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Homogenization of Boundary Value Problems in Plane Domains with Frequently Alternating Type of Nonlinear Boundary Conditions: Critical Case[1]

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Abstract— In the present paper we consider a boundary homogenization problem for the Poisson's equation in a bounded domain and with a part of the boundary conditions of highly oscillating type (alternating between homogeneous Neumman condition and a nonlinear Robin type condition involving a small parameter). Our main goal in this paper is to investigate the asymptotic behavior as $\varepsilon \to 0$ of the solution to such a problem in the case when the length of the boundary part, on which the Robin condition is specified, and the coefficient, contained in this condition, take so-called critical values. We show that in this case the character of the nonlinearity changes in the limit problem. The boundary homogenization problems were investigate for example in [1, 2, 4]. For the first time the effect of the nonlinearity character change via homogenization was noted for the first time in [5]. In that paper an effective model was constructed for the boundary value problem for the Poisson's equation in the bounded domain that is perforated by the balls of critical radius, when the space dimension equals to 3. In the last decade a lot of works appeared, e.g., [6–10], in which this effect was studied for different geometries of perforated domains and for different differential operators. We note that in [6–10] only perforations by balls were considered. In papers [11, 12] the case of domains perforated by an arbitrary shape sets in the critical case was studied.

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Let Ω be a bounded domain in $\mathbb{R}^2 \cap \{x_2 > 0\}$, the boundary of which consists of two smooth parts $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where $\Gamma_2 = \partial\Omega \cap \{x_2 = 0\} = [-l, l]$, $l > 0$, $\Gamma_1 = \partial\Omega \cap \{x_2 > 0\}$. The consideration of the case concerning higher dimensions will be the object of a separated work by the authors.

Denote by $Y_1 = \left\{(y_1, 0) | -\frac{1}{2} < y_1 < \frac{1}{2}\right\}$, $\hat{l}_0 = \{(y_1, 0) | -l_0 < y_1 < l_0\} \subset Y_1$, $l_0 \in \left(0, \frac{1}{2}\right)$. For a small parameter $\varepsilon > 0$ and $0 < a_\varepsilon \ll \varepsilon$ we introduce the sets

$$\tilde{G}_\varepsilon = \bigcup_{j \in \mathbb{Z}'} (a_\varepsilon \hat{l}_0 + \varepsilon j) = \bigcup_{j \in \mathbb{Z}'} l_\varepsilon^j,$$

where $\mathbb{Z}' = \mathbb{Z} \times \{0\}$ is a set of vectors $j = (j_1, 0)$ and j_1 is integer. Denote by $\Upsilon_\varepsilon = \{j \in \mathbb{Z}' \mid \overline{l_\varepsilon^j} \subset \{x = (x_1, 0): x_1 \in [-l + 2\varepsilon, l - 2\varepsilon] \times \{0\}\}$. Consider $Y_\varepsilon^j = \varepsilon Y_1 + \varepsilon j$ and

$$l_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} l_\varepsilon^j.$$

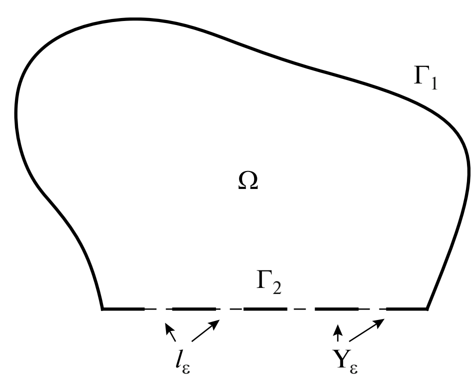

Fig. 1. Domain Ω.

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It is easy to see that $\overline{l_\varepsilon^j} \subset Y_\varepsilon^j$. Denote by $\gamma_\varepsilon = \Gamma_2 \backslash \overline{l_\varepsilon}$. Note that, for $\forall j \in \mathbb{Z}'$, $|l_\varepsilon^j| = 2a_\varepsilon l_0$, $|l_\varepsilon| \cong da_\varepsilon \varepsilon^{-1}$.

