Homogenization of a net of periodic critically scaled boundary obstacles related to reverse osmosis “nano-composite” membranes

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Abstract

One of the main goals of this paper is to extend some of the mathematical techniques of some previous papers by the authors showing that some very useful phenomenological properties which can be observed to the nano-scale can be simulated and justified mathematically by means of some homogenization processes when a certain critical scale is used in the corresponding framework. Here the motivating problem in consideration is formulated in the context of the reverse osmosis. We consider, on a part of the boundary of a domain $\Omega \subset \mathbb{R}^n$, a set of very small periodically distributed semipermeable membranes having an ideal infinite permeability coefficient (which leads to Signorini type boundary conditions) on a part Γ_1 of the boundary. We also assume that a possible chemical reaction may take place on the membranes. We obtain the rigorous convergence of the problems to a homogenized problem in which there is a change in the constitutive nonlinearities. Changes of this type are the reason for the big success of the nanocomposite materials. Our proof is carried out for membranes not necessarily of radially symmetric shape. The definition of the associated critical scale depends on the dimension of the space (and it is quite peculiar for the special case of $n = 2$). Roughly speaking, our result proves that the consideration of the critical case of the scale leads to an homogenized formulation which is equivalent to have a global semipermeable membrane, at the whole part of the boundary Γ_1, with a “finite permeability coefficient of this virtual membrane” which is the best we can get, even if the original problem involves a set of membranes of any arbitrary finite permeability coefficients.

keywords homogenization, critical scale, reverse osmosis, Signorini boundary conditions, elliptic partial differential equations, strange term
MSC (2010) 35B27, 76M50, 35M86, 35J87

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1 Introduction and statement of results

We present new results concerning the asymptotic behavior, as $\varepsilon \to 0$, of the solution u_ε of a family of boundary value problems formulated in a cavity (or plant) represented by a bounded domain $\Omega \subset \mathbb{R}^n$, in which a linear diffusion equation is satisfied. The boundary $\partial\Omega$ is split into two regions. On one the regions, homogeneous Dirichlet conditions are specified. On the other one, some small subsets G_ε is ε-periodically distributed, and some unilateral boundary condition are specified on them. We also assume that a possible "reaction" may take place on a net G_ε of small pieces of the boundary given by the periodic repetition of a rescaled particle G_0.

There are several relevant problems in a wide spectrum of applications leading to such type of formulations, ranging from water and wastewater treatment, to food and textile engineering, as well as pharmaceutical and biotechnology applications (for a recent review see [29]). One of them concerns the reverse osmosis when we apply it, for instance, to desalination processes (see, e.g. [24] and its references). Without intending to use here a "realistic model" we shall present an over-simplified formulation that, nonetheless, preserves most of the mathematical difficulties concerning the passing to the limit as $\varepsilon \to 0$. Some examples of more complex formulations, covering different aspects of the problems considered here can be found, for instance, in [17] and its many references.

We start by recalling that, roughly speaking, semipermeable membranes allow the passing of certain type of molecules (the so called as "solvents") but block other type of molecules (the "solutes"). The solvents flow from the region of smaller concentration of solute to the region of higher concentration (the difference of concentration produces the phenomenon known as *osmotic pressure*). Nevertheless, by creating a very high pressure it is possible to produce an inverse flow, such as it is used in desalination plants: it is the so called "reverse osmosis". Since in many cases the semipermeable membrane contains some chemical products (e.g. polyamides; see [18]), our formulation will contain also a nonlinear kinetic reaction term in the flux given by a continuous nondecreasing function $\sigma(s)$. Let us call w_ε the solvent concentration corresponding to the membrane periodicity scale ε. Let us modulate the intensity of the reaction in terms of a factor ε^{-k}, with k to be analyzed later. So, for a critical value of the solvent concentration ψ (associated to the osmotic pressure) the flux (including the reaction kinetic term) is an incoming flux with respect to the solvents plant Ω if the concentration of the solvent molecules $w(x)$ on the semipermeable membrane $G_\varepsilon \subset \partial\Omega$ is smaller or equal to this critical value, but it remains isolated (with no boundary flow, excluding the reaction term, on the membrane, i.e. when the concentration is $w(x) < \psi$). So, if ν is the exterior unit normal vector to the membrane surface we have

$$\begin{array}{ll} \text{on } \{x \in G_\varepsilon \subset \partial\Omega,\, w_\varepsilon(x) > \psi\} & \partial_\nu w_\varepsilon + \varepsilon^{-k}\sigma(\psi - w_\varepsilon) = 0, \\ \text{on } \{x \in G_\varepsilon,\, w_\varepsilon(x) \le \psi\} & \partial_\nu w_\varepsilon + \varepsilon^{-k}\sigma(\psi - w_\varepsilon) = -\varepsilon^{-k}\mu(\psi - w_\varepsilon) \end{array}$$

