ON A PARABOLIC SYSTEM WITH STRONG ABSORPTION MODELING DRYLAND VEGETATION

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Abstract. We consider a variant of a nonlinear parabolic system, proposed in 2007 by E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak and E. Meron, in desertification studies, in which there is a strong absorption. The system models the mutual interaction between the biomass b, the soil-water content w and the surface-water height h which is diffused by means of the degenerate operator Δh^m with $m \geq 2$. The main novelty in this paper is that the absorption is given in terms of an exponent $\alpha \in (0,1)$, in contrast to the case $\alpha = 1$ considered in the previous literature. Thanks to this, some new qualitative behavior of the dynamics of the solutions can be justified.

After proving the existence of non-negative solutions for the system with Dirichlet and Neumann boundary conditions, we demonstrate the possible extinction in finite time and the finite speed of propagation for the surface-water height component $h(t,x)$. Finally, we prove, for the associate stationary problem, that if the precipitation datum $p(x)$ grows near the boundary of the domain $\partial\Omega$ as $d(x,\partial\Omega)^{\frac{2\alpha}{m-\alpha}}$ then $h^m(x)$ grows, at most, as $d(x,\partial\Omega)^{\frac{2}{m-\alpha}}$. This property also implies the infinite waiting time property when the initial datum $h_0(x)$ grows at most as $d(x,\partial S(h_0))^{\frac{2m}{m-\alpha}}$ near the boundary of its support $S(h_0)$.

1. Introduction

We study a parabolic system which captures the interactions between vegetation and water in arid and semi-arid porous areas such as modeled in [16]. A slight variation in the modeling is introduced in order to get some new qualitative behavior by its solutions. The system, in non-dimensionalized form, that it is considered in this paper is the following:

$$\begin{cases} \partial_t b = d_b \Delta b + w G_1(b)(1-b)b - b, \\ \partial_t w = d_w \Delta w - (L(b) + G_2(b))\, w + I(b) h^{\alpha}, \\ \partial_t h = d_h \Delta h^m - I(b) h^{\alpha} + p. \end{cases} \tag{1.1}$$

Here, b represents the concentration of the above ground biomass, w the soil water content and h the height of a thin surface water layer per unit area. The equation for the evolution of biomass involves a water dependent growth rate $G_1(b)$, a mortality term with constant loss rate and a linear diffusion term modeling growth due to seeds dispersal or clonal growth. In the equation for the soil water, we have a loss term which consists of the water up-take rate by the plant roots denoted by $G_2(b)$ and the biomass dependent evaporation rate $L(b)$. Moreover, the equation contains the source term $I(b)h^{\alpha}$ representing the infiltrated surface water, which is discussed in more detail below, and

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a linear diffusion term modeling the soil water transport. The main novelty in this paper is that the exponent α will be assumed such that $\alpha \in (0,1)$, in contrast to the case $\alpha = 1$ considered in the previous literature. The third equation models the surface water flow and how this infiltrates into the ground.

The variable h corresponds to the non-dimensionless quantity $H = \rho d$, where $d(\tilde{t}, \tilde{x})$ represents the depth in meters of the surface water where $\tilde{x} \in \Omega \subset \mathbb{R}^2$ and ρ is the constant density of the fluid. In fact, the third equation in dimensional quantities (and $m \geq 2$) can be derived from the continuity equation

$$\frac{\partial H}{\partial \tilde{t}} + \operatorname{div}(H\vec{u}) = P - \mathcal{I}H$$

and the shallow water momentum equation

$$\frac{D\overrightarrow{u}}{\partial \tilde{t}} = -g\nabla(\zeta + d) + \frac{1}{\rho}F,$$

where $\vec{u}$ is the horizontal velocity of the fluid, F represents ground surface friction, P stands for the precipitation rate and $\mathcal{I}H$ is the infiltration rate of water through the soil surface. Moreover, ζ denotes the height of the soil surface and g stands for the acceleration of gravity. For non-trivial land topographies $\zeta(\tilde{x})$ is a nonnegative function of the space variable $\tilde{x}$ and it is convenient to set $Z = \rho\zeta$. We consider a friction term of the form $F = -k\vec{u}/d^l$, for $l \geq 0$ and $k > 0$, a biomass and surface water dependent infiltration rate term of the form $\mathcal{I}(B,H) = \mathcal{I}_B(B)\mathcal{I}_H(H)$ and we let $\zeta = 0$ which corresponds to a region with flat topography. Then

$$\frac{\partial H}{\partial \tilde{t}} - c\Delta H^m = P - \mathcal{I}(B,H)H,$$

where $m = l + 2$ and $c = \dfrac{g}{mk\rho^l}$. The biomass dependent infiltration rate $\mathcal{I}_B$ captures the infiltration contrast between vegetated regions and bare soil due to the formation of biogenic crusts in unvegetated regions which reduce the infiltration of surface water. Therefore, this term is monotonically increasing with B approaching a constant infiltration for high biomass concentrations. The counterpart $\mathcal{I}_H$ of the infiltration rate in this paper is chosen to be a decreasing function of H, taking the explicit form $H^{\alpha-1}$ for $\alpha \in (0,1)$. Other models related to desertification studies can be found, for instance, in [1] and [21].

From the mathematical viewpoint, we mention the study of the corresponding dynamical system in the case $\alpha = 1$ made in [18, 17]. Notice that, curiously for the associated stationary system (considered in [10] and [9]) the assumption $\alpha \in (0,1)$ does not introduce any big change in the problem, since the change of variables $\widehat{h} = h^m$ leads to the stationary equation $-d_h\Delta\widehat{h} + \mathcal{I}_b\left(\widehat{h}\right)^{\alpha/m} = p$ which involves an exponent $\alpha/m < 1$ even for $\alpha = 1$. So, the modifications implied by the assumption $\alpha \in (0,1)$ mainly affect the dynamics of solutions of (1.1).

The article is organized as follows. In Section 2 we complete the mathematical formulation of the system (1.1). In particular, we consider two cases of boundary conditions: the Dirichlet and Neumann boundary conditions. For the first case, we define a regularized approximating system which possesses positive bounded solutions. This allows us to pass to the limit of the approximating problem proving the existence of solutions for the original problem. For the second case, we use a different approach, specifically, the existence of solutions is given by a fixed point argument employing a fixed point theorem for sequentially weakly continuous mappings in Banach spaces. Section 3 is devoted to the qualitative behavior of solutions. We examine the behavior, in time, of the vanishing set of the surface water component h in the absence of precipitation during sufficiently long time intervals. The spatial location of the vanishing set of h is also analyzed.

2. Existence of solutions

In what follows, we denote by Ω a bounded domain in $\mathbb{R}^2$ with regular boundary $\partial\Omega$ and for $T > 0$ we let $Q_T = \Omega \times (0,T)$ and $S_T = \partial\Omega \times (0,T)$. Our purpose is to prove the existence of a solution $U = (b, w, h)$ of the system

$$\begin{cases} \partial_t b = d_b \Delta b + wG_1(b)(1-b)b - b, & \text{in } Q_T, \\ \partial_t w = d_w \Delta w - (L(b) + G_2(b))\, w + I(b)h^\alpha, & \text{in } Q_T, \\ \partial_t h = d_h \Delta h^m - I(b)h^\alpha + p, & \text{in } Q_T, \end{cases} \tag{2.1}$$

together with the initial conditions,

$$b(x,0) = b_0(x),\ w(x,0) = w_0(x),\ h(x,0) = h_0(x) \text{ for } x \in \Omega, \tag{2.2}$$

and some boundary conditions which are either of Dirichlet type

$$b = w = h = 0, \text{ on } \partial\Omega \times (0,T), \tag{2.3}$$

or of Neumann type

$$\frac{\partial b}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial h^m}{\partial n} = 0, \text{ on } \partial\Omega \times (0,T). \tag{2.4}$$

We shall mainly assume

$$b_0, w_0, h_0 \in L^\infty(\Omega), \tag{2.5}$$

and to get more regularity we will additionally assume that,

$$b_0, w_0, h_0 \in C(\bar{\Omega}). \tag{2.6}$$

In any case, we are specifically interested in the case in which the initial data satisfy

$$0 \le b_0 \le 1, w_0 \ge 0, h_0 \ge 0, \text{ on } \Omega. \tag{2.7}$$

Concerning the precipitation term p, we assume that $p \in L^\infty(Q_T)$ is nonnegative. Moreover, we suppose the structural conditions

$$I(b) = \theta \frac{b + r/c}{b + r}, \tag{2.8}$$

$$L(b) = \frac{\nu}{1 + \rho b}, \tag{2.9}$$

$$G_1(b) = \nu(1 + \eta b)^2, \tag{2.10}$$

$$G_2(b) = \gamma b(1 + \eta b)^2, \tag{2.11}$$

and that $d_b, d_w, d_h, \eta, \rho, r, \nu, \theta$ are given positive constants and that $c \ge 1$. For later use we also note that for $s \in [0,1]$, $I(s)$, $G_1(s)$, $G_2(s)$ are nondecreasing functions and $L(s)$ is nonincreasing function, so

$$I(0) \le I(s) \le I(1), \text{ for } s \in [0,1] \tag{2.12}$$

and

$$L(1) \le L(s) + G_2(s) \le L(0) + G_2(1), \text{ for } s \in [0,1]. \tag{2.13}$$

In what follows we refer to the Dirichlet problem (2.1), (2.3) and (2.2) as Problem (P_D), and to the Neumann problem (2.1), (2.2) and (2.4) as Problem (P_N). To begin with we define the notions of weak solutions of both problems.