In the domain Ω we consider the following boundary value problem

$$\begin{cases} -\Delta u_\varepsilon = f, \quad x \in \Omega \\ \partial_{x_2} u_\varepsilon = \beta(\varepsilon)\sigma(u_\varepsilon), \quad x \in l_\varepsilon \\ \partial_{x_2} u_\varepsilon = 0, \quad x \in \gamma_\varepsilon \\ u_\varepsilon = 0, \quad x \in \Gamma_1, \end{cases} \tag{1}$$

where $a_\varepsilon = C_0 \varepsilon e^{\frac{-\alpha^2}{\varepsilon}}$, $\beta(\varepsilon) = e^{\frac{\alpha^2}{\varepsilon}}$, $\alpha \neq 0, C_0 > 0$, $\sigma(u) \in C^1(\mathbb{R})$ and (for simplicity) $\sigma(0) = 0$ and there exist positive constants k_1 and k_2, $k_1 < k_2$ such that

$$k_1 \le \sigma'(u) \le k_2. \tag{2}$$

The consideration of much more general nonlinear terms σ will be the object of a separated work by the authors.

Remark 1. There are many applied models that lead to similar formulations as the stated in problem (1). For instance, the consideration of energy balance climate models for a deep ocean (see, e.g., [14]) leads to a quite similar problem where now boundary Γ_2 corresponds to a part of the atmospheric surface (in this case the x_1 coordinate must be substituted as $x_1 = L_1 - \hat{x}_1$ with $\hat{x}_1 \in (0, L_1)$; $\hat{x}_1 = L_1$ representing the atmosphere surface and $\hat{x}_1 = 0$ the bottom of the deep ocean). If such domain is taken in a neighborhood of the ice sheet, it is well-know, that there is a *mushy region* on the atmosphere surface in which there is a highly alternating coexistence of ice pieces and water (see, e.g., [15]). In a first approach to the modeling of the phenomenon it can be assumed that the formation of this set of infinitely many very small pieces of ice is due to the presence of a nonlinear Robin type condition which is surrounded by pure homogeneous Neumann boundary conditions in the exterior to the small pieces.

The weak solution to the problem (1) is defined as a function $u_\varepsilon \in H^1(\Omega, \Gamma_1)$ such that it satisfies integral identity

$$\int_\Omega \nabla u_\varepsilon \nabla \psi dx + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(u_\varepsilon) \psi dx_1 = \int_\Omega f \psi dx, \tag{3}$$

where $\psi \in H^1(\Omega, \Gamma_1)$ is an arbitrary test function.

It is well known (see [13]) that the problem (1) has a unique weak solution and for that solution we have following estimate

$$\|u_\varepsilon\|_{H^1(\Omega)} + e^{\frac{\alpha^2}{2\varepsilon}} \|u_\varepsilon\|_{L^2(l_\varepsilon)} \le K, \tag{4}$$

where K, here and below, is a positive constant that does not depend on ε.

The estimate (4) implies that there exists a subsequence (we preserve for it the notation of the original sequence) such that as $\varepsilon \to 0$ we have

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in} \quad H^1(\Omega), \tag{5}$$

$$u_\varepsilon \to u_0 \quad \text{strongly in} \quad L^2(\Omega). \tag{6}$$

Theorem 1. *Let* $a_\varepsilon = C_0 \varepsilon e^{-\frac{\alpha^2}{\varepsilon}}$, $\beta(\varepsilon) = e^{\frac{\alpha^2}{\varepsilon}}$, $\alpha \neq 0$, $C_0 > 0$, *and* u_ε *be solution to the problem* (1).