for some parameter $\mu > 0$ called as the "finite permeability coefficient of the membrane" (usually, in practice, μ takes big values). We assume a simplified linear diffusion equation on the solvent concentration

$$-\Delta w = F \text{ in } \Omega,$$

and some boundary conditions on the rest of the boundary $\partial\Omega$. For instance, we can distinguish some subregions where Dirichlet or Neumann types of boundary conditions hold, and so, if we introduce the partition $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and assume that, in fact, $G_\varepsilon \subset \Gamma_1$, then we can imagine that

$$\partial_\nu w_\varepsilon(x) = h(x) \text{ on } x \in \Gamma_1 \diagdown \overline{G_\varepsilon},$$

and

$$w_\varepsilon(x) = g(x) \text{ on } x \in \Gamma_2.$$

Figure 1 presents a simplified case of the above mentioned framework.

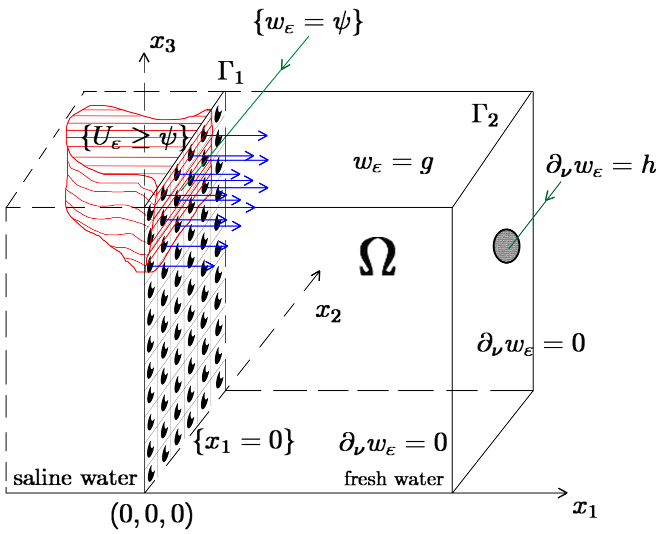

Figure 1: A simple illustration of a plant with a reverse osmosis membrane

We are specially interested in the study of new behaviours arising in the reverse osmosis membranes having a periodicity ε of the order of nanometers (see e.g. [4] and its references). Mathematically, we shall give a sense to those extremely small scales by asking that the diameter of these subsets included in G_ε let of order a_ε, where $a_\varepsilon \ll \varepsilon$.

Some times, it is interesting to consider semipermeable membranes with an "infinite permeability coefficient" (formally $\mu = +\infty$, but only for the case $w_\varepsilon(x) = \psi$) and thus ψ becomes an obstacle which is periodically repeated in G_ε. Following the approach presented in [16], this can be formulated as

$$\begin{aligned}
&\text{on } \{x \in G_\varepsilon,\, w_\varepsilon(x) > \psi\} \implies \partial_\nu w_\varepsilon + \varepsilon^{-k}\sigma(\psi - w_\varepsilon) = 0,\\
&\text{on } \{x \in G_\varepsilon,\, w_\varepsilon(x) = \psi\} \Longrightarrow \partial_\nu w_\varepsilon + \varepsilon^{-k}\sigma(\psi - w_\varepsilon) \leq 0,\\
&\text{and } (w_\varepsilon - \psi)(\partial_\nu w_\varepsilon + \varepsilon^{-k}\sigma(\psi - w_\varepsilon) = 0) \text{ on } G_\varepsilon.
\end{aligned}$$

Now, to carry out our mathematical treatment, it is quite convenient to work with the new unknown

$$u_\varepsilon(x) := \psi - w_\varepsilon(x)$$

and thus, if we assume (again for simplicity) that $h = g = 0$ and $f := -F$ we simplify the formulation to arrive at the following formulation which will be the object of study in this paper:

$$\begin{cases} -\Delta u_\varepsilon = f(x), & x \in \Omega, \\ u_\varepsilon \geq 0, \\ \partial_\nu u_\varepsilon + \varepsilon^{-k}\sigma(u_\varepsilon) \geq 0, \\ u_\varepsilon(\partial_\nu u_\varepsilon + \varepsilon^{-k}\sigma(u_\varepsilon)) = 0, & x \in G_\varepsilon, \\ \partial_\nu u_\varepsilon = 0, & x \in \Gamma_1 \setminus \overline{G}_\varepsilon, \\ u_\varepsilon = 0, & x \in \Gamma_2. \end{cases} \tag{1}$$

