(x) = 0$ if $\delta(x) < \varepsilon/2$.

One has

$$|D^\alpha h_\varepsilon(x)| \leqslant c\,\varepsilon^{-|\alpha|},\ \text{for a constant } c > 0 \text{ independent of } x \text{ and } \varepsilon.$$

Since we have, by Leibniz's formula

$$D^\alpha\big(\varphi(1-h_\varepsilon)\big)(x) = \sum_{\beta+\gamma=\alpha} c_{\gamma\beta}D^\beta\varphi(x)D^\gamma(1-h_\varepsilon)(x), \tag{78}$$

($c_{\gamma\beta}$ are constant depending only on $\gamma,\ \beta$) and for $\gamma \neq 0$.

$$\delta^r(x)\big|D^\gamma h_\varepsilon(x)\big| \leqslant c\,\varepsilon^{-|\gamma|+r}, \tag{79}$$

we then deduce, that

$$\delta^r(x)\big|D^\alpha\big(\varphi(1-h_\varepsilon)\big)(x)\big| \leqslant c\left[\sum_{\beta+\gamma=\alpha,\,\gamma\neq 0}|D^\beta\varphi(x)|\varepsilon^{-|\gamma|+r}+\delta^r|D^\alpha\varphi|(1-h_\varepsilon)\right].$$

Therefore

$$\sup_{x\in\Omega}\delta^r(x)\big|D^\alpha\varphi(1-h_\varepsilon)(x)\big| \leqslant c\,\varepsilon^{-|\alpha|+r}. \tag{80}$$

Taking $\varepsilon = \dfrac{1}{n}$, and $\varphi_n = h_{\frac{1}{n}}\varphi$ is convenient for large $n \geqslant n_0$. If furthermore $\varphi \in W_0^{1,\infty}(\Omega)$ then

$$|\varphi(x)| \leqslant \delta(x)||\nabla\varphi||_\infty.$$

Hence,

$$\begin{aligned}&\delta^r\Big|D^\alpha\big(\varphi(1-h_\varepsilon)\Big)(x)\Big|\\ &\leqslant c\,\delta^{r+1}(x)\varepsilon^{-|\alpha|}+c\sum_{\beta\neq 0,\beta+\gamma=\alpha}\big|D^\beta\varphi|(x)\big|D^\gamma(1-h_\varepsilon)|\delta^r(x) \leqslant c\,\varepsilon^{-|\alpha|+r+1}.\end{aligned}$$

On the other hand

$$\begin{cases}\text{on } \delta(x)\leqslant\varepsilon & \big|\nabla(\varphi h_\varepsilon)(x)\big| \leqslant |\varphi(x)|\nabla h_\varepsilon(x)|+2||\nabla\varphi||_\infty \leqslant c\,||\nabla\varphi||_\infty\left[1+\dfrac{\delta(x)}{\varepsilon}\right]\\ & \qquad\qquad\qquad\quad \leqslant c\,||\nabla\varphi||_\infty,\\ \text{on } \delta(x)>\varepsilon & \big|\nabla\varphi_n(x)\big| \leqslant 2||\nabla\varphi||_\infty.\end{cases}$$

Moreover, one has

$$\delta^r(x)\Big|\nabla\big(\varphi(1-h_\varepsilon)\big)\Big|(x) \leqslant \delta^r|D\varphi|(x)\big(1-h_\varepsilon(x)\big)+c\,\delta^{r+1}(x)||\nabla\varphi||_\infty|\nabla h_\varepsilon| \leqslant c\,\varepsilon^r.$$

□

Thanks to the above approximation lemma we can modify the set of the test functions in the Kato's inequality as follows

Corollary 4. (of Theorem 4.4: Variant of Kato's inequality) *Let* $\overline{\omega}$ *be in* $W^{1,1}_{\mathrm{loc}}(\Omega)\cap L^{N',\infty}(\Omega)$, $\overline{\omega}\in L^1(\Omega;\delta^{-r})$ *for* $r>1$ *and* $\vec{u}\in L^{N,1}(\Omega)^N$ *with* $\operatorname{div}(\vec{u})=0$, $\vec{u}\cdot\vec{n}=0$. *Assume furthermore that* $\mathrm{L}\overline{\omega}=-\Delta\overline{\omega}+\operatorname{div}(\vec{u}\overline{\omega})$ *is in* $L^1(\Omega;\delta)$.

