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The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach

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ABSTRACT

We study the Dirichlet problem for the stationary Schrödinger fractional Laplacian equation $(-\Delta)^s u + Vu = f$ posed in bounded domain $\Omega \subset \mathbb{R}^n$ with zero outside conditions. We consider general nonnegative potentials $V \in L^1_{loc}(\Omega)$ and prove well-posedness of very weak solutions when the data are chosen in an optimal class of weighted integrable functions f. Important properties of the solutions, such as its boundary behaviour, are derived. The case of super singular potentials that blow up near the boundary is given special consideration since it leads to so-called flat solutions. We comment on related literature.

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1. Introduction

Over the last decades there has been a strong research effort devoted to extend the theory of elliptic and parabolic equations to models in which the Laplacian operator or its elliptic equivalents are replaced by different types of nonlocal integro-differential operators, most notably those called fractional Laplacian operators, given by the formula

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \,, \tag{1.1}$$

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with parameter $s \in (0,1)$ and a precise constant $c_{n,s} > 0$ that we do not need to make explicit. In this formula the domain of definition is assumed to be $\mathbb{R}^n$. The operator can also be defined via the Fourier transform on $\mathbb{R}^n$, see the classical references [52,70]. With an appropriate value of the constant $c_{n,s}$, the limit $s \to 1$ produces the classical Laplace operator $-\Delta$, while the limit $s \to 0$ is the identity operator. An equivalent definition of this fractional Laplacian uses the so-called extension method, that was well-known for $s = \frac{1}{2}$ and has been extended to all $s \in (0,1)$ by Caffarelli and Silvestre [15]. In view of its interest in different applications, many authors have taken part in such an effort from different points of view: probability, potential theory, and PDEs. We are interested in the PDE point of view and its connection with questions of Functional Analysis.

In this paper we study the fractional elliptic equation of Schrödinger type

$$\begin{cases} (-\Delta)^s u + Vu = f & \Omega, \\ u = 0 & \mathbb{R}^n \setminus \Omega. \end{cases} \tag{P}$$

Here, Ω is a bounded subdomain of the space $\mathbb{R}^n$, $n \geq 2$, with C^2 boundary, and the fractional Laplacian operator $(-\Delta)^s$ is the so-called restricted or natural version given by the formula (1.1), where now $x \in \Omega$ while y extends to the whole space. The potential $V \geq 0$ is a measurable function satisfying mild integrability assumptions. The aim is to treat general classes of data f and potentials V, in particular very singular potentials that blow up near the boundary and appear in important applications.

We will start from the Dirichlet problem for the fractional Laplacian equation

$$\begin{cases} (-\Delta)^s u = f & \Omega, \\ u = 0 & \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{P^0}$$

i.e., the case of zero potential. This problem and variants thereof have been well studied by many authors, we refer to the excellent survey [66], which contains many basic references, see also [5,19,21,67]. Here, it will serve to introduce concepts and results and pose the theory for optimal classes of data f. We abandon the usual weak solutions of the energy theory and consider locally integrable functions f. This leads to the theory of *very weak* and *dual solutions*. The optimal class of data turns out to be the class of weighted integrable functions

$$L^1(\Omega; \delta^s) = \{f \text{ measurable in } \ \Omega : \ f\delta^s \in L^1(\Omega)\}, \tag{1.2}$$

where $\delta(x) = \mathrm{dist}\,(x, \Omega^c)$. The problem of well-posedness with weighted integrable data has been considered in the case $s = 1$, which we will call classical case hereafter, by Brezis in [10,12] in the framework of very weak solutions in weighted spaces, and has been treated recently by several authors, [33,34,59,65]. Extending that work to the fractional case is an important issue that we address. We recall the existence and uniqueness of very weak solutions, and establish the main properties in the case where $f \in L^1(\Omega; \delta^s)$, like comparison, accretivity, boundary behaviour, Hopf principle, and optimality of data. We devote special attention to the question of clarifying the existence of traces of the very weak solutions: we find the condition on f for the trace to exist in an integral sense.

We then address a key issue of this paper, i. e., the study of the stationary Schrödinger equation. We want to solve Problem (P) with general $V \geq 0$ and f. Here the concept of very weak solution plays an important role. The theory is simple in the class of bounded or integrable potentials. Besides, if the potential is moderately singular, in a sense to be specified later, Hardy's inequality and Lax–Milgram implies uniqueness of weak solutions. However, we are interested precisely in a type of potentials that diverges at the boundary, and this leads to a delicate analysis. Theorems 4.5 and 4.11 settle the well-posedness of the Dirichlet problem in the class of very weak solutions for general data and count among the main results of this paper. See the whole Section 4 for further results. By the way, our results also improve what was known for $s = 1$.

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Here is a main motivation for the interest in general potentials. For the classical Laplacian ($s = 1$), it was first shown by Sir Nevill Francis Mott in his 1930 book [57], inspired in the pioneering paper by Gamow [43] on the tunnelling effect, that for certain families of potentials the Schrödinger equation, which is naturally posed in $\mathbb{R}^n$, can be localized to a bounded domain of $\mathbb{R}^n$, which is given by the nature of V. On the other hand, the quasi-relativistic approach to bounded states of the Schrödinger equation, leads to the fractional operator corresponding to $s = 1/2$, $\sqrt{(-\Delta) + m^2 u}$ (which is also known as *Klein–Gordon square root operator*, see,e.g. [41,45], see also [38,47] and references). In the case of massless particles we obtain $(-\Delta)^{\frac{1}{2}} u$, also called in this context *ultra-relativistic operator*. Some illustrative examples of the class of singular potentials to which we want to apply our results are the ones given, for instance, by the attractive Coulomb case for a charge distributed over $\partial\Omega$: $V(x) = C/\delta(x)$ with $C > 0$, or even by more singular functions as it is the case of the Pösch–Teller potential (see [62])

$$V(x) = V(|x|) = \frac{1}{2} V_0 \left(\frac{k(k-1)}{\sin^2\alpha\,|x|} + \frac{\mu(\mu-1)}{\cos^2\alpha\,|x|} \right) \tag{1.3}$$

for some $V_0, \alpha > 0$, $k, \mu \geq 0$, intensively studied since 1933. Notice that this potential blows up in a sequence of spheres. Some other singular potentials, in the class of the so called *super-symmetric potentials* can be found, e.g. in [23]. We refer to [3] as a classical paper on the mathematical study of the time independent Schrödinger equation for the standard Laplacian. See also [63] for a recent reference.

Actually, another key point of this paper is studying the sense in which the solutions of (P) satisfy the boundary condition $u = 0$ on $\partial\Omega$, see Section 5 where so-called *flat solutions* are discussed, and also its interplay with the way in which the extended function (defined in the whole space $\mathbb{R}^n$) satisfies (or not) the same partial differential equation. This plays an important role in many applications as for instance Quantum Mechanics, as already mentioned. Furthermore, since in bounded domains there are several different choices of $(-\Delta)^s$ present in the literature (see, e.g., [5,74]), it is relevant to study which choice represents the correct localization of a global problem (see Section 6).

Our data f belong to an optimal class of locally integrable functions. There is a simple extension of the theory to cover the case where integrable functions f are replaced by measures μ. The precise space is $\mathcal{M}(\Omega, \delta^s)$ consisting of locally bounded signed Radon measures μ such that $\int_\Omega \delta^s(x) d|\mu|(x) < +\infty$. Actually, the results of [21] that cover the zero-potential case are written in that generality. Our existence and uniqueness theory, contained in Theorems 4.5 and 4.11, is valid in that context. We have refrained from that generality in our presentation because using functions makes most of our calculations and consequences easier to formulate.

Regarding potentials, we have considered general nonnegative potentials $V \in L^1_{loc}(\Omega)$. This class allows for extensions in two directions: considering signed potentials, and considering locally bounded measures as potentials. Both are present in the literature, but both lead to problems that we did not want to consider here.

Comment. The paper surveys topics that are treated, at least in part, in the recent literature, but it is also a research paper and many results are new, specially in Sections 3–6. We have tried to mention suitable references to relevant and related known results. Since the literature on elliptic problems with fractional Laplacians is so numerous, we refer to specialized monographs for more complete bibliographical information and beg excuse for possible undue omissions.

2. Preliminaries

We introduce the fractional seminorm

$$[v]^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}} dy dx, \tag{2.1}$$

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and then the fractional Hilbert spaces H^s defined by

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : [v]_{H^s(\mathbb{R}^n)} < +\infty\}, \tag{2.2}$$

with the norm

$$\|v\|^2_{H^s(\mathbb{R}^n)} = \|v\|^2_{L^2(\mathbb{R}^n)} + [v]^2_{H^s(\mathbb{R}^n)}. \tag{2.3}$$

We point out that

$$\|v\|_{H^s(\mathbb{R}^n)} \asymp \|v\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)}. \tag{2.4}$$

where the symbol $a \asymp b$ means that there are constants $c_1, c_2 > 0$ such that $c_1 a \le b \le c_2 a$.

When working in a bounded domain Ω, and in order to take into account the boundary and exterior conditions, we define the Hilbert spaces

$$H^s_0(\Omega) = \overline{\mathcal{C}^\infty_c(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}}. \tag{2.5}$$

Classical texts on Sobolev spaces to be consulted are [2,11,53,54,72]. For a concise introduction to $H^s(\mathbb{R}^n)$ we refer the reader to [7,26].

By analogy to the classical case $s = 1$, a "formula of integration by parts" (or "Green's formula") holds

Proposition 2.1. *Let $u, v \in C^\infty_c(\mathbb{R}^n)$ and $0 < s \le 1$. Then*

$$\int_{\mathbb{R}^n} v(-\Delta)^s u = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v. \tag{2.6}$$

Proof. Since the operator $(-\Delta)^{\frac{s}{2}}$ is self-adjoint (see, e.g., [21]) and $(-\Delta)^{s+t} = (-\Delta)^s(-\Delta)^t$ for $0 < s, t, s + t \le 1$ we have

$$\int_{\mathbb{R}^n} v(-\Delta)^s u = \langle v, (-\Delta)^{\frac{s}{2}} (-\Delta)^{\frac{s}{2}} u\rangle = \langle (-\Delta)^{\frac{s}{2}} v, (-\Delta)^{\frac{s}{2}} u\rangle = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v. \tag{2.7}$$

This proves the result. □

More general integration by parts results can be found in [1], that treats a general integration by parts formula that includes terms accounting for a non-zero value of u in Ω^c and a precise limit on the boundary.

Remark 2.2. By density, the formula is true for any $u \in H^{2s}(\Omega) \cap H^s_0(\Omega)$ and $v \in H^s_0(\Omega)$. Particularizing for $s = 1$ and making $u = v$ we deduce the classic formula

$$\|\nabla v\|_{L^2(\Omega)^n} = \|(-\Delta)^{\frac{1}{2}} v\|_{L^2(\mathbb{R}^n)}. \tag{2.8}$$

Remark 2.3. Some authors prefer the following presentation:

$$\int_{\mathbb{R}^n} v(-\Delta)^s u = c_{n,s} P.V. \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{n+2s}} dy dx. \tag{2.9}$$

This operator also has a Kato inequality. In [19], is presented simply as: $(-\Delta)^s|u| \le \operatorname{sign} u\,(-\Delta)^s u$ holds in the distributional sense. A precise expression can be found in Lemma 3.8. In Section 9 we provide for the reader's benefit a simple proof which is useful for our presentation.

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3. Dirichlet problem without potentials and general data

3.1. Weak solutions. A survey on existence, uniqueness and properties

A weak solution of the Dirichlet problem (P^0) (with zero potential $V = 0$) can be obtained by an energy minimization method using the appropriate fractional Sobolev spaces, as introduced in the previous section. Applying (2.6) we introduce the concept of weak solution as:

Definition 3.1. $f \in L^2(\Omega)$. A weak solution of (P^0) is a function $u \in H_0^s$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi \, dx = \int_\Omega f \varphi \, dx, \qquad \varphi \in H_0^s(\Omega). \tag{P$^0_{\rm w}$}$$

Existence and uniqueness of weak solutions is easy for $f \in L^2(\Omega)$, by the Lax–Milgram theorem. This is a basic result on which the extended theory is based. Actually, f can be taken in the dual space $(H_0^s(\Omega))'$.

Another option is pursued in [16–18] where the authors proved existence and regularity of viscosity solutions. Both classes of solutions coincide in the common class of data. We will not deal with viscosity solutions in this paper. See also [14].

In [22,50], the authors prove that the solution operator is given by an integral representation in terms of a Green kernel

$$[(-\Delta)^s]^{-1} f = \int_\Omega \mathbb{G}_s(x,y) f(y) dy \tag{3.1}$$

where

$$\mathbb{G}_s(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(\frac{\delta(x)}{|x-y|} \wedge 1 \right)^s \left(\frac{\delta(y)}{|x-y|} \wedge 1 \right)^s. \tag{3.2}$$

Using these bounds, many estimates of integrability and regularity can be given by suitably applying Hölder's inequality.

The following regularity results are proved by Ros-Oton and Serra in [67] and will be essential in what follows.

Proposition 3.2. *Let Ω be a bounded $C^{1,1}$, $f \in L^\infty(\Omega)$, and let u be a weak solution of* (P^0). *Then, the following holds:*

(1) *We have $u \in C^s(\mathbb{R}^n)$ and*

$$\|u\|_{C^s(\mathbb{R}^n)} \le C \|f\|_{L^\infty(\Omega)}, \tag{3.3}$$

where C is a constant depending only on Ω and s.

(2) *Moreover, if $\delta(x) = \operatorname{dist}(x, \Omega^c)$, then for all $x \in \Omega$*

$$|u(x)| \le C_1 \|f\|_{L^\infty(\Omega)} \delta(x)^s, \tag{3.4}$$

where C_1 is a constant depending only on Ω and s. Besides. $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$, we have $u/\delta^s \in C^\alpha(\Omega)$ and

$$\left\| \frac{u}{\delta^s} \right\|_{C^\alpha(\Omega)} \le C_2 \|f\|_{L^\infty(\Omega)} \tag{3.5}$$

for some $\alpha > 0$ satisfying $\alpha < \min\{s, 1-s\}$. The constants α and C_1, C_2 depend only on Ω and s.

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Remarks 3.3. (1) Existence and uniqueness of weak solutions in the classical case $s = 1$ is standard. There are also many results about the regularity of the weak solutions with data in Lebesgue spaces. For instance, for $p = 1$ and $n = 2$, we have the very sharp results of [40].