Notice that in the reaction kinetics we made emerge a re-scaling factor $\beta(\varepsilon) := \varepsilon^{-k}$ where $k \in \mathbb{R}$. The relation between the exponent k and the diameter of the chemical particles (which we shall assume to be given by $a_\varepsilon = C_0\varepsilon^\alpha$, where $C_0 > 0$ and $\alpha > 1$) will be discussed later. This relation will depend on the dimension of the space $n \geq 3$. The case $n = 2$ is rather special and will require a different treatment: we shall assume that $a_\varepsilon = C_0\varepsilon e^{-\frac{\alpha^2}{\varepsilon}}$ and $\beta(\varepsilon) = e^{\frac{\alpha^2}{\varepsilon}}$.

Homogenization results for boundary value problems with alternating type of boundary conditions, including Robin type condition, were widely considered in the literature. We refer, for instance to the papers [35, 8, 5, 1] which already contain an extensive bibliography on the subject. Huge attention was drawn to the similar homogenization problems but in a domains perforated by the tiny sets on which some nonlinear Robin type condition is specified on their boundaries. Some pioneering works in this direction are the papers by Kaizu [25, 26]. In this works where investigated all the possible relations between parameters except one the case of the "critical" relation between parameters α and $\beta(\varepsilon)$, i.e. $\alpha = k = n/(n-2)$. Later on, this critical case was considered in [22] for $n = 3$ and for the sets G_ε given by balls. It seems that it was in the paper [22] where the effect of "nonlinearity change due to the homogenization process" was discovered for the first time. After that, by using some different method of proof, the critical case was solved for $n \geq 3$ in [36]. The consideration of the case $n = 2$ and for an arbitrary shape domains G_ε was carried out in [34]. More recently, many results concerning the asymptotic behavior of solutions of problems similar to (1) were published in the literature [36, 23, 19, 20, 9, 10, 11, 12]. Nevertheless, in all the above mentioned works the particles (or perforations, according to the physical model used as motivation of the mathematical formulation) subsets G_ε where assumed to be balls (having a critical radius). We also mention here the paper [14] that describes the asymptotic behavior of some related problem for the case of arbitrary shape sets G_ε and for $n \geq 3$. One of the main goals of this paper is to extend some of the techniques of [14] to the problem (1) where the periodically distributed reactions arise merely on some part of the global boundary $\partial\Omega$ always for the critical scaled case. This is the case for which some phenomenological properties which arise to the nano-scale can be simulated and justified by means of homogenization processes.

Case $n \geq 3$ In order to present the main results of this paper, and their application to the reverse osmosis framework, we need to introduce some auxiliary notations. We start by considering the case $n \geq 3$. We assume that Ω that it is bounded domain in $\mathbb{R}^n \cap \{x_1 > 0\}$, $n \geq 3$, with a piecewise-smooth boundary $\partial\Omega$ that consists of two parts Γ_1 and Γ_2, with the property that

$$\Gamma_1 = \partial\Omega \cap \{x \in \mathbb{R}^n : x_1 = 0\} \neq \emptyset.$$

We consider a model G_0 such that $\overline{G_0} \subset \{x \in \mathbb{R}^n : x_1 = 0, |x| < 1/4\}$ with $\overline{G_0}$ diffeomorphic to a ball. We define $\delta B = \{x : \delta^{-1} x \in B\}$, $\delta > 0$. Let

$$\widetilde{G_\varepsilon} = \bigcup_{j \in \mathbb{Z}'} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \mathbb{Z}'} G_\varepsilon^j,$$

where

$$\mathbb{Z}' = \{0\} \times \mathbb{Z}^{n-1},$$

and

$$a_\varepsilon = C_0 \varepsilon^k, \qquad k = \frac{n-1}{n-2} \qquad \text{and} \qquad C_0 > 0. \tag{2}$$

A justification of the above choice of exponent k can be found, for instance, in [28] (see also [33]). We define the net of sets G_ε as the union of sets $G_\varepsilon^j \subset \widetilde{G_\varepsilon}$ such that $\overline{G_\varepsilon^j} \subset \Gamma_1$ and $\rho(\partial\Gamma_1, \overline{G_\varepsilon^j}) \geq 2\varepsilon$, i.e.