Then the function $u_0 \in H^1(\Omega, \Gamma_1)$ *defined in* (5) *and* (6) *is a weak solution to the following boundary value problem*

$$\begin{gathered} -\Delta u_0 = f, \quad x \in \Omega, \\ \partial_{x_2} u_0 - \frac{\pi}{\alpha^2} H(u_0) = 0, \quad x \in \Gamma_2, \\ u_0 = 0, \quad x \in \Gamma_1, \end{gathered} \tag{7}$$

where $H(u)$ *is the unique Lipschiz continuous and increasing function satisfying*

$$\pi H(u) = 2 l_0 \alpha^2 C_0 \sigma(u - H(u)). \tag{8}$$

Proof. We introduce the auxiliary functions w_ε^j and q_ε^j as weak solution to the following problems

$$\begin{gathered} \Delta w_\varepsilon^j = 0, \quad x \in T_{\varepsilon/4}^j \backslash \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 1, \quad x \in \partial T_{a_\varepsilon}^j, \\ w_\varepsilon^j = 0, \quad x \in \partial T_{\varepsilon/4}^j \end{gathered} \tag{9}$$

and

$$\begin{gathered} \Delta q_\varepsilon^j = 0, \quad x \in T_{\varepsilon/4}^j \backslash \overline{l_\varepsilon^j}, \\ q_\varepsilon^j = 1, \quad x \in l_\varepsilon^j, \\ q_\varepsilon^j = 0, \quad x \in \partial T_{\varepsilon/4}^j. \end{gathered} \tag{10}$$

Here, T_r^j denotes the ball of radius r centered at the point $(\varepsilon j, 0)$.

Note that, due to symmetry, w_ε^j and q_ε^j are also a solutions of the corresponding boundary value problems in the domains $(T_{\varepsilon/4}^j)^+ \backslash \overline{T_{a_\varepsilon}^j}$ and $(T_{\varepsilon/4}^j)^+$ respectively, where A^+ denotes set $A \cap \{x_2 > 0\}$ for $A \in \mathbb{R}^2$,

$$\begin{gathered} \Delta w_\varepsilon^j = 0, \quad x \in (T_{\varepsilon/4}^j)^+ \backslash \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 0, \quad x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ w_\varepsilon^j = 1, \quad x \in \partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} w_\varepsilon^j = 0, \quad x \in \{x_2 = 0\} \cap (T_{\varepsilon/4}^j \backslash \overline{T_{a_\varepsilon}^j}). \end{gathered} \tag{11}$$

Note that we can find explicit solution to this problem

$$w_\varepsilon^j(x) = \frac{\ln(4r/\varepsilon)}{\ln(4a_\varepsilon/\varepsilon)}.$$

Analogously

$$\begin{aligned} \Delta q_\varepsilon^j &= 0, \quad x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, \\ q_\varepsilon^j &= 1, \quad x \in l_\varepsilon^j, \\ q_\varepsilon^j &= 0, \quad x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} q_\varepsilon^j &= 0, \quad x \in (T_{\varepsilon/4}^j \cap \{x_2 = 0\}) \setminus \overline{l_\varepsilon^j}, \end{aligned} \tag{12}$$

where $j \in \Upsilon_\varepsilon$, $l_\varepsilon^j = a_\varepsilon \hat{l}_0 + \varepsilon j$.

Define

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, \ j \in \Upsilon_\varepsilon \\ 1, & x \in (T_{a_\varepsilon}^j)^+ \\ 0, & x \in \Omega \setminus \bigcup\limits_{j \in \Upsilon_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}, \end{cases} \tag{13}$$

where $\mathbb{R}_+^2 = \{x_2 > 0\}$,

$$Q_\varepsilon(x) = \begin{cases} q_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+, \quad j \in \Upsilon_\varepsilon \\ 0, & x \in \Omega \setminus \bigcup\limits_{j \in \Upsilon_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}. \end{cases} \tag{14}$$

We have $W_\varepsilon, Q_\varepsilon \in H^1(\Omega, \Gamma_1)$ and

$$W_\varepsilon \rightharpoonup 0, \quad \text{weakly in} \quad H^1(\Omega, \Gamma_1), \quad \varepsilon \to 0. \tag{15}$$

