Existence and uniqueness of solutions of Schrödinger type stationary equations with very singular potentials without prescribing boundary conditions and m-accretivity of the operator

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1 Introduction

The main goal of this paper is to improve some of the results of a previous paper by the authors in collaboration with R. Temam [9] as well as the recent researches presented in [19] concerning the Schrödinger type stationary equations equation with very singular potentials

$$-\Delta\omega + \vec{u}(x)\cdot\nabla\omega + V(x)\omega = f(x) \text{ in } \Omega, \tag{1}$$

where Ω will be an open subset of of R^n and $f \in L^1(\Omega,\delta)$, with

$$\delta(x) := d(x, \partial\Omega).$$

We assume given a flux transport term $\vec{u} \in L^n(\Omega)^n$ such that

$$\begin{cases} \operatorname{div}\vec{u} = 0 & \Omega, \\ \vec{u}\cdot\vec{n} = 0 & \partial\Omega, \end{cases} \tag{2}$$

and a potential in the general class of functions satisfying $V \in L^1_{loc}(\Omega), V \geq 0$ a.e. on Ω. Our main motivation is deal with "very singular potentials" in the sense that they satisfy

$$V(x) \geq \frac{C}{\delta(x)^r} \text{ for some } r \geq 2, \text{ near } \partial\Omega. \tag{3}$$

We send the reader to [9] for considerations and references concerning the case of "moderate singular" potentials corresponding to $r \in (0,2)$. Notice that our purpose, as already indicated in the title of the paper, is to prove the existence and uniqueness of a suitable class of solutions of

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(1) without prescribing any boundary condition in an explicit way although we shall demand to the solutions a certain integrability condition which implicitly assumes some behaviour on them on $\partial\Omega$: we shall enter into details later.

In our previous paper [9] we offered a set of relevant applications leading to the consideration of problem (1). In the special case of $\vec{u} = \vec{0}$ some of those motivations where: linearization of singular and /or degenerate nonlinear equations, shape optimization in Chemical Engineering and, very specially, the study of ground solutions $\boldsymbol{\psi}(t,x) = e^{-iEt}\omega(x)$ of the Schrödinger equation

$$\begin{cases} i\dfrac{\partial \boldsymbol{\psi}}{\partial t} = -\Delta\boldsymbol{\psi} + V(x)\boldsymbol{\psi} & \text{in } (0,\infty)\times\mathbb{R}^n \\ \boldsymbol{\psi}(0,x) = \boldsymbol{\psi}_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{4}$$

for singular potentials (satisfying (3)) which tray to confine the wave function of the particle in the domain Ω of R^n . A very interesting source of concrete singular potentials examples was described in the long paper [5] where only asymptotic technics were sketched for the treatment of the problems. We recall that the confinement takes place once that we prove that the solutions of (1) are, in fact, "flat solutions" (in the sense that $\omega = \frac{\partial \omega}{\partial n} = 0$ on $\partial\Omega$).

Concerning the case $\vec{u} \neq \vec{0}$ the main motivation mentioned in [9] was the study of the vorticity equation in fluid mechanics. Schroedinger equations involving also a flux term motivated by some questions in Control Theory where already considered also by several authors when proving the "unique continuation property" (see, e.g. [13] and its references). Notice that the existence of flat solutions to this equation implies the failure of the "unique continuation property" for such very singular class of potentials.

So, roughly speaking, the aim of this paper is to study the equation

$$A\omega = f \text{ in } \Omega, \tag{5a}$$

$$\omega = 0 \text{ on } \partial\Omega, \tag{5b}$$

where

$$A\omega = -\Delta\omega + \vec{u}\cdot\nabla\omega + V\omega. \tag{6}$$

2 Notations, definitions and previous results

We shall adopt the same notations as in our previous paper [9].

We set

$$L^0(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \text{ Lebesgue measurable} \Big\}$$

and we denote by $L^p(\Omega)$ the usual Lebesgue space $1 \leqslant p \leqslant +\infty$. Although it is not too standard, we shall use the notation $W^{1,p}(\Omega) = W^1L^p(\Omega)$ for the associate Sobolev space. We need the following definitions:

Definition 2.1 (of the distribution function and monotone rearrangement)**.** Let $u \in L^0(\Omega)$. The distribution function of u is the decreasing function

$$\begin{aligned} m = m_u : \mathbb{R} \quad &\to \quad [0, |\Omega|] \\ t \quad &\mapsto \quad \text{measure}\big\{x : u(x) > t\big\} = |\big\{u > t\big\}|. \end{aligned}$$

The generalized inverse u_* of m is defined by, for $s \in [0, |\Omega|[$,

$$u_*(s) = \inf\Big\{ t : |\big\{u > t\big\}| \leqslant s \Big\},$$

and is called the decreasing rearrangement of u. We shall set $\Omega_* =]0, |\Omega|\,[$.

Definition 2.2. Let $1 \leqslant p \leqslant +\infty,\ 0 < q \leqslant +\infty$:

- If $q < +\infty$, one defines the following norm for $u \in L^0(\Omega)$

$$||u||_{p,q} = ||u||_{L^{p,q}} := \left[\int_{\Omega_*} \left[t^{\frac{1}{p}} |u|_{**}(t)\right]^q \frac{dt}{t}\right]^{\frac{1}{q}} \text{ where } |u|_{**}(t) = \frac{1}{t}\int_0^t |u|_*(\sigma) d\sigma.$$

- If $q = +\infty$,

$$||u||_{p,\infty} = \sup_{0<t\leqslant|\Omega|} t^{\frac{1}{p}} |u|_{**}(t).$$

  The space

$$L^{p,q}(\Omega) = \Big\{u \in L^0(\Omega) : ||u||_{p,q} < +\infty\Big\} \tag{7}$$

  is called a **Lorentz space**.

- If $p = q = +\infty$, $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$.
- The dual of $L^{1,1}(\Omega)$ is called $L_{\exp}(\Omega)$

Remark 1. We recall that $L^{p,q}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega)$ for any $p > 1,\ q \geqslant 1$.

Definition 2.3. If X is a Banach space in $L^0(\Omega)$, we shall denote the Sobolev space associated to X by

$$W^1X = \Big\{\varphi \in L^1(\Omega) : \nabla\varphi \in X^n\Big\}$$

or more generally for $m \geqslant 1$,

$$W^mX = \Big\{\varphi \in W^1X,\ \forall\, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n,\ |\alpha| = \alpha_1 + \dots + \alpha_n \leqslant m,\ D^{|\alpha|}\varphi \in X\Big\}.$$

We also set

$$W_0^1X = W^1X \cap W_0^{1,1}(\Omega).$$

We shall often use the principal eigenvalue $\varphi_1 \in W_2$ of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

where

$$W_2 = \left\{\varphi \in C^2(\bar\Omega) : \varphi = 0 \text{ in } \partial\Omega\right\}. \tag{9}$$

We also need to recall the Hardy's inequality in $L^{n',\infty}$ (see, e.g. ...) saying that

$$\int_\Omega \frac{|\omega|}{\delta} \leq C\|\nabla\omega\|_{L^{n',\infty}} \qquad \forall\omega \in W_0^1L^{n',\infty}(\Omega). \tag{10}$$

Definition 2.4. In the weak setting, by (2) we will mean

$$\int_\Omega \varphi\nabla\phi\cdot\vec{u} = -\int_\Omega \phi\nabla\varphi\cdot\vec{u} \qquad \forall\phi,\varphi \in W_2. \tag{11}$$

In fact we will consider either of this assumptions

$$\begin{cases} V \in L^1_{loc}(\Omega), V \geq 0, \\ \vec{u} \in L^{p,1}(\Omega)^n, \text{ for some } p > n, \text{ and such that (11).} \end{cases} \tag{H$_1$}$$

or

$$\begin{cases} V \in L^1_{loc}(\Omega), V \geq 0, \\ \vec{u} \in L^{n,1}(\Omega)^n, \text{ with small norm as in [9], and such that (11).} \end{cases} \tag{H$_2$}$$

Most frequently we will assume that

$$\text{either (H}_1\text{) or (H}_2\text{) holds} \tag{H}$$

Definition 2.5. Under assumption (H), the local very weak formulation of (5a) results

$$\int_\Omega \omega(-\Delta\phi - \overrightarrow{u} \cdot \nabla\phi + V\phi) = \int_\Omega f\phi \qquad \forall \varphi \in \mathcal{C}^2_c(\Omega). \tag{12}$$

For $V \in L^1(\Omega, \delta)$, we say that ω is a very weak solution in the sense of Brezis of (5) if

$$\begin{cases} V\omega\delta \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega \omega(-\Delta\phi - u \cdot \nabla\phi + V\phi) = \int_\Omega f\phi \qquad \forall\varphi \in W_2. \end{cases} \tag{13a}$$

When V is only in $L^1_{loc}(\Omega)$, we will say that ω is a very weak distributional solution of (5) if

$$\begin{cases} V\omega\delta \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega \omega(-\Delta\phi - \vec{u} \cdot \nabla\phi + V\phi) = \int_\Omega f\phi \qquad \forall\varphi \in \mathcal{C}^2_c(\Omega). \end{cases} \tag{13b}$$