(2) There are many references to the variational treatment of equations with nonlocal operators, both linear and nonlinear, see [56].

3.2. Very weak solutions. A survey on existence, uniqueness and properties

However, our purpose is to deal with a larger class of data f, which are locally integrable functions, otherwise as general as possible. A more general definition of solution is necessary. We take an old idea by Brezis (see [10]).

Definition 3.4. Let $f \in L^1(\Omega, \delta^s)$. We say that u is a *very weak solution* of (P^0) if

$$\begin{cases} u \in L^1(\Omega), \\ u = 0 \text{ a.e. } \mathbb{R}^n \setminus \Omega \text{ and} \\ \displaystyle\int_\Omega u(-\Delta)^s \varphi \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in X^s_\Omega, \end{cases} \tag{P$^0_{\mathrm{vw}}$}$$

where

$$X^s_\Omega = \{\varphi \in \mathcal{C}^s(\mathbb{R}^n) : \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega \ \text{ and } \ (-\Delta)^s \varphi \in L^\infty(\Omega)\}. \tag{3.6}$$

This type of solution is also known as very *weak solution in the sense of Brezis.*

Remarks 3.5. (1) Applying identity (2.6) again we can prove that any weak solution in the sense of Definition 3.1 satisfies our definition of very weak solution.

(2) This definition allows us to take f to be outside L^1, but rather with a weighted integrability condition, $f \in L^1(\Omega; \delta^s)$, which will turn out to be the correct class. About the weight, Ros-Oton and Serra proved in [67, Lemma 3.9] that $\delta^s \in \mathcal{C}^\alpha(\Omega \cap \{\delta < \rho_0\})$ for $\alpha = \min\{s, 1 - s\}$ and

$$|(-\Delta)^s \delta^s| \le C_\Omega \qquad \text{in } \Omega \cap \{\delta < \rho_0\}. \tag{3.7}$$

In order to simply the calculations, it is convenient to replace δ^s by the first eigenfunction of the fractional Laplacian φ_1, which is positive and smooth everywhere inside Ω and satisfies exactly the same boundary behaviour ($\varphi_1 \asymp \delta^s$).

(3) In X^s_Ω we can only ask for $\mathcal{C}^s(\Omega)$ smoothness, because, when $\Omega = B_R$, we will want to approximate

$$\begin{cases} (-\Delta)^s \varphi = 1 & B_R \\ \varphi = 0 & \mathbb{R}^n \setminus B_R \end{cases}$$

which is

$$\varphi(x) = C\Big(R^2 - |x|^2\Big)^s$$

only of class $\mathcal{C}^s$. Nonetheless, φ/δ^s can be shown to be smoother. In this case, it is infinitely differentiable, whereas φ is not.

(4) Definition 3.4 corresponds to the notion of *weak dual solution* proposed and used in [7]:

$$\int_\Omega u\psi = \int_\Omega f[(-\Delta)^s]^{-1}\psi \tag{3.8}$$

where $[(-\Delta)^s]^{-1}$ is the solution operator. We will make a detailed comment about the interpretation of this kind of solution in Section 7.1.

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(5) By the formula of integration by parts, it is clear that any very weak solution with $f \in L^2(\Omega)$ that is also in $H_0^s(\Omega)$ is a weak solution.

Chen and Véron [21] seem to have been the first to apply this approach to the fractional case. They proved the following results:

Theorem 3.6 *([21]).* *Let $f\delta^s \in L^1(\Omega)$. Then, there exists exactly one very weak solution $u \in L^1(\Omega)$ of Problem (P^0_{vw}). If $f \geq 0$, then $u \geq 0$. Hence, the Maximum Principle holds.*

Remarks 3.7. (1) We point out that the authors also treat semilinear problems of the form $(-\Delta)^s u + g(u) = f$, and that their work does not apply Green function estimates. The authors work with the more general class of measure data $\mathcal{M}(\Omega, \delta^s)$.

(2) The Maximum Principle allows for the definition of super- and subsolutions that can be useful in getting estimates.

(3) A reference to optimal regularity for the fractional case is [51] which also includes nonlinear fractional elliptic problems with p-Laplacian type growth. The right-hand side data are locally bounded measures. When f has further regularity the solutions are smooth to different degrees by the representation via the Green kernel Equation (3.1).

(4) More properties of the solutions will be examined below and in the study of the Schrödinger equation.

With the formulation $(\mathrm{P}^0_{\mathrm{vw}})$ we can precisely state the Kato inequality in a general way. The proof for the fractional operator is also due to Chen and Véron [21]

Lemma 3.8 (*Kato's Inequality [21]*)**.** *Let $f \in L^1(\Omega, \delta^s)$ and $u \in L^1(\Omega)$ be a solution of (P^0_{vw}). Then*

$$\int_\Omega |u|(-\Delta)^s \varphi \leq \int_\Omega \mathrm{sign}(u) f \varphi, \tag{3.9}$$

$$\int_\Omega u_+(-\Delta)^s \varphi \leq \int_\Omega \mathrm{sign}_+(u) f \varphi \tag{3.10}$$

hold for all $\varphi \in X^s_\Omega$, $\varphi \geq 0$.

3.3. A quantitative lower Hopf principle for data in $L^1(\Omega, \delta^s)$

By uniqueness and approximation we easily see that the preceding solutions admit a representation via the Green kernel (3.1), and estimates (3.2) we can prove an adapted lower Hopf inequality. The classical case $s = 1$ was first stated in this form in [32].

Proposition 3.9. *Let $0 \leq f \in L^1(\Omega, \delta^s)$ and let $u \in L^1(\Omega)$ be the unique very weak solution of (P^0_{vw}). Then*

$$u(x) \geq c\delta(x)^s \int_\Omega f(y)\delta(y)^s \tag{3.11}$$

a.e. $x \in \Omega$, where $c > 0$ depends only on Ω.

Proof. We first show that

$$\mathbb{G}_s(x, y) \geq c(x)\delta(y)^s \tag{3.12}$$

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for all $x, y \in \Omega$, and $c > 0$ in Ω. Let $x \in \Omega$. Applying (3.2), it is clear that

$$c(x) = \inf_{y \in \bar{\Omega}} \frac{1}{\delta(y)^s |x-y|^{n-2s}} \left(\frac{\delta(x)}{|x-y|} \wedge 1 \right)^s \left(\frac{\delta(y)}{|x-y|} \wedge 1 \right)^s \tag{3.13}$$

is reached at some point y^*. It is easy to see that $c(x) \geq c\delta(x)^s$ where $c > 0$. Therefore

$$u(x) = \int_\Omega \mathbb{G}_s(x,y) f(y) dy \geq c\delta(x)^s \int_\Omega f(y)\delta^s(y) dy. \tag{3.14}$$

This completes the proof. □

The strict positivity of solutions with nonnegative data, in particular the behaviour near the boundary, has been studied in [5]. There exists also a wide literature for parabolic equations, both linear and nonlinear.

3.4. $L^1(\Omega; \delta^s)$ as an optimal class of data

Through the Hopf inequality it is easy to show that this is largest space to look for solutions if we want to keep the class of weak solutions for bounded data and the maximum principle. The nonexistence of solution for such data is a consequence of the following blow-up result.

Proposition 3.10. *Let f be a nonnegative function such that $f \notin L^1(\Omega, \delta^s)$, and let f_k be a sequence of approximations by bounded functions, $f_k \leq f$, $f_k \to f$ a.e. in Ω. Then, $u_k = (-\Delta)^{-s} f_k \to \infty$ in Ω.*

Proof. Applying Proposition 3.9 we have that

$$u_k(x) \geq c\delta(x)^s \int_\Omega f_k(y)\delta^s(y) dy. \tag{3.15}$$

Passing to the limit $k \to \infty$ we would arrive at $u(x) \geq \lim_k u_k(x) = +\infty$. □

Remark 3.11. In the limit this optimality can be extended to measure data in the class $\mathcal{M}(\Omega, \delta^s)$.

3.5. Traces of very weak solutions and boundary weighted integrability

The definition of weak solution includes a very clear sense of zero boundary trace since $u \in H_0^s(\Omega)$. However, the definition of very weak solution merely requires $u \in L^1(\Omega)$. Clearly, since the space $L^1(\Omega)$ does not have a boundary trace operator there is a question about the sense in which the solution u takes null boundary data on $\partial\Omega$. In the classical case $s = 1$ some authors have proposed to study local solutions of the Laplace equation inside Ω that satisfy a generalized 0 boundary condition of the form $u/\delta \in L^1(\Omega)$. Always for $s = 1$, Kufner [49] was amongst the first to notice that this kind of singular weights give significant boundary information. We recall that, for $p > 1$, the classical Hardy inequality implies that

$$u \in W_0^{1,p}(\Omega) \iff \begin{cases} u \in W^{1,p}(\Omega) \text{ and} \\ \dfrac{u}{\delta} \in L^p(\Omega). \end{cases} \tag{3.16}$$

For $p = 1$ this result is no longer true.

The convenience of using the integral condition $u/\delta \in L^1(\Omega)$ as a kind of generalized boundary condition instead of a standard trace condition has been observed recently (see [30]). Moreover, Rakotoson showed in [65] the following equivalence:

$$\frac{u}{\delta} \in L^1(\Omega) \iff f\delta(1 + |\log \delta|) \in L^1(\Omega). \tag{3.17}$$

When considering the fractional Dirichlet problem we have found that the appropriate weight is δ^s. To begin with, there is a Hardy inequality for these operators

Proposition 3.12 ([46]). *Let $1 < p < \infty$ and $0 < s < 1$ be such that $sp < n$. Let $\Omega \subset \mathbb{R}^n$ be an open set, $n \geq 2$, with regular boundary. Then, for, every $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{sp}} \leq c_0 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dydx, \tag{3.18}$$

where u is extended by 0 outside Ω.

Note that for $p = 2$ then $\frac{n}{p} \geq 1$, so the condition $0 < s < \frac{n}{p}$ is trivial. Hence, for $u \in H_0^s(\Omega)$, we know that $u/\delta^s \in L^2$.

In order to present our results we need to introduce a new test function:

$$\begin{cases} (-\Delta)^s \varphi_\delta = \dfrac{1}{\delta^s} & \Omega \\ \varphi_\delta = 0 & \Omega^c. \end{cases} \tag{3.19}$$

We prove the following theorem:

Proposition 3.13. *Let $f\delta^s \in L^1(\Omega)$ and let $u \in L^1(\Omega)$ be the solution of (P^0_{vw}). Then, $u/\delta^s \in L^1(\Omega)$ if and only if $f\varphi_\delta \in L^1(\Omega)$.*

Remark. The difficulty with this function is that, since $1/\delta^s \notin L^\infty$, we know $\varphi_\delta \notin X_\Omega^s$.

Proof. Let us consider the auxiliary functions

$$\begin{cases} (-\Delta)^s \varphi_{\delta,k} = \min\left\{\dfrac{1}{\delta^s}, k\right\} & \Omega \\ \varphi_{\delta,k} = 0 & \Omega^c. \end{cases} \tag{3.20}$$

Then

$$\int_\Omega u \min\left\{\frac{1}{\delta^s}, k\right\} = \int_\Omega u(-\Delta)^s \varphi_{\delta,k} = \int_\Omega f\varphi_{\delta,k}. \tag{3.21}$$

We will prove the case $f \geq 0$, and the sign changing case follows directly. Since $f \geq 0$ then $u \geq 0$. It is clear that $\varphi_{\delta,k}$ is a nondecreasing sequence, the limit of which is φ_δ. Since $f, u \geq 0$, by the Monotone Convergence Theorem

$$\int_\Omega \frac{|u|}{\delta^s} = \int_\Omega \frac{u}{\delta^s} = \lim_{k\to\infty} \int_\Omega u \min\left\{\frac{1}{\delta^s}, k\right\} = \lim_{k\to\infty} \int_\Omega f\varphi_{\delta,k} = \int_\Omega f\varphi_\delta = \int_\Omega |f|\varphi_\delta. \tag{3.22}$$

One integral is finite if and only the other integral is finite. □

We can characterize the behaviour of φ_δ near the boundary:

Lemma 3.14. *There exist constants, $c, C > 0$ such that*

$$c\delta^s |\log \delta| \leq \varphi_\delta \leq C\delta^s(1 + |\log \delta|). \tag{3.23}$$

This result is technical but simple, we give the proof in Section 8. For $s = 1$ the result is due to Rakotoson [64].

Through this estimate, we provide an extension of (3.17) to the fractional case:

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Proposition 3.15 (*Necessary and Sufficient Condition for $u/\delta^s \in L^1(\Omega)$*). *Let $f\delta^s \in L^1(\Omega)$ and $u \in L^1(\Omega)$ be the very weak solution of* (P^0_{vw}). *Then,*

$$\frac{u}{\delta^s} \in L^1(\Omega) \iff f\delta^s(1+|\log\delta|) \in L^1(\Omega). \tag{3.24}$$

Proof. We will only give the proof for $f \geq 0$. Then $u \geq 0$.

$$\int_\Omega \frac{u}{\delta^s} = \int_\Omega f\varphi_\delta. \tag{3.25}$$

Thus

$$c_1 \int_\Omega f\delta^s|\log\delta| \leq \int_\Omega \frac{u}{\delta^s} \leq c_2 \int_\Omega f\delta^s(1+|\log\delta|). \tag{3.26}$$

The $\Longleftarrow$ part is then proved.

On the other hand, if $u/\delta^s \in L^1(\Omega)$, then $f\delta^s|\log\delta| \in L^1(\Omega)$. For the first eigenfunction of $(-\Delta)^s$ it is well known that $c_1\delta^s \leq \varphi_1 \leq c_2\delta^s$. Hence,

$$c_1 \int_\Omega f\delta^s \leq \int_\Omega f\varphi_1 = \int_\Omega u(-\Delta)^s\varphi_1 = \lambda_1 \int_\Omega u\varphi_1 \leq \left\| \frac{u}{\delta^s} \right\|_{L^1} \|\varphi_1\delta^s\|_{L^\infty} \tag{3.27}$$

Adding both computations, the $\Longrightarrow$ part of the theorem is proved. □

3.6. A note on local very weak solutions

In this theory, it is natural to define local solutions, if we are not concerned with the boundary information:

Definition 3.16. We say that u is a very weak *local* solution if

$$\begin{cases} u \in L^1(\Omega), \\ u = 0 \text{ a.e. } \mathbb{R}^n \setminus \Omega \text{ and} \\ \displaystyle\int_\Omega u(-\Delta)^s\varphi\, dx = \int_\Omega f\varphi\, dx, \qquad \forall \varphi \in X^s_\Omega \cap C_c(\Omega). \end{cases} \tag{$\mathrm{P}^0_{\mathrm{loc}}$}$$

Notice that the difference with $(\mathrm{P}^0_{\mathrm{vw}})$ lies in space where the test functions are taken. It is clear that, even with the extra requirement $u \in H^s(\Omega)$, there is no uniqueness of solutions. The reason why $(\mathrm{P}_{\mathrm{vw}})$ has uniqueness of solutions is the fact, in the very weak formulation, that the test function "sees" the boundary information. This is due to the integration by parts formula.