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}' : \rho(\partial\Omega, \overline{G_\varepsilon^j}) \geq 2\varepsilon\}.$$

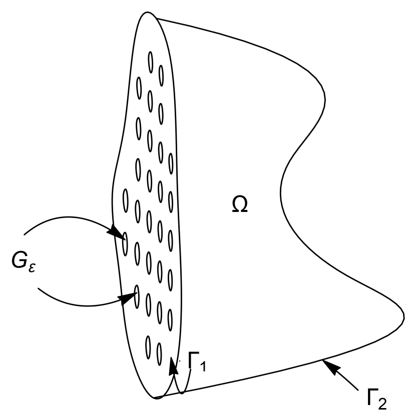

Figure 2: Domain Ω when $n \geq 3$

Notice that $|\Upsilon_\varepsilon| \cong d\varepsilon^{1-n}$, $d = const > 0$. It will be useful later to observe that if we denote by $T_r(x_0)$ the ball in $\mathbb{R}^n$ of radius r centered at a point x_0, and if we define the boundary points

$$P_\varepsilon^j = \varepsilon j = (0, P_{\varepsilon,2}^j, \dots, P_{\varepsilon,n}^j) \qquad \text{for } j \in \mathbb{Z}',$$

and the set $T_{\varepsilon/4}^j = T_{\varepsilon/4}(P_\varepsilon^j)$, then we have $\overline{G_\varepsilon^j} \subset T_{\varepsilon/4}^j$. In this geometrical setting, the so called strong formulation" of the problem for which we want to study the asymptotic behavior of its solutions is the following:

$$\begin{cases} -\Delta u_\varepsilon = f(x), & x \in \Omega, \\ u_\varepsilon \geq 0, & \\ \partial_\nu u_\varepsilon + \varepsilon^{-k}\sigma(u_\varepsilon) \geq 0, & \\ u_\varepsilon(\partial_\nu u_\varepsilon + \varepsilon^{-k}\sigma(u_\varepsilon)) = 0, & x \in G_\varepsilon, \\ \partial_\nu u_\varepsilon = 0, & x \in \Gamma_1 \setminus \overline{G}_\varepsilon, \\ u_\varepsilon = 0, & x \in \Gamma_2, \end{cases} \tag{3}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous non-decreasing function and, at most, super-linear at the infinity: i.e., such that

$$k_1|s-t| \leq |\sigma(t) - \sigma(s)| \leq K_1|t-s|^{\rho_1} + K_2|s-t|^{\rho_2}, \text{ for some } \rho_1, \rho_2 \in (0, 2], \tag{4}$$

for all $t, s \geq 0$ where $k_1, K_1, K_2 > 0$, $\sigma(0) = 0$. In problem (3) $\nu = (-1, 0, \cdots, 0)$ is the unit outward normal vector Ω at $\{x_1 = 0\}$ and $\partial_\nu u = -\frac{\partial u}{\partial x_1}$ is the normal derivative of u at this part of the boundary.

Example 1.1. Notice that examples of such functions cover

$$\sigma(s) = \sqrt{s}. \tag{5}$$

Furthermore, the behaviour at infinity may be superlinear as, for example, in

$$\sigma(s) = \begin{cases} \sqrt{s} & 0 \leq s \leq s_0, \\ \sqrt{s_0} + (s - s_0)^2 & s > s_0. \end{cases} \tag{6}$$

The weak formulation of (3) is the following (see, e.g. [16]):

Definition 1.1. We say that u_ε is a weak solution of (1) if

$$u_\varepsilon \in K_\varepsilon = \Big\{ g \in H^1(\Omega, \Gamma_2) : \ g \geq 0 \text{ a.e. on } G_\varepsilon \Big\} \tag{7}$$

and

$$\int\limits_\Omega \nabla u_\varepsilon \nabla(\varphi - u_\varepsilon) dx + \varepsilon^{-k} \int\limits_{G_\varepsilon} \sigma(u_\varepsilon)(\varphi - u_\varepsilon) dx' \geq \int\limits_\Omega f(\varphi - u_\varepsilon) dx, \tag{8}$$

for all $\varphi \in K_\varepsilon$.

By $H^1(\Omega, \Gamma_2)$ we denote the closure in $H^1(\Omega)$ of the set of infinitely differentiable functions in $\overline{\Omega}$, vanishing on the boundary Γ_2.