Then for all $\phi\in C^2(\overline{\Omega})$, $\phi=0$ *on* $\partial\Omega$, $\phi\geqslant 0$ *one has*

1. $\displaystyle\int_\Omega \overline{\omega}_+\mathrm{L}^*\phi dx \leqslant \int_\Omega \phi\,\mathrm{sign}_+(\overline{\omega})\mathrm{L}(\overline{\omega})dx$,
2. $\displaystyle\int_\Omega |\omega|\mathrm{L}^*\phi dx \leqslant \int_\Omega \phi\,\mathrm{sign}(\overline{\omega})\mathrm{L}(\overline{\omega})dx$,

where $\mathrm{L}^*\phi=-\Delta\phi-\vec{u}\cdot\nabla\phi=-\Delta\phi-\operatorname{div}(\vec{u}\phi)$.

Proof. Let $\phi\geqslant 0$ be in $C^2(\overline{\Omega})$ with $\phi=0$ on $\partial\Omega$. Then according to Lemma 4.4, we have a sequence $\phi_n\in C^2_c(\Omega)$, $\phi\geqslant 0$, such that

$$\begin{cases}\delta^r\Delta\phi_n\to\delta^r\Delta\phi & \text{in } \ C(\overline{\Omega}) \text{ for } r>1,\\ \delta^r\nabla\phi_n\to\delta^r\nabla\phi & \text{in } \ C(\overline{\Omega})^N,\ ||\nabla\phi_n||_\infty\leqslant c||\nabla\phi||_\infty.\end{cases}$$

Therefore

$$\lim_{n\to+\infty}\int_\Omega\overline{\omega}_+\Delta\phi_n dx=\lim_{n\to+\infty}\int_\Omega\overline{\omega}_+\delta^{-r}\cdot\delta^r\Delta\phi_n dx=\int_\Omega\overline{\omega}_+\Delta\phi dx,$$

since $\overline{\omega}_+\in L^1(\Omega;\delta^{-r})$ and $r>1$.

By the Lebesgue dominated convergence theorem, one has

$$\lim_{n\to+\infty}\int_\Omega \vec{u}\cdot\nabla\phi_n\varpi_+dx = \int_\Omega \vec{u}\cdot\nabla\phi\varpi_+dx, \text{ since } \vec{u}\cdot\varpi_+\in L^1(\Omega)^N.$$

Therefore

$$\begin{aligned}\int_\Omega \varpi_+\mathrm{L}^*\phi dx &= \lim_{n\to\infty}\int_\Omega \varpi_+\mathrm{L}^*\phi_n dx \leqslant \lim_{n\to+\infty}\int_\Omega \phi_n \operatorname{sign}_+\varpi \operatorname{sign}_+\mathrm{L}\varpi dx\\ &= \int_\Omega \phi \operatorname{sign}_+\varpi\mathrm{L}\varpi \ (\text{since } \frac{|\phi_n|}{\delta}\leqslant ||\nabla\phi_n||_\infty\leqslant c||\nabla\phi||_\infty).\end{aligned}$$

Now we come to the proof of the uniqueness result stated in Theorem 4.2.

Proof of Theorem 4.2. Since the v.w.s. ω satisfies $V\omega\in L^1(\Omega;\delta)$, so if $V\geqslant c\,\delta^{-r}$, for $r>2$, we have in a neighborhood U of $\partial\Omega$

$$\int_\Omega |\omega|\delta^{-(r-1)}dx\leqslant c\int_U V|\omega|\delta dx + c_1\int_\Omega|\omega|dx<+\infty.$$

Thus $\omega\in L^1(\Omega;\delta^{-\widetilde{r}})$ with $\widetilde{r}=r-1>1$ for $r>2$.

If $\omega_1,\ \omega_2$ are two v.w.s. then $\omega=\omega_1-\omega_2$

$$\mathrm{L}\omega=\mathrm{L}(\omega_1-\omega_2)=-\Delta\omega+\operatorname{div}(\vec{u}\omega)=-V\omega\in L^1(\Omega;\delta).$$

We deduce from the Corollary 4 of Theorem 4.4 that $\forall\,\phi\geqslant0,\ \phi\in C^2(\overline{\Omega}),\ \phi=0$ on $\partial\Omega$

$$\int_\Omega|\omega|\mathrm{L}^*\phi dx\leqslant-\int_\Omega\phi\operatorname{sign}(\omega)V\omega dx=-\int_\Omega\phi\, V|\omega|dx\leqslant0.$$