Lemma 1. *Let W_ε be a function defined by the formula* (13), *Q_ε be a function defined by the formula* (14). *Then*

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \le K\sqrt{\varepsilon}.$$

Proof. Note that for an arbitrary function $\psi \in H^1(T_{\varepsilon/4}^j)$ such that $\psi = 0$ on l_ε^j we have

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla q_\varepsilon^j \nabla \psi \, dx_1 dx_2 = 0.$$

We consider $\psi = w^j - q_\varepsilon^j$ as a test function in the above equality and get

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla q_\varepsilon^j \nabla (w_\varepsilon^j - q_\varepsilon^j) \, dx_1 dx_2 = 0. \tag{16}$$

In addition, we have

$$\begin{aligned} &\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla (w_\varepsilon^j - q_\varepsilon^j) \, dx_1 dx_2 \\ &= \int\limits_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} \partial_\nu w_\varepsilon^j (w_\varepsilon^j - q_\varepsilon^j) \, ds. \end{aligned} \tag{17}$$

By subtracting (16) from (17) we derive

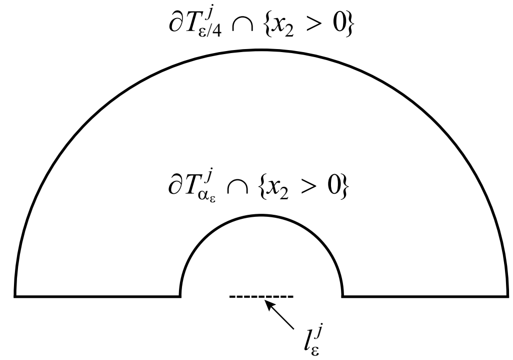

Fig. 2. Domain $T_{\varepsilon/4}^{+,j} \setminus \overline{T_{a_\varepsilon}^j}$ and l_ε^j.

$$\begin{aligned} &\int\limits_{(T_{\varepsilon/4}^j)^+} |\nabla (w_\varepsilon^j - q_\varepsilon^j)|^2 dx_1 dx_2 \\ &= \int\limits_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} \partial_\nu w_\varepsilon^j (w_\varepsilon^j - q_\varepsilon^j) \, ds. \end{aligned} \tag{18}$$

Note that $w_\varepsilon^j(x) = -\dfrac{\ln(4r/\varepsilon)}{\ln(4a_\varepsilon/\varepsilon)}$ and $\partial_\nu w_\varepsilon^j|_{\partial T_{a_\varepsilon}^j} = -\dfrac{1}{a_\varepsilon \ln(4a_\varepsilon/\varepsilon)}$. Hence, (17) implies that

$$\begin{aligned} \|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T_{\varepsilon/4}^j)^+)}^2 &\le \frac{1}{a_\varepsilon |\ln(4a_\varepsilon/\varepsilon)|} \int\limits_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} |w_\varepsilon^j - q_\varepsilon^j| \, ds \\ &= \frac{1}{|\ln(4a_\varepsilon/\varepsilon)|} \int\limits_{\partial T_1^j \cap \{y_2 > 0\}} |w_\varepsilon^j - q_\varepsilon^j| \, ds_y \equiv J_\varepsilon. \end{aligned}$$

Given that $w_\varepsilon^j - q_\varepsilon^j = 0$ if $y \in \hat{l}_0$ and using the embedding theorem, we get

$$\begin{aligned} J_\varepsilon &\le \frac{K}{|\ln(4a_\varepsilon/\varepsilon)|} \left(\int\limits_{(T_1^j)^+} |\nabla_y (w_\varepsilon^j - q_\varepsilon^j)|^2 dy \right)^{1/2} \\ &\le K\varepsilon \|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T_{\varepsilon/4}^j)^+)}. \end{aligned} \tag{19}$$

From here we derive an estimate

$$\|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T_{\varepsilon/4}^j)^+)} \le K\varepsilon.$$

From this estimate it follows that

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \le K\sqrt{\varepsilon}. \tag{20}$$

This completes the proof.