Definition 1. *We call (b, w, h) a weak solution of Problem (P_D) on $[0, T]$, if it satisfies*

1. $(b, w, h) \in C([0, T] : L^1(\Omega)^3) \cap L^\infty(Q_T)^3$ *and* $b,\ w,\ h^m \in L^\infty(0, T; H_0^1(\Omega))$
2. *for all* $\psi \in C^1(\overline{Q}_T) \cap L^2(0, T; H_0^1(\Omega))$

$$\int_\Omega b(t)\psi(t) + d_b \int_0^t \int_\Omega \{\nabla b \cdot \nabla\psi - b\psi_t\}\, dxd\tau = \int_\Omega b_0\psi(0) + \\ + \int_0^t \int_\Omega \{G_1(b)w(1-b)b - b\}\psi, \tag{2.14}$$

$$\int_\Omega w(t)\psi(t) + d_w \int_0^t \int_\Omega \{\nabla w \cdot \nabla\psi - w\psi_t\} dxd\tau = \int_\Omega w_0\psi(0) + \\ + \int_0^t \int_\Omega \{-\left(L(b) + G_2(b)\right) w + I(b)h^\alpha\}\psi, \tag{2.15}$$

$$\int_\Omega h(t)\psi(t) + d_h \int_0^t \int_\Omega \{\nabla h^m \cdot \nabla\psi - h\psi_t\} dxd\tau = \int_\Omega h_0\psi(0) + \\ + \int_0^t \int_\Omega \{p - I(b)h^\alpha\}\psi\, dxd\tau. \tag{2.16}$$

Definition 2. *We call (b, w, h) a weak solution of Problem (P_N) on $[0, T]$, if it satisfies*

1. $U \in C([0, T] : L^1(\Omega)^3) \cap L^\infty(Q_T)^3$ *and* $b,\ w,\ h^m \in L^\infty(0, T; H^1(\Omega))$,
2. *for all* $\psi \in C^1(\overline{Q}_T) \cap L^2(0, T; H^1(\Omega))$ b, w, h *satisfy (2.14)–(2.16).*

As a matter of fact, in the case of Dirichlet boundary conditions we shall be able to prove, additionally, that the weak solutions are in fact continuous functions if (2.6) holds true. Other regularity properties could be obtained by different techniques (see, e.g. [23]).

2.1. **The regularized system to (P_D).**

The main difficulty for the study of Problem (P_D) is the fact that the equation for h is degenerate. Here, we overcome this difficulty by defining a sequence of approximating uniformly parabolic problems for which classical solutions exist. Finally, we prove existence of Problem (P_D) by passing to the limit thanks to some a priori estimates.

For $\epsilon \in (0, 1)$, $\kappa \geq 1$ and $0 < \alpha < 1$, we let

$$\phi_\epsilon(s) := (s + \epsilon)^m - \epsilon^m,$$

and

$$f_\epsilon(s) := (s + \epsilon)^\alpha - \epsilon^\alpha.$$

We consider the regularized system

$$\begin{cases} \partial_t b_\epsilon - d_b \Delta b_\epsilon + b_\epsilon = G_1(b_\epsilon) w_\epsilon (1-b_\epsilon) b_\epsilon, & \text{in } Q_T, \\ \partial_t w_\epsilon - d_w \Delta w_\epsilon + (L(b_\epsilon) + G_2(b_\epsilon))\, w_\epsilon = I(b_\epsilon) f_\epsilon(h_\epsilon), & \text{in } Q_T, \\ \partial_t h_\epsilon - d_h \Delta(\phi_\epsilon(h_\epsilon)) + I(b_\epsilon) f_\epsilon(h_\epsilon) = p_\epsilon, & \text{in } Q_T, \\ b = w = h = 0, & \text{on } S_T, \\ b_\epsilon(x,0) = b_{0,\epsilon}(x),\ w_\epsilon(x,0) = w_{0,\epsilon}(x), & \\ h_\epsilon(x,0) = h_{0,\epsilon}(x), & \text{for } x \in \Omega, \end{cases} \tag{2.17}$$

where $p_\epsilon \in C^\infty$ such that

$$0 \le p_\epsilon \le \|p\|_{L^\infty(Q_T)}, \tag{2.18}$$

$$\|p_\epsilon - p\|_{L^1(Q_T)} \to 0 \text{ as } \epsilon \to 0, \tag{2.19}$$

for $T > 0$ arbitrary and the initial conditions $b_{0,\epsilon}, w_{0,\epsilon}, h_{0,\epsilon}$, with

$$b_\epsilon(0) = b_{0,\epsilon},\ w_\epsilon(0) = w_{0,\epsilon},\ h_\epsilon(0) = h_{0,\epsilon} \in C_c^\infty(\Omega),$$

such that

$$0 \le b_{0,\epsilon}(x) \le \|b_0\|_{L^\infty(\Omega)},\ 0 \le w_{0,\epsilon}(x) \le \|w_0\|_{L^\infty(\Omega)},\ 0 \le h_{0,\epsilon}(x) \le \|h_0\|_{L^\infty(\Omega)} \tag{2.20}$$

for a.e $x \in \Omega$ and

$$(b_{0,\epsilon}, w_{0,\epsilon}, h_{0,\epsilon}) \to (b_0, w_0, h_0) \text{ in } L^1(\Omega)^3 \text{ as } \epsilon \to 0. \tag{2.21}$$

We also note for later use that (2.18), (2.19) and (2.20), (2.21) imply

$$p_\epsilon \to p \text{ in } L^q(Q_T) \text{ as } \epsilon \to 0, \tag{2.22}$$

and

$$(b_{0,\epsilon}, w_{0,\epsilon}, h_{0,\epsilon}) \to (b_0, w_0, h_0) \text{ in } L^q(\Omega)^3 \text{ as } \epsilon \to 0, \tag{2.23}$$

for all $q > 1$.

Under the above considerations the following result holds

Theorem 1. *For every $\epsilon \in (0,1)$, problem (2.17) possesses a unique classical solution $(b_\epsilon, w_\epsilon, h_\epsilon)$ such that*

$$0 \le b_\epsilon \le 1,\ \textit{in } Q_T \tag{2.24}$$

and there exists a positive constant $\bar{C}$ such that

$$0 \le w_\epsilon,\ h_\epsilon \le \bar{C},\ \textit{in } Q_T, \tag{2.25}$$

where $\bar{C}$ does not depend on ϵ.

Proof. The existence of classical solution of (2.17) for the non-negative initial data $(b_{0,\epsilon}, w_{0,\epsilon}, h_{0,\epsilon})$ follows from [20]. Moreover, from the classical maximum principle we have that for $b_{\epsilon,0} \in [0,1]$, $0 \le b_\epsilon \le 1$. Similarly, we can show that $w_\epsilon, h_\epsilon \ge 0$. Next we prove that h_ϵ is bounded from above. We first recall the definition of the negative and positive parts of a function f, namely $(f)_+ = \max\{f, 0\}$, $(f)_- = \max\{-f, 0\}$. We set, $\hat{h} = h_\epsilon - \bar{h}$, with $\bar{h}$ an arbitrary positive constant to be determined later. We multiply the equation for h_ϵ in (2.17), by $\hat{h}_+$ and integrate over Ω to obtain

$$\int_\Omega \frac{\partial h_\epsilon}{\partial t} \hat{h}_+\, dx - d_h \int_\Omega \Delta\phi_\epsilon(h_\epsilon) \hat{h}_+\, dx + \int_\Omega I(b_\epsilon) f_\epsilon(h_\epsilon) \hat{h}_+\, dx = \int_\Omega p_\epsilon \hat{h}_+\, dx, \tag{2.26}$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\hat{h}_+|^2\,dx + d_h\int_\Omega \phi_\epsilon'(h_\epsilon)|\nabla \hat{h}_+|^2\,dx + \int_\Omega I(b_\epsilon)f_\epsilon(h_\epsilon)\hat{h}_+\,dx = \int_\Omega p_\epsilon \hat{h}_+\,dx. \tag{2.27}$$