For $\vec{u}\in L^{p,1}(\Omega)^N$, ($p\geqslant N$ as in the statement of Theorem 4.2) let us consider $\phi_0\in H^1_0(\Omega)$ solution of

$$\mathrm{L}^*\phi_0=-\Delta\phi_0-\vec{u}\,\nabla\phi_0=1.$$

Then $\phi_0\geqslant0,\ \phi_0\in W^2L^{p,1}(\Omega)$ according to the above regularity result, (see Propositions 11 or 12) and ϕ_0 can be approximated by a sequence $\phi_{0j}\in C^2(\overline{\Omega}),\ \phi_{0j}\geqslant0$, $\phi_{0j}=0$ on $\partial\Omega$ satisfying

$$\mathrm{L}^*_j\phi_{0j}=-\Delta\phi_{0j}-\vec{u_j}\cdot\nabla\phi_{0j}=1,\ \vec{u}_j\to\vec{u}\text{ in }L^{p,1},\ \vec{u}_j\in\mathcal{V},$$

so that

$$||\phi_{0j}||_{W^2L^{p,1}}\leqslant c.$$

Indeed, we may assume that ϕ_{0j} converges weakly to a function $\overline{\phi}_0$ in $W^2L^{p,1}(\Omega)$,

$$\nabla\phi_{0j}(x)\to\nabla\overline{\phi}_0(x)\text{ and }\phi_{0j}(x)\to\overline{\phi}_0(x)\ a.e.\ x\in\Omega.$$

Since

$$\int_\Omega|\omega|\,|\vec{u_j}-\vec{u}|\leqslant||\vec{u_j}-\vec{u}||_{L^{N,1}}|\omega|_{L^{N',\infty}},$$

and

$$||\nabla\phi_{0j}||_\infty\leqslant c,$$

we deduce that

$$\lim_{j\to+\infty}\int_\Omega|\omega|\vec{u}_j\cdot\nabla\phi_{0j}=\int_\Omega|\omega|\vec{u}\cdot\nabla\overline{\phi}_0dx.$$

Thus

$$\mathrm{L}^*\overline{\phi}_0=1,\quad\overline{\phi}_0\in W^2L^{N,1}(\Omega)\cap H^1_0(\Omega).$$

By uniqueness $\overline{\phi}_0 = \phi_0$ and then $\mathrm{L}_j^*\phi_{0j} \rightharpoonup \mathrm{L}^*\phi_0$ weakly in $L^{N,1}$. Since, we have

$$\int_\Omega |\omega| \mathrm{L}^*\phi_{0j} dx \leqslant 0.$$

$$0 \leqslant \int_\Omega |\omega| dx = \int_\Omega |\omega| \mathrm{L}_j^*\phi_{0j} dx \leqslant \int_\Omega |\omega| (\mathrm{L}_j^*\phi_{0j} - \mathrm{L}^*\phi_{0j}) dx \xrightarrow[j \to +\infty]{} 0,$$

we arrive to $\omega = 0$. □

Remark 7. In Theorem 4.3 and Theorem 4.4, if $\vec{u} \equiv 0$ (or $\vec{u} \in C^1(\overline{\Omega})^N$) then we can weaken the conditions on ϖ reducing it to ϖ belongs to $L^1(\Omega; \delta^{-r})$, $r > 1$. Then the above conclusions hold true.

Remark 8. In fact, in Corollary 3, we can state that the unique solution of (1) (without any indication of the boundary condition) **must** satisfy that $\omega = 0$ on $\partial\Omega$ at least if ω is differentiable. Indeed, a consequence of Lemma 7 we have

$$L^1(\Omega, \delta^{-r}) \cap W^1 L^{p,q}(\Omega) = W_0^1 L^{p,q}(\Omega) \text{ if } r > 1 \ (1 \leqslant p, q \leqslant +\infty).$$

Remark 9. There is a large amount of works in the literature in which the uniqueness of solutions of suitable elliptic problems is established without indicating any boundary condition but these previous papers deal with degenerate elliptic operators (see, e.g. [3], [4], [21] and the references therein). We point out that the main reason to get this type of results in our case (in which the diffusion operator is the simplest one and is not degenerate) is the presence of a very singular coefficient of the zero order term (the potential $V(x)$) which is "pathological" since it is more singular on the boundary of the domain than what the Hardy inequality may allow.

4.3. **Boundedness in $L^{N'}(\Omega)$ of the v.w.s., regularity and blow-up in absence of any potential ($V = 0$).** Since the very weak solutions found in Theorem 4.1 needs not be in $L^1(\Omega)$ our main goal now (assuming $V \equiv 0$) is to analyze under which conditions ω is globally integrable. We have

Theorem 4.5. (Integrability in $L^{N'}(\Omega)$.) *Let f be in $L^1\Big(\Omega; \delta\big(1+|\mathrm{Log}\,\delta|\big)^{\frac{1}{N'}}\Big)$, $\dfrac{1}{N} + \dfrac{1}{N'} = 1$, $V = 0$, $\vec{u} \in (L^N(\mathrm{Log}\,L)^{\frac{\beta}{N}})^N$, with $\beta > N - 1$, $\mathrm{div}\,(\vec{u}) = 0$ in Ω and $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega$. Then the unique very weak solution ω of equation (1) belongs to $L^{N'}(\Omega)$.*