Due to the integration by parts formula and density, any solution $(\mathrm{P}^0_{\mathrm{loc}})$ such that $u \in H^s_0(\Omega)$ is a solution of $(\mathrm{P}^0_{\mathrm{vw}})$. This raises the question: how much extra information does one need to show uniqueness of $(\mathrm{P}^0_{\mathrm{loc}})$. In this section, we will show that

$$\frac{u}{\delta^s} \in L^1(\Omega) \tag{3.28}$$

is sufficient information (i.e., there is, at most, one solution of $(\mathrm{P}^0_{\mathrm{loc}})$ such that (3.28) holds). Proposition 3.13 shows that this (3.28) is only possible if $f\varphi_\delta \in L^1(\Omega)$, and in this case it is always true. This produces an equivalent formulation of $(\mathrm{P}_{\mathrm{vw}})$ when $f\varphi_\delta \in L^1(\Omega)$.

3.6.1. A lemma of approximation of test functions

Since the only difference between $(\mathrm{P}^0_{\mathrm{vw}})$ and $(\mathrm{P}^0_{\mathrm{loc}})$ is the space of test functions, let us study further these spaces. It is clear that one way to pass from $(\mathrm{P}^0_{\mathrm{loc}})$ to $(\mathrm{P}^0_{\mathrm{vw}})$ will be to select, for each $\varphi \in X^s_\Omega$ a sequence

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$\varphi_k \in X^s_\Omega \cap \mathcal{C}^\infty_c(\Omega)$ such that $\varphi_k \to \varphi$. This convergence must be good enough to preserve the equation. In this direction we introduce the following cut-offs.

Let η be a $C^2(\mathbb{R})$ function such that $0 \le \eta \le 1$ and

$$\eta(t) = \begin{cases} 0 & t \le 0, \\ 1 & t \ge 2. \end{cases} \tag{3.29}$$

We define the functions

$$\eta_\varepsilon(x) = \eta\left(\frac{\varphi_1(x) - \varepsilon^s}{\varepsilon^s}\right). \tag{3.30}$$

where φ_1 is the first eigenfunction of $(-\Delta)^s$. Notice that $\varphi_1 \asymp \delta^s$. We prove the following approximation result:

Lemma 3.17. *For $\varphi \in X^s_\Omega$ we have that $\eta_\varepsilon\varphi \in X^s_\Omega \cap C_c(\Omega)$ and*

$$\delta^s(-\Delta)^s(\varphi\eta_\varepsilon) \rightharpoonup \delta^s(-\Delta)^s\varphi \tag{3.31}$$

$$\frac{\varphi\eta_\varepsilon}{\delta^s} \rightharpoonup \frac{\varphi}{\delta^s} \tag{3.32}$$

in L^∞-weak-$\star$ as $\varepsilon \to 0$.

To prove Lemma 3.17 we can use the following decomposition:

Theorem 3.18 (*Eilertsen Formula (See [35])*)**.** *Let $u, v \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ and $0 < s < 1$. Then*

$$(-\Delta)^s(uv)(x) = u(x)(-\Delta)^s v(x) + v(x)(-\Delta)^s u(x) - A_s \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dy, \tag{3.33}$$

where $A_s \asymp s(1-s)$.

The difficult term will be the first one.

Lemma 3.19. *There exists a constant C independent of ε such that*

$$|\delta^{2s}(x)(-\Delta)^s\eta_\varepsilon(x)| \le C. \tag{3.34}$$

Proof. In order to use [21, Lemma 2.3] we write $\eta_\varepsilon(x) = \gamma_\varepsilon(\varphi_1(x))$, where

$$\gamma_\varepsilon(t) = \eta\left(\frac{t - \varepsilon^s}{\varepsilon^s}\right). \tag{3.35}$$

Due to (3.7), we have that $\eta_\varepsilon \in X^s_\Omega$ and, for every $x \in \overline{\Omega}$, there exists $z_x \in \overline{\Omega}$

$$(-\Delta)^s\eta_\varepsilon(x) = (-\Delta)^s(\gamma_\varepsilon \circ \varphi_1)(x) \tag{3.36}$$

$$= \gamma'_\varepsilon(\varphi_1(x))(-\Delta)^s\varphi_1(x) + \frac{\gamma''(\varphi_1(z_x))}{2s} \int_\Omega \frac{|\varphi_1(x) - \varphi_1(y)|^2}{|x-y|^{n+2s}} dy \tag{3.37}$$

$$= \frac{1}{\varepsilon^s}\eta'\left(\frac{\varphi_1(x) - \varepsilon^s}{\varepsilon^s}\right)\lambda_1\varphi_1(x) \tag{3.38}$$

$$+ \frac{\eta''\left(\frac{\varphi_1(z_x) - \varepsilon^s}{\varepsilon^s}\right)}{2s\varepsilon^{2s}} \int_\Omega \frac{|\varphi_1(x) - \varphi_1(y)|^2}{|x-y|^{n+2s}} dy. \tag{3.39}$$

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From this, applying the regularity of φ_1

$$|(-\Delta)^s \eta_\varepsilon(x)| \le \varepsilon^{-2s}, \qquad \forall x \in \overline{\Omega}. \tag{3.40}$$

Consider, in particular, that $\delta(x) \ge 3\varepsilon$. Let $d = d(x, \{\varepsilon < \delta < 2\varepsilon\}) > 0$, so $\eta_\varepsilon(y) = 1 = \eta_\varepsilon(x)$ if $|x-y| \le d$. We have that

$$|(-\Delta)^s \eta_\varepsilon(x)| = \left| \int_{\mathbb{R}^n} \frac{\eta_\varepsilon(x) - \eta_\varepsilon(y)}{|x-y|^{n+2s}} dy \right| \tag{3.41}$$

$$= \left| \int_{|x-y|>d} \frac{\eta_\varepsilon(x) - \eta_\varepsilon(y)}{|x-y|^{n+2s}} dy \right| \tag{3.42}$$

$$\le \int_{|x-y|>d} \frac{2}{|x-y|^{n+2s}} dy \tag{3.43}$$

$$= C \int_d^{+\infty} \frac{1}{r^{n+2s}} r^{n-1} dr \tag{3.44}$$

$$= \frac{1}{d^{2s}}. \tag{3.45}$$

Since $d \ge \delta(x) - 2\varepsilon \ge \delta(x)/3$. We have that

$$|(-\Delta)^s \eta_\varepsilon(x)| \le \frac{C}{\delta(x)^{2s}}, \qquad \forall x \text{ such that } \delta(x) \ge 3\varepsilon. \tag{3.46}$$

On the other hand,

$$|\delta^{2s}(x)(-\Delta)^s \eta_\varepsilon(x)| \le 3^{2s}\varepsilon^{2s} C \varepsilon^{-2s} = C \qquad \forall x \text{ such that } \delta(x) \le 3\varepsilon. \tag{3.47}$$

This proves the result. □

Lemma 3.20. *For all $\varphi \in X_\Omega^s$*

$$\left| \delta^s(x) \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))(\eta_\varepsilon(x) - \eta_\varepsilon(y))}{|x-y|^{n+2s}} dy \right| \le C, \tag{3.48}$$

where C does not depend on ε.

Proof. For any $x \in \overline{\Omega}$

$$\left| \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))(\eta_\varepsilon(x) - \eta_\varepsilon(y))}{|x-y|^{n+2s}} dy \right| = \left| \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))\eta'\left(\frac{\varphi_1(z_y)-\varepsilon^s}{\varepsilon^s}\right)\frac{\varphi_1(x)-\varphi_1(y)}{\varepsilon^s}}{|x-y|^{n+2s}} dy \right|$$

$$\le C\varepsilon^{-s} \left| \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))(\varphi_1(x) - \varphi_1(y))}{|x-y|^{n+2s}} dy \right|$$

$$\le C\varepsilon^{-s}. \tag{3.49}$$

We compute, for $\delta(x) \ge 3\varepsilon$

$$\left| \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))(\eta_\varepsilon(x) - \eta_\varepsilon(y))}{|x-y|^{n+2s}} \right| \le \left(\int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+2s}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{|x-y|^{n+2s}} \right)^{\frac{1}{2}} \tag{3.50}$$

$$\le C\|\varphi\|_{H^s(\mathbb{R}^n)} \left(\int_{|x-y|\ge d} \frac{1}{|x-y|^{n+2s}} \right)^{\frac{1}{2}} \tag{3.51}$$

$$\le \frac{C\|\varphi\|_{H^s(\mathbb{R}^n)}}{d^s}. \tag{3.52}$$

From this estimate and (3.49) we conclude, as before, the result. □

Proof of Lemma 3.17. We write Eilertsen's formula in our case

$$(-\Delta)^s(\varphi\eta_\varepsilon)(x) = \eta_\varepsilon(x)(-\Delta)^s\varphi(x) + \varphi(x)(-\Delta)^s\eta_\varepsilon(x) - A_s\int_{\mathbb{R}^n}\frac{(\varphi(x)-\varphi(y))(\eta_\varepsilon(x)-\eta_\varepsilon(y))}{|x-y|^{n+2s}}dy. \quad (3.53)$$

We have proven that the second and third terms are bounded when multiplied by $\delta^{2s}(x)$ and $\delta^s(x)$, respectively. They converge pointwise to 0. Hence, up to a subnet,

$$\delta^{2s}(x)(-\Delta)^s\eta_\varepsilon \rightharpoonup 0, \quad (3.54)$$

$$\delta^s(x)\int_{\mathbb{R}^n}\frac{(\varphi(x)-\varphi(y))(\eta_\varepsilon(x)-\eta_\varepsilon(y))}{|x-y|^{n+2s}}dy \rightharpoonup 0 \qquad \text{in } L^\infty\text{-weak-}\star. \quad (3.55)$$

Since $\varphi \in X^s_\Omega$ we know that $\frac{\varphi}{\delta^s} \in L^\infty$. Hence, up to a subnet,

$$\delta^s\varphi(-\Delta)^s\eta_\varepsilon = \frac{\varphi}{\delta^s}\delta^{2s}(-\Delta)^s\eta_\varepsilon \rightharpoonup 0 \text{ in } L^\infty\text{-weak-}\star. \quad (3.56)$$

On the other hand η_ε is bounded, and converges pointwise to 1. Hence, up to a subnet,

$$\eta_\varepsilon \rightharpoonup 1 \qquad \text{in } L^\infty\text{-weak-}\star. \quad (3.57)$$

Thus, up to a subnet,

$$\delta^s\eta_\varepsilon \rightharpoonup \delta^s \qquad \text{in } L^\infty\text{-weak-}\star. \quad (3.58)$$

All the above are bounded net, such that every subnet have the same limit. All nets converge. This proves (3.31).

On the other hand

$$\frac{\varphi\eta_\varepsilon}{\delta^s} = \frac{\varphi}{\delta^s}\eta_\varepsilon \rightharpoonup \frac{\varphi}{\delta^s} \qquad \text{in } L^\infty\text{-weak-}\star, \quad (3.59)$$

proving (3.32). This concludes the proof. □

3.6.2. A local solution which is integrable with a suitable boundary weight is a v.w.s.

Proposition 3.21. *Any solution of* (P^0_{loc}) *such that* $\frac{u}{\delta^s} \in L^1(\Omega)$ *is a solution of* (P^0_{vw}).

Proof. Let $\varphi \in X^s_\Omega$. Consider an approximation $\eta_{\frac{1}{k}}\varphi \in X^s_\Omega \cap C_c(\Omega)$. Since $u/\delta^s \in L^1(\Omega)$ then

$$\int_\Omega u(-\Delta)^s(\eta_{\frac{1}{k}}\varphi) = \int_\Omega f\varphi\eta_{\frac{1}{k}}, \quad (3.60)$$

$$\int_\Omega \frac{u}{\delta^s}\delta^s(-\Delta)^s(\eta_{\frac{1}{k}}\varphi) = \int_\Omega f\delta^s\frac{\varphi\eta_{\frac{1}{k}}}{\delta^s}. \quad (3.61)$$

By passing to the limit applying Lemma 3.17

$$\int_\Omega u(-\Delta)^s\varphi = \int_\Omega f\varphi. \quad (3.62)$$

This proves the result. □

As a corollary of the uniqueness of $(\mathrm{P}^0_{\mathrm{vw}})$ we have the following

Proposition 3.22. *There is, at most, one solution of* (P^0_{loc}) *such that* $u/\delta^s \in L^1(\Omega)$.

Summary. It is obvious that $(\mathrm{P}^0_{\mathrm{vw}}) \implies (\mathrm{P}^0_{\mathrm{loc}})$. Proposition 3.13 states that

$$\begin{cases}(\mathrm{P}^0_{\mathrm{vw}}) \text{ and} \\ f\varphi_\delta \in L^1(\Omega)\end{cases} \implies \begin{cases}(\mathrm{P}^0_{\mathrm{vw}}) \text{ and} \\ \dfrac{u}{\delta^s} \in L^1(\Omega)\end{cases} \quad (3.63)$$

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Proposition 3.21 states:

$$(\mathrm{P}^0_{\mathrm{vw}}) \quad \Longleftarrow \quad \begin{cases} (\mathrm{P}^0_{\mathrm{loc}}) \text{ and} \\ \dfrac{u}{\delta^s} \in L^1(\Omega) \end{cases} \tag{3.64}$$

Combining both facts:

$$\begin{cases} (\mathrm{P}^0_{\mathrm{vw}}) \text{ and} \\ f\varphi_\delta \in L^1(\Omega) \end{cases} \quad \Longleftrightarrow \quad \begin{cases} (\mathrm{P}^0_{\mathrm{loc}}) \text{ and} \\ \dfrac{u}{\delta^s} \in L^1(\Omega). \end{cases} \tag{3.65}$$

Finally, let us state the following comparison result.