From (2.12), (2.27) and the fact that, $\phi_\epsilon'(h_\epsilon) > 0$, $0 \le p_\epsilon(t,x) \le \|p\|_{L^\infty(Q_T)}$, we have that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\hat{h}_+|^2 + I(0)\int_\Omega f_\epsilon(h_\epsilon)\hat{h}_+ \le \|p\|_{L^\infty(Q_T)}\int_\Omega \hat{h}_+\,dx, \tag{2.28}$$

from which we infer that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\hat{h}_+|^2 + I(0)\int_\Omega \left(f_\epsilon(h_\epsilon) - f_\epsilon(\bar{h})\right)(h_\epsilon - \bar{h})_+ \\ \le \|p\|_{L^\infty(Q_T)}\int_\Omega \hat{h}_+\,dx,\end{aligned} \tag{2.29}$$

which we may write as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\hat{h}_+|^2 + I(0)\int_\Omega \left(f_\epsilon(h_\epsilon) - f_\epsilon(\bar{h})\right)(h_\epsilon - \bar{h})_+ \\ \le \left(\|p\|_{L^\infty(Q_T)} - I(0)f_\epsilon(\bar{h})\right)\int_\Omega \hat{h}_+\,dx,\end{aligned} \tag{2.30}$$

Thanks to the monotonicity of $\phi_\epsilon(\cdot)$, the second term on the left hand side of the above inequality is nonnegative. Next, we look for $\bar{h} > \|h_0\|_{L^\infty(\Omega)}$ such that

$$\|p\|_{L^\infty(Q_T)} - I(0)f_\epsilon(\bar{h}) \le 0. \tag{2.31}$$

Since, $-f_\epsilon(\bar{h}) \le (1 - \bar{h}^\alpha)$, we may choose

$$\bar{h} := \max\left\{\left(\frac{\|p\|_{L^\infty(Q_T)}}{I(0)} + 1\right)^{1/\alpha}, \|h_0\|_{L^\infty(\Omega)}\right\}, \tag{2.32}$$

so that

$$\frac{d}{dt}\int_\Omega |\hat{h}_+|^2(t)\,dx \le 0, \tag{2.33}$$

which in turn implies that

$$|\hat{h}_+(t)|^2_{L^2(\Omega)} \le |\hat{h}_+(0)|^2_{L^2(\Omega)} = |(h_0 - \bar{h})_+|^2_{L^2(\Omega)} = 0\,, \tag{2.34}$$

for $\bar{h}$ given by (2.32), and so,

$$h_\epsilon \le \bar{h} \text{ in } Q_T. \tag{2.35}$$

To obtain an upper bound for w_ϵ we work similarly. We set $\hat{w} = w_\epsilon - \bar{w}$, where $\bar{w}$ is a positive constant to be determined later, we multiply the equation for w_ϵ in (2.17) by $\hat{w}_+$ and integrate over Ω to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\hat{w}_+|^2\,dx + \delta_w\int_\Omega |\nabla \hat{w}_+|^2\,dx + \\ + \int_\Omega (L(b_\epsilon) + G_2(b_\epsilon))(|\hat{w}_+|^2 + \bar{w}\hat{w}_+)\,dx = \int_\Omega I(b_\epsilon)f_\epsilon(h_\epsilon)\hat{w}_+\,dx.\end{aligned} \tag{2.36}$$

where we have used the equality $w_\epsilon = (w_\epsilon - \bar{w})_+ - (w_\epsilon - \bar{w})_- + \bar{w}$. Then, from (2.12), (2.35) and the fact that $f_\epsilon(s) \le s^\alpha$ for $s \ge 0$, we have that

$$\int_\Omega I(b_\epsilon) f_\epsilon(h_\epsilon) \hat{w}_+ \, dx \le I(1)\bar{h}^\alpha \int_\Omega \hat{w}_+ \, dx. \tag{2.37}$$

So, using (2.13) and dropping the appropriate non-negative terms on the left hand side of (2.36), we end up with

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\hat{w}_+|^2 \, dx + \left(\bar{w}L(1) - I(1)\bar{h}^\alpha\right)\int_\Omega \hat{w}_+ \, dx \le 0. \tag{2.38}$$

Therefore, arguing as before we may choose

$$\bar{w} := \max\left\{\frac{I(1)\bar{h}^\alpha}{L(1)}, \|w_0\|_{L^\infty(\Omega)}\right\},$$

so that

$$w_\epsilon \le \bar{w}, \text{ in } Q_T. \tag{2.39}$$

□

Next we remark that b_ϵ satisfies a problem of the form

$$\begin{cases} b_{\epsilon t} = d_b \Delta b_\epsilon + F_\epsilon & \text{in } Q_T \\ b_\epsilon = 0 & \text{on } S_T \\ b_\epsilon(x,0) = b_{0,\epsilon}(x) & \text{in } \Omega \end{cases} \tag{2.40}$$

where

$$0 \le b_{0,\epsilon} \le \|b_0\|_{L^\infty(\Omega)} \le 1$$

and

$$F_\epsilon \in L^\infty(Q_T). \tag{2.41}$$

Multiplying the equation by b_ϵ and integrating by parts, we deduce that

$$\|b_\epsilon\|_{L^2(0,T;H_0^1(\Omega))} \le C, \tag{2.42}$$

Further taking the duality product $< \cdot,\cdot >_{(H^{-1},H_0^1)}$ of $b_{\epsilon t}$ with an arbitrary test function from $L^2(0,T;H_0^1(\Omega))$, we deduce that

$$\|b_{\epsilon t}\|_{L^2(0,T;H^{-1}(\Omega))} \le C. \tag{2.43}$$

Then, the inequalities (2.42) and (2.43) imply that

$$\{b_\epsilon\} \text{ is relatively compact in } L^2(Q_T). \tag{2.44}$$

(cf. [22, Theorem 2.1 p. 27]. We deduce that there exist a function $b \in L^2(0,T;H_0^1(\Omega))$ with $b_t \in L^2(0,T;H^{-1}(\Omega))$ and a subsequence $\{b_{\epsilon_j}\}$ of $\{b_\epsilon\}$ such that

$$b_{\epsilon_j} \to b \text{ strongly in } L^2(Q_T) \tag{2.45}$$

$$b_{\epsilon_j t} \to b_t \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{2.46}$$

Moreover, it follows from [22, Lemma 1.2 p.260] that

$$b \in C([0,T];L^2(\Omega)). \tag{2.47}$$

Finally, it is clear that $0 \leq b \leq 1$ for all $t \in [0,T]$ and a.e. $x \in \Omega$. Since, w_ϵ satisfies the equation it follows in a similar way that w_ϵ converges along a subsequence to a limit w strongly in $L^2(Q_T)$ and weakly in $L^2(0,T;H_0^1(\Omega))$ as $\epsilon \to 0$ where $w \in C([0,T];L^2(\Omega))$ and $0 \leq w \leq \bar{w}$ for all $t \in [0,T]$ and a.e. $x \in \Omega$.

Next we consider the problem for h_ϵ, namely

$$\begin{cases} \partial_t h_\epsilon = \Delta\phi_\epsilon(h_\epsilon) - I(b_\epsilon) f_\epsilon(h_\epsilon) + p_\epsilon & \text{in } Q_T, \\ h_\epsilon = 0 & \text{on } S_T, \\ h_\epsilon(x,0) = h_{0,\epsilon}(x) & \text{for } x \in \Omega, \end{cases} \tag{2.48}$$

We first prove the following estimate

Lemma 1. *We have that*

$$\frac{1}{2}\int_\Omega (h_\epsilon)^2(t)\,dx + \int_0^T \int_\Omega |\nabla \psi_\epsilon(h_\epsilon)|^2\,dx\,dt \leq C(T) \tag{2.49}$$

where $\psi_\epsilon(s) = \int_0^s \sqrt{\phi_\epsilon'(s)}\,ds$, *which in turn implies that*

$$\int_0^T \int_\Omega |\nabla \phi_\epsilon(h_\epsilon(t))|^2 \leq C(T). \tag{2.50}$$