We recall the

Lemma 4.5. (see [37]) *Let Ω be a bounded open Lipschitz set and $\alpha > 0$. Then, there exists a constant $c_\alpha(\Omega) > 0$ such that $\forall\, \phi \in W_0^1 L^\alpha_{exp}(\Omega)$*

$$|\phi(x)| \leqslant c_\alpha(\Omega)\delta(x)(1 + |\mathrm{Log}\,\delta(x)|)^\alpha ||\nabla\phi||_{L^\alpha_{exp}(\Omega)}.$$

Proof of Theorem 4.5 (boundedness in $L^{N'}(\Omega)$). Let ω be the very weak solution found in Theorem 4.1 and assume that

$$f \in L^1\big(\Omega; \delta(1 + |\mathrm{Log}\,\delta|)^{\frac{1}{N'}}\big).$$

We know that there exists a sequence $\vec{u}_j \in \mathcal{V}$ such that the corresponding sequence $(\omega_j)_j$ satisfying relation (58) verifies $\omega_j \rightharpoonup \omega$ weak-* in $L^{N',\infty}$ and that $\forall\, \phi \in H_0^1 \cap W^2 L^N(\Omega)$

$$\int_\Omega \omega_j \big[-\Delta\phi - \vec{u_j}\nabla\phi\big] dx = \int_\Omega f\phi dx. \tag{81}$$

Here $\vec{u}_j$ converges in $(L^N(\operatorname{Log} L)^{\frac{\beta}{N}})^N = \Lambda$ to $\vec{u}$ strongly where $\beta > N-1$. Let $g \in L^N(\Omega)$and let ϕ_j be the solution of

$$\phi_j \in W^2L^N(\Omega) \text{ such that } -\Delta\phi_j - \vec{u}_j\nabla\phi_j = g \text{ in } \Omega,\ \phi_j = 0 \text{ on } \partial\Omega.$$

Then according to Theorem 3.1, we have

$$||\phi_j||_{W^2L^N(\Omega)} \leqslant K_\varepsilon \frac{1+||\vec{u}_j||_\Lambda}{1-\varepsilon||\vec{u}_j||_\Lambda}||g||_{L^N(\Omega)},$$

with

$$\varepsilon\sup_j ||\vec{u}_j||_\Lambda \leqslant \frac{1}{2}, \text{ for some } \varepsilon > 0.$$

Thus

$$||\phi_j||_{W^2L^N(\Omega)} \leqslant K(\Omega)||g||_{L^N(\Omega)}. \tag{82}$$

By the Trudinger's type inclusion (see Lemma 3.1)

$$||\nabla\phi_j||_{L_{exp}^{\frac{1}{N'}}} \leqslant K_{10}||\phi_j||_{W^2L^N(\Omega)} \leqslant K_{11}||g||_{L^N(\Omega)}. \tag{83}$$

Therefore, considering equation (81), we have

$$\int_\Omega \omega_j g dx = \int_\Omega f\phi_j dx, \tag{84}$$

with the help of Lemma 4.5 with $\alpha = \dfrac{1}{N'}$ and estimate (83), this relation gives:

$$\int_\Omega \omega_j g dx \leqslant K_{12}||g||_{L^N}\int_\Omega |f|\delta(x)(1+|\operatorname{Log}\delta(x)|)^{\frac{1}{N'}}dx. \tag{85}$$

Hence

$$\sup_{||g||_{L^N}=1}\int_\Omega \omega_j g dx \leqslant K_{12}\int_\Omega |f|\delta(x)(1+|\operatorname{Log}\delta(x)|)^{\frac{1}{N'}}dx, \tag{86}$$

which shows that :

$$||\omega||_{L^{N'}(\Omega)} \leqslant K_{12}\int_\Omega |f|\delta(x)(1+|\operatorname{Log}\delta(x)|)^{\frac{1}{N'}}dx, \tag{87}$$

proving the result. □

For the case $V \equiv 0$, we can always obtain the $W^{1,q}(\Omega)$-regularity, for $q \geqslant 1$, provided some integrability on f but also on $\vec{u}$. Here is a first result in that direction :

Theorem 4.6. *Let f be in $L^1(\Omega;\delta(1+|\operatorname{Log}\delta|))$, $V=0$, and $\vec{u}$ in $\mathrm{bmo}_r(\Omega)^N$. Then, the very weak solution found in Theorem 4.1 belongs to $W_0^{1,1}(\Omega)$.*

Proof. As before we consider the approximating problem (57) with $\vec{u}_j = \vec{u}$, say

$$\begin{cases} -\Delta\omega_j + \vec{u}\cdot\nabla\omega_j = f_j & \text{in} \quad \Omega, \\ \omega_j \in H_0^1(\Omega)\cap W^2L^{p,1}(\Omega) & \forall p < +\infty. \end{cases}$$