Proposition 3.23 (*Comparison Principle*). *Assume that*

$$\begin{cases} \displaystyle\int_\Omega u(-\Delta)^s \varphi \le 0 \qquad \forall 0 \le \varphi \in X^s_\Omega \cap C_c(\Omega), \\ \dfrac{u}{\delta^s} \in L^1(\Omega). \end{cases} \tag{3.66}$$

Then $u \le 0$.

Proof of Proposition 3.23. Let $\varphi \in X^s_\Omega$. Take $\varphi_k = \varphi \eta_{\frac{1}{k}} \in X^s_\Omega \cap C_c(\Omega)$. Then, by Lemma 3.17,

$$0 \ge \int_\Omega u(-\Delta)^s \varphi_k = \int_\Omega \frac{u}{\delta^s} \delta^s (-\Delta)^s \varphi_k \to \int_\Omega \frac{u}{\delta^s} \delta^s (-\Delta)^s \varphi = \int_\Omega u(-\Delta)^s \varphi. \tag{3.67}$$

Hence, taking the test function solution of

$$\begin{cases} (-\Delta)^s \varphi = \operatorname{sign}_+ u & \Omega, \\ \varphi = 0 & \Omega^c, \end{cases} \tag{3.68}$$

we deduce that

$$\int_\Omega u_+ \le 0. \tag{3.69}$$

Hence $u_+ = 0$. This completes the proof. □

Remark 3.24. We notice that for $0 < s < 1$ we have $\delta^{-s} \in L^1(\Omega)$, unlike for $s = 1$. This makes $u/\delta^s \in L^1$ not entirely a “boundary condition”. For $s = 1$, (3.66) was shown in [30]. On the other hand, in the limit $s = 0$ it says

$$\begin{cases} u = (-\Delta)^0 u \le 0 & \Omega \\ u = \dfrac{u}{\delta^0} \in L^1(\Omega) \end{cases} \tag{3.70}$$

The second item gives no information, but still the result is trivially true for $s = 0$, and all the information comes from the operator. For $s = 1$ most of the information came from the integral condition. For the interpolation $0 < s < 1$, the responsibility needs to be shared.

To give an intuition on how much more information the fractional Laplacian $s < 1$ has with respect to the classical Laplacian ($s = 1$) we provide the following example:

Example 3.25. Let $u_c = c\chi_\Omega$ (where χ is the characteristic function). Then:

(i) if $c < 0$ then $(-\Delta)^s u_c(x) < 0$ in Ω,
(ii) if $c > 0$ then $(-\Delta)^s u_c(x) > 0$ in Ω.

For the proof note that for every $x \in \Omega$ we have

$$(-\Delta)^s u_c(x) = c \int_{\Omega^c} \frac{dy}{|x-y|^{n+2s}}.$$

Both signs are reversed for $x \in \Omega^c$. This property is obviously false for $s = 1$.

3.7. Accretivity

In the study of evolution equations associated to elliptic operators the property of *accretivity* plays an important role since it can be used as a basic tool in the solution of associated parabolic problems and the generation of the corresponding semigroups, [11]. We say that a (possibly unbounded or nonlinear) operator A acting in a Banach space X is accretive if for every $u_1, u_2 \in D(A)$, the domain of the operator $D(A) \subset X$, and every $\lambda > 0$ we have

$$\|u_1 - u_2\|_X \le \|f_1 - f_2\|_X \,, \tag{3.71}$$

where $f_i = u_i + \lambda L u_i$, $i = 1, 2$. This is a contractivity property. Moreover, an accretive operator is called m-accretive if the problem $f = u + \lambda L u$ can be solved for every $f \in X$ and every $\lambda > 0$ and the solution u lies in $D(A)$. The Crandall–Liggett Theorem [24] implies that, when L is an m-accretive operator in a Banach space X, we can solve the evolution problem

$$\partial_t u(t) + L u(t) = f(t)$$

for every initial data $u(0) \in X$ for every $f \in L^1(0, \infty; X)$, and find a unique generalized solution $u \in C([0, \infty); X)$ that solves this initial-value problem in the so-called mild sense.

A further concept is T-accretivity, that incorporates the maximum principle and applies to ordered Banach spaces, like spaces of real functions. It reads

$$\|(u_1 - u_2)_+\|_X \le \|(f_1 - f_2)_+\|_X. \tag{3.72}$$

under the same assumptions as in (3.71).

The results of the preceding subsections allow to prove the first part of the following statement.

Proposition 3.26. *The fractional Laplacian operator $L = (-\Delta)^s$ is m-T-accretive in the space $L^1(\Omega)$ and also in the spaces $L^1(\Omega; \phi)$ for all positive weights $\phi \in X^s$, such that $(-\Delta)^s \phi \ge 0$. The restricted Laplacian operator is also m-T-accretive in the spaces $L^p(\Omega)$ with $1 < p \le \infty$.*

For the accretivity in $L^1(\Omega; \phi)$ we have to check that

$$\int_\Omega L u \operatorname{sign}(u) \phi \, dx = \int_\Omega L|u| \phi \, dx = \int_\Omega |u| L\phi \, dx \ge 0,$$

where we use Kato's inequality, see Lemma 3.8, and the symmetry implied by (2.9).

The last statement for finite $p > 1$ admits an easy proof that uses the Stroock–Varopoulos inequality for weak solutions that we quote from [61], Lemma 5.1:

Lemma 3.27 (*Stroock–Varopoulos' Inequality*)**.** *Let $0 < s < 1$, $p > 1$. Then*

$$\int_{\mathbb{R}^N} (|v|^{p-2} v)(-\Delta)^s v \ge \frac{4(p-1)}{p^2} \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} |v|^{p/2} \right|^2 \tag{3.73}$$

for all $v \in L^p(\mathbb{R}^n)$ such that $(-\Delta)^s v \in L^p(\mathbb{R}^n)$.

This is done for functions defined in $\mathbb{R}^n$, when working in a bounded domain we recall that $v = 0$ outside of Ω. The inequality not only shows that the operator is accretive but it measures its amount in terms of a square norm of the fractional operator of half order.

The application for very weak solutions is obtained by passage to the limit. For $p = \infty$ pass to the limit in the result for finite p.

3.8. Comparison with the class of large solutions

The theory we have described asks for zero boundary conditions, but only in some generalized sense. An immediate extension of our class of solutions is the class of large solutions that has been studied by [1]. These solutions blow-up at the boundary, which is explained by the presence of some singular boundary measure in the weak formulation. See also [39]. The typical example is

$$u_{1-s}(x) = \frac{c(n,s)}{(1-|x|^2)^{1-s}} \qquad \text{in } B_R(0)$$

with $u_{1-s}(x) = 0$ outside. This function is found in [4], where it is proved that u_{1-s} satisfies

$$(-\Delta)^s u_{1-s} = 0 \qquad \text{pointwise in } B.$$

Since $u_{1-s}(x)/\delta(x)^s \asymp c/\delta(x)$, which is not integrable near the boundary, we are sure that this is a large solution and not a very weak solution of the Dirichlet Problem as in the preceding theory. The divergence $\delta(x)^{-1}$ is just borderline for our class of very weak solutions, and this is another proof of optimality for our theory.

4. Schrödinger problem with positive potentials

Here we extend previous results by authors in [30,31] dealing with the classical stationary Schrödinger equation to fractional operators, i.e. to problem (P). As a preliminary, we start by the easier case of bounded potentials and functions.

By analogy to Definitions 3.1 and 3.4 we introduce

Definition 4.1. Let $f \in L^2(\Omega)$. A *weak solution* of (P) if a function $u \in H_0^s(\Omega)$, $Vu \in L^2(\Omega)$, and such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi \tag{P$_\mathrm{w}$}$$

for all $\varphi \in H_0^s(\Omega)$. This definition can be extended by asking that $f, Vu \in (H_0^s(\Omega))'$.

Definition 4.2. We assume that $f \in L^1(\Omega, \delta^s)$. We say that u is a *very weak solution* of (P) if

$$\begin{cases} u \in L^1(\Omega), \\ u = 0 \text{ a.e. } \mathbb{R}^n \setminus \Omega \text{ and} \\ V u \delta^s \in L^1(\Omega), \\ \displaystyle\int_\Omega u(-\Delta)^s \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \qquad \forall \varphi \in X_\Omega^s, \end{cases} \tag{P$_\mathrm{vw}$}$$

where X_Ω^s is given by (3.6).

As seen in the previous section, the concept of weak solution will not be sufficient to solve the Dirichlet Problem with general data. Moreover, through Proposition 2.1 it is trivial to show that

Lemma 4.3. *If $u \in H_0^s(\Omega)$ is a weak solution of (P) with $f \in L^2(\Omega)$ in the sense of (P$_w$), then it is a very weak solution of (P) in the sense of (P$_{vw}$).*

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Remark 4.4. The converse implication, which we indicated as true for (P^0) in Remarks 3.5.3), escapes the interest of this paper. However, since $u \in H_0^s(\Omega)$ and $f \in L^2(\Omega)$ then, it seems natural, although it requires a rigorous proof, that $Vu = f - (-\Delta)^s u \in (H_0^s(\Omega))'$. Even if we do not prove that $Vu \in L^2(\Omega)$, this would be enough to say that u is a weak solution (in a natural sense).

The following result confirms that the class of very weak solutions is not too general

Theorem 4.5. *Let $f \in L^1(\Omega, \delta^s)$. There is, at most, one solution of* (P_{vw}).

Proof. Let u_1, u_2 be two solutions. Let $u = u_1 - u_2$. Therefore,

$$\int_\Omega u(-\Delta)^s \varphi = -\int_\Omega V u \varphi, \qquad \forall \varphi \in X_\Omega^s, \tag{4.1}$$

Therefore, through Lemma 3.8 and Kato's inequality Proposition 9.1 we have that

$$\int_\Omega |u|(-\Delta)^s \varphi \le 0, \qquad \forall\, 0 \le \varphi \in X_\Omega^s, \tag{4.2}$$

In particular, $|u| \le 0$. This completes the proof. □

4.1. The case $V \in L^\infty(\Omega)$

We now address the question of existence. The simplest case concerns bounded potentials. When $V \in L^\infty(\Omega)$, $(-\Delta)^s + V$ is a self-adjoint operator in $L^2(\Omega)$, as $(-\Delta)^s$, which has a positive first eigenvalue $\lambda_1 > 0$. It is easy to see, through the Lax–Milgram theorem, that there exists a unique weak solution.

When $V, f \in L^\infty(\Omega)$ we can apply the regularity estimates in Proposition 3.2 by bootstrapping. For a fixed solution, we define $g = f - Vu \in L^\infty(\Omega)$, and we know that $u \in \mathcal{C}^s(\bar{\Omega})$.

Lemma 4.6. *Let $V, f \in L^\infty(\Omega)$. There exists a unique solution u of* (P_w). *It satisfies*

$$\|u\|_{L^1} \le C \|f\delta^s\|_{L^1} \tag{4.3a}$$

$$\|Vu\delta^s\|_{L^1} \le C\|f\delta^s\|_{L^1}, \tag{4.3b}$$

where C does not depend on u. Furthermore,

$$\left\| \frac{u}{\delta^s} \right\|_{L^1} \le \|f\varphi_\delta\|_{L^1}, \tag{4.4a}$$

$$\|Vu\varphi_\delta\|_{L^1} \le \|f\varphi_\delta\|_{L^1}. \tag{4.4b}$$

Moreover,

$$\|u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)} \tag{4.5a}$$

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\Omega)}. \tag{4.5b}$$

If $f \ge 0$, then $u \ge 0$.

Notice that (4.4a) and (4.4b) hold with constant 1. In order for (4.3a) and (4.3b) to also hold with constant 1 we can choose the first eigenfunction of $(-\Delta)^s$, φ_1, as a weight.

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Proof of Lemma 4.6. The existence of a weak solution $u \in H_0^s(\Omega)$ follows from the Lax–Milgram theorem. It is a very weak solution. The fact that, if $f \geq 0$, then $u \geq 0$ follows as for the $(-\Delta)^s$ operator.

To compute the estimates, we start by considering $f \geq 0$. Then $u \geq 0$. By considering as test function the unique solution of problem

$$\begin{cases} (-\Delta)^s \varphi_0 = 1 & \Omega \\ \varphi_0 = 0 & \Omega^c. \end{cases} \tag{4.6}$$

From the representation formula (3.1) and (3.2) we know that $\varphi_0 \geq c\delta^s$ and, from the results in [67], that $\varphi_0 \in X_\Omega^s$. Therefore

$$\int_\Omega u + \int_\Omega V u \delta^s \leq \int_\Omega u + \frac{1}{c}\int_\Omega V u \varphi_0 \leq C \int_\Omega (u + V u \varphi_0) \leq C \int_\Omega f \delta^s \frac{\varphi_0}{\delta^s} \leq C \left\| \frac{\varphi_0}{\delta^s} \right\|_{L^\infty} \| f \delta^s \|_{L^1} \tag{4.7}$$

so (4.3a) and (4.3b) hold. Using φ_δ as a test function in the very weak formulation

$$0 \leq \int_\Omega \frac{u}{\delta^s} + V u \varphi_\delta \leq \int_\Omega u (-\Delta)^s \varphi_\delta + V u \varphi_\delta = \int_\Omega f \varphi_\delta. \tag{4.8}$$

Hence,

$$\left\| \frac{u}{\delta^s} \right\|_{L^1} \leq \| f \varphi_\delta \|_{L^1}. \tag{4.9}$$

To obtain (4.5a) we can take as a test function in the very weak formulation the solution of

$$\begin{cases} (-\Delta)^s \varphi = u & \Omega, \\ \varphi = 0 & \partial\Omega. \end{cases} \tag{4.10}$$

It is clear that $\varphi \geq 0$ and, due to Green kernel estimates, $\|\varphi\|_{L^2} \leq C \|u\|_{L^2}$. Thus

$$\int_\Omega u^2 \leq \int_\Omega u^2 + \int_\Omega V u \varphi = \int_\Omega f \varphi \leq \|f\|_{L^2} \|\varphi\|_{L^2} \leq C \|f\|_{L^2} \|u\|_{L^2}. \tag{4.11}$$

This concludes (4.5a). Since u is a weak solution, (4.5b) can be obtained by using u as a test function

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 + \int_\Omega V u^2 = \int_\Omega f u \leq \|f\|_{L^2} \|u\|_{L^2} \leq C \|f\|_{L^2}^2. \tag{4.12}$$

Finally, if f changes sign, we can decompose it as $f = f^+ - f^-$, and apply twice the previous result to complete the proof. $\square$

Remark 4.7. Due to Proposition 3.12 applied to the case $p = 2$, we know that $u \in H_0^s(\Omega) \mapsto u/\delta^s \in L^2(\Omega)$ is well-defined and continuous. Hence, for $0 \leq V \leq C\delta^{-2s}$ the following bilinear map is continuous

$$H_0^s(\Omega) \times H_0^s(\Omega) \longrightarrow \mathbb{R} \tag{4.13}$$

$$(u, \varphi) \longmapsto \int_\Omega V u \varphi \tag{4.14}$$

because

$$\left| \int_\Omega V u \varphi \right| = \left| \int_\Omega V \delta^{-2s} \frac{u}{\delta^s} \frac{\varphi}{\delta^s} \right| \leq C \left\| \frac{u}{\delta^s} \right\|_{L^2(\Omega)} \left\| \frac{\varphi}{\delta^s} \right\|_{L^2(\Omega)} \leq C \|u\|_{H^s(\mathbb{R}^n)} \|\varphi\|_{H^s(\mathbb{R}^n)}. \tag{4.15}$$

Thus, when $0 \leq V \leq C\delta^{-2s}$ and $f \in L^2(\Omega)$, we can also use the Lax–Milgram Theorem to show existence and uniqueness of weak solutions.