It is well known (see, e.g. some references in [11]) that problem (8) has a unique weak solution $u_\varepsilon \in K_\varepsilon$. From (8), we immediately deduce that

$$\|\nabla u_\varepsilon\|_{L_2(\Omega)} \leq K \tag{9}$$

where, here and below, constant K is independent of ε. Hence, there exists a subsequence (denoted as the original sequence by $\widetilde{u}_\varepsilon$) such that, as $\varepsilon \to 0$, we have

$$\begin{aligned} &u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega, \Gamma_2), \\ &u_\varepsilon \to u_0 \text{ strongly in } L_2(\Omega). \end{aligned} \tag{10}$$

By using the monotonicity of function $\sigma(u)$ one can show (see some references in [11]) that u_ε satisfies the following "very weak formulation"

$$\int\limits_\Omega \nabla\varphi\nabla(\varphi - u_\varepsilon)dx + \varepsilon^{-k}\int\limits_{G_\varepsilon} \sigma(\varphi)(\varphi - u_\varepsilon)dx' \geq \int\limits_\Omega f(\varphi - u_\varepsilon)dx, \tag{11}$$

where φ is an arbitrary function from K_ε.

The main goal of this paper is to consider this critical relation between parameters. This scale is characterized by the fact that the resulting homogenized problem will contain a so-called "strange term" expressing the fact that the character of some nonlinearity arising in the homogenized problem differs from the original nonlinearity appearing in the boundary condition of (3). Still focusing first on the case $n \geq 3$, it can be shown that critical scale of the size of the holes is given by

$$\alpha = \frac{n-1}{n-2} \tag{12}$$

(see, e.g., the arguments used in [28] and [11]). The appropriate scaling of the reaction term so that both the diffusive and nonlinear characters are preserved at the limit the limit is, as usual, driven $\varepsilon^k \sim |G_\varepsilon|$. Therefore

$$k = \frac{n-1}{n-2}. \tag{13}$$

In the present paper we construct homogenized problem with a nonlinear Robin type boundary condition, that contains a new nonlinear term, and prove the corresponding theorem stating that the solution of the original problem converges as $\varepsilon \to 0$ to the solution of the homogenized problem.

We point out that the main difficulties to get an homogenized problem associated to (3) come from the following different aspects:

i) the low differentiability assumed on function σ (since it is non-Lipschitz continuous at $u = 0$ and it has quadratic growth at infinity),

ii) the unilateral formulation of the boundary conditions on G_ε,

iii) the general shape assumed on the sets G_ε, and,

iv) the critical scale of the sets G_ε^j.

Some of those difficulties where already in the previous short presentation paper by the authors ([13]) but only for $n = 2$, without ii) and by assuming σ Lipschitz continuous. Our main goal is to extend our techniques to the above more general mentioned framework.

To build the homogenized problem we still need to introduce some "capacity type" auxiliary problems. Given $u \in \mathbb{R}$, for $y \in (\mathbb{R}^n)^+ = \mathbb{R}^n \cap \{y_1 > 0\}$, we introduce the new auxiliary function $\widehat{w}(y; u)$, depending also on G_0 and σ, as the (unique) solution of the exterior problem

$$\begin{cases} -\Delta \widehat{w} = 0 & y \in (\mathbb{R}^n)^+, \\ \partial_\nu \widehat{w} - C_0 \sigma(u - \widehat{w}) = 0 & y \in G_0, \\ \partial_\nu \widehat{w} = 0 & y \notin G_0, y_1 = 0, \\ \widehat{w} \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{14}$$

Remember that $C_0 > 0$ was given in the structural assumption (2). The existence and uniqueness of $\widehat{w}(y; u)$ is given in Lemma 2.2 below. Let us introduce also the auxiliary function $\widehat{\kappa}(y), y \in (\mathbb{R}^n)^+$, as the unique solution of the problem

$$\begin{cases} \Delta \widehat{\kappa} = 0 & y \in (\mathbb{R}^n)^+, \\ \widehat{\kappa} = 1 & y \in G_0, \\ \partial_\nu \widehat{\kappa} = 0 & y \notin G_0, y_1 = 0, \\ \widehat{\kappa} \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{15}$$

We then define the possibly nonlinear function, for $u \in \mathbb{R}$,

$$H_{G_0}(u) := \int_{G_0} \partial_\nu \widehat{w}(u, y') dy' = C_0 \int_{G_0} \sigma(u - \widehat{w}(y'; u)) dy', \tag{16}$$

and the scalar

$$\lambda_{G_0} := \int_{G_0} \partial_\nu \widehat{\kappa}(y') \, dy', \tag{17}$$

where in both definitions $y' = (0, y_2, \cdots, y_n)$. Notice that $\widehat{\kappa}(y)$ can be extended by symmetry to $\mathbb{R}^n \backslash \overline{G}_0$ as an harmonic function. Moreover, by the maximum principle, $\widehat{\kappa}(y)$ reaches its maximum in G_0, and so by the strong maximum principle $\partial_\nu \widehat{\kappa} = -\partial_{x_1} \widehat{\kappa} > 0$. Then, we know that

$$\lambda_{G_0} > 0.$$

Some properties of the function H_{G_0} will be presented later. In particular, in Lemma 2.6 below we will show that H_{G_0} is always a Lipschitz continuous function (even if σ is merely Hölder continuous) such that

$$0 \leq H'_{G_0} \leq \lambda_{G_0}. \tag{18}$$

The following theorem gives a description of the limiting function u_0 obtained in (10).