We proceed to the proof of theorem. By using monotonicity of the function $\sigma(u)$ we derive that u_ε satisfies the integral inequality

$$\int_{\Omega} \nabla v \nabla (v - u_\varepsilon) dx + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(v)(v - u_\varepsilon) dx_1 \geq \int_{\Omega} f(v - u_\varepsilon) dx \quad (21)$$

for an arbitrary test function $v \in H^1(\Omega, \Gamma_1)$.

We take $v = \psi - Q_\varepsilon H(\psi)$ as a test function in (21), where $\psi \in C^\infty(\overline{\Omega})$, $\psi(x) = 0$ in neighbourhood of Γ_1 and $H(u)$ is solution to the functional equation (8). We get

$$\int_{\Omega} \nabla(\psi - Q_\varepsilon H(\psi)) \nabla(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(\psi - H(\psi))(\psi - H(\psi) - u_\varepsilon) dx_1 \geq \int_{\Omega} f(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx. \quad (22)$$

We rewrite inequality (22) in the following way

$$\int_{\Omega} \nabla(\psi - W_\varepsilon H(\psi)) \nabla(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx - \int_{\Omega} \nabla((Q_\varepsilon - W_\varepsilon) H(\psi)) \nabla(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx \quad (23)$$

$$+ e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(\psi - H(\psi))(\psi - H(\psi) - u_\varepsilon) dx_1 \geq \int_{\Omega} f(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx.$$

From the fact, that $Q_\varepsilon \rightharpoonup 0$ as $\varepsilon \to 0$ weakly in $H^1(\Omega, \Gamma_1)$, we have

$$\lim_{\varepsilon \to 0} \int_{\Omega} f(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx = \int_{\Omega} f(\psi - u_0) dx, \quad (24)$$

$$\lim_{\varepsilon \to 0} \int_{\Omega} \nabla \psi \nabla(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx = \int_{\Omega} \nabla \psi \nabla(\psi - u_0) dx. \quad (25)$$

Lemma 1 implies that

$$\lim_{\varepsilon \to 0} \int_{\Omega} \nabla((Q_\varepsilon - W_\varepsilon) H(\psi)) \nabla(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx = 0.$$

Consider the remaining integrals in (23). Denote by

$$I_\varepsilon \equiv -\int_{\Omega} \nabla(W_\varepsilon H(\psi)) \nabla(\psi - Q_\varepsilon H(\psi) - u_\varepsilon) dx = -\int_{\Omega} \nabla W_\varepsilon \nabla \{H(\psi)(\psi - Q_\varepsilon H(\psi) - u_\varepsilon)\} dx + \alpha_\varepsilon,$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$.

It is easy to see that

$$I_\varepsilon = -\int_{\Omega} \nabla W_\varepsilon \nabla \{H(\psi)(\psi - W_\varepsilon H(\psi) - u_\varepsilon)\} dx + \tilde{\alpha}_\varepsilon$$

$$= -\sum_{j \in \Upsilon_\varepsilon} \int_{(T^j_{\varepsilon/4})^+ \setminus \overline{T^j_{a_\varepsilon}}} \nabla w^j_\varepsilon \nabla \{H(\psi)(\psi - w^j_\varepsilon H(\psi) - u_\varepsilon)\} dx + \tilde{\alpha}_\varepsilon$$

$$= -\sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4} \cap \{x_2 > 0\}} \partial_\nu w^j_\varepsilon H(\psi)(\psi - u_\varepsilon) ds \quad (26)$$

$$- \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{a_\varepsilon} \cap \{x_2 > 0\}} \partial_\nu w^j_\varepsilon H(\psi)(\psi - H(\psi) - u_\varepsilon) ds + \tilde{\alpha}_\varepsilon,$$

where $\tilde{\alpha}_\varepsilon \to 0$, $\varepsilon \to 0$.