Thus, taking $\phi \in W_0^1\mathrm{bmo}_r(\Omega)$ we have

$$\begin{aligned}&\int_\Omega \nabla\omega_j\cdot\nabla\phi\, dx + \int_\Omega \vec{u}\cdot\nabla\omega_j\phi\, dx \\ &= \int_\Omega f_j\phi dx \Longleftrightarrow \int_\Omega \left[\nabla\omega_j\cdot\nabla\phi - \vec{u}\cdot\nabla\phi\,\omega_j\right]dx = \int_\Omega f_j\phi\, dx.\end{aligned}$$

Let $F_j = \dfrac{\nabla\omega_j}{|\nabla\omega_j|}$ if $\nabla\omega_j \neq 0$, and 0 otherwise, $F_j \in L^\infty(\Omega)^N$, $||F_j||_\infty \leqslant 1$. According to Proposition 10, there exists a function $\phi_j \in W_0^1\mathrm{bmo}_r(\Omega)$ such that

$$-\Delta\phi_j - \vec{u}\nabla\phi_j = -\mathrm{div}\,(F_j), \text{ and } ||\phi_j||_{W_0^1L^q} \leqslant c_9||F_j||_{L^q} \leqslant c_q < +\infty \;\forall\, q > 1,$$

$$\Longleftrightarrow \int_\Omega \nabla\phi_j\nabla\varphi\,dx - \int_\Omega \vec{u}\nabla\phi_j\varphi\,dx = \int_\Omega F_j\nabla\varphi\,dx \quad \forall\,\varphi \in H_0^1(\Omega).$$

Choosing $\varphi = \omega_j$, we have

$$\int_\Omega |\nabla\omega_j|dx = \int_\Omega \nabla\phi_j\cdot\nabla\omega_j\,dx - \int_\Omega \vec{u}\cdot\nabla\phi_j\omega_j\,dx = \int_\Omega f_j\phi_j\,dx. \tag{88}$$

From Lemma 4.5, and by the John-Nirenberg inequality (see [47]) we have :

$$\begin{aligned}|\phi_j(x)| &\leqslant c(\Omega)\delta(x)(1 + |\mathrm{Log}\,\delta(x)|)||\nabla\phi_j||_{L_{exp}} \\ &\leqslant c(\Omega)\delta(x)(1 + |\mathrm{Log}\,\delta(x)|)||\nabla\phi_j||_{\mathrm{bmo}_r(\Omega)}.\end{aligned} \tag{89}$$

We recall that

$$||\nabla\phi_j||_{\mathrm{bmo}_r} \leqslant K(||F_j||_\infty + ||\vec{u}\phi_j||_{\mathrm{bmo}_r}) \leqslant c, \tag{90}$$

since $\phi_j \to \phi$ strongly in $C^{0,\alpha}(\overline{\Omega})$ (see Proposition 10).

Combining (88) to (90), we have

$$\int_\Omega |\nabla\omega_j|dx \leqslant c\int_\Omega |f_j|\delta(x)(1 + |\mathrm{Log}\,\delta|)dx \leqslant K\int_\Omega |f|\delta(1 + \mathrm{Log}\,\delta|)dx; \tag{91}$$

using also the fact that

$$\begin{cases}\omega_j \to \omega \text{ strongly } \text{ in } L^q(\Omega)\ q < N', \\ \omega_j \rightharpoonup \omega \text{ weakly in } W^{1,q}_{\mathrm{loc}}(\Omega)\ 1 < q < 1 + \dfrac{1}{N},\end{cases}$$

we deduce that :

$$\int_\Omega |\nabla\omega|dx \leqslant c\int_\Omega |f|\delta(1 + |\mathrm{Log}\,\delta|)dx.$$

□

Let us prove that if we enhance the integrability condition on f to $f \in L^1(\Omega, \delta^\alpha)$ for some $\alpha \in]\,0, 1\,[$ then we can weaken the condition on $\vec{u}$ to $\vec{u} \in L^{\frac{N}{1-\alpha}}(\Omega)^N$ and in that case we have

Theorem 4.7. *Let f be in $L^1(\Omega, \delta^\alpha)$ for some $\alpha \in]\,0, 1\,[$, $V = 0$, $\vec{u} \in L^{\frac{N}{1-\alpha}}(\Omega)$ with* $\mathrm{div}\,(\vec{u}) = 0$, $\vec{u}\cdot\vec{n} = 0$ *on $\partial\Omega$. Then, the very weak solution ω found in Theorem 4.1 belongs to $W_0^1L^{\frac{N}{N-1+\alpha}}(\Omega)$. Moreover, there exists a constant $K(\alpha;\Omega) > 0$ such that*

$$||\omega||_{W_0^1L^{\frac{N}{N-1+\alpha}}(\Omega)} \leqslant K(\alpha;\Omega)\big(1 + ||\vec{u}||_{L^{\frac{N}{1-\alpha}}}\big)||f||_{L^1(\Omega,\delta^\alpha)}.$$

The proof of Theorem 4.7 relies on the following result, dual of Proposition 8.