Proof. The function h_ϵ satisfies the initial value problem

$$h_{\epsilon t} = d_h \Delta \phi_\epsilon(h_\epsilon) + G_\epsilon \tag{2.51}$$

together with zero Dirichlet boundary conditions where we have set $G_\epsilon = -I(b_\epsilon)f_\epsilon(h_\epsilon) + p_\epsilon$, and so $\|G_\epsilon\|_{L^\infty(Q_T)} \leq C$. We multiply the equation (2.51) by h_ϵ and integrate by parts to deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (h_\epsilon)^2(t)\,dx + d_h \int_\Omega \nabla\phi_\epsilon(h_\epsilon)\cdot\nabla h_\epsilon\,dxdt = \int_\Omega G_\epsilon h_\epsilon\,dx \tag{2.52}$$

which implies that

$$\begin{aligned} &\int_\Omega (h_\epsilon)^2(t)\,dx + 2d_h \int_0^T \int_\Omega \phi_\epsilon'(h_\epsilon)|\nabla h_\epsilon|^2\,dxdt \\ &\quad \leq \int_\Omega G_\epsilon^2\,dxdt + \int_0^T \int_\Omega h_\epsilon^2\,dxdt + \int_\Omega h_{0\epsilon}^2\,dx. \end{aligned} \tag{2.53}$$

Since

$$\int_0^T \int_\Omega \phi_\epsilon'(h_\epsilon)|\nabla h_\epsilon|^2\,dxdt = \int_0^T \int_\Omega (\sqrt{\phi_\epsilon'(h_\epsilon)}\nabla h_\epsilon)^2 = \int_0^T \int_\Omega |\nabla\psi_\epsilon(h_\epsilon)|^2\,dxdt, \tag{2.54}$$

we deduce that

$$\begin{aligned} \int_\Omega (h_\epsilon(T))^2\,dx + 2d_h \int_0^T \int_\Omega |\nabla\psi_\epsilon(h_\epsilon)|^2 \leq \int_0^T \int_\Omega G_\epsilon^2\,dxdt + \\ + \int_0^T \int_\Omega h_\epsilon^2\,dxdt + \int_\Omega h_{0\epsilon}^2\,dx, \end{aligned} \tag{2.55}$$

which in turn yields inequality (2.49). To prove (2.50), we observe that

$$
\begin{aligned}
\int_0^T\int_\Omega |(\nabla\phi_\epsilon(h_\epsilon(t))|^2 &= \int_0^T\int_\Omega \phi_\epsilon'(h_\epsilon(t))^2|\nabla h_\epsilon(t)|^2\,dxdt \\
&\le \sup|\phi_\epsilon'(h_\epsilon(x,t))|\int_0^T\int_\Omega \phi_\epsilon'(h_\epsilon(t))|\nabla h_\epsilon(t)|^2\,dxdt \\
&\le M\int_0^T\int_\Omega\left(\sqrt{\phi_\epsilon'(h_\epsilon(t))}\nabla h_\epsilon(t)\right)^2,
\end{aligned}
\tag{2.56}
$$

where M is independent of ϵ and so (2.50) follows from (2.54) and (2.49). □

Next we set $U_\epsilon = \phi_\epsilon(h_\epsilon)$ and $\beta_\epsilon(\cdot) = \phi_\epsilon^{-1}(\cdot)$, to apply the result of [15, Theorem 6.2].

Lemma 2. (i) *For all $\tau > 0$, the function U_ϵ is equicontinuous in $\bar{Q}_T^\tau$. Precisely, there exists a continuous nondecreasing function $\omega_\tau(\cdot)$ with $\omega_\tau(0) = 0$, such that*

$$
|U_\epsilon(x_1,t_1) - U_\epsilon(x_2,t_2)| \le \omega_\tau(|x_1-x_2| + |t_1-t_2|^{1/2})
\tag{2.57}
$$

for all $(x_i,t_i)\in\bar{Q}_T^\tau$, $i=1,2$. The function ω_τ does not depend on ϵ.

(ii) *If in addition $U(0,x) = U_0(x)\in C(\bar\Omega)$, then $\{U_\epsilon\}$ is equicontinuous on $\bar{Q}_T$.*

We deduce from Lemma 2(i) that for all $\tau > 0$, U_{ϵ_j} is precompact in $C(\bar{Q}_T^\tau)$ and thus there exists a subsequence that we denote again by U_{ϵ_j} and a function $\zeta\in C(\bar{Q}_T^\tau)$ such that

$$
U_{\epsilon_j}\to\zeta,
$$

uniformly in $\bar{Q}_T^\tau$ as $\epsilon_j\to 0$. Then,

$$
\begin{aligned}
|h_{\epsilon_j} - \zeta^{1/m}| &= |\beta_\epsilon(U_{\epsilon_j}) - \zeta^{1/m}| \\
&\le |\beta_{\epsilon_j}(U_{\epsilon_j}) - (U_{\epsilon_j})^{1/m}| + |(U_{\epsilon_j})^{1/m} - \zeta^{1/m}|,
\end{aligned}
\tag{2.58}
$$

Therefore, since for all $\epsilon > 0$, $|\beta_\epsilon(U_\epsilon) - (U_\epsilon)^{1/m}| < 2\epsilon$, setting $h = \zeta^{1/m}$ we have that

$$
\begin{aligned}
&h_{\epsilon_j}\to h, \text{ uniformly in } \bar{Q}_T^\tau,\\
&\phi_\epsilon(h_{\epsilon_j})\to h^m \text{ uniformly in } \bar{Q}_T^\tau,
\end{aligned}
\tag{2.59}
$$

as $\epsilon_j\to 0$, for all $\tau > 0$. Moreover, from Lemma 1 there exists a subsequence of $\{h_{\epsilon_j}\}$ which we denote again by h_{ϵ_j} and a function $\chi\in L^2((0,T);H_0^1(\Omega))$ such that

$$
\phi_{\epsilon_j}(h_{\epsilon_j}) \rightharpoonup \chi \text{ weakly in } L^2(0,T;H_0^1(\Omega)),
\tag{2.60}
$$

as $\epsilon_j\to 0$. Since, $L^2(\Omega)\subset H^{-1}(\Omega)$ we further deduce that $\phi_{\epsilon_j}(h_{\epsilon_j})\rightharpoonup\chi$ weakly in $L^2(Q_T)$. On the other hand, $\phi_{\epsilon_j}(h_{\epsilon_j}) \le (h_{\epsilon_j})^m \le \bar{h}^m$ and from (2.59) we have that $\phi_{\epsilon_j}(h_{\epsilon_j})\to h^m$ a.e in Q_T. Then, by the dominated convergence theorem we deduce that $\phi_{\epsilon_j}(h_{\epsilon_j})\to\phi(h)$ strongly in $L^2(Q_T)$. Therefore, $\phi(h_{\epsilon_j})\rightharpoonup h^m$ weakly in $L^2(Q_T)$ and uniqueness of the weak limits implies that $\chi = h^m$. Hence, we conclude that

$$
\phi_{\epsilon_j}(h_{\epsilon_j}) \rightharpoonup h^m \text{ weakly in } L^2(0,T;H_0^1(\Omega)),
\tag{2.61}
$$

as $\epsilon_j\to 0$.

Next we prove that (b,w,h) is a weak solution of Problem (P). We multiply the three partial differential equations in (2.17) by $\psi \in C^1(\overline{Q}_T) \cap L^2(0,T;H_0^1(\Omega))$ and integrate by parts to obtain (here for simplicity $\epsilon = \epsilon_j$)

$$\int_\Omega b_\epsilon(t)\psi(t) + d_b \int_0^t \int_\Omega \{\nabla b_\epsilon \cdot \nabla\psi - b_\epsilon \psi_t\}\, dx d\tau = \int_\Omega b_{0,\epsilon}\psi(0) + \int_0^t \int_\Omega \{G_1(b_\epsilon) w_\epsilon (1-b_\epsilon) b_\epsilon - b_\epsilon\}\psi, \tag{2.62}$$

$$\int_\Omega w_\epsilon(t)\psi(t) + d_w \int_0^t \int_\Omega \{\nabla w_\epsilon \cdot \nabla\psi - w_\epsilon \psi_t\} dx d\tau = \int_\Omega w_{0,\epsilon}\psi(0) + \int_0^t \int_\Omega \{-\left(L(b_\epsilon) + G_2(b_\epsilon)\right) w_\epsilon + I(b_\epsilon) f_\epsilon(h_\epsilon)\}\psi, \tag{2.63}$$

$$\int_\Omega h_\epsilon(t)\psi(t) + d_h \int_0^t \int_\Omega \{\nabla \phi_\epsilon(h_\epsilon) \cdot \nabla\psi - h_\epsilon \psi_t\} dx d\tau = \int_\Omega h_{0,\epsilon}\psi(0) + \int_0^t \int_\Omega \{p_\epsilon - I(b_\epsilon) f_\epsilon(h_\epsilon)\}\psi\, dx d\tau. \tag{2.64}$$