When $f \in L^1(\Omega, \delta)$ the natural setting for such solutions is

$$\omega \in L^{n',\infty}(\Omega). \tag{14}$$

In our previous paper [9] we proved that:

Theorem 2.1. *Let $f \in L^1(\Omega, \delta)$ and* (H) *hold. Then, there exists $\omega \in L^{n',\infty}(\Omega)$ such that* (13b) *holds. Furthermore if $V \in L^1(\Omega, \delta)$ then* (13a) *is satisfied.*

Moreover we also proved the following uniqueness result:

Theorem 2.2. *There exists, at most, one solution ω of* (13b) *such that $\frac{\omega}{\delta^r} \in L^1(\Omega)$, for some $r > 1$.*

One of the main aims of this paper is to show that this exponent $r > 1$ is not optimal in the conclusion ii) because, in fact, $r = 1$ suffices. That improves a remark (following different arguments) pointed out by H. Brezis to the second author concerning the case $\overrightarrow{u} = \overrightarrow{0}$ (see [12]). Moreover, we shall present here a numerous of other improvements with respect to our previous paper [9], as, for instance, the study of the associated eigenvalue problem, the consideration of flat solutions, the accretiveness in $L^1(\Omega, \delta^\alpha)$ of the operator when $\alpha \in [0, 1)$, the consideration of the associated evolution problem, the confinement for the solution of complex Schroedinger problem, etc.

3 Statement of new existence, uniqueness and regularity results

First, we show the equivalent of the Brezis and distributional formulations, in the space $L^1(\Omega, \delta^{-1})$.

Lemma 3.1 (equivalence of (13a) and (13b))**.** *Assume that* $f \in L^1(\Omega, \delta)$*,* (H) *and let* $\omega \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$*. Then* (13a) *if and only if* (13b)*.*

First we prove an existence result in $L^{n',\infty}$ with additional bounds

Theorem 3.1 (general existence result)**.** *Assume that* $f \in L^1(\Omega, \delta)$ *and* (H)*. Then there exists* $\omega \in L^{n',\infty}(\Omega)$ *such that* (13a)*. Furthermore, if* $f \geq 0$*, then* $\omega \geq 0$*. Besides*

$$\int_\Omega V|\omega|\delta \leq C_u \int_\Omega |f|\delta. \tag{15}$$

where C_u *does not depend on* V *and* f*.*

Then we will extend our uniqueness result

Theorem 3.2 (uniqueness in $L^1(\Omega, \delta^{-1})$)**.** *Assume that* $f \in L^1(\Omega, \delta)$ *and* (H)*. Then, there exists at most one* $\omega \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$ *such that* (13a)*.*

From this, several existence and uniqueness results follow:

Theorem 3.3. *Assume that* $f \in L^1(\Omega, \delta)$*,* (H) *and* $V \geq C\delta^{-2}$ *for some* $C > 0$*. Then there exists a unique* $\omega \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$ *such that* (13a)*.*

Theorem 3.4. *Assume that* $f \in L^1(\Omega)$ *and* (H)*. Then, there exists exactly one* $\omega \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$ *such that* (13a)*. Furthermore,* $\omega \in W_0^1 L^{n',\infty}(\Omega)$ *and*

$$\int_\Omega V|\omega| \leq C \int_\Omega |f|, \tag{16}$$

$$\int_\Omega V|\omega|\delta \leq c_\Omega (1 + \|\vec{u}\|_{L^{n,1}}) \int_\Omega |f|\delta, \tag{17}$$

$$\|\nabla\omega\|_{L^{n',\infty}} \leq C \int_\Omega |f| \tag{18}$$

Theorem 3.5. *Assume that* $f \in L^1(\Omega; \delta(1 + |\log\delta|))$ *and* (H_1)*. Then there exists a unique* $\omega \in L^{n',\infty}(\Omega) \cap L^1(\Omega; \delta^{-1})$ *such that* (13a)*.*

Theorem 3.6. *Let* $0 < \alpha < 1$*. Assume that* (H_1)*,* $f \in L^1(\Omega, \delta^\alpha)$ *and* $\vec{u} \in L^{\frac{n}{1-\alpha}}(\Omega)$*. Then, there exists a unique solution* $\omega \in L^{n',\infty}(\Omega) \cap L^1(\Omega; \delta^{-1})$ *of* (13a)*. Furthermore,* $\omega \in W_0^1 L^{\frac{n}{n+1+\alpha}}$ *and*

$$\int_\Omega V|\omega|\delta^\alpha \leq \int_\Omega |f|\delta^\alpha \tag{19}$$

4 Proof of the equivalence of (13a) and (13b)

The proof is based on the following lemma, which improves [9].

Lemma 4.1 (approximation of test functions in W_2)**.** *Let* $\phi \in W_2$*. Then, there exists a sequence* $\phi_j \in C_c^\infty(\Omega)$ *such that*

1. *There exists $C > 0$ such that $\|\nabla \phi_j\|_{L^\infty} \le C$ for all $j \ge 1$.*
2. *$\|\phi_j - \phi\|_{L^\infty} + \|\nabla \phi_j - \nabla \phi\|_{L^1} \to 0$.*
3. *$\delta \Delta \phi_j \rightharpoonup \delta \Delta \phi$ in L^∞-weak-$\star$.*
4. *$\frac{\phi_j}{\delta} \rightharpoonup \frac{\phi}{\delta}$ in L^∞-weak-$\star$.*

Proof. Following [9], we shall consider $h \in C^\infty(\mathbb{R})$ such that

$$h(t) = \begin{cases} 1 & \text{if } t \geqslant 2, \\ 0 & \text{if } t \leqslant 1, \end{cases}$$

for $j \in \mathbb{N}^*$ set $\varepsilon = \frac{1}{j}$ and let $h_j(x) = h\left(\frac{\delta(x)-\varepsilon}{\varepsilon}\right),\ x \in \Omega$. Setting

$$E_j = \Big\{x \in \Omega : \frac{2}{j} \leqslant \delta(x) \leqslant \frac{3}{j}\Big\}, \qquad E_j^c = \Omega \backslash E_j.$$

One has the following properties of h_j:

1. $\Delta h_j(x) = |\nabla h_j(x)| = 0$ for $x \in E_j^c$,
2. $h_j(x) \xrightarrow[j\to+\infty]{} 1$ for any $x \in \Omega$ since $h_j(x) = 1$ if $\delta(x) \geqslant \dfrac{3}{j}$,
3. $||\delta h_j - \delta||_\infty = \underset{x\in\overline{\Omega}}{\mathrm{Max}}\ |\delta(x)h_j(x) - \delta(x)| \leqslant 3(1 + ||h||_\infty)\varepsilon$,
4. on Ω, $\delta(x)|\nabla h_j(x)| \leqslant 3||h'||_\infty$ and $\delta^2(x)|\Delta h_j(x)| \leqslant c_{ih}$ is constant depending on h and Ω.

Let $\phi \in W_2$, the sequence $\varphi_j = h_j\phi$ is in $C_c^2(\Omega)$ and enjoy the following property,

$$\text{there is a constant } c > 0 \text{ such } ||\nabla\varphi_j||_\infty \leqslant c||\nabla\phi||_\infty. \tag{20}$$

Indeed

$$|\nabla\varphi_j(x)| \leqslant 3||h'||_\infty||\nabla\phi||_\infty + ||h||_\infty||\nabla\phi||_\infty.$$

Moreover, one has

$$||h_j\phi - \phi||_\infty \leqslant c\varepsilon||\nabla\phi||_\infty \tag{21}$$

and

$$\int_\Omega |\nabla\varphi_j - \nabla\phi|(x)dx \leqslant c\,\mathrm{meas}\Big\{x \in \Omega : \delta(x) \leqslant \frac{3}{j}\Big\} \xrightarrow[j\to+\infty]{} 0. \tag{22}$$

More,

$$|\delta(x)\Delta\varphi_j(x) - \delta(x)\Delta\phi(x)| \leqslant ||\delta h_j - \delta||_\infty|\Delta\phi(x)| \text{ for } x \in E_j^c. \tag{23}$$

For $x \in E_j$, we have

$$\begin{aligned} |\delta(x)\Delta\varphi_j(x) - \delta\Delta\phi(x)| \leqslant{}& ||\delta h_j - \delta||_\infty|\Delta\phi(x)| + \delta^2(x)||\nabla\phi||_\infty|\Delta h_j(x)| \\ &+ 2\delta(x)|\nabla h_j(x)|||\nabla\phi||_\infty \end{aligned} \tag{24}$$

The statements (23) and (24) are obtained with a straightforward computation. From those statements, we deduce that there is a constant $c_\phi > 0$ such that

$$||\delta\Delta\varphi_j - \delta\Delta\phi||_\infty \leqslant c_\phi. \tag{25}$$

Since

$$\text{meas}\,(E_j) \xrightarrow[j\to+\infty]{} 0 \text{ and } ||\delta h_j - \delta||_\infty \xrightarrow[j\to+\infty]{} 0.$$

We then have

$$\int_\Omega |\delta\Delta\varphi_j - \delta\Delta\phi|(x)dx \xrightarrow[j\to+\infty]{} 0. \tag{26}$$

Indeed, we have

$$\begin{aligned} \int_\Omega |\delta\Delta\varphi_j - \delta\Delta\phi|(x)dx &\leqslant \int_{E_j^c} |\delta\Delta\varphi_j - \delta\Delta\phi|(x)dx + c_\phi \text{meas}\,(E_j) \\ &\leqslant ||\delta h_j - \delta||_\infty ||\Delta\phi||_\infty + c_\phi \text{meas}\,(E_j) \xrightarrow[j\to+\infty]{} 0. \end{aligned} \tag{27}$$

One deduces from relations (25) and (26) that

$$\delta\Delta\varphi_j \text{ converges to } \delta\Delta\phi \text{ in weakly-}\star \text{ in } L^\infty(\Omega)$$

Since $C_c^\infty(\Omega)$ is dense in $C_c^2(\Omega)$, we obtain the desired result. □

With this technique we can now move the proof of the equivalence.