Theorem 1. *Let $n \geq 3$, $\alpha = k = \frac{n-1}{n-2}$ and u_ε be a weak solution of the problem* (3)*. Then function u_0 defined in* (10) *is a weak solution of the following problem*

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega, \\ \partial_\nu u_0 + C_0^{n-2} H_{G_0}(u_{0,+}) - \lambda_{G_0} C_0^{n-2} u_{0,-} = 0, & x \in \Gamma_1, \\ u_0 = 0, & x \in \Gamma_2, \end{cases} \tag{19}$$

where H_{G_0} is defined by (16) *and λ_{G_0} by* (17)*. Here, as usual $u_{0,+} := \max\{u_0, 0\}$ and $u_{0,-} = \max\{-u_0, 0\}$, so that $u_0 = u_{0,+} - u_{0,-}$.*

Case $n = 2$ As mentioned before (see also [13]) the case of $n = 2$ requires to introduce some slight changes. The domain is given now in the following way: we consider Ω be a bounded domain in $\mathbb{R}^2 \cap \{x_2 > 0\}$, the boundary of which consists of two parts $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 = \partial\Omega \cap \{x_2 = 0\} = [-l, l], l > 0$, $\Gamma_2 = \partial\Omega \cap \{x_2 > 0\}$. We denote

$$Y_1 = \{(y_1, 0) | -1/2 < y_1 < 1/2\}, \quad \widehat{l}_0 = \{(y_1, 0) | -l_0 < y_1 < l_0\} \subset Y_1, \quad l_0 \in (0, 1/2).$$

For a small parameter $\varepsilon > 0$ and $0 < a_\varepsilon < \varepsilon$ we introduce the sets

$$\widetilde{G}_\varepsilon = \bigcup_{j \in \mathbb{Z}'} (a_\varepsilon \widehat{l}_0 + \varepsilon j) = \bigcup_{j \in \mathbb{Z}'} l_\varepsilon^j,$$

where $\mathbb{Z}'$ is a set of vectors $j = (j_1, 0)$ and j_1 is a whole number. Denote $\Upsilon_\varepsilon = \{j \in \mathbb{Z}' | \overline{l_\varepsilon^j} \subset \{x = (x_1, 0) : x_1 \in [-l + 2\varepsilon, l - 2\varepsilon]\}$. Consider $Y_\varepsilon^j = \varepsilon Y_0 + \varepsilon j$ and

$$l_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} l_\varepsilon^j.$$

It is easy to see that $\overline{l_\varepsilon^j} \subset Y_\varepsilon^j$. Denote $\gamma_\varepsilon = \Gamma_1 \setminus \overline{l_\varepsilon}$. Note that for $\forall j \in \mathbb{Z}'$, $|l_\varepsilon^j| = 2a_\varepsilon l_0$, $|l_\varepsilon| \cong d a_\varepsilon \varepsilon^{-1}$.

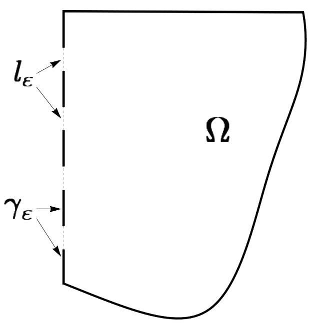

Figure 3: Domain Ω in two dimensions

The formulation of the problem starts by searching

$$u_\varepsilon \in K_\varepsilon = \Big\{ g \in H^1(\Omega, \Gamma_2) : \, g \geq 0 \text{ a.e. on } l_\varepsilon \Big\} \tag{20}$$

be a solution to the following variational inequality

$$\int\limits_{\Omega} \nabla u_\varepsilon \nabla(\varphi - u_\varepsilon) dx + e^{\frac{\alpha^2}{\varepsilon}} \int\limits_{G_\varepsilon} \sigma(u_\varepsilon)(\varphi - u_\varepsilon) dx_1 \geq \int\limits_{\Omega} f(\varphi - u_\varepsilon) dx, \tag{21}$$

where φ is an arbitrary function from K_ε. This time, for simplicity, we assume merely that function $\sigma : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $\sigma(0) = 0$ and and there exists positive constants k_1, k_2 such that condition is satisfied

$$k_1 \leq \partial_u \sigma(u) \leq k_2, \qquad \forall u \in \mathbb{R}. \tag{22}$$