Remark 4.8. About regularity for weak solutions of nonlocal Schrödinger equations in an open set of $\mathbb{R}^n$ subject to exterior Dirichlet, recently Fall [37] proves Hölder regularity estimates for general nonlocal operators defined via Dirichlet forms, by symmetric kernels $K(x, y)$ bounded from above and below by $|x - y|^{-(N+2s)}$, $0 < s < 1$. See also [42].

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4.2. General potentials $V \in L^1_{loc}(\Omega)$

We now consider the problem for $0 \le V \in L^1_{loc}(\Omega)$. For solutions of (P$_{\rm vw}$), $Vu\delta^s$ could, in principle, not be in $L^1(\Omega)$. We introduce the following definition

Definition 4.9. We say that u is a very weak *local* solution of (P) if

$$\begin{cases} u \in L^1(\Omega), u = 0 \text{ a.e. } \mathbb{R}^n \setminus \Omega, \\ Vu \in L^1_{loc}(\Omega) \text{ and} \\ \displaystyle\int_\Omega u[(-\Delta)^s \varphi + V\varphi] = \int_\Omega f\varphi, \qquad \forall \varphi \in X^s_\Omega \cap C_c(\Omega). \end{cases} \tag{P$_{\rm loc}$}$$

Note that for a local solution, $Vu\delta^s \in L^1(\Omega)$ does not seem like a natural part of the definition. It is clear that (P$_{\rm loc}$) is a weaker concept than (P$_{\rm vw}$) because it lacks the information on the boundary, so it cannot produce uniqueness. But it is a very convenient step into existence.

In spaces with traces, solutions of (P$_{\rm loc}$) with trace 0 are solutions of (P$_{\rm vw}$). The following theorem shows what a local solution is missing to become a very weak solution

Theorem 4.10. *Let $V \in L^1_{loc}(\Omega)$ and $f\delta^s \in L^1(\Omega)$. Any solution $u \in L^1(\Omega)$ of (P$_{loc}$) such that $Vu\delta^s \in L^1(\Omega)$ and $u/\delta^s \in L^1(\Omega)$ is a solution of (P$_{vw}$).*

Proof. If $Vu\delta^s \in L^1(\Omega)$, then $g = f - Vu \in L^1(\Omega, \delta^s)$. Hence, we can apply Proposition 3.21. □

We are ready to state one of the main results of the paper.

Theorem 4.11 (*Existence theorem*)**.** *Let $V \in L^1_{loc}(\Omega)$ and $f\delta^s \in L^1(\Omega)$. Then*

(i) There exists a very weak solution of (P$_{vw}$). It satisfies (4.3a) and (4.3b).

(ii) If $f \ge 0$, then $u \ge 0$.

(iii) Furthermore, if $f\varphi_\delta \in L^1(\Omega)$ then (4.4a) and *(4.4b) hold and, hence, $u/\delta^s \in L^1(\Omega)$.*

(iv) Moreover, if $f \in L^2(\Omega)$, then u is in $H^s_0(\Omega)$.

Remark 4.12. This result extends previous results by the two first authors for the classical case ($s = 1$) (see [30,31]) to the fractional case. Furthermore, the argument we provide allows us to improve the results for the classical case. In the present text, we have proved that the definition of *very weak solution in the weighted sense* used in previous papers is not necessary as a concept of solution, but rather as a intermediate step.

Proof of Theorem 4.11. We proceed in several steps.

(1) We start by assuming $f \ge 0$ and bounded. Let $V_k = \min\{V, k\}$. Let u_k be the solution of

$$\begin{cases} (-\Delta)^s u_k + V_k u_k = f & \Omega, \\ u = 0 & \Omega^c. \end{cases} \tag{4.16}$$

We know that

$$\|u_k\|_{L^1(\Omega)} \le \|f\delta^s\|_{L^1(\Omega)} \tag{4.17}$$

$$\|V_k u_k \delta^s\|_{L^1(\Omega)} \le \|f\delta^s\|_{L^1(\Omega)}. \tag{4.18}$$

It is easy to prove that for $k_1 < k_2$ we have

$$0 \le u_{k_2} \le u_{k_1}. \tag{4.19}$$

Hence, by the Monotone Convergence Theorem we know that there exists $u \in L^1(\Omega)$ such that

$$u_k \to u \qquad \text{a.e. and in } L^1(\Omega). \tag{4.20}$$

Furthermore,

$$\|u_k\|_{L^\infty} \le c\|f\|_{L^\infty}. \tag{4.21}$$

Hence,

$$u_k \rightharpoonup u \qquad L^\infty\text{-weak-}\star\,. \tag{4.22}$$

Let $K \Subset \Omega$ be a compact set. We have that

$$\|V_k u_k\|_{L^1(K)} \le c\|V\|_{L^1(K)}\|f\|_{L^\infty}. \tag{4.23}$$

Notice that this is not true if K is replaced by Ω. Also

$$0 \le V_k u_k \delta^s \le V u_k \delta^s \le V u_0 \delta^s. \tag{4.24}$$

By the Dominated Convergence Theorem

$$V_k u_k \delta^s \to V u \delta^s \qquad L^1(K). \tag{4.25}$$

We have proved, therefore, that

$$\int_\Omega u(-\Delta)^s \varphi + \int_\Omega V u \varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{4.26}$$

This completes the proof of existence of a solution u of ($\mathrm{P}_{\mathrm{loc}}$) for $f \ge 0$ and bounded.

(2) We improve the result, still keeping f bounded. Since $0 \le f\varphi_\delta \in L^1(\Omega)$, then (4.4a) and (4.4b) hold for u_k and $V_k u_k$. It is easy to check, applying Fatou's lemma, that the estimates hold for u and Vu. In particular, $u/\delta^s \in L^1(\Omega)$ and $Vu\delta^s \in L^1(\Omega)$. Applying Theorem 4.10 we deduce that it is a solution of (P_{vw}). Hence,

$$\int_\Omega u(-\Delta)^s \varphi + \int_\Omega V u \varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in X_\Omega^s. \tag{4.27}$$

(3) Assume now that $0 \le f \in L^1(\Omega, \delta^s)$. Let $f_m = \min\{f, m\}$ and let u_m be the solution of

$$\begin{cases} (-\Delta)^s u_m + V u_m = f_m & \Omega, \\ u = 0 & \Omega^c. \end{cases} \tag{4.28}$$

Since f_m is a pointwise nondecreasing sequence, then u_m, Vu_m and $Vu_m\delta^s$ are pointwise nondecreasing sequences. If $m_1 < m_2$ then

$$u_{m_1} \le u_{m_2}. \tag{4.29}$$

The sequence of functions u_m converges in $L^1(\Omega)$ due to the Monotone Convergence Theorem since it is uniformly bounded above in L^1. We have

$$\|u\|_{L^1(\Omega)} = \lim_{m\to\infty} \|u_m\|_{L^1(\Omega)} \le c \lim_{m\to\infty} \|f_m \delta^s\|_{L^1(\Omega)} = \|f\delta^s\|_{L^1(\Omega)}. \tag{4.30}$$

Likewise

$$\|Vu\delta^s\|_{L^1(\Omega)} = \lim_{m\to\infty} \|Vu_m\delta^s\|_{L^1(\Omega)} \le c \lim_{m\to\infty} \|f_m \delta^s\|_{L^1(\Omega)} = \|f\delta^s\|_{L^1(\Omega)} \tag{4.31}$$

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and $Vu_m \to Vu$ in $L^1(\Omega;\delta^s)$ by monotone convergence. We can now pass to the limit in the very weak formulations to show that

$$\int_\Omega u(-\Delta)^s\varphi + \int_\Omega Vu\varphi = \int_\Omega f\varphi, \qquad \forall\varphi \in C_c^\infty(\Omega). \tag{4.32}$$

This proves existence of a very weak solution when $f \geq 0$ and also positivity (ii).

(4) To prove item (iii) when $f \geq 0$, we assume that $0 \leq f\varphi_\delta \in L^1(\Omega)$. Then (4.4a) and (4.4b) hold for the sequences u_k, u_m and $V_k u_k, Vu_m$ that appear in steps (1)–(3) of the previous proof. It is easy to check that, in each of the limits, the estimates hold.

(5) In all the limits, applying (4.5a) and (4.5b) we know that $\|u_m\|_{H^s(\mathbb{R}^n)}, \|u_k\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{L^2}$, and so it converges weakly in $H^s(\mathbb{R}^n)$. In particular, the limit $u \in H_0^s(\Omega)$ and (4.5a) and (4.5b) hold.

(6) In order to prove items (i), (ii) and (iii) when f changes sign, we can split $f = f_1 - f_2$ where $f_i \geq 0$. We apply the previous part of the proof for f_i to construct u_1 and u_2. We define $u = u_1 - u_2$. This concludes the proof. □

Remark 4.13. An analogous way to complete step (2) is to realize that $V_k u_k \varphi_\delta \in L^1(\Omega)$ with uniform bounds. By splitting the integrals near and far from the boundary, and using the sharp estimates for φ_δ, we can check that $V_k u_k \delta^s$ converges in $L^1(\Omega)$.

This result can be extended to measures as data, in the space $\mathcal{M}(\Omega,\delta^s)$ by taking limits. See comments on Section 11.

4.3. Accretivity and counterexample

The results of the preceding subsections allow to prove the following extension of the results for the operator without potential.

Corollary 4.14. *The fractional operator $L_V = (-\Delta)^s + V$ with $V \geq 0$, $V \in L^1_{loc}(\Omega)$ is m-T-accretive in the space $L^1(\Omega)$ and also in the spaces $L^1(\Omega;\phi)$ for all positive weights $\phi \in X^s$, such that $(-\Delta)^s\phi \geq 0$. Moreover, L_V is accretive in all the spaces $L^p(\Omega)$, $1 \leq p \leq \infty$.*

As a negative result for operators with potentials, we want to show that for unbounded potentials $V \geq 0$ the requirement that $f \in L^\infty(\Omega)$ does not imply, in general, that $Vu \in L^\infty(\Omega)$, where u is the solution of $(-\Delta)^s u + Vu = f$.

Construction of a counterexample. (i) We consider a nice bounded, positive and smooth function $f \geq 0$ defined in Ω, we may also assume that f has compact support in Ω; we also take a nice bounded potential $V_1 \geq 0$, and consider the solutions of the Dirichlet problem in Ω

$$(-\Delta)^s u_0 = f, \quad (-\Delta)^s u_1 + V_1(x)u_1 = f. \tag{4.33}$$

By the theory of preceding sections, both solutions are bounded and nonnegative in Ω and $0 \leq u_1 \leq u_0$. Since $V_1 u_1$ is bounded the theory says that the solution u_1 is also C^s in Ω.

Take any point $x_0 \in \Omega$ where u_1 is strictly positive $u_1(x_0) = c_0 > 0$. By continuity we can take a small ball $B_{r_0}(x_0) \subset \Omega$ where $u_1(x) = c_0/2 > 0$.

(ii) We now take a perturbation $g(x) = G(|x - x_0|) \geq 0$ which is radially symmetric and decreasing around x_0 and is supported maybe in $B_{r_0}(x_0)$. Consider now the solution $u_2 \geq 0$ of the Dirichlet problem in Ω

$$(-\Delta)^s u_2 + V_2(x)u_2 = f \quad\ V_2(x) = V_1(x) + g(x). \tag{4.34}$$

We have $0 \leq u_2(x) \leq u_1(x)$ in Ω.

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(iii) Let us prove that $u_2(x)$ is uniformly positive if $g \in L^p(\Omega)$ with a small bound. In fact if $u = u_1 - u_2$ we have

$$(-\Delta)^s u + V_1(x)u = g(x)u_2 \leq Cg(x). \tag{4.35}$$

By the known embedding theorems or using the bounds for the Green function, we conclude that when p is large enough, $p > p(s,n)$ we have $u \in C(\Omega)$ and

$$0 \leq u(x) \leq c_1(p,s,n)\|g\|_p.$$

Therefore, if $\|g\|_p$ is small enough we have $u(x) \leq c_0/4$. Note that c_0 was defined before and does not depend on the perturbation g. It follows that

$$u_2(x) = u_1(x) - u(x) \geq c_0/4 \qquad \text{in } B_r(x_0).$$

(iv) We now impose the last requirement, $g(x_0) = +\infty$. Then we have

$$V_2(x_0)u_2(x_0) = +\infty.$$

Moreover, we can easily find functions $g \notin L^q(B_r(x_0))$ with $q > p > p(s,n)$. Therefore, in that case

$$\|V_2 u_2\|_q = +\infty.$$

Remark 4.15. The construction can be generalized to cases with blow-up of Vu at many points; and maybe the requirement $q > p(n,s)$ can be eliminated, so that bound of Vu by means of f in L^p is false for $p > 1$.