Proof of Lemma 3.1. Let ϕ be in W_2. Then we have a sequence $\phi_j \in C_c^\infty(\Omega)$ such that

$$\int_\Omega \omega\Big[-\Delta\phi_j + \vec{u}\cdot\nabla\phi_j + V\,\phi_j\Big]dx = \int_\Omega f\,\phi_j dx. \tag{28}$$

and with the convergence stated in Lemma 4.1. Therefore, we have

$$\lim_{j\to+\infty} \int_\Omega \omega\Delta\phi_j dx = \lim_j \int_\Omega \frac{\omega}{\delta}(\delta\Delta\phi_j)dx = \int_\Omega \omega\Delta\phi\,dx, \tag{29}$$

since $\dfrac{\omega}{\delta} \in L^1(\Omega)$ and $\delta\Delta\phi_j \underset{j}{\rightharpoonup} \delta\Delta\phi$ in $L^\infty(\Omega)$-weak-$\star$.

For the same reason, one has:

$$\lim_j \int_\Omega \omega\,\vec{u}\cdot\nabla\phi_j dx = \int_\Omega \omega\,\vec{u}\cdot\nabla\phi dx$$

since $\vec{u}\cdot\omega \in L^1$ and $\nabla\phi_j \rightharpoonup \nabla\phi$ in L^∞-weak-$\star$.

$$\lim \int_\Omega \omega\,V\phi_j dx = \int_\Omega \omega\,V\phi dx \text{ since } V\,\omega\delta \in L^1(\Omega) \text{ and } \frac{\phi_j}{\delta} \rightharpoonup \frac{\phi}{\delta} \text{ in } L^\infty(\Omega)\text{-weak-}\star.$$

We easily pass to the limit in equation (28). □

5 Proof of the existence and regularity results

We will consider the approximating sequence

$$\begin{cases} -\Delta\omega_j + \vec{u}_j\cdot\nabla\omega_j + V_j\omega_j = f_j \\ \omega_j \in W_0^{1,1}(\Omega)\cap W^2L^{p,1}(\Omega) \end{cases} \tag{30}$$

i.e.

$$\int_\Omega \omega_j(-\Delta\varphi - \vec{u}_j\cdot\nabla\varphi + V_j\varphi) = \int_\Omega f_j \qquad \forall\varphi\in W_2. \tag{31}$$

where

$$V_j(x) = \min(V(x), j), \tag{32}$$
$$f_j(x) = \operatorname{sign}(f(x))\min(|f(x)|, j) \tag{33}$$

and $\vec{u}_j \in \mathcal{C}_c^\infty(\Omega)^n$, such that (2) and

$$\vec{u}_j \to \vec{u} \text{ in } L^{p,1}(\Omega)^n. \tag{34}$$

First we recall our result in [9] about the approximation of solutions

Theorem 5.1 (existence and approximation of solutions when $f \in L^1(\Omega;\delta)$)**.** *Assume $f \in L^1(\Omega,\delta)$ and* (H)*. Then, there is a unique solution $\omega_j \in W_0^{1,1}(\Omega)\cap W^2L^{p,1}(\Omega)$ of* (31) *and there exists ω such that:*

1. *ω is a solution of* (13b)
2. *$\omega_j \to \omega$ a.e. in Ω,*
3. *$\omega_j \rightharpoonup \omega$ in $L^{n',\infty}$-weak-$\star$ and $W^{1,q}(\Omega,\delta)$-weak, for $q <$???.*
4. *$\omega_j \to \omega$ in $L^r(\Omega)$ for $r < n'$.*
5. *$\omega_j\vec{u}_j \to \omega\vec{u}$ in $L^1(\Omega)^n$*
6. *$\displaystyle\int_\Omega V_j|\omega_j|\delta dx \leqslant c\Big(1+||\vec{u}_j||_{L^{n,1}}\Big)\int_\Omega |f_j|\delta dx.$*
7. *$V_j\omega_j\delta \rightharpoonup V\omega\delta$ weakly in $L^1_{loc}(\Omega)$*

We can make some additional estimations if we restriction the set of datum f:

Proposition 5.2 (existence of solutions when $f \in L^1(\Omega)$)**.** *Assume that $f\in L^1(\Omega)$ and* (H)*. Then, the sequence ω_j satisfies*

$$\|\nabla\omega_j\|_{L^{n',\infty}} \le C\int_\Omega |f_j|, \tag{35}$$

$$\int_\Omega V_j|\omega_j| \le C\int_\Omega |f_j|. \tag{36}$$

Hence

$$\omega_j \rightharpoonup \omega \text{ in } W_0^1L^{n',\infty}(\Omega). \tag{37}$$

and the (35) *and* (36) *hold for ω, V and f.*

Proof. Let $k>0$. Then the sequence given in Theorem 5.1 satisfies

$$\int_\Omega \vec{u}_j\cdot\nabla\omega_j T_k(\omega_j)dx = 0 \text{ and } \int_\Omega V_j\omega_j T_k(\omega_j)dx \geqslant 0. \tag{38}$$

Therefore, we can use $T_k(\omega_j)$ as a test function in equation (30) and derive

$$\int_\Omega |\nabla T_k(\omega_j)|^2 dx \leqslant k\int_\Omega |f_j|dx \leqslant k\int_\Omega |f|(x)dx. \tag{39}$$

From relation (39), we deduce (see [1] or [15]) that

$$||\nabla\omega_j||_{L^{n',\infty}} \leqslant c|f|_{L^1(\Omega)}. \tag{40}$$

While to obtain relation (36), we choose as a test function for $t > 0$,

$$\Phi(t;\omega_j) = (|\omega_j| - t)_+ \operatorname{sign}(\omega_j).$$

Knowing as before that

$$\int_\Omega \vec{u}_j \cdot \nabla\omega_j \Phi(t;\omega_j)dx = 0. \tag{41}$$

One obtains from equation (30)

$$\int_{|\omega_j|>t} |\nabla\omega_j|^2 dx + \int_\Omega V_j\omega_j\Phi(t;\omega_j)dx = \int_\Omega f_j\Phi(t,\omega_j)dx. \tag{42}$$

We derive with respect with respect to t this equation

$$-\frac{d}{dt}\int_{|\omega_j|>t} |\nabla\omega_j|^2 dx + \int_{|\omega_j|>t} V_j|\omega_j|dx = \int_{|\omega_j|>t} f(x)\operatorname{sign}(\omega_j)dx \tag{43}$$

Since the first term is non negative, we derive from relation (43) that for all $t > 0$

$$\int_{|\omega_j|>t} V_j|\omega_j|dx \leqslant \int_{|\omega_j|>t} |f(x)|dx \tag{44}$$

letting $t \to 0$, we get the desired relation (36). Since $V_j\omega_j \to V\omega$ e.e in Ω, The Fatou's lemma yields

$$\int_\Omega V|\omega|dx \leqslant \int_\Omega |f(x)|dx. \tag{45}$$

Since $\nabla\omega_j \rightharpoonup \nabla\omega$ in $L^{n',\infty}$-weak-$\star$, we derive

$$||\nabla\omega||_{L^{n',\infty}} \leqslant c|f|_{L^1(\Omega)} \tag{46}$$

and the pointwise convergence $(V_j\omega_j)(x) \to (V\omega)(x)$ a.e with Fatou's lemma implies

$$\int_\Omega V|\omega|dx \leqslant \int_\Omega |f|(x)dx \tag{47}$$

That (13a) is satisfied is a consequence of Lemma 3.1, since by the Hardy's inequality we have

$$\left|\frac{\omega}{\delta}\right|_{L^1(\Omega)} \leqslant c||\nabla\omega||_{L^{n',\infty}} < +\infty. \tag{48}$$