(more general terms $\sigma(u)$ where considered in [13] but without the Signorini type contraints). Therefore, we have

$$k_1 u^2 \leq u\sigma(u) \leq k_2 u^2, \quad \forall u \in \mathbb{R}.$$

Note that (20), (21) is a weak formulation of the following strong formulation of the problem:

$$\begin{cases} -\Delta u_\varepsilon = f, & x \in \Omega, \\ u_\varepsilon \geq 0, & \\ \quad \partial_{x_2} u_\varepsilon - \beta(\varepsilon)\sigma(u_\varepsilon) \geq 0 & \\ \quad u_\varepsilon(\partial_{x_2} u_\varepsilon - \beta(\varepsilon)\sigma(u_\varepsilon)) = 0, & x \in l_\varepsilon, \\ \partial_{x_2} u_\varepsilon = 0, & x \in \gamma_\varepsilon, \\ u_\varepsilon = 0, & x \in \Gamma_1, \end{cases} \tag{23}$$

Our homogenized result in this case is the following:

Theorem 2. *Let* $a_\varepsilon = C_0 \varepsilon e^{-\frac{\alpha^2}{\varepsilon}}$, $\beta(\varepsilon) = e^{\frac{\alpha^2}{\varepsilon}}$, $\alpha \neq 0$, $C_0 > 0$ *and* u_ε *be a solution to the problem* (23). *Then, there exists a subsequence such that* $u_\varepsilon \to u_0$, *strongly in* $L^2(\Omega)$, *and weakly in* $H^1(\Omega, \Gamma_1)$, *as* $\varepsilon \to 0$, *and the function* $u_0 \in H^1(\Omega, \Gamma_1)$ *is a weak solution to the following boundary value problem*

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega, \\ \partial_{x_2} u_0 = \frac{\pi}{\alpha^2}(H_{l_0}(u_{0,+}) - u_{0,-}), & x \in \Gamma_1, \\ u_0 = 0, & x \in \Gamma_2, \end{cases} \tag{24}$$

where $H_{l_0}(u)$ *verifies the functional equation*

$$\pi H_{l_0}(u) = 2 l_0 \alpha^2 C_0 \sigma(u - H_{l_0}(u)). \tag{25}$$

The special case of dimension $n = 2$ is illustrative in order to get a complete identification of the function H_{G_0}. Curiously enough, the characterization condition for H_{G_0} and H_{l_0} (see e.g. the functional equations (16) and (25), respectively) is quite related (but not exactly the same) than the one obtained in [11], although the problem under consideration in that paper was not the same than problem (3). It was shown in [11] if $H\colon \mathbb{R} \to \mathbb{R}$ is the solution of a problem

$$H(u) = C\sigma(u - H(u)) \tag{26}$$

then H is given by

$$H(u) = (I + (C\sigma)^{-1})^{-1}(u). \tag{27}$$

for $u \in \mathbb{R}$. When the Signorini condition is include, σ can be generalized to the maximal monotone graph

$$\widetilde{\sigma}(u) = \begin{cases} \sigma(u) & u > 0, \\ (-\infty, 0] & u = 0, \\ \emptyset & u < 0. \end{cases} \tag{28}$$

It was proven there that the corresponding zero order term is given by

$$\widetilde{H}(u) = \begin{cases} H(u) & u > 0, \\ u & u \leq 0. \end{cases} \tag{29}$$

This behaviour is interesting, because $\widetilde{H}$ matches (27) formally. The maximal monotone graph

$$\gamma(u) = \begin{cases} 0 & u > 0, \\ (-\infty, 0] & u = 0, \\ \emptyset & u < 0. \end{cases} \tag{30}$$

has, formally, inverse

$$\gamma^{-1}(u) = \begin{cases} \emptyset & u > 0, \\ [0, +\infty) & u = 0, \\ 0 & u < 0. \end{cases} \tag{31}$$

In particular

$$H_\gamma(u) := (I + (C\gamma)^{-1})^{-1}(u) = \begin{cases} \emptyset & u > 0, \\ [0, +\infty) & u = 0, \\ u & u < 0. \end{cases} \tag{32}$$

In this way, it is clear that

$$\widetilde{H}(u) = \begin{cases} H(u) & u > 0, \\ H_\gamma(u) & u < 0. \end{cases} \tag{33}$$

For the case $n = 3$ we will make further comments in this direction in Section 6.1.