Given that $\partial_\nu w^j_\varepsilon \big|_{\partial T^j_{\varepsilon/4}} = \dfrac{4}{\varepsilon \ln(4a_\varepsilon/\varepsilon)} = \dfrac{4}{-\alpha^2 + \varepsilon \ln(4C_0)}$ and using the results of the paper [3] we derive

$$-\lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4} \cap \{x_2 > 0\}} \partial_\nu w^j_\varepsilon H(\psi)(\psi - u_\varepsilon) ds$$

$$= \lim_{\varepsilon \to 0} \frac{4}{\alpha^2 - \varepsilon \ln(4C_0)} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4} \cap \{x_2 > 0\}} H(\psi)(\psi - u_\varepsilon) ds \quad (27)$$

$$= \frac{\pi}{\alpha^2} \int_{\Gamma_2} H(\psi)(\psi - u_0) ds.$$

Let us find the limit of the expression

$$- \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{a_\varepsilon} \cap \{x_2 > 0\}} \partial_\nu w^j_\varepsilon H(\psi)(\psi - H(\psi) - u_\varepsilon) ds + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(\psi - H(\psi))(\psi - H(\psi) - u_\varepsilon) dx_1$$

$$= \sum_{j \in \Upsilon_\varepsilon} \frac{(\alpha^2 C_0)^{-1} e^{\alpha^2/\varepsilon}}{1 - \varepsilon \alpha^{-2} \ln(4C_0)} \times \int_{\partial T^j_{a_\varepsilon} \cap \{x_2 > 0\}} H(\psi)(\psi - H(\psi) - u_\varepsilon) ds$$

$$+ e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \sigma(\psi - H(\psi))(\psi - H(\psi) - u_\varepsilon) dx_1$$

$$= e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \sigma(\psi - H(\psi))(\psi - H(\psi) - u_\varepsilon) dx_1$$

$$- \frac{e^{\alpha^2/\varepsilon}}{\alpha^2 C_0} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{a_\varepsilon} \cap \{x_2 > 0\}} H(\psi)(\psi - H(\psi) - u_\varepsilon) ds + \hat{\alpha}_\varepsilon \equiv D_\varepsilon + \hat{\alpha}_\varepsilon, \quad (28)$$

where $\hat{\alpha}_\varepsilon \to 0$, $\varepsilon \to 0$.

We introduce function $m(y) \in H^1((T_1^0)^+)$, with $y = (y_1, y_2)$, as a weak solution to the following boundary value problem

$$\begin{cases} \Delta_y m = 0, & y \in (T_1^0)^+ \\ \partial_{y_2} m = 1, & y \in \hat{l}_0 \\ \partial_\nu m = \dfrac{2l_0}{\pi}, & y \in (\partial T_1^0)^+ \\ \partial_{y_2} m = 0, & y \in (\partial (T_1^0)^+)\setminus(\hat{l}_0 \cup (\partial T_1^0)^+). \end{cases} \tag{29}$$

Consider

$$m_\varepsilon^j(x) = \varepsilon m\left(\frac{x - P_\varepsilon^j}{a_\varepsilon}\right), \quad x \in (T_{a_\varepsilon}^j)^+. \tag{30}$$

The function $m_\varepsilon^j(x)$ is a solution to the problem

$$\begin{cases} \Delta_x m_\varepsilon^j = 0, & x \in (T_{a_\varepsilon}^j)^+ \\ \partial_\nu m_\varepsilon^j = \dfrac{\varepsilon a_\varepsilon^{-1} 2l_0}{\pi}, & x \in (\partial T_{a_\varepsilon}^j)^+ \\ \partial_{x_2} m_\varepsilon^j = \varepsilon a_\varepsilon^{-1}, & x \in l_\varepsilon^j \\ \partial_{x_2} m_\varepsilon^j = 0, & x \in (\partial (T_{a_\varepsilon}^j)^+)\setminus(\hat{l}_\varepsilon^j \cup (\partial T_{a_\varepsilon}^j)^+). \end{cases} \tag{31}$$