To summarize we have that b_ϵ,w_ϵ, h_ϵ are positive and bounded. Moreover, there exists a subsequence of $(b_\epsilon, w_\epsilon, h_\epsilon)$, which converges strongly to (b,w,h) in $L^2(Q_T)^3$, and a.e in Q_T, and $(b_\epsilon, w_\epsilon, \phi_\epsilon(h_\epsilon))$ converges weakly to $(b, w, \phi(h))$ in $L^2(0,T;H_0^1)^3$. To pass to the limit in the terms involving $f_\epsilon(h_\epsilon)$ we notice that

$$|f_\epsilon(s) - s^\alpha| \le 2\epsilon^\alpha, \tag{2.65}$$

and thus,

$$\begin{aligned} |I(b_\epsilon)f_\epsilon(h_\epsilon) - I(b_\epsilon)f_\epsilon(h_\epsilon)| \le & |I(b_\epsilon)||(f_\epsilon(h_\epsilon) - f(h_\epsilon)| + |I(b_\epsilon) - I(b)||f(h_\epsilon)| \\ & + |I(b)||f(h_\epsilon) - f(h)| \\ \le & 2I(1)\epsilon^\alpha + |I(b_\epsilon) - I(b)||f(\bar{h})| \\ & + |I(1)||f(h_\epsilon) - f(h)|\,. \end{aligned} \tag{2.66}$$

Moreover, recalling (2.22) and (2.23), we can let $\epsilon \to 0$ in (2.62)-(2.64), to obtain the integral identities (2.14)-(2.16). Finally, if $h_0 \in C(\bar{\Omega})$ from Lemma 2(ii) $h \in C(\bar{Q}_T)$.

In fact, a weak solution of Problem (P_D) exists even if h_0 is just essentially bounded, since working as above we know that there exists $h \in C((0,T] : L^1(\Omega))$. We would also like to know if $\|h(t)\|_{L^1(\Omega)}$ is continuous at 0. To this end let $h_{0,n}$ be a sequence of smooth bounded functions which converges to h_0 in $L^1(\Omega)$. Working as above and using Lemma 2(ii) there exists a solution of the system, denoted by h_n, obtained as a limit of the approximating system such that $h_n \in C(\bar{Q}_T)$. Next note that

$$\|h(t) - h_0\|_{L^1(\Omega)} \le \|h(t) - h_n(t)\|_{L^1(\Omega)} + \|h_n(t) - h_{0,n}\|_{L^1(\Omega)} + \|h_{0,n} - h_0\|_{L^1(\Omega)}, \tag{2.67}$$

where the second term on the right hand side goes to zero as t tends to 0, while the last term becomes arbitrarily small for n large enough. On the other hand, for any $\epsilon > 0$ and $h_1(0)$ and $h_2(0)$

smooth initial data, the solutions $h_{1,\epsilon}$ and $h_{2,\epsilon}$ of the corresponding approximating problems satisfy

$$\begin{aligned}\|h_{1,\epsilon}(t)-h_{2,\epsilon}(t)\|_{L^1(\Omega)} &\le \|h_{1,\epsilon}(0)-h_{2,\epsilon}(0)\|_{L^1(\Omega)}+\int_0^t \|b_{1,\epsilon}(s)-b_{2,\epsilon}(s)\|_{L^1(\Omega)}\,ds\\ &\le \|h_{1,\epsilon}(0)-h_{2,\epsilon}(0)\|_{L^1(\Omega)}+tD.\end{aligned}$$

where D is a positive constant. Letting $\epsilon\to 0$, this in turn implies that

$$\|h_1(t)-h_2(t)\|_{L^1(\Omega)} \le \|h_1(0)-h_2(0)\|_{L^1(\Omega)}+tD. \tag{2.68}$$

Finally, by (2.67) and (2.68)

$$\begin{aligned}\|h(t)-h_0\|_{L_1(\Omega)} \le &\|h(0)-h_n(0)\|_{L^1(\Omega)}\\ &+tD+\|h_n(t)-h_{0,n}\|_{L_1(\Omega)}+\|h_{0,n}-h_0\|_{L_1(\Omega)}\end{aligned} \tag{2.69}$$

Then $t\mapsto \|h(t)\|_{L^1(\Omega)}$ is continuous at zero and so $h\in C([0,T];L^1(\Omega))$. We thus have the following result

Theorem 2. *If the initial condition (h_0,b_0,w_0) satisfies (2.5) and (2.7), then there exists a weak solution $(b,\,w,\,h)$ of Problem (P_D) such that $0\le b\le 1$, $0\le w\le \bar{w}$ and $0\le h\le \bar{h}$. If in addition $h_0,b_0,w_0\in C(\bar{\Omega})$, then $h\in C(\bar{Q}_T)$ and $b,w\in C(\bar{\Omega}\times[\delta,T])$ for all $\delta>0$.*

Proof. To complete the proof we note that from (2.41) and [20, Theorem 9.1 p. 341] it follows that

$$\|b_\epsilon\|_{W_q^{2,1}(Q_T^\delta)} \le C(\delta,T,q,\Omega) \tag{2.70}$$

for all $\delta\in(0,T)$ and all $q\in(1,\infty)$, where $Q_\delta^T=(\delta,T)\times\Omega$ and $W_q^{2,\,1}(Q_\delta^T)=W^{1,\,q}(\delta,T;L^q(\Omega))\cap L^q(\delta,T;W^{2,\,q}(\Omega)\cap W_0^{1,\,q}(\Omega))$. This in turn implies that

$$\|b_\epsilon\|_{C^{\alpha,\,\frac{\alpha}{2}}(\bar{\Omega}\times[\delta,T])} \le C, \tag{2.71}$$

for $\alpha=2-\frac{N+2}{q}$ and $q\ne N+2$ [6, Lemma 3.5]. Therefore, we can conclude (passing if necessary to another subsequence) that $b_{\epsilon_j}\to b$ uniformly in $\bar{Q}_T^\delta$ for all $\delta>0$ and so $b\in C(\bar{\Omega}\times[\delta,T])$. Similarly, $w\in C(\bar{\Omega}\times[\delta,T])$. ☐

2.2. **The case of Neumann boundary conditions (P_N).** Although the above strategy can also be adapted to obtain the existence of weak solutions in the case of the Neumann problem (P_N), in this section, we exploit a different approach which is based on a fixed point argument.

Theorem 3. *There exists a weak solution of Problem (P_N).*

Before, giving the proof of Theorem 3 it is useful to state a lemma related to the problem

$$\begin{cases}\partial_t u-\Delta\varphi(u)=v & \text{in } Q_T,\\ \frac{\partial\varphi(u)}{\partial n}=0 & \text{on } S_T,\\ u(x,0)=u_0(x) & \text{for } x\in\Omega.\end{cases} \tag{2.72}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is nondecreasing continuous function with $\varphi(0)=0$, $u_0\in L^1(\Omega)$ and $v\in L^1(0,T;L^1(\Omega))$. It is known that problem (2.72) possesses a unique weak solution (see, e.g., [5] and [23]). For fixed u_0, let us denote by u_v the unique weak solution of (2.72) for some $v\in L^1(0,T;L^1(\Omega))$.

Lemma 3. *Suppose that $\varphi:\mathbb{R}\to\mathbb{R}$ is a strictly increasing continuous function with $\varphi(0)=0$, then*

(i) *for each fixed* $u_0 \in L^1(\Omega)$ *and a weakly relatively compact set* $\mathcal{K}$ *in* $L^1(0,T;L^1(\Omega))$*, the set* $\{u_v : v \in \mathcal{K}\}$ *is relatively compact in* $C([0,T];L^1(\Omega))$*,*

(ii) *for each fixed* $u_0 \in L^\infty(\Omega)$ *and a bounded set* $\mathcal{K}$ *in* $L^\infty(0,T;L^\infty(\Omega))$ *the mapping* $v \mapsto u_v$*, is sequentially continuous from* $\mathcal{K}$ *endowed with the weak topology of* $L^1(0,T;L^1(\Omega))$ *into* $C([0,T];L^p(\Omega))$ *endowed with the strong topology, for all* $p \in [1,\infty)$*.*

For the proof of Lemma 3 (i) we refer to the results of Diaz-Vrabie [12, 13]. It must be pointed out that although the compactness results of the above references concern the case of Dirichlet boundary conditions, the arguments are identical for the case of Neumann boundary conditions (see [13, Section 2] and [5]). On the other hand Lemma 3 (ii) is a consequence of the counterpart (i) thanks to the uniqueness of the weak solution (for details see [14, Corollary 1, Section 2] or [14, Corollary 3.1] for the case $p = 1$).

For convenience in what follows, if $\mathbf{u} = (u_1, u_2, u_3)$ is a vector function with $u_i \in X$, where X is a Banach space, we shall make use of the notation $\|\mathbf{u}\|_X := \max_{i=1,2,3}\{\|u_i\|_X\}$.