4.4. Solutions with measure data

As we pointed out in the introduction, there is a simple extension of our existence and uniqueness theory as reflected in Theorems 4.5 and 4.11 to right-hand side of the equation is a measure $\mu \in \mathcal{M}(\Omega, \delta^s)$. We leave it to the reader to prove that both mentioned theorems hold in that generality.

For (P^0), a nice theory can be done, as well, through the Green kernel. Many of the results and techniques in [55] still hold in this setting.

5. Super-singular potentials

In this section we discuss the influence on the theory of potentials $V \in L^1(\Omega)$ that blow up near the boundary. We are in particular interested in potentials $V \geq C/\delta^{2s}$ that we call *super-singular potentials*. This kind of potentials is very relevant in Physics (see, e.g., [23,62]). Surprisingly, potentials with large blow-up on $\partial\Omega$ are very good for the theory we have described above, as we will show next.

5.1. Definitions and first results

We want to address the question of how super-singular potentials regularize the solutions. A problem with the regularity of solutions of problems involving the fractional Laplacian operator $(-\Delta)^s$ is that, according to Proposition 3.2 and Proposition 3.9, the solutions are typically $u \asymp \delta^s$, which is not of class $\mathcal{C}^1$ in a neighbourhood in Ω of $\partial\Omega$. However, for super-singular potentials, solutions such that $u/\delta^{s+\varepsilon} \in L^\infty(\Omega)$ may be found (see Theorem 5.6). Hence, we have a higher Hölder exponent at $\partial\Omega$. In this sense, super-singular potentials force the solution u to be more regular in the proximity of the boundary. A natural definition of the concept of flat solution for $s < 1$ is the following

Definition 5.1. We say that $u \in L^1(\Omega)$ is an s-flat solution if, for every $y \in \partial\Omega$

$$\lim_{x \to y} \frac{u(x)}{\delta(x)^s} = 0. \tag{5.1}$$

Clearly, a sufficient condition for s-flatness is that $u/\delta(x)^{s+\varepsilon} \in L^\infty(\Omega)$ for some $\varepsilon > 0$.

We first obtain a result on existence of flat solutions in a weaker integral sense that follows directly from our results in previous sections:

Proposition 5.2. *If $V \geq C/\delta^{2s+\varepsilon}$ for some $C > 0$, $\varepsilon \geq 0$, then for every $f \in L^1(\Omega; \delta^s)$ we have $u/\delta^{s+\varepsilon} \in L^1(\Omega)$, even if $f\varphi_\delta \notin L^1(\Omega)$.*

Indeed, we have proved that $V u \delta^s \in L^1(\Omega)$, so that the lower bound for V implies the conclusion.

Remarks 5.3. (1) When $V \geq c\delta^{-2s}$, the equivalence (3.24) no longer holds. The sufficiency part still holds, but this result here shows that the extra condition on f is no longer necessary.

(2) The integral sense is not a very strong concept of flat solution, but it is nevertheless credited in the literature for $s = 1$. In that case ($s = 1$) super-singular potentials have been shown to "flatten" the solution, in the sense that $\partial u/\partial n = 0$ on $\partial\Omega$. For large powers, higher order derivatives vanish. For further reference, see [27–29,59].

5.2. Pointwise flatness estimates of solutions through barrier functions

We give next conditions for s-flatness in the everywhere sense. In order to prove this fact we will construct some clever barrier function (as in [27]). We first need a technical result.

Lemma 5.4. *Let $\nu_\beta(x) = |x|^\beta$ with $\beta > 0$. Then*

$$(-\Delta)^s \nu_\beta = \gamma_\beta |x|^{-2s} \nu_\beta, \qquad \textit{in } \mathbb{R}^n, \tag{5.2}$$

where

$$\gamma_\beta = 2^{2s} \frac{\Gamma\left(\frac{n+\beta}{2}\right) \Gamma\left(s - \frac{\beta}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right) \Gamma\left(-s + \frac{\beta+n}{2}\right)} \tag{5.3}$$

is a constant.

The computation of the result above can be found in [73, p. 798] and [36]. It can be obtained by applying the Fourier transform formula of a radial function given in [71, Theorem 4.1]. Note that $\gamma_{s+\varepsilon} < 0$ for $0 < \varepsilon < s$, while γ_{2s} diverges.

Lemma 5.5. *Let $0 < \varepsilon < s$, $0 \leq f \in L^\infty$, $V \geq C_V |x - x_0|^{-2s}$ with $C_V > -\gamma_{s+\varepsilon} > 0$, and let $x_0 \in \partial\Omega$. Then,*

$$\frac{u(x)}{|x - x_0|^{s+\varepsilon}} \leq \frac{\|f\|_{L^\infty}}{(\gamma_{s+\varepsilon} + C_V)} R(x_0)^{s-\varepsilon}, \tag{5.4}$$

a.e. in Ω, where $R(x_0) = \max_{x \in \bar{\Omega}} d(x, x_0)$ (i.e. such that $\Omega \subset B_R(x_0)$).

Proof. Since $0 \leq f \in L^\infty$ we have that $0 \leq u \in L^\infty$.

Let us consider $U(x) = C_U \nu_{s+\varepsilon}(x - x_0)$ where

$$C_U = \frac{\|f\|_{L^\infty}}{(\gamma_{s+\varepsilon} + C_V)} R^{s-\varepsilon} > 0. \tag{5.5}$$

We compute

$$
\begin{aligned}
(-\Delta)^s U + VU &= \gamma_{s+\varepsilon}|x-x_0|^{-2s}U + VU && (5.6)\\
&\geq (\gamma_{s+\varepsilon} + C_V)|x-x_0|^{-2s}U && (5.7)\\
&= C_U(\gamma_{s+\varepsilon} + C_V)|x-x_0|^{-s+\varepsilon} && (5.8)\\
&\geq C_U(\gamma_{s+\varepsilon} + C_V)R^{-s+\varepsilon} && (5.9)\\
&= \|f\|_{L^\infty} && (5.10)
\end{aligned}
$$

a.e. in Ω, since $-s+\varepsilon<0$. Since also $U \geq 0 = u$ on Ω^c we have that $U \geq u$ a.e. in Ω. Therefore,

$$
\frac{u}{|x-x_0|^{s+\varepsilon}} \leq \frac{U}{|x-x_0|^{s+\varepsilon}} = C_U. \tag{5.11}
$$

a.e. in Ω. This completes the proof. □

Theorem 5.6. *Let $0<\varepsilon<s$, $0 \leq f \in L^\infty$, $V(x) \geq C_V \delta(x)^{-2s} \geq 0$ with $C_V > -\gamma_{s+\varepsilon}$. Then,*

$$
\frac{u}{\delta^{s+\varepsilon}} \in L^\infty(\Omega). \tag{5.12}
$$

Proof. Since $\delta(x) = \min_{x_0 \in \partial\Omega}|x-x_0|$ we have that

$$
\frac{u(x)}{\delta(x)^{s+\varepsilon}} = \max_{x_0\in\partial\Omega} \frac{u(x)}{|x-x_0|^{s+\varepsilon}} \leq \frac{\|f\|_{L^\infty}}{(\gamma_{s+\varepsilon}+C_V)} \max_{x_0\in\bar{\Omega}} R(x_0)^{s-\varepsilon}. \tag{5.13}
$$

This last maximum if finite because Ω is a bounded set. This completes the proof. □

Remarks 5.7. (1) We have that

$$
\frac{u(x)}{\delta^s(x)} \leq C\delta^\varepsilon(x) \to 0 \tag{5.14}
$$

uniformly as $x \to \partial\Omega$. These functions are *uniformly s*-flat.

Notice that this implies the unique continuation property (see, e.g., [38]) fails for super-singular negative potentials.

(2) Notice that the Pösch–Teller potential (1.3) is $L^1_{loc}(\Omega)$ and behaves like $V \geq cd(x,\partial\Omega)^{-2}$ in any annulus of the form

$$
\Omega_k = \left\{x \in \mathbb{R}^n : k\pi < \alpha|x| \leq k\pi + \frac{\pi}{2}\right\}. \tag{5.15}
$$

or

$$
\Omega_k = \left\{x \in \mathbb{R}^n : k\pi + \frac{\pi}{2} < \alpha|x| \leq (k+1)\pi\right\}. \tag{5.16}
$$

(3) The results presented here are part of an ongoing research and must be improved.

5.3. Pointwise flatness estimates for radially symmetric data

There is still the question of getting solutions as flat as desired if V is singular enough near the border. We do not have a general result in that direction, since it needs more tools and space, but we do have a convincing example. It is as follows:

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Theorem 5.8. *Let $\Omega = B_R(0)$ be a ball, $f \in L^1(\Omega)$ positive, radially symmetric and decreasing, and let the potential V be positive, radially symmetric and increasing. Then the solution is nonnegative, radially symmetric and non-increasing. If moreover $V(x) \geq C_V \delta(x)^{-p}$ for some $C_V > 0$ and $p > 1$, then we have*

$$u(x) \leq \frac{C_1}{C_V} \|f\|_1 \delta(x)^{p-1}, \tag{5.17}$$

when $\frac{R}{2} < |x| \leq R$, where $C_1 > 0$ depends on n, s. The same conclusion holds if $f \geq 0$ is not radially symmetric but $f \leq g$ where g is positive, integrable, radially symmetric and decreasing. Then formula (5.17) holds if we replace $\|f\|_1$ with $\|g\|_1$.

Proof. (i) We assume first that f is positive, radial and decreasing, hence $f = g$. In that case we already know that $u \geq 0$. The fact that u is radially symmetric follows from uniqueness and the invariance of the problem under rotations. The fact that $u(r)$ is nonincreasing can be proved by a modification of the Aleksandrov reflection principle proved in [73], Section 15.

To get the estimate we first integrate in Ω we get

$$\int_\Omega (-\Delta)^s u(x)\, dx + \int_\Omega V(x) u(x)\, dx = \int_\Omega f(x)\, dx$$

It is easy to prove that for $u \geq 0$ there is the inequality $\int_\Omega (-\Delta)^s u(x)\, dx \geq 0$. We recall that such an inequality is quite standard in the classical Laplacian case. In order to prove it we may use the formula for the operator and get

$$\int_\Omega (-\Delta)^s u(x)\, dx = \int_{x\in\Omega} \int_{y\notin\Omega} \frac{u(x)}{|x-y|^{n+2s}}\, dxdy \geq 0,$$

since the remaining integral cancels by symmetry:

$$\int_{x\in\Omega} \int_{y\in\Omega} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dxdy = 0.$$

We may use all this to conclude that

$$C_V \int_B \frac{u(x)}{(R-|x|)^p} \leq \int_B V(x) u(x) dx \leq \|f\|_1. \tag{5.18}$$

Note that, in this case, $\delta(x) = R - |x|$ for $x \in \Omega = B_R(0)$. Take now a fixed point $x_0 \in \Omega$ such that $\frac{R}{2} < |x_0| \leq R$ (i.e., near the border) and consider the annulus A with outer radius $r_2 = |x_0|$ and inner radius $r_1 = r_2 - \delta(x_0) = 2|x_0| - R > 0$. Then, by monotonicity and radial symmetry

$$\int_A \frac{u(x)}{(R-|x|)^p}\, dx \geq \frac{u(x_0)}{(R-r_1)^p} |A| = \frac{u(x_0)}{(2\delta(x_0))^p} |A| = c u(x_0) \delta(x_0)^{1-p}. \tag{5.19}$$

Since $A \subset \Omega$, combining (5.18) and (5.19) we have

$$u(x_0) \leq \frac{C_1}{C_V} (R - |x_0|)^{p-1} \|f\|_1. \tag{5.20}$$

This completes the proof under the stated assumptions.

(ii) In the case where f is not necessarily radially symmetric but it is bounded above by g, we can solve the problem with right-hand side g, obtain a solution v that satisfies the desired estimate and then we can use the comparison theorem, $u \leq v$. □

We may also take a nonradial V that has a supersingular radial lower bound V_1 like in the Theorem, and apply again the maximum principle to conclude the same type of bound; we leave the easy detail to the reader. We also note that the argument can be extended to other equations and more general domains, but since other techniques are needed this extension will not be discussed here.

Though this example does not give optimal rates, it shows the existence of solutions that are as flat as we like near the boundary, always depending on the divergence of the potential. In that sense, we have the following interesting consequence.

Corollary 5.9. *Under the assumptions of Theorem 5.8, if moreover V satisfies*

$$\lim_{\delta(x)\to 0} V(x)\delta(x)^p = +\infty \quad \textit{for every } p > 1,$$

then the solution $u(x)$ vanishes at the boundary to infinite order in the sense that

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\delta(x)^p} = 0 \quad \textit{for every } p > 1. \tag{5.21}$$

6. The restricted fractional laplacian as the natural limit of the schrödinger equation in $\mathbb{R}^n$ for the super-singular potential

Let us consider the singular infinite well potential (see the exposition in [28,29] for $s = 1$ and [27] for $0 < s < 1$)

$$V(x) = \begin{cases} d(x, \partial\Omega)^{-2s} & \Omega, \\ +\infty & \Omega^c. \end{cases} \tag{6.1}$$

To avoid the ambiguity of the definition of Vu in Ω^c, the solutions of the associated Schrödinger problem can be understood as the limit of the solutions of the corresponding finite-well potentials

$$V_k(x) = k \wedge V(x). \tag{6.2}$$

The stationary Schrödinger equation over its natural domain, the whole space, corresponds to finding $u_k \in H^s(\mathbb{R}^n)$ such that

$$\begin{cases} (-\Delta)^s u_k + V_k(x) u_k = f & \mathbb{R}^n, \\ u_k \to 0 & |x| \to +\infty, \end{cases} \tag{6.3}$$

for some function $0 \le f \in L^\infty(\Omega)$, $f = 0$ in Ω^c. Here, all the usual formulations are equivalent. Hence $u_k \ge 0$. Furthermore, $0 \le u_k$ is a decreasing sequence, and hence has limit in $L^1(\Omega)$, $0 \le u \in L^1(\mathbb{R}^n)$ which is also an a.e. pointwise limit, due to the Monotone Convergence Theorem.

Theorem 6.1. *Assume (6.1). As $k \to \infty$ the solutions of the approximate problems in $\mathbb{R}^n$ converge to the solution of Problem (P). In particular $u = 0$ in Ω^c.*

Proof. Using the solution of

$$\begin{cases} (-\Delta)^s \varphi_0 = 1 & \mathbb{R}^n, \\ \varphi \to 0 & |x| \to \infty \end{cases} \tag{6.4}$$

we deduce that

$$(1 + k \min_{\Omega_c} \varphi_0) \int_{\Omega^c} u_k \le \int_\Omega f \varphi_0 \tag{6.5}$$

Hence $u = 0$ in Ω^c.