This concludes the proof. □

With this we proceed

Proof of Theorem 3.4. According to Proposition 5.2, the sequence ω_j is in a bounded set of $W_0^1L^{n',\infty}(\Omega)$ and since the sequence converges to a solution ω of the equation (13b) given in Theorem 2.1, we deduce that this solution ω is in $W_0^1L^{n',\infty}(\Omega)$ and satisfies the same kind of estimates as ω_j. Moreover, $\frac{\omega}{\delta} \in L^1(\Omega)$ according to relation (48). Now we may appeal Theorem 3.2 to conclude that ω is unique. □

Finally we can prove

Proof of Theorem 3.1. Let f be in $L^1(\Omega;\delta)$ and consider $f_j = \operatorname{sign}\big(f(\cdot)\big)\min\big(|f|;j\big)$, $j\geqslant 0$. Then according to the above result Theorem 3.4, there exists a unique $\widetilde{\omega}_j \in W_0^1L^{n',\infty}(\Omega)$ satisfying

$$\int_\Omega \widetilde{\omega}_j\big[-\Delta\phi-\vec{u}\cdot\nabla\phi+V\,\phi\big]dx=\int_\Omega f_j\phi dx,\quad \forall\,\phi\in W_2. \tag{$(2)_j$}$$

Since $f_j-f_k\in L^1(\Omega)$ for k and j in $\mathbb{N}$, by the same corollary 1 of Theorem 3.2 and Theorem 5.2, we deduce that $\widetilde{\omega}_j-\widetilde{\omega}_k$ is the unique solution of $(2)_j-(2)_k$, then it satisfies

$$\int_\Omega V|\widetilde{\omega}_j-\widetilde{\omega}_k|\delta\,dx\leqslant c_u\int_\Omega|f_j-f_k|\delta\,dx$$

and

$$||\widetilde{\omega}_j-\widetilde{\omega}_k||_{L^{n',\infty}}\leqslant c_u\int_\Omega|f_j-f_k|\delta\,dx. \tag{49}$$

Thus $(\widetilde{\omega}_j)_j$ is a Cauchy sequence in $L^{n',\infty}(\Omega)$ and $(V\,\widetilde{\omega}_j)_j$ is also a Cauchy one in $L^1(\Omega;\delta)$. Therefore one has easily $\widetilde{\omega}\in L^{n',\infty}(\Omega)$ with $V\,\widetilde{\omega}\in L^1(\Omega;\delta)$ such that $\widetilde{\omega}$ satisfies equation (13a). Moreover, $\int_\Omega V|\widetilde{\omega}|\delta\,dx\leqslant c\int_\Omega f\delta\,dx$ and if $f\geqslant 0$ then $f_j\geqslant 0$ therefore $\widetilde{\omega}_j\geqslant 0$ which yields that $\widetilde{\omega}\geqslant 0$. □

6 Proof of the uniqueness results

To complete the proof of the results we only need to proof the uniqueness of equations. Once we complete the proof of Theorem 3.2 the rest of the proofs will follow as a corollary. The main tool in this proof will a Kato inequality up to the boundary.

6.1 Kato's inequality

Theorem 6.1 (Variant of Kato's inequality)**.** *Let $\overline{\omega}$ be in $W^{1,1}_{loc}(\Omega)\cap L^{n',\infty}(\Omega)$ with $\dfrac{\overline{\omega}}{\delta}\in L^1(\Omega)$ and $\vec{u}\in L^{n,1}(\Omega)^n$ with $\operatorname{div}(\vec{u})=0$ in $\mathcal{D}'(\Omega)$, $\vec{u}\cdot\vec{\nu}=0$ on $\partial\Omega$.*
Assume that $L\overline{\omega}=-\Delta\,\overline{\omega}+\operatorname{div}(\vec{u}\,\overline{\omega})\in L^1(\Omega;\delta)$.
Then for all $\phi\in W_2$, $\phi\geqslant 0$ one has

1. $\displaystyle\int_\Omega \overline{\omega}_+L^*\,\phi\,dx\leqslant\int_\Omega\phi\,\operatorname{sign}_+(\overline{\omega})\,L\,\overline{\omega}\,dx$
2. $\displaystyle\int_\Omega |\overline{\omega}|L^*\,\phi\,dx\leqslant\int_\Omega\phi\,\operatorname{sign}(\overline{\omega})L\overline{\omega}\,dx,$

where $L^*\phi=-\Delta\phi-\vec{u}\cdot\nabla\phi=-\Delta\phi-\operatorname{div}(\vec{u}\,\phi)$,

$$\operatorname{sign}_+(\sigma)=\begin{cases}1 & \text{if } \sigma>0,\\ 0 & \text{otherwise},\end{cases}\qquad \text{and}\qquad \operatorname{sign}(\sigma)=\begin{cases}1 & \text{if } \sigma>0,\\ 0 & \text{if } \sigma=0,\\ -1 & \text{if } \sigma<0.\end{cases}$$

The proof of both theorem (Theorem 6.1 and Theorem 3.2 below) follow the same argument as we did in [9] (Corollary 4 Theorem 10, Theorem 8). The only difference is the use of the approximation Lemma 4.1. For convenience we sketch those proofs :

Sketch of the proof of Theorem 6.1. Let $\phi \geqslant 0$, $\phi \in W_2$. Then according Lemma 4.1 one has a sequence $\phi_j \in C_c^\infty(\Omega)$ such that 1. $\delta\Delta\phi_j \rightharpoonup \delta\Delta\phi$ in $L^\infty(\Omega)$-weak-$\star$, this implies with the hypothesis that $\dfrac{\omega_+}{\delta} \in L^1(\Omega)$ that

$$\lim_{j\to+\infty}\int_\Omega \overline{\omega}_+\Delta\phi_j dx = \int_\Omega \overline{\omega}_+\Delta\phi dx. \tag{50}$$

For the same reason

$$\lim_{j\to+\infty}\int_\Omega \vec{u}\cdot\nabla\phi_j\overline{\omega}_+ dx = \int_\Omega \vec{u}\cdot\nabla\phi\overline{\omega}_+ dx. \tag{51}$$

We conclude as in [9], knowing that the local Kato's inequality (Theorem 10 in [9]) holds true. □

One of the consequence of the Kato's inequality is the following maximum principle.

Corollary 6.2 (of Theorem 6.1)**.** *Under the same hypothesis as for Theorem 6.1, assume that* $L\overline{\omega} = f(x) - G(x;\overline{\omega}) \in L^1(\Omega;\delta)$, *with* $G : \Omega\times\mathbb{R}\to\mathbb{R}$ *a Caratheodory function (i.e for a.e* x*,* $\sigma \to G(x;\sigma)$ *is continuous, and* $x \to G(x;\sigma)$ *is measurable* $\forall\, x$*), satisfying the sign-function condition*

$$\mathrm{sign}(\sigma)G(x;\sigma) \geqslant 0 \quad \forall\,\sigma\in\mathbb{R}\ a.e\ x\in\Omega.$$

Then, if $f \leqslant 0$ *one has* $\overline{\omega} \leqslant 0$.

Proof. One has $\forall\,\phi \geqslant 0,\ \phi \in W_2$

$$\int_\Omega \overline{\omega}_+ L^*\phi\, dx \leqslant \int_\Omega \phi\ \mathrm{sign}_+(\overline{\omega})f(x)dx - \int_\Omega \phi G(x;\overline{\omega}_+)dx. \tag{52}$$

(Since $G(x;0) = 0$ and $\mathrm{sign}_+(\sigma)G(x;\sigma) = G(x;\sigma_+) \geqslant 0$).
Therefore, from this last inequality (52), knowing that

$$-\phi G(x;\overline{\omega}_+) \leqslant 0, \quad f(x)\,\mathrm{sign}_+(\overline{\omega}) \leqslant 0,$$

we deduce that

$$\forall\,\phi \geqslant 0,\ \phi\in W_2 : \int_\Omega \overline{\omega}_+ L^*\phi dx \leqslant 0 \tag{53}$$

Since $\overline{\omega} \in L^{n',\infty}(\Omega)$ and $L^*\phi = -\Delta\phi - \vec{u}\cdot\nabla\phi$ is in $L^{n,1}(\Omega)$ for $\phi \in W^2L^{n,1}(\Omega)\cap H_0^1(\Omega)$, thus, a density argument leads from equation (53),

$$\int_\Omega \overline{\omega}_+ L^*\phi dx \leqslant 0\ \forall\,\phi\in W^2L^{n,1}(\Omega)\cap H_0^1(\Omega), \quad \phi \geqslant 0. \tag{54}$$

Thus, we get:

$$\overline{\omega}_+ = 0.$$

This completes the proof. □

6.2 Proof of the uniqueness results

Proof of Theorem 3.2. Let $\overline{\omega} = \omega_1 - \omega_2$ where ω_i are in $L^{n',\infty}(\Omega)\cap L^1(\Omega;\delta^{-1})$and are two solutions of equation (**??**) (with test function $\phi \in W_2$ or $\phi \in C_c^2(\Omega)$), these formulations are equivalent if $\omega_i \in L^1(\Omega;\delta^{-1})$ (see Lemma **??**). Then

$$L\overline{\omega} = -V\overline{\omega} \in L^1(\Omega;\delta).$$

From Theorem 6.1 one has $\forall\,\phi \in W_2,\ \phi \geqslant 0$

$$\int_\Omega |\overline{\omega}| L^* \phi dx \leqslant -\int_\Omega \phi\,\mathrm{sign}(\omega) V \overline{\omega} = -\int_\Omega \phi V |\overline{\omega}| dx \leqslant 0. \tag{55}$$