Proof of Theorem 3. Let us start with the existence of a local (in time) weak solution of (P_N). We introduce the reaction functions $\mathbf{R} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{R}(b,w,h) = (R_1(b,w,h), R_2(b,w,h), R_3(b,w,h))$$

with

$$\begin{cases} R_1(b,w,h) = wG_1(b)(1-b)b - b \\ R_2(b,w,h) = -(\tilde{L}(b) + G_2(b))w + \tilde{I}(b)h|h|^{\alpha-1} \\ R_3(b,w,h) = -\tilde{I}(b)h|h|^{\alpha-1}. \end{cases} \tag{2.73}$$

where $\tilde{I}(b)$ (respectively $\tilde{L}(b)$) is a truncation of $I(b)$ (respectively $L(b)$) extending it continuously by a constant equal to $I(0)$ (respectively $L(0)$) for $b < 0$. We choose $K > 0$ such that

$$\max(\|b_0\|_{L^\infty(\Omega)}, \|w_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\Omega)}) + 1 \le K.$$

Since the functions $R_i : \mathbb{R}^3 \to \mathbb{R}$ are continuous it is possible to find $M > 0$, such that

$$\max\{|R_1(b,w,h)|, |R_2(b,w,h)|, |R_3(b,w,h)| + \|p\|_{L^\infty(Q_T)}\} \le M$$

assumed that

$$0 \le b, w, h \le K.$$

Now we define the "solution operator" $\mathcal{S} : L^1(0,T : L^1(\Omega))^3 \to C([0,T];L^2(\Omega))^3$ by $\mathcal{S}(f,g,v) = (b,w,h)$ where b, w, h are the unique weak solutions of the decoupled system

$$\begin{cases} \partial_t b - d_b \Delta b = f & \text{in } Q_T, \\ \partial_t w - d_w \Delta w = g & \text{in } Q_T, \\ \partial_t h - d_h \Delta h^m = v & \text{in } Q_T, \\ \frac{\partial b}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial h}{\partial n} = 0 & \text{on } S_T, \\ b(x,0) = b_0(x),\ w(x,0) = w_0(x),\ h(x,0) = h_0(x) & \text{for } x \in \Omega. \end{cases}$$

Next, to control some a priori estimates it is useful to introduce the following convex set (adapted to the reaction terms $\mathbf{R}(b,w,h)$):

$$\mathcal{K}_{r,T_0} = \{(f,g,v) : f,g,v \in L^1(0,T_0 : L^1(\Omega)), \|(f,\ g,\ v)\|_{L^\infty(Q_{T_0})} \le r\},$$

where $Q_{T_0} := (0,T_0) \times \Omega$, $r \ge M$ and $T_0 \in (0,T]$ is such that

$$\mathcal{S}(\mathcal{K}_{r,T_0}) \subset B_{L^\infty(Q_{T_0})}(\mathbf{0},K)$$

with $B_{L^\infty(Q_{T_0})}(\mathbf{0},K) := \{\mathbf{u} \in L^\infty(Q_{T_0})^3 : \|\mathbf{u}\|_{L^\infty(Q_{T_0})}) \leq K\}$. Recall that M depends on K through the properties of $\mathbf{R}$. Moreover, it is not difficult to see that $\mathcal{K}_{r,T_0}$ is nonempty and weakly compact in $(L^1(0,T_0 : L^1(\Omega)))^3$. Next let us define the restriction of the solution operator on $\mathcal{K}_{r,T_0}$:

$$\widehat{\mathcal{S}} = \mathcal{S}|_{\mathcal{K}_{r,T_0}} : \mathcal{K}_{r,T_0} \to L^\infty(Q_{T_0})^3.$$

We also define the composition of the realization operator associated to $\mathbf{R}$ and $\widehat{\mathcal{S}}$, namely, the operator $\mathcal{R} : \mathcal{K}_{r,T_0} \to C([0,T]; L^2(\Omega))^3$ defined by

$$\mathcal{R}(f,g,v) = (R_1(\widehat{\mathcal{S}}(f,g,v)), R_2(\widehat{\mathcal{S}}(f,g,v)), R_3(\widehat{\mathcal{S}}(f,g,v)) + p)$$

i.e. $\mathcal{R}(f,g,v) = \mathbf{R}(b,w,h)$ with $(b,w,h) = \widehat{\mathcal{S}}(f,g,v)$. Then, from the choice of the set $\mathcal{K}_{r,T_0}$ we know that $\mathcal{R}$ maps $\mathcal{K}_{r,T_0}$ into $\mathcal{K}_{r,T_0}$.

Next we prove that there exists at least one fixed point of $\mathcal{R} : \mathcal{K}_{r,T_0} \to \mathcal{K}_{r,T_0}$. This will be a consequence of a variant of the Schauder fixed point theorem given in [24, Theorem 1.2.11], which requires $\mathcal{R}$ to be weakly-weakly sequentially continuous. It is actually enough to show that the graph of $\mathcal{R}$, is weakly-weakly sequentially closed [24, Corollary 1.2.5]. To this end, let $\{(f_n, g_n, v_n)\}_{n\in\mathbb{N}} \in \mathcal{K}_{r,T_0}$ and $\{(F_n, G_n, V_n)\}_{n\in\mathbb{N}} \in \mathcal{R}(f_n, g_n, v_n)$ be sequences which converge weakly in $(L^1((0,T); L^1(\Omega)))^3$ to (f,g,v) and (F,G,V), respectively. Then from Lemma 3 (ii) $\widehat{\mathcal{S}}$ is weakly-strongly sequentially continuous from $L^1(0,T_0; L^1(\Omega))$ into $C([0,T_0]; L^p(\Omega))$ and so we may assume without loss of generality (taking a subsequence if necessary) that

$$\widehat{\mathcal{S}}(f_n, g_n, v_n) \to \widehat{\mathcal{S}}(f,g,v) \text{ a.e in } Q_{T_0}, \tag{2.74}$$

which, by continuity of R_i, implies that

$$R_i(\widehat{\mathcal{S}}(f_n, g_n, v_n)) \to R_i(\widehat{\mathcal{S}}(f,g,v)) \text{ a.e in } Q_{T_0}. \tag{2.75}$$

Moreover, $R_i(\widehat{\mathcal{S}}(f_n, g_n, v_n))$ is a.e bounded in Q_{T_0} and therefore by the dominated convergence theorem we have that $\mathcal{R}(f_n, g_n, v_n) \to \mathcal{R}(f,g,v)$ strongly in $(L^1(Q_{T_0}))^3$. Consequently, by uniqueness of weak limits we conclude that $(F,G,V) = \mathcal{R}(f,g,v)$.

Therefore, the graph of $\mathcal{R}$ is weakly-weakly sequentially closed and so $\mathcal{R}$ has at least one fixed point (f,g,v). Since $(b,w,h) = \widehat{\mathcal{S}}(f,g,v)$ we conclude that (b,w,h) is a weak solution of the problem (P_N) on the cylinder $Q_{T_0} := (0,T_0) \times \Omega$, i.e. a local (in time) solution of (P_N) on Q_{T_0}.

Now it only remains to prove that no possible blow-up of the norm in $C([0,T]; L^2(\Omega))^3$ may arise to get the continuation of the local weak solution to the whole cylinder Q_T. But for the reaction terms $\mathbf{R}(b,w,h)$ given by (2.73) and for positive initial conditions satisfying (2.5), (2.7) this is an easy task: indeed, similar arguments to the ones of the proof of Theorem 1 show that the local weak solution satisfies that

$$0 \leq b \leq 1, 0 \leq w \leq \overline{C} \text{ a.e. in } Q_{T_0},$$

where $\overline{C} > 0$ is independent of T_0 and, by well-known estimates for the porous medium with monotone absorption

$$0 \leq h \leq \|h_0\|_{L^\infty(\Omega)} + T\|p\|_{L^\infty(Q_T)} \text{ a.e. in } Q_{T_0}$$

which is also independent of T_0. Therefore, the local weak solution can be extended, by taking T_0 as initial time and the values of b, w, h, at $t = T_0$ as new initial data, to the complete cylinder Q_T producing at least one global weak solution of (P_N) in view of the fact that b, w, h are nonnegative.