On the other hand,

$$\int_{\mathbb{R}^n} u_k(-\Delta)^s\varphi + \int_{\mathbb{R}^n} V_k u_k \varphi = \int_\Omega f\varphi. \tag{6.6}$$

As before, for $K \subset \Omega$ compact $V_k u_k \to V u$ in $L^1(K)$ by the Dominated Convergence Theorem.

Finally, for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $(-\Delta)^s\varphi \in L^\infty(\mathbb{R}^n)$, we pass to the limit to obtain

$$\int_{\mathbb{R}^n} u(-\Delta)^s\varphi + \int_{\mathbb{R}^n} V u \varphi = \int_\Omega f\varphi. \tag{6.7}$$

For φ the restricted fractional Laplacian and the fractional Laplacian in $\mathbb{R}^n$ coincide.

Since $u = 0$ in Ω_c and φ is supported in Ω, this is precisely

$$\int_\Omega u(-\Delta)^s\varphi + \int_\Omega V u \varphi = \int_\Omega f\varphi. \tag{6.8}$$

By density and using the divergence rate of V, we have the previous formulation for all $\varphi \in X_\Omega^s$. $\square$

This shows that the natural fractional Laplacian to deal with the Schrödinger equation with the singular infinite-well potential problem is the restricted fractional Laplacian over Ω. We point out that, physically, the Schrödinger equation a priori must be defined over the whole space, $\mathbb{R}^n$, and that any other constraint (as, for instance, to assume a localization to a subset Ω) must be justified.

7. Another perspective on the results

7.1. The Green operator's viewpoint

Let us start for the case $V = 0$. As mentioned, for regular f we know that the unique solution of

$$\begin{cases} (-\Delta)^s u = f & \Omega, \\ u = 0 & \Omega^c, \end{cases} \tag{7.1}$$

is written in the form

$$u(x) = \int_\Omega \mathbb{G}_s(x,y) f(y)\,dy \tag{7.2}$$

where $\mathbb{G}_s$ satisfies (3.2). In Section 3.4 we have shown that the optimal set of data functions for the Green kernel is given by:

$$\mathrm{G}_s : \mathrm{Dom}(\mathrm{G}_s) = L^1(\Omega, \delta^s) \longrightarrow L_0^1(\Omega) \tag{7.3a}$$

$$f \longmapsto \chi_\Omega(\cdot)\int_\Omega \mathbb{G}_s(\cdot, y) f(y)\,dy, \tag{7.3b}$$

where χ_Ω is the characteristic function of Ω, and

$$L_0^1(\Omega) = \{u \in L^1(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\}. \tag{7.4}$$

In this sense we can characterize

$$X_\Omega^s = \mathrm{G}_s(L^\infty(\Omega)). \tag{7.5}$$

For this, let us read the very weak formulation of (P^0) in terms of G_s. The very weak formulation (P_{vw}) reduces to

$$\int_\Omega u(-\Delta)^s\varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in X_\Omega^s. \tag{7.6}$$

First, for $f \in L^\infty(\Omega)$, we point out that, as the unique solution is $u = \mathrm{G}_s(f)$, we can write

$$\int_\Omega \mathrm{G}_s(f)(-\Delta)^s \varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in X_\Omega^s. \tag{7.7}$$

Since, $X_\Omega^s = \mathrm{G}_s(L^\infty)$, we can write $\varphi = \mathrm{G}_s(\psi)$ for some $\psi \in L^\infty(\Omega)$, and so $(-\Delta)^s\varphi = \psi$. Therefore, (7.6) is equivalent to

$$\int_\Omega \mathrm{G}_s(f)\psi = \int_\Omega f \mathrm{G}_s(\psi), \qquad \forall \psi \in L^\infty(\Omega). \tag{7.8}$$

Thus, the very weak formulation for $f \in L^\infty$ is equivalent to the fact that G_s is self-adjoint.

The following result gives a direct answer:

Proposition 7.1. *Let $L^\infty(\Omega) \subset Y \subset \mathrm{Dom}(\mathrm{G}_s)$ be such that*

$$\mathrm{G}_s : Y \to L^1(\Omega) \text{ is continuous} \tag{7.9}$$

and assume that $L^\infty(\Omega)$ is dense Y. Then $\mathrm{G}_s(f)$ is a very weak solution of (P^0).

Proof. Let $f \in Y$ and $f_k \in L^\infty(\Omega)$ be a sequence converging to f in Y. Then $\mathrm{G}_s(f_k) \to \mathrm{G}_s(f)$ in $L^1(\Omega)$. On the other hand $\mathrm{G}_s(f_k)$ is a very weak solution of (P^0). By passing to the limit in (7.8), we deduce that $\mathrm{G}_s(f)$ also satisfies (7.8), and so it is a very weak solution. □

It was shown in [67] $\mathrm{G}_s : L^\infty(\Omega) \to \mathcal{C}^s(\Omega)$ is continuous. In [21] the authors showed that $\mathrm{G}_s : L^1(\Omega, \delta^s) \to L^1(\Omega)$ is also continuous. We have shown here that $\mathrm{G}_s : L^1(\Omega, \varphi_\delta) \to L^1(\Omega, \delta^{-s})$ is also continuous. Furthermore,

$$\mathrm{G}_s^{-1}\Big(L^1(\Omega, \delta^{-s}) \cap \mathrm{Im}(\mathrm{G}_s)\Big) = L^1(\Omega, \varphi_\delta). \tag{7.10}$$

Problem (P) with $V \neq 0$ is also linear, and could allow for another Green kernel. However, we can write (P) as a fixed point problem for the Green operator of $(-\Delta)^s$ as:

$$u = \mathrm{G}_s(f - Vu). \tag{7.11}$$

Let u be a solution, and let $g = f - Vu$. In Lemma 4.6 we show that, if $V, f \in L^\infty$, then

$$\|g\delta^s\|_{L^1} \le C\|f\delta^s\|_{L^1} \tag{7.12a}$$
$$\|g\varphi_\delta\|_{L^1} \le 2\|f\varphi_\delta\|_{L^1}. \tag{7.12b}$$

The results in Section 4 of this paper lead to corresponding properties of the Green operator for (P)

$$\mathrm{G}_{s,V} : f \mapsto u. \tag{7.13}$$

So far, we have proved that:

1. If $V \in L^1_{loc}$ then $\mathrm{G}_{s,V} : L^1(\Omega, \delta^s) \to L^1(\Omega)$ and $\mathrm{G}_{s,V} : L^1(\Omega, \varphi_\delta) \to L^1(\Omega, \delta^{-s})$.
2. If $V \ge \delta^{-2s}$ then $\mathrm{G}_{s,V} : L^1(\Omega, \delta^s) \to L^1(\Omega, \delta^{-s})$.

It is easy to show that

$$\mathrm{G}_{s,V}(x, y) \le \mathrm{G}_s(x, y) \qquad \forall x, y \in \Omega. \tag{7.14}$$

For $V \in L^\infty$ it is likely that

$$\mathrm{G}_{s,V}(x, y) \asymp \mathrm{G}_s(x, y). \tag{7.15}$$

However, the additional integrability for the case $V \ge c\delta^{-2s}$ guaranties that

$$\mathrm{G}_{s,V}(x, y) \not\asymp \mathrm{G}_s(x, y). \tag{7.16}$$

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7.2. What is $(-\Delta)^s$ of a very weak solution?

Let us think about $(-\Delta)^s$ as a functional operator. It is natural to define

$$\mathrm{L}_s : \mathrm{Dom}(\mathrm{L}_s) \subset L^0_0(\Omega) \longrightarrow L^0(\Omega) \tag{7.17a}$$

$$u \longmapsto c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(\cdot) - u(y)}{|\cdot - y|^{n+2s}} dy, \tag{7.17b}$$

where $L^0(\Omega)$ is the set of measurable functions in Ω and

$$L^0_0(\Omega) = \{u \in L^0(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\}. \tag{7.18}$$

It is easy to show that

$$\mathrm{L}_s : \mathcal{C}^{2s}_0(\overline{\Omega}) \longrightarrow \mathcal{C}^s(\bar{\Omega}) \tag{7.19}$$

where

$$\mathcal{C}^{2s}_0(\overline{\Omega}) = \{u \in \mathcal{C}^{2s}(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\}. \tag{7.20}$$

However, working with integrable rather than smooth functions f, we do not expect $u \in \mathcal{C}^{2s}_0(\overline{\Omega})$. Nonetheless, our aim is to solve the problem (P), so we are interested in the definition of $(-\Delta)^s u$.

By the regularization results obtained through Hörmander theory in [44,68], we have that

$$\mathrm{G}_s : \mathcal{C}^{\gamma}(\bar{\Omega}) \to \mathcal{C}^{\gamma+s}(\bar{\Omega}, \delta^{-s}), \tag{7.21}$$

if $\gamma + s \notin \mathbb{N}$. We point that $\mathcal{C}^{\gamma+s}(\bar{\Omega}, \delta^{-s}) \subset \mathcal{C}^{\gamma+s}(\Omega) \cap \mathcal{C}_0(\bar{\Omega})$. By uniqueness of solutions (P^0) it is clear that

$$u = \mathrm{G}_s \mathrm{L}_s u \qquad u \in \mathcal{C}^{2s}(\bar{\Omega}, \delta^{-s}), \tag{7.22}$$

$$f = \mathrm{L}_s \mathrm{G}_s f \qquad f \in \mathcal{C}^s(\bar{\Omega}). \tag{7.23}$$

We can extend this result to an abstract setting. In this direction we have:

Proposition 7.2. *Let $X \subset \mathrm{Dom}(\mathrm{L}_s)$ and $\mathcal{C}^s(\bar{\Omega}) \subset Y \subset \mathrm{Dom}(\mathrm{G}_s)$. Assume that $\mathrm{G}_s : Y \to X$ and $\mathrm{L}_s : \mathrm{G}_s(Y) \to Y$ are continuous and that $\mathcal{C}^s(\bar{\Omega})$ is dense in Y. Then*

$$f = \mathrm{L}_s \mathrm{G}_s f \text{ in } Y. \tag{7.24}$$

Proof. Let $f_k \in \mathcal{C}^s(\bar{\Omega})$ be a sequence such that $f_k \to f$ in Y. Then $\mathrm{G}_s f_k \to \mathrm{G}_s f$ in X. On the other hand, from (7.23) we know that $f_k = \mathrm{L}_s \mathrm{G}_s f_k$. Therefore $f = \mathrm{L}_s \mathrm{G}_s$. $\square$

If we get inspiration in the case of usual Laplacian we soon see that this pointwise construction, although natural, is not optimal working grounds. By looking again at the case of the usual Laplacian, we would like to study a distributional formulation. By Proposition 2.1 we have that $\mathrm{L}_s|_{\mathcal{C}^{2s}(\bar{\Omega})}$ is self-adjoint. We can define a self-adjoint extension as a distributional operator

$$\widetilde{\mathrm{L}_s} : L^1(\Omega) \to \mathcal{D}'(\Omega) \tag{7.25}$$

through the notion of very weak solution, i.e.

$$\widetilde{\mathrm{L}_s} u : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathbb{R} \tag{7.26a}$$

$$\varphi \mapsto \int_\Omega u(\mathrm{L}_s \varphi). \tag{7.26b}$$

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Through Proposition 2.1 we know that, for $u \in \mathcal{C}_0^{2s}(\overline{\Omega})$,

$$\langle \widetilde{\mathrm{L}_s u}, \varphi \rangle = \int_\Omega (\mathrm{L}_s u)\varphi \tag{7.27}$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$, i.e. $\widetilde{\mathrm{L}_s u}$ has a Riesz representation as a pointwise function (see, e.g., [70,71]). In this sense, we can ensure that any very weak solution of (P^0) satisfies

$$\widetilde{\mathrm{L}_s u} = f \qquad \text{in } \mathcal{D}'(\Omega). \tag{7.28}$$

In fact, this distributional extension is precisely the one that comes naturally from the very weak solutions used in this paper.

8. Auxiliary result. Boundary behaviour of φ_δ

Proof of Lemma 3.14. We know that

$$\varphi_\delta(x) = \int_\Omega \frac{\mathbb{G}_s(x,y)}{\delta^s(y)} dy. \tag{8.1}$$

Due to (3.2)

$$\mathbb{G}_s(x,y) \le \frac{c}{|x-y|^{n-2s}} \min\left(\frac{\delta^s(x)}{|x-y|^s}, 1 \right). \tag{8.2}$$

To estimate the behaviour of φ_δ near the boundary we take a point x near $\partial\Omega$ and consider the integral in a small ball B with centre x and radius $\frac{\delta(x)}{2}$. We split $\varphi_\delta(x) = I_1 + I_2$ by splitting the integral (8.1) into integrals in B and $\Omega \setminus B$. We have

$$I_1 \doteq \int_B \frac{\mathbb{G}_s(x,y)}{\delta^s(y)} dy \le \int_B \frac{c}{|x-y|^{n-2s}\delta^s(y)} dy. \tag{8.3}$$

On the other hand, in B, $\delta(y) \ge \delta - \frac{\delta(x)}{2} \ge c\delta(x)$ and hence

$$I_1 \le \frac{c}{\delta^s(x)} \int_B \frac{1}{|x-y|^{n-2s}} dy. \tag{8.4}$$

Integrating in spherical coordinates

$$I_1 \le \frac{c}{\delta^s(x)} \int_0^{\frac{\delta(x)}{2}} \frac{1}{r^{n-2s}} r^{n-1} dr = \frac{c}{\delta^s(x)} r^{2s} \Big|_0^{\frac{\delta(x)}{2}} \le c\delta^s(x). \tag{8.5}$$

On the other hand we have that

$$\begin{aligned} I_2 &\doteq \int_{\Omega\setminus B} \frac{\mathbb{G}_s(x,y)}{\delta(y)^s} dy && (8.6)\\ &\le c \int_{|x-y|\ge \frac{\delta(x)}{2}} \frac{1}{|x-y|^{n-2s}\delta(y)^s} \frac{\delta(x)^s}{|x-y|^s} dy && (8.7)\\ &= c\delta^s(x) \int_{|x-y|\ge \frac{\delta(x)}{2}} \frac{1}{|x-y|^{n-s}\delta(y)^s} dy && (8.8)\\ &\le c\delta^s(x) \int_{|x-y|\ge \frac{\delta(x)}{2}} \frac{1}{|x-y|^{n}} dy. && (8.9) \end{aligned}$$

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Let $R = \max_{y\in\Omega}|x-y|$. We can integrate radially to compute

$$\int_{|x-y|\geq \frac{\delta(x)}{2}} \frac{1}{|x-y|^n} dy \leq c \int_{\frac{\delta(x)}{2}}^R \frac{1}{r^n} r^{n-1} dr \tag{8.10}$$

$$= c\left(\log R - \log \frac{\delta(x)}{2}\right) \tag{8.11}$$

$$\leq c(1+|\log \delta(x)|). \tag{8.12}$$

Thus

$$I_2 \leq c\delta^s(x)(1+|\log\delta(x)|). \tag{8.13}$$

This concludes the upper bound for φ_δ.