As before one has :

$$\int_\Omega |\overline{\omega}| L^* \phi dx \leqslant 0 \qquad \forall\,\phi \in W^2 L^{n,1}(\Omega) \cap H_0^1(\Omega), \quad \phi \geqslant 0. \tag{56}$$

Considering $\overline{\phi}_0 \in W^2 L^{n,1}(\Omega) \cap H_0^1(\Omega),\ \overline{\phi}_0 \geqslant 0$ solution of $L^\star \overline{\phi}_0 = 1$, we deduce $\displaystyle\int_\Omega |\overline{\omega}| dx \leqslant 0$ thus $\overline{\omega} = 0$. □

Proof of Theorem 3.5. First let us assume that $f \geq 0$. Since f is nonnegative function in $L^1(\Omega;\delta)$, the existence of a solution $\omega \geqslant 0$ is a consequence of Theorem 3.1. To prove the uniqueness result, let us show that exists a $c > 0$ independent of ω, f and V such that

$$\int_\Omega \frac{\omega}{\delta} dx + \int_\Omega V\,\omega\delta(1 + |\log\delta|) dx \leqslant c \int_\Omega f(x)(1 + |\log\delta|)\delta dx \tag{57}$$

For this, we use the argument introduced in [17] by choosing as a test function

$$\phi = \varphi_1 \log(\varphi_1 + \varepsilon),\ \varepsilon > 0,$$

φ_1 the first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition.
One obtains

$$-\int_\Omega \omega\Delta(\varphi_1 \log(\varphi_1+\varepsilon))dx - \int_\Omega \vec{u}\,\omega\cdot\nabla(\varphi_1 \log(\varphi_1+\varepsilon))dx + \int_\Omega V\omega\varphi_1 \log(\varphi_1+\varepsilon)dx = \int_\Omega f\varphi_1 \log(\varphi_1+\varepsilon)dx. \tag{58}$$

We develop each term in relation (58) as we did in [17] knowing that φ_1 is equivalent to the distance function (say $\exists c_0 > 0,\ c_1 > 0,\ c_0\delta \leqslant \varphi_1 \leqslant c_1\delta$). We derive

$$\begin{aligned} \int_\Omega |\nabla\varphi_1|^2 \frac{\omega}{\varphi_1 + \varepsilon} dx - \int_\Omega V\,\omega\varphi_1 \log(\varphi_1 + \varepsilon) dx \\ \leqslant c\left[\int_\Omega \omega(x)dx + \int_\Omega f(x)(1 + |\log\delta|)\delta dx\right] \\ + c\int_\Omega ||\vec{u}||\,|\log\delta|\omega dx + c\int_\Omega ||\vec{u}||(x)\omega(x)dx. \end{aligned} \tag{59}$$

Since $\vec{u} \in L^{p,1}(\Omega),\ p > 1$ then $||\vec{u}||\log\delta \in L^{n,1}(\Omega)$ and there exists a constant $c > 0$.

$$\Big\|\,|\vec{u}|\log\delta\Big\|_{L^{n,1}} \leqslant c||\vec{u}||_{L^{p,1}(\Omega)}.$$

Therefore, we have

$$c\int_\Omega ||\vec{u}||\,|\log\delta|\omega\,dx + c\int_\Omega ||\vec{u}||(x)\omega(x)dx \leqslant c_u||\omega||_{L^{n',\infty}} \leqslant c\int_\Omega f(x)\delta(x)dx \tag{60}$$

From relations (59) and (60), we deduce

$$\int_\Omega |\nabla\varphi_1|^2 \frac{\omega}{\varphi_1 + \varepsilon} dx - \int_\Omega V\,\omega\varphi_1 \log(\varphi_1 + \varepsilon) dx \leqslant c\int_\Omega f(x)(1 + |\log\delta|)\delta dx \tag{61}$$

As in [17] we write

$$\int_\Omega V\,\omega\varphi_1|\log(\varphi_1+\varepsilon)|dx = -\int_\Omega V\,\omega\varphi_1\log(\varphi_1+\varepsilon)dx + 2\int_{\varphi_1+\varepsilon>1} V\,\omega\varphi_1\log(\varphi_1+\varepsilon)dx. \quad (62)$$

Combining these two last relations, we get

$$\int_\Omega |\nabla\varphi_1|^2\frac{\omega}{\varphi_1+\varepsilon}dx + \int_\Omega V\,\omega\varphi_1|\log(\varphi_1+\varepsilon)|dx \ \leqslant c\int_\Omega f(x)(1+|\log\delta|)\delta dx + c\int_\Omega V\,\omega\delta dx. \quad (63)$$

Noticing that in a neighborhood of the boundary $\partial\Omega\subset U\subset\overline{\Omega}$ one has $\inf\limits_{x\in U}|\nabla\varphi_1|^2(x)>0$, we derive from relation (63) the inequality (57).

Let f be in $L^1\big(\Omega;\delta(1+\log\delta|)\big)$, we decompose $f=f_+-f_-$ where $f_+,f_-\geq 0$. Due to the first part of the proof, we have ω_1 (resp. ω_2) a nonnegative solution of (13a) associated to f_+ (resp. f_-). One has according to relation (57) for $i=1,2$

$$\int_\Omega \frac{\omega_i}{\delta}dx + \int_\Omega V\,\omega_i\delta(1+|\log\delta|)dx \leqslant c\int_\Omega |f|(1+|\log\delta|)\delta dx. \quad (64)$$

By linearity we deduce that $\widetilde{\omega}=\omega_1-\omega_2$ is a solution of equation (13b) and satisfies $\frac{\widetilde{\omega}}{\delta}\in L^1(\Omega)$. We conclude with Theorem 3.2 to obtain the result.

$\square$

7 Estimates when the datum f is $L^1(\Omega;\delta^\alpha)$, $0\leqslant\alpha\leqslant 1$

Lemma 7.1. *Under the same assumptions as for Theorem 3.5, if furthermore $f\in L^1(\Omega;\delta^\alpha)$, $0\leqslant\alpha<1$ then the function $\widetilde{\omega}$ solution of equation* (13a) *verifies*

$$\int_\Omega (V|\widetilde{\omega}|\delta^\alpha)(x)dx \leqslant c_\alpha\int_\Omega |f(x)|\delta^\alpha(x)dx.$$

Proof. For $k\geqslant 0$, let us consider $V_k=\min(V;k)$ and define the linear operator T_k on $L^1(\Omega;\delta)$ by setting $T_kf=V_k\widetilde{\omega}_{kf}$, where $\widetilde{\omega}_{kf}$ is the unique solution of

$$\int_\Omega \widetilde{\omega}_{kf}\left[-\Delta\phi+\vec{u}\cdot\nabla\phi+V_k\phi\right]dx = \int_\Omega f\phi dx \qquad \forall\,\phi\in W_k. \quad (65)$$

The existence and uniqueness follows from ([9] Theorem 7).

According to Corollary 3.4 of Theorem 3.2 and Proposition 5.2. T_k maps $L^1(\Omega)$ into itself with

$$|T_kf|_{L^1(\Omega)} = \int_\Omega V_k|\widetilde{\omega}_{kf}|dx \leqslant |f|_{L^1(\Omega)}, \quad (66)$$

and T_k maps $L^1(\Omega;\delta)$ into itself with

$$|T_kf|_{L^1(\Omega;\delta)} \leqslant c(1+||\vec{u}||_{L^{n,1}})|f|_{L^1(\Omega;\delta)}. \quad (67)$$

Since $L^1(\Omega;\delta^\alpha)$ is the interpolation space in the sense of Peetre between $L^1(\Omega;\delta)$ and $L^1(\Omega)$, that is

$$L^1(\Omega,\delta^\alpha) = \Big(L^1(\Omega;\delta),L^1(\Omega)\Big)_{\alpha,1},$$

we derive from Marcinkewicz's interpolation theorem (see[?] or [15]) that T_k maps $L^1(\Omega;\delta^\alpha)$ into itself and

$$|T_k f|_{L^1(\Omega;\delta^\alpha)} \leqslant c^\alpha(1+||\vec{u}||_{L^n,1})^\alpha |f|_{L^1(\Omega,\delta^\alpha)}, \quad \forall f\in L^1(\Omega;\delta^\alpha).$$