□

Remark. The above type of arguments can be applied to prove the convergence of some numerical algorithms that relies in suitable decoupling of the system and applying the Díaz-Vrabie ([12]) *ad hoc* compactness argument. For instance we can consider the following iterative argument: we solve the uniformly parabolic equations by prescribing the h-component

$$\left\{ \begin{array}{ll} \partial_t b_n = d_b \Delta b_n + w_n G_1(b_n)(1-b_n)b_n - b_n & \text{in } Q_T, \\ \partial_t w_n = d_w \Delta w_n - (L(b_n) + G_2(b_n))w_n + I(b_n)h_{n-1}^{\alpha} & \text{in } Q_T, \end{array} \right. \tag{2.76}$$

together with the initial conditions,

$$b_n(x,0) = b_0(x),\ w_n(x,0) = w_0(x),\ \text{ for } x \in \Omega, \tag{2.77}$$

and Neumann boundary conditions

$$\frac{\partial b_n}{\partial n} = \frac{\partial w_n}{\partial n} = 0,\ \text{ on } \partial\Omega \times (0,T). \tag{2.78}$$

Then we solve the degenerate equation by prescribing the b-component

$$(P_{h,n}) = \left\{ \begin{array}{lr} \partial_t h_n = d_h \Delta h_n^m - I(b_{n-1})h_n^{\alpha} + p & \text{in } Q_T, \\ \frac{\partial h_n}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ h_n(x,0) = h_0(x) & \text{for } x \in \Omega. \end{array} \right.$$

Obviously the iteration starts with the initial data. The existence of weak solutions for the decoupled problems are easy modifications of previous results in the literature (or they can be obtained by following some ideas of the preceding section for the treatment of the Dirichlet case). The convergence of the algorithm is a small variant of the proof of Theorem 3.

3. Some qualitative properties of the surface water component

In this section, we focus on the qualitative properties of the surface water component h investigating the impact of dry periods on the zero set of h. More precisely, we start by assuming that precipitation is negligible for sufficiently long time, in the sense that

$$p(t) = 0 \text{ for } t \in (0,T), \tag{3.1}$$

with T large enough. Then, we will show that h vanishes after a finite time for the Dirichlet boundary conditions. As a second qualitative property, we will consider a compactly supported initial condition h_0 and we will show that h has a compact support (which defines a free boundary during a dry period in which $p = 0$).

In what follows, without loss of generality we suppose that $\delta_h = 1$. We let (b,w,h) be a solution of system (2.1) for a non-negative and bounded initial datum (b_0, w_0, h_0), with $0 \le b_0 \le 1$. In order to determine the properties of h it suffices to study the following scalar equation

$$\partial_t h - \Delta h^m + I(b(t,x))h^{\alpha} = 0 \text{ in } Q_T, \tag{3.2}$$

which involves the bounded solution component b. We consider (3.2), subject to the homogeneous Dirichlet boundary conditions

$$h(t,x) = 0,\ \text{ on } (0,T) \times \partial\Omega, \tag{3.3}$$

and a given non-negative initial datum

$$h_0(x) = h(0,x),\ x \in \Omega. \tag{3.4}$$

At the end of the section some remarks are given concerning the homogeneous Neumann boundary conditions as well as the non-homogeneous Dirichlet boundary conditions. We point out that these qualitative behavior properties can be proved by means of some energy methods (see, e.g. [4]) but here we shall use some comparison arguments because they are simpler and lead to sharper estimates.

3.1. **Extinction in finite time.** We recall that $0 \leq b_0 \leq 1$ implies $0 \leq b \leq 1$, and thus $I(0) \leq I(b(t,x) \leq I(1)$. Therefore, if h satisfies (3.2)–(3.4) and $\bar{U}$ is such that

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial t} - \Delta \bar{U}^m + I(0)\bar{U}^\alpha \geq 0 & \text{in } \Omega \\ \bar{U}(t,x) \geq h(t,x) & \text{on } (0,T) \times \Omega \\ \bar{U}(0,x) \geq h(0,x) & \text{in } \Omega \end{cases} \tag{3.5}$$

since

$$\frac{\partial h}{\partial t} - \Delta h^m + I(0)h^\alpha \leq \frac{\partial h}{\partial t} - \Delta h^m + I(b(t,x))h^\alpha = 0. \tag{3.6}$$

by comparison we have that $h \leq \bar{U}$ in Q_T [8]. This simple observation leads to the following

Theorem 4. *Let (3.1) hold true and let (b,w,h) be a solution of problem (2.1)–(2.3) in the time interval $(0,T)$. Then, if $T > 0$ is large enough, there exists $T^* \in (0,T)$ such that $h(t,x) = 0$ for all $t > T^*$.*

Proof. Let U be uniform in space satisfying the non-linear ODE:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \lambda U^\alpha = 0, \\ U(0) = \|h_0\|_{L^\infty(\Omega)}, \end{cases} \tag{3.7}$$

for $\lambda > 0$. Then, for $\alpha < 1$, (3.7) possesses the following explicit solution

$$U(t;\lambda) = \left(\max\{0, \|h_0\|^{1-\alpha}_{L^\infty(\Omega)} - \lambda(1-\alpha)t\}\right)^{1/(1-\alpha)}. \tag{3.8}$$

Obviously, $U(t; I(0)) = \bar{U}(t)$ satisfies (3.5). As a result, letting

$$T^*(\|h_0\|_{L^\infty(\Omega)}, I(0), \alpha) = \frac{(\|h_0\|_{L^\infty(\Omega)})^{(1-\alpha)}}{I(0)(1-\alpha)},$$

by comparison $0 \leq h(t) \leq \bar{U}(t)$, $h(t) = 0$ for all $t \geq T^*(\|h_0\|_{L^\infty(\Omega)}, I(0), \alpha)$. □

3.2. **Estimates on the support of $h(t,.)$.** First, let us introduce the following notation. If f is a real-valued function defined on Ω, we denote by $\text{supp}(f)$ the support of f in Ω, that is

$$\text{supp}(f) := \overline{\{x \in \Omega |\, f(x) \neq 0\}},$$

and by $N(f)$ the complement of the support, namely,

$$N(f) := \bar{\Omega} - \text{supp}(f).$$

Next we estimate the location of the support of $h(t,.)$ in Ω, which is equivalent to study the location of the set $N(h(t,.))$.

Theorem 5. *Let $\sigma = \alpha/m < 1$ and suppose that $h_0 \in L^\infty(\Omega)$, $h_0 \geq 0$ and with compact support. Then $N(h(t,.)) \subset N(h_0(.))$ for all $t \in (0,T)$. In particular, if we set $M = \|h_0\|_{L^\infty(\Omega)}$, $L^* = \left(\frac{M^m}{K}\right)^{\frac{2}{1-\sigma}}$ and $K = \left(\frac{I(0)(1-\sigma)^2}{2(2\sigma+N(1-\sigma))}\right)^{\frac{1}{1-\sigma}}$, we have that*

$$N(h(t,.)) \subset \{x \in (\Omega - supp(h_0)) \text{ such that } dist(x, supp(h_0)) \geq L^*\}.$$

Proof. We look for local supersolutions which may vanish at points of the zero set of the initial datum h_0. Letting $\sigma = \alpha/m$, we have that for $0 < \sigma < 1$ and $\lambda > 0$, the function $V(x) = K(\lambda)|x - x_0|^{\frac{2}{1-\sigma}}$, with $K(\lambda) = \left(\frac{\lambda(1-\sigma)^2}{2(2\sigma+N(1-\sigma))}\right)^{\frac{1}{1-\sigma}}$ satisfies the equation $-\Delta V + \lambda V^\sigma = 0$ (see [7]).

Now, let $x_0 \in \Omega - \text{supp}(h_0)$, $R := \text{dist}\{x_0, \text{supp}(h_0)\}$ and $\tilde{\Omega} := (B_R(x_0) \cap \Omega)$. Then for $\bar{u}(t,x) = (V(x))^{1/m}$ we have that

$$\begin{cases} \partial_t \bar{u} - \Delta \bar{u}^m + I(0)\bar{u}^\sigma = 0 & \text{in } (0,T) \times \tilde{\Omega}, \\ \bar{u}(x) \geq 0 = h_0(x) & \text{on } \tilde{\Omega}, \end{cases} \tag{3.9}$$

and

$$\bar{u} \geq 0 \text{ on } (0,T) \times (\partial\Omega \cap \tilde{\Omega}). \tag{3.10}$$

By (3.6), $\bar{u}$ is a local super solution of (3.2) as long as the inequality $h(t,x) \leq \bar{u}(t,x)$ is also satisfied for all x in $\partial B_R(x_0) \cap \text{int}(\Omega)$ and $t \in (0,T)$.