Proposition 15. *Let $\vec{u} \in L^{p,q}(\Omega)$, $p > N, q \in [1, +\infty]$, $V = 0$, and $F \in L^{p',q'}(\Omega)^N$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1 = \dfrac{1}{q} + \dfrac{1}{q'}$. Then there exists $\varpi \in W_0^1L^{p',q'}(\Omega)$ such that*

$$-\Delta\varpi + \vec{u}\cdot\nabla\varpi = -\mathrm{div}\,(F), \tag{92}$$

which is equivalent to

$$a(\varpi;\phi) = \int_\Omega \nabla\varpi\cdot\nabla\phi\,dx + \int_\Omega \vec{u}\cdot\nabla\varpi\phi\,dx = \int_\Omega F\cdot\nabla\phi\,dx \tag{93}$$

$\forall\, \phi \in W_0^1 L^{p,q}(\Omega)$. *Moreover*

$$||\nabla \omega||_{L^{p',q'}} \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F||_{L^{p',q'}}$$

Proof. Let G be in $L^{p,q}(\Omega)^N$, $p > N$. Following Proposition 8, there exists a function $\phi_0 \in W_0^1 L^{p,q}(\Omega)$ such that

$$\int_\Omega \nabla \phi_0 \cdot \nabla \varphi\, fx - \int_\Omega \vec{u} \cdot \nabla \phi_0 \varphi\, dx = \int_\Omega G \cdot \nabla \varphi\, dx \qquad \forall\, \varphi \in C_c^\infty(\Omega).$$

Since

$$-\int_\Omega \vec{u} \cdot \nabla \phi_0 \varphi\, dx = \int_\Omega \vec{u} \cdot \nabla \varphi \phi_0\, dx,$$

by using a density argument over the set of test functions there exists

$$a(\varphi, \phi_0) = \int_\Omega \nabla \phi_0 \cdot \nabla \varphi + \int_\Omega \vec{u} \cdot \nabla \varphi \phi_0 = \int_\Omega G \cdot \nabla \varphi\, dx \qquad \forall\, \varphi \in W_0^1 L^{p',q'}(\Omega). \tag{94}$$

Let $F_k \in L^\infty(\Omega)^N$, with $|F_k(x)| \leqslant |F(x)|$ in Ω. Then we have that $\overline{\omega}_k \in W_0^1 L^{p',q'}(\Omega) \cap H^1(\Omega)$ such that

$$a(\overline{\omega}_k, \phi) = \int_\Omega F_k \cdot \nabla \phi\, dx \qquad \forall \phi \in W_0^1 L^{p,q}(\Omega). \tag{95}$$

Choosing $\phi = \phi_0$ in this last equation, we find that

$$\int_\Omega G \cdot \nabla \overline{\omega}_k\, dx = a(\overline{\omega}_k, \phi_0) = \int_\Omega F_k \cdot \nabla \phi_0\, dx. \tag{96}$$

Following Proposition 8, we have

$$||\nabla \phi_0||_{L^{p,q}} \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||G||_{L^{p,q}}. \tag{97}$$

From relation (96) and (97), we have

$$\int_\Omega G \cdot \nabla \overline{\omega}_k\, dx \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F_k||_{L^{p',q'}}||G||_{L^{p,q}}. \tag{98}$$

So that we have

$$\sup_{||G||_{L^{p,q}}=1} \int_\Omega G \cdot \nabla \overline{\omega}_k dx \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F||_{L^{p',q'}} \tag{99}$$

$$||\nabla \overline{\omega_k}||_{L^{p',q'}} \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F||_{L^{p',q'}}. \tag{100}$$

By standard argument, we derive the existence of $\overline{\omega}$ satisfying (92) as a weak limit of $\overline{\omega}_k$ in $W^1 L^{p',q'}(\Omega)$. □

Proof of Theorem 4.7. Since $f \in L^1(\Omega; \delta^\alpha)$, according to [18], there exists $F = \nabla v \in L^{\frac{N}{N-1+\alpha}}(\Omega)^N$, $f = -\mathrm{div}\,(F)$. Moreover, the function $F_k = \nabla v_k$ satisfying $-\Delta v_k = T_k(f)$ converge to F strongly in $L^{\frac{N}{N-1+\alpha}}(\Omega)^N$ (v_k and v are in $W_0^1 L^{\frac{N}{N-1+\alpha}}(\Omega)$).