Considering the unique solution $\widetilde{\omega}_{kj}$ for j fixed in $\mathbb{N}$, of the equation

$$\int_\Omega \widetilde{\omega}_{kj}\Big[-\Delta\phi-\vec{u}\cdot\nabla\phi+V_k\phi\Big]dx=\int_\Omega f_j\phi dx \qquad (2)_{kj}$$

where $f_j=\operatorname{sign}(f)\min|f|;j)$. Applying Theorem 5.1 with the sequence $(\widetilde{\omega}_{kj})_k$, and due to the uniqueness result we deduce that, when $k\to+\infty$, $\widetilde{\omega}_{kj}\to\widetilde{\omega}_j$ in $L^{n',\infty}(\Omega)$ and $\widetilde{\omega}_j$ is the solution of $(2)_j$. Therefore, one has

$$\int_\Omega V|\widetilde{\omega}_j|\delta^\alpha dx \leqslant \lim_{k\to+\infty}|T_kf_j|_{L^1(\Omega;\delta^\alpha)}\leqslant c_\alpha|f_j|_{L^1(\Omega;\delta^\alpha)}. \tag{68}$$

As we have shown in the proof of Theorem 3.1, $\widetilde{\omega}_j$ converges to $\widetilde{\omega}$ as $j\to+\infty$; we deduce the desired inequality. □

The proof needs the following Lemma is given in ([9], Theorem 13)

Lemma 7.2. *Let $0<\alpha<1$, $g\in L^1(\Omega;\delta^\alpha)$, $\vec{u}$ in $L^{\frac{n}{1+\alpha}}(\Omega)^n$, (2). Then, there exists a unique solution $\overline{\omega}\in L^{n',\infty}(\Omega)$ satisfying*

$$\int_\Omega \overline{\omega}\Big[-\Delta\phi-\vec{u}\cdot\nabla\phi\Big]dx=\int_\Omega g\phi dx \qquad \forall\,\phi\in W_2. \tag{69}$$

Moreover, there exists a constant $K(\alpha;\Omega)>0$ such that

$$||\overline{\omega}||_{W_0^1L^{\frac{n}{n-1+\alpha}}(\Omega)}\leqslant K(\alpha;\Omega)\left(1+||u||_{L^{\frac{n}{1-\alpha}}}\right)|g|_{L^1(\Omega;\delta^\alpha)}. \tag{70}$$

Proof of Theorem 3.6. Let ω be the unique solution (2) given by Theorem 3.5 when $f\in L^1(\Omega;\delta^\alpha)$, $0<\alpha<1$. We set $g=V\omega-f$. Then following Lemma 7.1, one has $g\in L^1(\Omega;\delta^a)$ and ω satisfies the same type equation (69). Therefore, we can apply Lemma 7.2 to conclude. □

8 Some consequences: principal eigenvalue and eigenfunction of $-\Delta+\vec{u}\cdot\nabla$ and of the operator A, the m -accretivity of A and the complex Schroedinger problem in the whole space

8.1 Principal eigenvalue and eigenfunction for $-\Delta+\vec{u}\cdot V$ and the m-accretivity of $-\Delta+\vec{u}\nabla+V$

Theorem 8.1 (Krein-Rutman's theorem)**.** *Let X be a Banach space, K a cone whose interior $\mathring{K}$ is non void, $T:X\to X$ a compact linear operator which is strongly positive, i.e $Tf>0$ if $f>0$. Then, the spectral radius of T, $r(T)>0$ and is a simple eigenvalue with an eigenvector $\psi_1\in\mathring{K}$.*

We recall the following definition of an m-accretive operator.

Definition 8.1 (m-accretive operator)**.** Let X be a Banach space. A linear unbounded operator

$$A : D(A) \subset X \to X$$

is called m-accretive if

(a) The domain $D(A)$ is dense in X : $\overline{D(A)} = X$.

(b) $\forall \widetilde{\omega} \in D(A),\ \forall \lambda > 0 \quad ||\widetilde{\omega}||_X \leqslant ||\widetilde{\omega} + \lambda A\,\widetilde{\omega}||_X$.

(c) $\forall \lambda > 0,\ \forall f \in X,\ \exists \widetilde{\omega} \in D(A) : \widetilde{\omega} + \lambda A\,\widetilde{\omega} = f : R(I + \lambda A) = X$.

Let us consider $\vec{u} \in L^{p,1}(\Omega)^n,\ p > n$ (or in $L^{n,1}(\Omega)^n$ but with a small norm as in [9]), we define a compact operator

$$T : C(\overline{\Omega}) \to W_0^1 L^{p,1}(\Omega) \hookrightarrow C(\overline{\Omega})$$

by setting

$$Tf = \omega \text{ if and only if } \begin{cases} -\Delta\omega - \vec{u}\cdot\nabla\omega = f \\ \omega \in W_0^1 L^{p,1}(\Omega), p > n \end{cases}$$

(The existence, uniqueness and regularity of ω in given in [9]).
Using the Bony's maximum principle or Stapamcchia's argument, we have for $f > 0$ the solution $\omega > 0$.
Since the positive cone $K = C_+(\overline{\Omega}) = \Big\{\varphi \in C(\overline{\Omega}) : \varphi \geqslant 0\Big\}$ has its interior $\mathring{K}$ non void, we may apply the Krein-Rutman's theorem (see Theorem 8.3) to derive the

Theorem 8.2. *There exist a real $\lambda_1 > 0$ and a positive function $\psi_1 \in W^2L^{p,1}(\Omega) \cap H_1^0(\Omega)$ such that*

$$-\Delta\psi_1 - \vec{u}\cdot\nabla\psi_1 = \lambda_1\psi_1.$$

Moreover, $L^1(\Omega;\delta) \hookrightarrow L^1(\Omega;\psi_1)$ and if $\vec{u} \in L^\infty(\Omega)^n$ then $\psi_1 \geqslant c\delta$ so that

$$L^1(\Omega;\delta) = L^1(\Omega;\psi_1).$$

Remark 2.
The fact that $L^1(\Omega;\delta) \hookrightarrow L^1(\Omega;\psi_1)$ comes from the fact

$$0 < \psi_1(x) \leqslant \delta(x)||\nabla\psi_1||_\infty \leqslant c||\psi_1||_{W^2L^{p,1}}\delta(x) < +\infty,\ x \in \Omega.$$

Next, we want to prove Theorem 8.3 concerning the m-accretivity of $A = -\Delta + \vec{u}\cdot V + V$ in the Banach space $L^1(\Omega;\delta^\alpha),\ 0 \leqslant \alpha \leqslant 1$. The argument is similar to the one given in [16].
First, we endow the space $L^1(\Omega;\delta^\alpha)$ with the following equivalent norm

$$||f||_\alpha = \int_\Omega |f|(x)\psi_1^\alpha(x)dx,$$

ψ_1 is given by Theorem 8.2.
We shall introduce the following definition

Definition 8.2. Let ϖ be in $L^1(\Omega,;\delta^\alpha)$ with $V\varpi \in L^1(\Omega;\delta^\alpha)$. We will say that $A\varpi \in L^1(\Omega;\delta^\alpha)$ if there exists a function $f \in L^1(\Omega;\delta^\alpha)$ such that $A\varpi = f$ and

$$\int_\Omega \phi f dx = \int_\Omega \varpi\Big[-\Delta\phi - \vec{u}\nabla\phi + V\phi\Big]dx, \quad \forall\,\phi \in C_c^2(\Omega). \tag{71}$$

Here, $V \geqslant 0$ locally integrable and $\vec{u}$ is as in Theorem 2.1.

So we can define the operator $A : D(A) \subset L^1(\Omega;\delta^\alpha) \to L^1(\Omega;\delta^\alpha)$, where the domain of A is

$$D(A) = \Big\{\overline{\omega} \in L^{n',\infty}(\Omega) \cap L^1(\Omega;\delta^{-1}) \cap L^1(\Omega;V\delta) : A\overline{\omega} \in L^1(\Omega;\delta^\alpha)\Big\}.$$

Therefore, we always have $C^2_c(\Omega) \subset D(A) \subset L^1(\Omega;\delta^\alpha)$ this implies that $D(A)$ is dense in $L^1(\Omega;\delta^\alpha),\ 0 \leqslant \alpha \leqslant 1$.
Moreover, one has the :

Lemma 8.1.
If $V \geqslant 0$, locally integrable, $\vec{u}$ bounded with $\operatorname{div}(\vec{u}) = 0$ and $\vec{u}\cdot\vec{\nu} = 0$ on $\partial\Omega$, $0 \leqslant \alpha < 1$, then $\forall\lambda > 0,\ \forall f \in L^1(\Omega;\delta^\alpha)$, there exists a unique function $\omega \in D(A)$ such that

$$\omega + \lambda A\,\omega = f.$$

Proof. Indeed, since $L^1(\Omega;\delta^\alpha) \subset L^1\big(\Omega;\delta(1+|\log\delta|)\big)$, we may apply Theorem 3.5 to derive that for all $\lambda > 0$ all $f \in L^1(\Omega;\delta^\alpha)$ we have a unique function $\omega \in L^{n',\infty}(\Omega)$ with $\dfrac{\omega}{\delta} \in L^1(\Omega)$, $V\omega \in L^1(\Omega;\delta^\alpha)$ and for all $\phi \in W^2L^{n,1}(\Omega) \in H^1_0(\Omega)$,

$$\int_\Omega f\phi dx = \int_\Omega \omega\big[\phi + \lambda(-\Delta\phi - \vec{u}\cdot\nabla\phi + V\phi)\big]dx. \tag{72}$$