Denote by $h_\varepsilon = H(\psi)(\psi - H(\psi) - u_\varepsilon)$. Then

$$\left| \frac{2l_0 \varepsilon a_\varepsilon^{-1}}{\pi} \int_{(\partial T_{a_\varepsilon}^j)^+} h_\varepsilon ds - \varepsilon a_\varepsilon^{-1} \int_{l_\varepsilon^j} h_\varepsilon dx_1 \right|$$

$$= \left| \int_{(T_{a_\varepsilon}^j)^+} \nabla_x m_\varepsilon^j \nabla h_\varepsilon dx \right| \le \|\nabla_x m_\varepsilon^j\|_{L^2((T_{a_\varepsilon}^j)^+)} \|\nabla h_\varepsilon\|_{L^2((T_{a_\varepsilon}^j)^+)}. \tag{32}$$

Due to the fact that

$$\|\nabla_x m_\varepsilon^j\|^2_{L^2((T_{a_\varepsilon}^j)^+)} = \varepsilon^2 \|\nabla_y m(y)\|^2_{L^2((T_1^0)^+)} \le K\varepsilon^2,$$

we have

$$\sum_{j \in \Upsilon_\varepsilon} \|\nabla_x m_\varepsilon^j\|^2_{L^2((T_{a_\varepsilon}^j)^+)} \le K\varepsilon. \tag{33}$$

From (32), (33) we derive

$$\left| e^{\alpha^2/\varepsilon} \frac{\pi}{2l_0} \int_{l_\varepsilon} h_\varepsilon dx_1 - e^{\alpha^2/\varepsilon} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} h_\varepsilon ds \right|$$

$$\le \delta^{-1} \sum_{j \in \Upsilon_\varepsilon} \|\nabla_x m_\varepsilon^j\|^2_{L^2((T_{a_\varepsilon}^j)^+)} + \delta \|\nabla h_\varepsilon\|^2_{L^2(\Omega)} \le K\sqrt{\varepsilon}, \tag{34}$$

if $\delta = \sqrt{\varepsilon}$.

Estimate (34) implies that the limit as $\varepsilon \to 0$ of the expression D_ε in (28) is equal to zero. Indeed, from the fact that $H(u)$ satisfies the functional equation (8) we derive

$$|D_\varepsilon| = \left| e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \sigma(\psi - H(\psi))(\psi - H(\psi) - u_\varepsilon) dx_1 \right.$$

$$\left. - \frac{e^{\alpha^2/\varepsilon}}{\alpha^2 C_0} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} H(\psi)(\psi - H(\psi) - u_\varepsilon) ds \right|$$

$$\le \left| \frac{\pi}{2l_0 \alpha^2 C_0} e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} H(\psi)(\psi - H(\psi) - u_\varepsilon) dx_1 \right.$$

$$\left. - \frac{e^{\alpha^2/\varepsilon}}{\alpha^2 C_0} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} H(\psi)(\psi - H(\psi) - u_\varepsilon) ds \right|$$

$$+ \left| e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \left\{ \sigma(\psi - H(\psi)) - \frac{\pi}{2l_0 \alpha^2 C_0} H(\psi) \right\} \right. \tag{35}$$

$$\left. \times (\psi - H(\psi) - u_\varepsilon)\, dx_1 \right| \le K\sqrt{\varepsilon}.$$

Here we used that the second module equals to zero due to Eq. (8).

Therefore, from (22)–(35) we conclude that $u_0 \in H^1(\Omega, \Gamma_1)$ satisfies the following inequality

$$\begin{gathered} \int_\Omega \nabla\psi \nabla(\psi - u_0) dx + \frac{\pi}{\alpha^2} \int_{\Gamma_2} H(\psi)(\psi - u_0) dx_1 \\ \ge \int_\Omega f(\psi - u_0) dx, \end{gathered} \tag{36}$$

where ψ is an arbitrary function from $H^1(\Omega, \Gamma_1)$, $H(u)$ satisfies the functional equation (8). This concludes the proof.

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