Since the very weak solution ω found in Theorem 4.1 is the weak-* limit of the solutions of the regularized problem

$$\begin{cases} -\Delta \omega_k + \vec{u} \cdot \nabla \omega_k = f_k = T_k(f) = -\mathrm{div}\,(F_k), \\ \omega_k \in W^2 L^q(\Omega) \cap H_0^1(\Omega) \text{ with } q = \dfrac{N}{1-\alpha} > N, \end{cases}$$

and

$$||\nabla \omega_k||_{L^{q'}(\Omega)} \leqslant K_q(1 + ||\vec{u}||_{L^q})||F_k||_{L^{q'}(\Omega)}, \quad q' = \frac{N}{N-1+\alpha},$$

letting $k \to +\infty$, we derive the result once we know that $||F||_{L^{q'}} \leqslant c||f||_{L^1(\Omega,\delta^\alpha)}$. $\square$

When $\alpha = 0$, that is $f \in L^1(\Omega)$, we can weaken the integrability assumption on $\vec{u}$ as we state in the following result :

Theorem 4.8. *Let f be in $L^1(\Omega)$, $V = 0$, $\vec{u} \in L^N(\Omega)^N$ with $\operatorname{div}(\vec{u}) = 0$ on $\partial\Omega$, $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Then, the very weak solution ω found in Theorem 4.1 belongs to $W_0^1 L^{N',\infty}(\Omega)$.*

Moreover, there exists a constant $c(\Omega) > 0$, independent of $\vec{u}$, such that

$$||\nabla\omega||_{L^{N',\infty}(\Omega)} \leqslant c(\Omega)||f||_{L^1(\Omega)}.$$

Proof. Let $\vec{u}_j \in \mathcal{V}$ be such that $\vec{u}_j \to \vec{u}$ in $L^N(\Omega)^N$, and let $f_j \in L^\infty(\Omega)$ be such that $|f_j(x) \leqslant |f(x)|$ and $f_j(x) \to f(x)$ a.e, $x \in \Omega$.

Let us consider the functions $\omega_j \in W^2L^m(\Omega) \cap H_0^1(\Omega)$ $\forall\, m < +\infty$ satisfying

$$-\Delta\omega_j + \vec{u}_j \cdot \nabla\omega_j = f_j.$$

Then

$$\int_\Omega |\nabla T_k(\omega_j)|^2 dx + \int_\Omega \vec{u}_j \cdot \nabla \int_0^{\omega_j} T_k(\sigma)d\sigma = \int_\Omega T_k(\omega_j)f_j(x)dx,$$

and since by integration by parts we have $\displaystyle\int_\Omega \vec{u}_j \cdot \nabla \int_0^{\omega_j} T_k(\sigma)d\sigma = 0$ we get

$$\int_\Omega |\nabla T_k(\omega_j)|^2 dx \leqslant k \int_\Omega |f(x)|dx. \tag{101}$$

By the Poincaré-Sobolev inequality

$$\int_\Omega |T_k(\omega_j)|^2 dx \leqslant c_\Omega k \int_\Omega |f(x)|dx.$$

By Proposition 13, we deduce that

$$||\nabla\omega_j||_{L^{N',\infty}(\Omega)} \leqslant c_\Omega \int_\Omega |f(x)|dx.$$

Since $\vec{u}_j \to \vec{u}$ in $L^N(\Omega)^N$ and by compactness $\omega_j \to \omega$ in $L^{N'}(\Omega)$ $\Big($note that $W^1L^{N',\infty}(\Omega) \hookrightarrow L^{\frac{N}{N-2},\infty}(\Omega)$ for $N \geqslant 3$ (see [35])$\Big)$, we then have for all $\phi \in C^2(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$,

$$\int_\Omega \omega_j \vec{u}_j \nabla\phi dx \xrightarrow[j\to+\infty]{} \int_\Omega \omega\vec{u}\cdot\nabla\phi dx,$$

so that ω solves (16) for $V \equiv 0$. $\square$

As for the case $\vec{u} = 0$, the additional regularity questions are numerous; for instance, does there exists a datum $f \in L^1(\Omega;\delta)$ for which we have

$$\int_\Omega |\nabla\omega|dx = +\infty \text{ or } \int_\Omega |\omega|^{N'} dx = +\infty?$$

For the explosion of the norm of ω in $L^{N'}$, we can adopt the same proof as for the explosion of the gradient in $L^1(\Omega)$. We have

Theorem 4.9. **(blow-up in $L^{N'}(\Omega)$)** *Assume that $N \geqslant 3$, $\vec{u} \in C^{0,\alpha}(\overline{\Omega})^N$, $\alpha > 0$, $V = 0$. Then there exists a function f in $L^1_+(\Omega;\delta)\backslash L^1(\Omega, \delta(1+|\mathrm{Log}\,\delta|)^{\frac{1}{N'}})$ such that the very weak solution ω found in Theorem 4.1 satisfies that ω does not belong to $L^{N'}(\Omega)$).*

First we recall the following result that can be proved as in [39] (see also [40]).