This is equivalent to say that $\omega + \lambda A\,\omega = f$ and $\omega \in D(A)$. □

So for $0 \leqslant \alpha < 1$, it remains to show that for all $\overline{\omega} \in D(A)$, for all $\lambda > 0$

$$||\overline{\omega}||_\alpha \leqslant ||\overline{\omega} + \lambda A\,\overline{\omega}||_\alpha. \tag{73}$$

That is to say, setting $f = \overline{\omega} + \lambda A\,\overline{\omega}$,

$$\int_\Omega |\overline{\omega}|\psi_1^\alpha dx \leqslant \int_\Omega |f|\psi_1^\alpha dx. \tag{74}$$

To prove such inequality, we introduce as in [16] the

Lemma 8.2. *Let $\varepsilon > 0$, $\psi_{1\varepsilon} = (\psi+\varepsilon)^\alpha - \varepsilon^\alpha \in W^2L^{n,1}(\Omega) \cap H^1_0(\Omega)$. Then for all $\overline{\omega} \in L^{n',\infty}(\Omega),\ \overline{\omega} \geqslant 0$, one has*

$$J_\varepsilon = \int_\Omega \overline{\omega}\Big[-\Delta_{1\varepsilon} - \vec{u}\cdot\psi_{1\varepsilon}\Big]dx \geqslant 0. \tag{75}$$

Proof. We develop the term $-\Delta\psi_{1\varepsilon} - \vec{u}\cdot\psi_{1\varepsilon}$ to derive the

$$\begin{aligned} J_\varepsilon &= \alpha\int_\Omega \overline{\omega}\Big[-\Delta\psi_1 - \vec{u}\cdot\nabla\psi_1\Big](\psi_1+\varepsilon)^{\alpha-1}dx + \alpha(1-\alpha)\int_\Omega |\nabla\psi_1|^2(\psi_1+\varepsilon)^{\alpha-2}\overline{\omega}dx \\ &= \alpha\lambda_1\int_\Omega \overline{\omega}\psi_1(\psi_1+\varepsilon)^{\alpha-1}dx + \alpha(1-\alpha)\int_\Omega |\nabla\psi_1|^2(\psi_1+\varepsilon)^{\alpha-2}\overline{\omega}dx \geqslant 0. \end{aligned}$$

□

Let us decompose $f = f_+ - f_-$, $f_+ \in L^1(\Omega; \delta^\alpha)$, $f_- \in L^1(\Omega; \delta^\alpha)$. By Theorem 3.5, we know that we have $\omega_1 \in D(A)$ (resp. $\omega_2 \in D(A)$ such that

$$\omega_1 + \lambda A\,\omega_1 = f_+ \qquad \omega_2 + \lambda A\,\omega_2 = f_-. \tag{76}$$

So by linearity and uniqueness, one has

$$\overline{\omega} = \omega_1 - \omega_2. \tag{77}$$

Therefore, it suffices to show that the inequality (74) holds for ω_1 (resp. ω_1). That is to say that is sufficient to prove the inequality for $f \geqslant 0$. But in that case, the unique solution of (72) is non negative : $\overline{\omega} \geqslant 0$ and we can choose as a test function $\phi = \psi_{1\varepsilon}$ given in Lemma 8.2. We then have

$$\int_\Omega f\psi_{1\varepsilon}dx = \int_\Omega \overline{\omega}\psi_{1\varepsilon}dx + \lambda \int_\Omega \overline{\omega}\Big[- \Delta\psi_{1\varepsilon} - \vec{u}\cdot\psi_{1\varepsilon}]dx + \lambda \int_\Omega V\,\psi_{1\varepsilon}\overline{\omega}dx. \tag{78}$$

According to Lemma 8.2 and the fact that $V\,\omega\,\psi_{1\varepsilon} \geqslant 0$ the two last integrals in relation (78) are non negative. Therefore,

$$\int_\Omega f\psi_{1\varepsilon}dx \geqslant \int_\Omega \overline{\omega}\,\psi_{1\varepsilon}dx, \qquad \varepsilon > 0. \tag{79}$$

Letting $\varepsilon \to 0$ in (79), we obtain

$$\int_\Omega \overline{\omega}\,\psi_1^\alpha dx \leqslant \int_\Omega f\psi_1^\alpha dx \tag{80}$$

whenever $f \in \overline{\omega} + \lambda A\,\overline{\omega}$, $\overline{\omega} \in D(A)$.

We have shown that the Schoedinger operator $A = -\Delta + \vec{u}\cdot V + V$ is m-accretive in $L^1(\Omega, \delta^\alpha)$, whenever $0 \leqslant \alpha < 1$, as in the first statement of Theorem 8.3. □

We have a similar result in $L^1(\Omega;\delta)$ provided that $V(x) \geqslant c\delta(x)^{-2}$ in a neighborhood U of the boundary. The argument is similar to the preceding one but we need to replace the use of Theorem 3.5 by Theorem 3.3. Indeed, if $f = f_+ - f_- \in L^1(\Omega;\delta)$ and $\overline{\omega} \in D(A)$ satisfies $\overline{\omega} + \lambda A\,\overline{\omega} = f$ then, Theorem 3.3 allows us to spleet $\overline{\omega} = \omega_2 - \omega_1$ with $\omega_i \in D(A)$ and $\omega_1 + \lambda A\,\omega_1 = f_+$ (idem $\omega_2 + \lambda A\,\omega_2 = f_-$). therefore, it suffices to show the inequality

$$\int_\Omega \overline{\omega}\,\psi_1 dx \leqslant \int_\Omega f\psi_1 dx \text{ for } f \geqslant 0,\ \overline{\omega} \geqslant 0.$$

To do so, we choose $\phi = \psi_1$ in equation (72) and derive

$$\int_\Omega f\psi_1 dx = (1 + \lambda\lambda_1)\int_\Omega \overline{\omega}\psi_1 dx + \int_\Omega V\overline{\omega}\,\psi_1 dx. \tag{81}$$

We drop the nonnegative term with V to derive

$$\int_\Omega \omega\psi_1 dx \leqslant \frac{1}{1+\lambda\lambda_1}\int_\Omega f\psi_1 dx \leqslant \int_\Omega f\psi_1 dx. \tag{82}$$

This show the desired inequality and implies that
$\forall\,\lambda > 0$, $\forall\overline{\omega} \in D(A)$, $\overline{\omega} + \lambda A\,\overline{\omega} = f \in L^1(\Omega; d)$

$$\int_\Omega |\overline{\omega}|\psi_1 dx \leqslant \int_\Omega |\overline{\omega} + \lambda A\,\overline{\omega}|\psi_1 dx. \tag{83}$$

♢

Therefore, we have shown the following theorem :

Theorem 8.3. *Let $\vec{u} \in L^\infty(\Omega)^n$ with $\operatorname{div}(\vec{u}) = 0$ in $\mathcal{D}'(\Omega)$, $\vec{u} \cdot \vec{\nu} = 0$ on $\partial\Omega$, $V \geqslant 0$ locally integrable.*
Then the Schroedinger operator $A = -\Delta + \vec{u} \cdot \nabla + V$ is m-accretive in $L^1(\Omega; \delta^\alpha)$ for any $0 \leqslant \alpha < 1$. If $\alpha = 1$ and $V(x) \geqslant c\delta(x)^{-2}$ in a neighborhood U of the boundary then the operator A is still m-accretive in $L^1(\Omega; \delta)$.

Remark 3. (Parabolic equation)
The m-accretivity property implies many results on parabolic equations (see [14, 3, **?**]) associated to the same operator.

For instance, we may have

Theorem 8.4. *Let $T > 0$, $\omega_0 \in D(A)$, $f \in W^{1,1}(0, T; L^1; \delta^\alpha))$, $a \in [0, 1]$. Then, there exists a function satisfying :*

$$\begin{cases} \omega \in C\Big([0,T]; D(A)\Big) \cap C^1\Big([0,T]; L^1(\Omega; \delta^\alpha)\Big) \\ \dfrac{d\omega}{dt}(t) + A\,\omega(t) = f(t) = f(t), \ \forall\, t \in [0, T], \quad \omega(0) = \omega_0. \end{cases}$$

8.2 Complex Schrödinger problem

Let us apply our previous results to the mathematical treatment of problem 4. In some sense, our main aim now is to show that the solution of this Schrödinger equation is *localized* for any $t > 0$, in the sense that if we start with a localized initial wave packet $\boldsymbol{\psi}_0 \in H^1(\mathbb{R}^n : \mathbb{C})$ (here $\mathbb{C}$ denotes the complex numbers), i.e. such that

$$\text{support } \boldsymbol{\psi}_0 \subset \overline{\Omega},$$

then the particle still remains permanently confined in Ω in the sense that

$$\text{support} \boldsymbol{\psi}(t, \cdot) \subset \overline{\Omega} \text{ for any } t > 0.$$