In fact, since $|x - x_0| = R$ on $\partial B_R(x_0)$, if

$$R \geq \left(\frac{M_h^m}{K(I(0))}\right)^{\frac{1-\sigma}{2}} \tag{3.11}$$

and $\|h\|_{L^\infty(0,T;\Omega)} \leq M_h$, we have that $\left(K(I(0))R^{\frac{2}{1-\sigma}}\right)^{1/m} \geq M_h$ which in turn implies that

$$\bar{u} \geq h, \text{ on } (0,T) \times \partial\tilde{\Omega} - \partial\Omega. \tag{3.12}$$

Therefore, when (3.11) holds true, $\bar{u}$ is a local supersolution thanks to (3.6), (3.9), (3.10) and (3.12). Finally, $\|h\|_{L^\infty(0,t;\Omega)} \leq \|h_0\|_{L^\infty(\Omega)}$, so we may set $M_h = \|h_0\|_{L^\infty(\Omega)}$ and since $0 \leq h(x_0) \leq \bar{u}(x_0) = 0$ the result follows. □

Remark 1. *In the case of homogeneous Neumann boundary conditions, a similar result is true due to the local nature of the supersolutions. In particular, in the proof above we may take $R := dist\{x_0, supp(h_0) \cup \partial\Omega\}$ so that the ball $B_R(x_0)$ for $x_0 \in \Omega - supp(h_0)$ is entirely contained in Ω, then if (3.11) is satisfied, (3.6), along with (3.9) and (3.12) ensure that the super-solution is appropriately defined.*

Remark 2. *The same result holds true for the problem with compactly supported inhomogeneous Dirichlet boundary conditions, i.e when $h(t,x) = g(t,x) \geq 0$ on $(0,T) \times (\partial\Omega \cap \tilde{\Omega})$ with $g(t,\cdot) > 0$ on a compact subset of $\partial\Omega$. In this case, we may take $R := dist\{x_0, supp(h_0) \cup (\cup_{\tau>0} supp\,(g(\tau,\cdot)))\}$.*

Remark 3. *It seems possible to extend most of the results of this paper to the case in which $\alpha \in (-1,0]$. See, e.g, the treatment made in* [11] *for a scalar equation.*

We shall end with a result which implies an infinite waiting time (see [4] and [23] for some general expositions on the subject). More precisely, as in [9] it is enough to consider the stationary problem this time for the condition $\alpha \in (0,m)$. Moreover, we shall not assume that $p = 0$ but that $p(x)$

vanishes outside a closed subset ω of $\mathbb{R}^2$ (the study could be extended to $\mathbb{R}^n$ for any $n \geq 1$). The case $\omega \subset\subset \Omega$ and $p(x) = p\chi_\omega(x)$ on Ω, where χ_ω denotes the characteristic function of ω (as well as with Neumann boundary conditions on $\partial\Omega$) was considered in [19]. In this paper we will extend the mentioned study to the case in which $\omega = \Omega$, i.e.

$$p(x) > 0 \text{ in } \Omega \text{ and } p = 0 \text{ on } \partial\Omega.$$

It is easy to see that in the stationary problem there exists a positive constant c_b such that

$$c_b d(x) \leq b(x) \leq 1 \text{ for any } x \in \Omega,$$

where $d(x) = d(x, \partial\Omega)$. Indeed, it suffices to apply the strong maximum principle to the stationary equation satisfied by $b(x)$.

We set

$$\Lambda(x) := \theta \frac{b(x) + r/c}{b(x) + r} \quad \text{in } \Omega.$$

Then

$$\theta \frac{(c_b d(x) + r/c)}{1 + r} \leq \Lambda(x) \leq \theta \frac{(1 + r/c)}{c_b d(x) + r} \text{ in } \Omega,$$

and the stationary version of the third equation of (1.1) can be written for $\widehat{h} = h^m$ as the stationary problem $-d_h \Delta \widehat{h} + \mathcal{I}_b \left(\widehat{h}\right)^{\alpha/m} = p$

$$\begin{cases} -\Delta \widehat{h} + \dfrac{\Lambda(x)}{d_h} \widehat{h}^{\alpha/m} = \phi(x) & \text{in } \Omega, \\ \widehat{h} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

with $\phi(x) := \dfrac{p(x)}{d_h}$. For b fixed (i.e., for a given $\Lambda(x)$) it is well-known that there is a unique solution $\widehat{h}$ of (3.13). The following result gives a sufficient condition on $p(x)$ in order to get that $\widehat{h}$ is a *flat solution* (in the sense that also $\frac{\partial \widehat{h}}{\partial n} = 0$ on $\partial\Omega$). In fact, the following result holds for

$$\alpha \in (0, m). \tag{3.14}$$

Theorem 6. *Assume (3.14), let $\sigma = \alpha/m$ and suppose that $p(x)$ is such that*

$$0 \leq p(x) \leq d_h K d(x)^{\frac{2\sigma}{1-\sigma}} \text{ in } \Omega, \tag{3.15}$$

for some $K > 0$ small enough. Then, there exists a constant $C_\sigma^* > 0$ such that

$$0 \leq \widehat{h}(x) \leq C_\sigma^* d(x)^{\frac{2}{1-\sigma}} \text{ in } \Omega. \tag{3.16}$$

In particular, $\widehat{h}$ is a flat solution.

Proof. As in the proof of Theorem 5, we shall apply the *method of local supersolutions* such as presented in [7]. Let $x_0 \in \partial\Omega$ and define $\Omega_{x_0,R} = \Omega \cap B_R(x_0)$ for some $R > 0$ to be determined later. Observe that since $d(x) \leq |x - x_0|$, we have

$$-\Delta \widehat{h} + \frac{\theta r}{d_h c(1 + r)} \widehat{h}^\sigma \leq \phi(x) \leq K |x - x_0|^{\frac{2\sigma}{1-\sigma}} \text{ in } \Omega_{x_0,R}.$$

Let $\overline{h}(x:x_0) = C\,|x - x_0|^{\frac{2}{1-\sigma}}$. As a consequence of Theorem 1.15 of [7], if we set $\xi = \dfrac{\theta r}{d_h c(1+r)}$ then we know that

$$-\Delta\overline{h} + \xi\overline{h}^{\sigma} = \left[\xi C^{\sigma} - \frac{2(2\sigma + N\,(1-\sigma))}{(1-\sigma)^2}C\right]|x - x_0|^{\frac{2\sigma}{1-\sigma}},$$

(in our model $N = 2$ but the result applies to any arbitrary $N \geq 1$). The function

$$\Psi(C) = \xi C^{\sigma} - \frac{2(2\sigma + N\,(1-\sigma))}{(1-\sigma)^2}C$$

takes nonnegative values for $C \in [0, C_{N,\xi,\sigma}]$ with

$$C_{N,\xi,\sigma} = \left[\frac{\xi(1-\sigma)^2}{2(2\sigma + N(1-\sigma))}\right]^{\frac{1}{1-\sigma}},$$

(notice that $\Psi(C_{N,\xi,\sigma}) = 0$). Moreover $\Psi(C)$ attains its maximum at some $C^*_{N,\xi,\sigma}$. Then, a good choice of the constant K mentioned in (3.15) is

$$K = \frac{\Psi(C^*_{N,\xi,\sigma})}{d_h}.$$

In that case we know that

$$-\Delta\widehat{h} + \xi\widehat{h}^{\sigma} \leq -\Delta\overline{h} + \xi\overline{h}^{\sigma} \text{ in } \Omega_{x_0,R}.$$

Moreover, clearly $\widehat{h} \leq \overline{h}$ on $\partial\Omega_{x_0,R} \cap \partial\Omega$ and we also have $\widehat{h} \leq \overline{h}$ on $\partial\Omega_{x_0,R} \setminus \partial\Omega$ if, for instance,

$$\left\|\widehat{h}\right\|_{L^\infty(\Omega)} \leq C^*_{N,\xi,\alpha} R^{\frac{2}{1-\sigma}}. \tag{3.17}$$

Finally, we assume R "large enough" so that

$$R \geq \left[\frac{\left\|\widehat{h}\right\|_{L^\infty(\Omega)}}{C^*_{N,\xi}}\right]^{(1-\sigma)/2}$$

and then (3.17) holds. In conclusion, by the maximum principle

$$0 \leq \widehat{h}(x) \leq C^*_{N,\xi,\sigma}\,|x - x_0|^{\frac{2}{1-\sigma}} \text{ in } \Omega_{x_0,R},$$

and since $x_0 \in \partial\Omega$ is arbitrary this implies (3.16). □

Remark 4. *An easy modification of the proof of Theorem 5, by using the special constant $C^*_{N,\xi,\sigma}$ of the above proof, allows to show the infinite waiting time property when the initial datum $h_0(x)$ grows at most as $d(x, \partial S(h_0))^{\frac{2}{m-\alpha}}$ near the boundary of its support $S(h_0)$. Indeed, it suffices to use the same arguments of Theorem 3.1 of* [3]*. Other qualitative behavior can be proved by using the methods of* [2, 3].

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