Lemma 4.6. *Let $N \geqslant 3$. There exists a function $g \in L^N_+(\Omega)$ such that the unique solution $\psi \in W^2L^N(\Omega) \cap H^1_0(\Omega)$ of $-\Delta\psi - \vec{u}\cdot\nabla\psi = g$ satisfies :*

1. $\psi(x) \geqslant c_1\delta(x),\ \forall x,$
2. $\sup\left\{\dfrac{\psi(x)}{\delta(x)},\ x \in \Omega\right\} = +\infty,$
3. $L^1_+(\Omega;\delta)\backslash L^1(\Omega,\psi)$ *is non empty.*

Arguing as in [39], [1], we consider $g_k = T_k(g)$, g given by Lemma 4.6 such that

$$\psi_k \in W^2L^q(\Omega) \cap H^1_0(\Omega) \text{ for all } q < +\infty,\ -\Delta\psi_k - \vec{u}\nabla\phi_k = T_k(g).$$

Now assume that for all $f \in L^1(\Omega;\delta)$, we have for the v.w.s. $||\omega||_{L^{N'}} < +\infty$. Then by the Banach-Steinhauss uniform boundedness theorem as in [1, 39], we derive the existence of a constant $c_0 > 0$ such that

$$||\omega||_{L^{N'}} \leqslant c_0 \int_\Omega |f|\delta dx \quad \forall\, f \in L^1(\Omega;\delta),$$

and

$$\int_\Omega \omega\big[-\Delta\phi - \vec{u}\cdot\nabla\phi\big]\,dx = \int_\Omega f\phi\,dx \qquad \forall\,\phi \in W^2L^{N,1}(\Omega) \cap H^1_0(\Omega).$$

Taking $\phi = \psi_k$, and $f \in L^1_+(\Omega;\delta)\backslash L^1_+(\Omega,\psi)$ we see that

$$0 \leqslant \int_\Omega f\psi_k = \int_\Omega \omega g_k\,dx \leqslant ||\omega||_{L^{N'}}||g||_{L^N} < +\infty. \tag{102}$$

Letting $k \to +\infty$, we have a contradiction since

$$\underline{\lim}_{k\to+\infty}\int_\Omega f\psi_k \geqslant \int_\Omega f\psi\,dx = +\infty,$$

which concludes the proof Theorem 4.9. □

Remark 10. We can give the more precise information that the function f in Theorem 4.9 is not in $L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^{\frac{1}{N'}})$ (due to Theorem 4.5).

4.4. **Some final conclusion.**

In the opinion of the authors, the results of this paper open many different further applications in different directions. Besides the consideration of the list of concrete problems mentioned in the Introduction other studies can be carried out. For instance, following the arguments of [19], it is not complicated to extend many of the results of this paper to the study of semilinear problems for which equation (1) is replaced by the equation

$$-\Delta\omega + \vec{u}\cdot\nabla\omega + V\omega + \beta(x,u,\nabla u) = f(x) \text{ on } \Omega,$$

when β is nondecreasing in u. Moreover the consideration of parabolic problems of the type

$$\omega_t - \Delta\omega + \vec{u}\cdot\nabla\omega + V\omega + \beta(x,u,\nabla u) = f(t,x) \textit{ on } \Omega\times(0,T),$$

can be carried out with the help of the results of this paper (mainly the $L^1(\Omega;\delta)$-accretiveness property of the associates operator). The details will be given in some separate work by the authors.

After this article was completed, we learned, during a presentation at a conference (March 29-30, 2017) in Poitiers, France, that L. Orsina and A. Ponce have obtained related results in the references [33, 34]. Their results deal essentially with the existence and the use of the normal derivative for any function in $W^{1,1}_0(\Omega)$. In the *improved version* [34] *that they sent to us by the authors after the conference, they add a new proposition (Proposition 2.7) which provides a complement to our results since it gives a qualitative property for* ω *solution of our problem (16) if the velocity* $\vec{u}$ *is zero when the solution is integrable on the whole domain (for a right hand side* f *in* $L^1(\Omega, \delta(1+|Log\delta|))$. *We note also that J.I.Díaz has already derived results similar to their Proposition 2.7 in* [15, 16].

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