As in [8] we start by considering the auxiliary eigenvalue problem

$$DP(V, \lambda, \Omega) \quad \begin{cases} -\Delta\omega + \vec{u} \cdot \nabla\omega + V(x)\omega = \lambda\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 8.5. *Assume (3), then there exists a sequence of eigenvalues $\lambda_m \to +\infty$, $\lambda_1 > \lambda_{1,\Omega}$ (the first eigenvalue for the Dirichlet problem associated to the operator $-\Delta + \vec{u} \cdot \nabla$ on Ω), λ_1 is isolated and $\omega_1 > 0$ on Ω.*

Proof. We start by arguing as in the proof of Proposition 3 of [9]. We introduce the space

$$W = \left\{\varphi \in H_0^1(\Omega) : V\varphi^2 \in L^1(\Omega)\right\}.$$

For any $h \in L^2(\Omega)$ we define the operator $Th = z \in W$ solution of the linear problem

$$\begin{cases} Az = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{84}$$

We recall that the existence and uniqueness of a solution was obtained in Proposition 3 when $h \in W'$ (the dual space of W) and that, trivially, $L^2(\Omega) \subset W'$.Then the composition with the (compact) embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is a selfadjoint compact linear operator $\widetilde{T} = i \circ T : L^2(\Omega) \to L^2(\Omega)$ for which we obtain in the usual way a sequence of eigenvalues $\lambda_m \to +\infty$. By well-known results (see e.g. [18] or [2]) we know that $\lambda_1 > 0$ (notice that $\lambda_1 = 0$ would imply that $z = 0$). In fact, since $V(x) \geq 0$, by the comparison principle we know that $\lambda_1 > \lambda_{1,\Omega}$. The positivity of the first eigenfunction ω_1 is an easy modification of Proposition 3.2 of [8]. Moreover a variant of the Krein-Rutman can be applied (see [6]) and so we know that λ_1 is isolated.

Remark 4. When $r = 2$ in (3) then, by the Hardy inequality, $W = H_0^1(\Omega)$.

A different, and useful, consequence of Proposition 3 of [9] is the following:

Proposition 8.6. *Assume (3), then the operator operator $A : D(A)(\subset L^2(\Omega)) \to L^2(\Omega)$ given by $D(A) = W = \left\{\varphi \in H_0^1(\Omega) : V\varphi^2 \in L^1(\Omega)\right\}$ and $A\omega = -\Delta\omega + \vec{u} \cdot \nabla\omega + V\omega$ if $\omega \in D(A)$ is a maximal monotone operator in $L^2(\Omega)$.*

Proof. Given $h \in L^2(\Omega)$, the existence and uniqueness of solution of the equation $A\omega + \omega = h$ was obtained in Proposition 3 of [9]. Moreover, thanks to the assumptions on $\vec{u}$, by Lemmas 2.6 and 2.7 of [9] we get that

$$\|\omega\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)}$$

which proves the monotonicity in $L^2(\Omega)$ (i.e. the operator is m-accretive in $L^2(\Omega)$).

Let us prove now that the singularity of the potential implies that all the eigenfunctions ω_m of operator A are flat solutions (in the sense that $\omega = \dfrac{\partial\omega}{\partial n} = 0$ on $\partial\Omega$). As usual in Quantum Mechanics we shall pay attention to the associate eigenfunctions with normalized L^2-norm, i.e. such that

$$\|\omega_m\|_{L^2(\Omega)} = 1. \tag{85}$$

Theorem 8.7. *Let ω_m be an eigenfunction associated to the eigenvalue λ_m. Then $\omega_m \in L^\infty(\Omega)$ and ω_m is a flat solution of $DP(V, \lambda_m, \Omega)$. In fact, there exists $\overline{K}_m > 0$ such that*

$$|\omega_m(x)| \leq \overline{K}_m d(x, \partial\Omega)^2 \quad a.e.\ x \in \Omega. \tag{86}$$

Proof. It suffices to repeat all the arguments of Theorem 2.1 of [8] (concerning the case $r = 2$ and $\vec{u} = \overrightarrow{0}$) since the the main idea of the proof consists in the use of a Moser-type iterative argument (as in [10]) and take as test functions

$$\varphi(x) = v_{m,M}^{2\kappa+1}(x), \text{ with } v_{m,M}(x) := \min\{|\omega_m(x)|, M\} \operatorname{sign}(\omega_m(x)), \tag{87}$$

for any arbitrary $M, \kappa > 0$. Then, by using (3) and Lemmas 2.6 and 2.7 of [9] we conclude that $\varphi \in H_0^1(\Omega)$ is an appropriate test function and

$$\begin{aligned}
(2\kappa+1) \int_\Omega \left|v_M^{2\kappa}(x)\right| |\nabla\omega_m|^2 \, dx + \int_\Omega \frac{C}{\delta(x)^2} \left|v_M^{2\kappa+1}(x)\right| |\omega_m| \, dx \\
\leq (2\kappa+1) \int_\Omega \left|v_M^{2\kappa}(x)\right| |\nabla\omega_m|^2 \, dx + \int_\Omega V(x) \left|v_M^{2\kappa+1}(x)\right| |\omega_m| \, dx \\
= \lambda_n \int_\Omega \left|v_M^{2\kappa+1}(x)\right| |\omega_m| \, dx
\end{aligned} \tag{88}$$

where we used the simplified notation $v_M = v_{m,M}$. This is exactly the same starting energy estimate than the one used in the proof of Theorem 2.1 of [8] and thus the rest of the proof (passing to the limit when $M \nearrow +\infty$) applies without any other modification.

Remark 5. The flatness of the eigenfunctions ω_m of operator A can be also proved by using Proposition 2.7 of [19] nevertheless the statement given here supplies some decay estimates on ω_m near $\partial\Omega$ which are not given in the mentioned reference.

Remark 6. The decay estimates (86) is not optimal if $r > 2$ in (3). It seems possible to adapt the formal exposition made in [5] developing asymptotically some Bessel functions to prove that in that case

$$|\omega_m(x)| \leq \overline{K}_m \sqrt{d(x,\partial\Omega)}^{r/4} \exp\left(-\frac{\widehat{K}_m}{(r-2)} d(x,\partial\Omega)^{-(r-2)/2}\right) \quad \text{a.e. } x \in \Omega, \tag{89}$$

for some positive constants $\overline{K}_m$ and $\widehat{K}_m$, but we shall not enter into the details here.

Remark 7. Arguing as in [8] it is easy to get several qualitative properties of solutions of the complex evolution Schroedinger problem

$$\begin{cases} \mathbf{i}\dfrac{\partial \boldsymbol{\psi}}{\partial t} = -\Delta\boldsymbol{\psi} + \vec{u}\cdot\nabla\boldsymbol{\psi} + V(x)\boldsymbol{\psi} & \text{in } (0,\infty)\times\mathbb{R}^n \\ \boldsymbol{\psi}(0,x) = \boldsymbol{\psi}_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{90}$$

for very singular potentials over Ω which are extended (for instance) in a finite way to the whole space. So, we assume now that there exists $q \in [0,+\infty)$ such that

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases} \tag{91}$$

and that (3) holds. We can study the time evolution of a localized initial wave packet $\psi_0 \in H^1(R^n : C)$ such that support $\psi_0 \subset \overline{\Omega}$.

Then we can prove that there exists a unique solution $\boldsymbol{\psi} \in C([0,+\infty) : L^2(\mathbb{R}^n : \mathbb{C}))$ with $\boldsymbol{\psi} \in C([0,+\infty) : H^1(\mathbb{R}^n : \mathbb{C}))$ and $V_{q,\Omega}(x)\psi \in L^2(0,T : L^2(\mathbb{R}^n : \mathbb{C}))\}$ for any $T > 0$, and that the Galerkin decomposition

$$\boldsymbol{\psi}_\Omega(t,x) = \sum_{m=1}^{\infty} \mathbf{a}_m e^{-\mathbf{i}\lambda_m t}\omega_m(x), \tag{92}$$

holds with convergence at least in $L^2(\mathbb{R}^n : \mathbb{C})$where where λ_m and ω_m are the eigenvalues and eigenfunctions given in Proposition 8.5 and

$$\mathbf{a}_m = \int_\Omega \boldsymbol{\psi}_0(x)\omega_m(x)dx.$$

For localizing purposes we assume that

$$\sum_{m=1}^{\infty} |\mathbf{a}_m| \overline{K}_m < +\infty, \tag{93}$$

where $\overline{K}_m > 0$ was given in Theorem 8.7. Thus, we conclude that

$$|\boldsymbol{\psi}(t,x)| \leq K d(x,\partial\Omega)^2 \quad \text{for any } t > 0 \text{ and a.e. } x \in \Omega, \tag{94}$$

for some $K > 0$, and in consequence the unique solution of (90) satisfies that support $\boldsymbol{\psi}(t,.) \subset \overline{\Omega}$ for any $t > 0$.

Concerning the existence of solutions, it is enough to apply the Hille-Yosida theorem (see, e.g. [18], [2]). For the Galerkin decomposition we can adapt the arguments given in [4].

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