On the other hand $I_1, I_2 \geq 0$. For the lower bound we look only at I_1. Due to (3.2) we also have that

$$\mathbb{G}_s(x,y) \geq \frac{c}{|x-y|^{n-2s}} \min\left(\frac{\delta(x)^s}{|x-y|^s}, 1\right) \min\left(\frac{\delta(y)^s}{|x-y|^s}, 1\right). \tag{8.14}$$

Here we have to be a bit more careful with the minimum. In B, $\frac{\delta(x)^s}{|x-y|^s} \geq 2^s \geq 1$. Also, $\delta(y) \geq \frac{\delta(x)}{2}$ and so $\frac{\delta(y)^s}{|x-y|^s} \geq 1$ in B. Hence

$$\mathbb{G}_s(x,y) \geq \frac{c}{|x-y|^{n-2s}}, \qquad \text{if } |x-y| \leq \frac{\delta(x)}{2}. \tag{8.15}$$

Therefore

$$I_1 \geq \int_B \frac{c}{|x-y|^{n-2s}\delta^s(y)} dy \geq \frac{c}{\delta^s(x)} \int_0^{\frac{\delta(x)}{2}} \frac{1}{r^{n-2s}} r^{n-1} dr = \frac{c}{\delta^s(x)} r^{2s}\Big|_0^{\frac{\delta(x)}{2}} \geq c\delta^s(x). \tag{8.16}$$

This concludes the proof. □

9. An alternative proof of Kato's inequality for the fractional Laplacian with weight

Proposition 9.1 (*Kato's Inequality*). *Let $u \in \mathcal{C}^{2s}(\mathbb{R}^n)$, then, for every $x \in \mathbb{R}^n$*

$$(-\Delta)^s u_+ \leq \mathrm{sign}_+ u\,(-\Delta)^s u \tag{9.1}$$

$$(-\Delta)^s |u| \leq \mathrm{sign} u\,(-\Delta)^s u. \tag{9.2}$$

Moreover, if $u \in L^1(\Omega)$, $f\delta^s \in L^1(\Omega)$ and assuming that

$$\int_\Omega u(-\Delta)^s\varphi = \int_\Omega f\varphi \qquad \forall 0 \leq \varphi \in X_\Omega^s \cap C_c(\Omega), \tag{9.3}$$

then, there exist $\xi_+ \in \widetilde{\mathrm{sign}}_+(u)$ and $\xi \in \widetilde{\mathrm{sign}}(u)$ such that

$$\int_\Omega u_+(-\Delta)^s\varphi \leq \int_\Omega \xi_+ f\,\varphi \tag{9.4}$$

$$\int_\Omega |u|(-\Delta)^s\varphi \leq \int_\Omega \xi f\,\varphi, \tag{9.5}$$

for all $0 \leq \varphi \in X_\Omega^s \cap C_c(\Omega)$, where $\widetilde{\mathrm{sign}}_+$ and $\widetilde{\mathrm{sign}}$ are the maximal monotone graphs given by

$$\widetilde{\mathrm{sign}}_+(s) = \begin{cases} 0 & s<0, \\ [0,1] & s=0, \\ 1 & s\geq 0. \end{cases} \qquad \widetilde{\mathrm{sign}}(s) = \begin{cases} -1 & s<0, \\ [-1,1] & s=0, \\ 1 & s\geq 0. \end{cases} \tag{9.6}$$

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Proof. First assume $u \in \mathcal{C}^{2s}(\mathbb{R}^n)$. Let $s_+(x) = \mathrm{sign}_+ u(x)$. We have that

$$u_+(y) \geq s_+(x)u(y), \tag{9.7}$$
$$u_+(x) = s_+(x)u(x), \tag{9.8}$$
$$\frac{u_+(x) - u_+(y)}{|x-y|^{n+2s}} \leq s_+(x)\frac{u(x)-u(y)}{|x-y|^{n+2s}} \tag{9.9}$$
$$\int_\Omega \frac{u_+(x) - u_+(y)}{|x-y|^{n+2s}} dy \leq s_+(x) \int_\omega \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \tag{9.10}$$
$$(-\Delta)^s u_+(x) \leq s_+(x)(-\Delta)^s u(x). \tag{9.11}$$

Applying this result to $-u$:

$$(-\Delta)^s(-u)_+(x) \leq \mathrm{sign}_+(-u)(-\Delta)^s(-u) \tag{9.12}$$
$$(-\Delta)^s u_-(x) \leq \mathrm{sign}_-(u)(-\Delta)^s u. \tag{9.13}$$

Therefore,

$$(-\Delta)^s|u| \leq \mathrm{sign}(u)\,(-\Delta)^s u. \tag{9.14}$$

If $0 \leq \varphi \in X^s_\Omega \cap C_c(\Omega)$ we have

$$\int_\Omega u_+(x)(-\Delta)^s\varphi(x)dx = \int_\Omega (-\Delta)^s u_+(x)\varphi(x)dx \leq \int_\Omega s_+(x)f(x)\varphi(x)dx. \tag{9.15}$$

Assume now that $u \in L^1(\Omega)$, $u = 0$ in Ω^c and (9.3) holds. Let $f_k = T_k(f)$ we have that $f_k\delta^s \to f\delta^s$ in $L^1(\Omega)$. Let u_k be the unique solutions of

$$\begin{cases} (-\Delta)^s u_k = f_k & \Omega, \\ u_k = 0 & \Omega^c. \end{cases} \tag{9.16}$$

Then, by the results in [21], we know that $u_k \to u$ in $L^1(\Omega)$, hence $(u_k)_+ \to u_+$ in $L^1(\Omega)$. On the other hand, by the previous part of the proof

$$\int_\Omega (u_k)_+(x)(-\Delta)^s\varphi(x)dx \leq \int_\Omega \mathrm{sign}_+(u_k(x))f_k(x)\varphi(x)dx, \qquad \forall 0 \leq \varphi \in X^s_\Omega \cap C_c(\Omega). \tag{9.17}$$

Let $0 \leq \underline{\gamma}_\varepsilon(s) \leq \mathrm{sign}_+(s) \leq \overline{\gamma}_\varepsilon(s) \leq 1$ be smooth functions

$$\overline{\gamma}_\varepsilon(s) = \begin{cases} 0 & s < -\varepsilon, \\ 1 & s > 0. \end{cases}, \qquad \underline{\gamma}_\varepsilon(s) = \begin{cases} 0 & s < 0, \\ 1 & s > \varepsilon. \end{cases} \tag{9.18}$$

Since $f(x) > 0$ if and only $f_k(x) > 0$, we have that

$$\int_\Omega (u_k)_+(x)(-\Delta)^s\varphi(x)dx \leq \int_{\{f\geq 0\}} \overline{\gamma}_\varepsilon(u_k(x))f_k(x)\varphi(x)dx + \int_{\{f<0\}} \underline{\gamma}_\varepsilon(u_k(x))f_k(x)\varphi(x)dx, \tag{9.19}$$

for all $0 \leq \varphi \in X^s_\Omega \cap C_c(\Omega)$. As $k \to \infty$ we have that

$$\int_\Omega u_+(x)(-\Delta)^s\varphi(x)dx \leq \int_{\{f\geq 0\}} \overline{\gamma}_\varepsilon(u(x))f(x)\varphi(x)dx + \int_{\{f<0\}} \underline{\gamma}_\varepsilon(u(x))f(x)\varphi(x)dx. \tag{9.20}$$

Up to a subsequence, there exists $\xi_+ \in L^\infty(\Omega)$ such that

$$\overline{\gamma}_\varepsilon(u(x))\chi_{\{f\geq 0\}} + \underline{\gamma}_\varepsilon(u(x))\chi_{\{f<0\}} \to \xi_+(x) \qquad \text{in } L^\infty\text{-weak-}\star. \tag{9.21}$$

By the pointwise limits $\xi_+(x) = \mathrm{sign}_+(u(x))$ when $u(x) \neq 0$ and $0 \le \xi_+ \le 1$. Thus $\xi_+(x) \in \widetilde{\mathrm{sign}}(u(x))$.

Hence

$$\int_\Omega u_+(x)(-\Delta)^s \varphi(x)dx \le \int_\Omega \xi_+(x) f(x)\varphi(x)dx, \qquad \forall 0 \le \varphi \in X^s_\Omega \cap C_c(\Omega). \tag{9.22}$$

As for the pointwise estimate, we can proceed analogously for u_- (where $u = u_+ - u_-$) to deduce that

$$\int_\Omega u_-(x)(-\Delta)^s \varphi(x)dx \le \int_\Omega \xi_- f(x)\varphi(x)dx, \qquad \forall 0 \le \varphi \in X^s_\Omega \cap C_c(\Omega), \tag{9.23}$$

with $\xi_-(x) \in \widetilde{\mathrm{sign}}_-(u(x))$. We then have

$$\int_\Omega |u(x)|(-\Delta)^s \varphi(x)dx \le \int_\Omega \xi(x) f(x)\varphi(x)dx, \qquad \forall 0 \le \varphi \in X^s_\Omega \cap C_c(\Omega), \tag{9.24}$$

where $\xi(x) = \xi_+(x) + \xi_-(x) \in \widetilde{\mathrm{sign}}(u(x))$. This concludes the proof. □

10. The weighted approach for related parabolic problems

The combination of our well-posedness results and a priori estimates allows us to immediately solve a number of related evolution problems, according to a general procedure of the evolution theory.

1. The initial-value parabolic problem

$$\begin{cases} \partial_t u + (-\Delta)^s u + V(x)u = f(x,t) & \Omega \times (0,T) \\ u = 0 & \Omega^c \times [0,T), \\ u = u_0 & \Omega \times \{0\}, \end{cases} \tag{10.1}$$

can be solved for every $u_0 \in L^1(\Omega;\phi)$, $f \in L^1(0,T;L^1(\Omega;\phi))$ under the conditions $0 < s < 1$, $V \in L^1_{loc}(\Omega)$, $V \le 0$ and ϕ is a positive weight in X^s such that $(-\Delta)^s\phi \ge 0$.

Using Corollary 4.14 and the Crandall–Liggett generation theorem [24] a contraction semigroup in all such spaces is generated and it satisfies the Maximum Principle.

Note that the fractional heat equation (case $V = 0$) has been studied in the whole space $\mathbb{R}^n$ in an optimal class of weighted integrable data in [8]. The optimal weighted space in which solutions of the Cauchy problem for $\partial_t u + (-\Delta)^s u = 0$ are well-posed is

$$\int \frac{|u_0(x)|}{(1+|x|^2)^{(n+2s)/2}}, dx < \infty.$$

The reader will notice that the weight decays at infinity in a precise way, to be compared with the behaviour δ^s of the bounded case.

The considerations made in [27] for the associated complex relativistic Schrödinger problem with potentials $V = \delta^{-2s}$ can be extended to the case of supersingular potentials $V \ge c\delta^{-2s}$, thanks to the results of Section 4 of this paper.

2. Fractional-PME The same project can be applied to the *fractional porous medium equation*

$$\partial_t u + (-\Delta)^s u^m = f,$$

with $m > 0$, $m \neq 1$, that has been studied in many works, mainly when $f = 0$. Thus, the non-weighted theory is done in [6,7,60,61,73]. The basic result of generation of a semigroup in L^1 goes back to [25] and was used in [9]. The weighted theory is to be done.

Much work remains to be done on these issues.

11. Comments, extensions, and open problems

Here are some issues motivated by the previous presentation.

11.1. More general potentials

In this paper we have considered nonnegative potentials $V \in L^1_{loc}(\Omega)$. This allows for extensions in two directions: considering signed potentials, and considering locally bounded measures as potentials. Both are present in the literature, but both lead to problems that we do not want to consider here.

11.2. Other fractionary and nonlocal operators

When working in bounded domains, there are several different choices of $(-\Delta)^s$ present in the literature (see, e.g., [5,7,58,69,74]. The main choices apart from the restricted Laplacian treated here are the spectral Laplacian and censored Laplacian, ... Many of our results can be extended to them and this is content of future work. Note that regularity for the equation $Lu = f$ in the case of the spectral Laplacian was studied in [20].

Another issue is the Klein–Gordon fractional operator considered in Quantum Mechanics $\sqrt{(-\Delta) + m^2}\, u$, and mentioned in the Introduction. The theory for this operator is quite similar to what we have exhibited above for $(-\Delta)^{1/2}$, see [27]. The theory of this paper can be developed for more related integro-differential operators that are being investigated like the integro-differential operators with irregular or rough kernels, as in [48].

The behaviour of the typical solutions of these operators near the boundary makes a difference. Thus, solutions of equations involving the spectral Laplacian satisfy the linear behaviour of the classical Hopf principle, i.e., linear growth near the boundary.

11.3. Associated eigenvalue problem

A main question for the Schrödinger equation is the eigenvalue problem, which comes from separation of variables. The eigenvalue theory works well in the sense of weak solutions in $L^2(\Omega)$. For the classical Schrödinger problem with $s = 1$, it is known that the eigenvalues of $L^1(\Omega)$ and $L^2(\Omega)$ are not the same. See [13]. It would be interesting to know if such difference remains being true for $s < 1$.

11.4. Open problem on further integrability of the solutions

For problem (P^0), via the estimates on the Green kernel (3.1), the natural space for integrability will be of the form $W^{s',p}(\Omega, \delta^s)$ for $s' < s$ and p small. These estimates can then be extended to problem (P) using maybe the methods of [33,34].

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