Coming back to the framework of the semipermeable membranes problems what we can conclude is that the homogenization of a set of periodic semipermeable membranes with an "infinite permeability coefficient", in the critical case, leads to an homogenized formulation which is equivalent to have a global semipermeable membrane, at Γ_1, with a "finite permeability coefficient of this virtual membrane" μ_∞ which is *the best we can get*, even if the original problem involves a finite permeability μ. Indeed, this comes from the properties of the function H_{G_0} (which can be also computed for the case of a microscopic membrane with finite permeability). For instance, on the subpart of the boundary $\{x \in \Gamma_1,\ w_0(x) \leq \psi\}$ (with $w_0(x) := \psi - u_0(x)$; i.e. where $u_0(x) \leq 0$). Moreover, by using that $H_{G_0}(0) = 0$ and the decomposition $u_0 = u_{0,+} - u_{0,-}$, we know now that $\partial_\nu w_0 = \lambda_{G_0} C_0^{n-2} w_0$ and thus, for the case of a microscopic finite permeability membrane, it is not difficult to show that we get an homogenized permeability membrane coefficient given by $\lambda_{G_0} C_0^{n-2}$, which is larger than μ (see more details in Subsection 6.1 below).

The plan of the rest of the paper is the following: Part I (containing Sections 2-6) is devoted to the study of the case $n \geq 3$ and contains also some comments and possible extensions (for instance, we give some link between the present homogenization results and the homogenization of some problems involving non-local fractional operators). Part II (containing Section 7) is devoted to the proof of the convergence result for $n = 2$.

Part I
The case $n \geq 3$

2 Estimates on the auxiliary functions

2.1 On the auxiliary function $\widehat{\kappa}$

Using the method of sub and supersolutions as in [14] the following result holds:

Lemma 2.1. *There exists a unique solution $\widehat{\kappa} \in \mathbb{X}$ of the problem* (14)*, where*

$$\mathbb{X} = \left\{ v \in L^2_{loc}(\overline{(\mathbb{R}^n)^+}) : \nabla v \in L^2((\mathbb{R}^n)^+)^n, |v| \le \frac{K}{|y|^{n-2}} \right\} \tag{34}$$

such that

$$0 \le \widehat{\kappa}(y) \le 1 \quad a.e.\ y \in \overline{(\mathbb{R}^n)^+}, \tag{35}$$

and

$$\widehat{\kappa}(y) \le \frac{K}{|y|^{n-2}} \quad a.e.\ y \in \overline{(\mathbb{R}^n)^+}. \tag{36}$$

We define also the family of auxiliary functions

$$\widehat{\kappa}^j_\varepsilon = \kappa \left(\frac{x - P^j_\varepsilon}{a_\varepsilon} \right). \tag{37}$$

2.2 On the auxiliary function $\widehat{w}$

Arguing, as above, in similar terms to the paper [14] we get:

Lemma 2.2. *Let σ be a maximal monotone graph. There exists a unique solution $\widehat{w}(\cdot, u) \in \mathbb{X}$ of problem* (14)*. Furthermore, it satisfies that*

- *If $u \ge 0$ then $0 \le \widehat{w}(y, u) \le u\widehat{\kappa}(y) \le u$.*
- *If $u \le 0$ then $0 \ge \widehat{w}(y, u) \ge u\widehat{\kappa}(y) \ge u$.*

Hence

$$|\widehat{w}(y; u)| \le |u|\widehat{\kappa}(y) \quad a.e.\ y \in \overline{(\mathbb{R}^n)^+}. \tag{38}$$

Concerning the dependence with respect to the parameter u we start by considering the case in which σ is a maximal monotone graph associated to a continuous function:

Lemma 2.3. *Let σ be a nondecreasing continuous function such that $\sigma(0) = 0$. Then*

$$|\widehat{w}(u_1, y) - \widehat{w}(u_2, y)| \le |u_1 - u_2|. \tag{39}$$

for all $u_1, u_2 \in \mathbb{R}$.

Proof. Let $\widehat{w}(u_1, y)$, $\widehat{w}(u_2, y)$ be two solutions of the problem (14) with parameters u_1, $u_2 \in \mathbb{R}$. Consider the function $v = w(u_1, y) - w(u_2, y)$. This function is a solution of the following exterior problem

$$\begin{cases} \Delta v = 0, & y \in (\mathbb{R}^n)^+, \\ \partial_{\nu_y} v = C_0 \left(\sigma(u_1 - \widehat{w}(u_1, y)) - \sigma(u_2 - \widehat{w}(u_2, y)) \right), & y \in G_0, \\ v \to 0 & \text{as } |y| \to \infty, \end{cases} \tag{40}$$

First consider the case $u_1 > u_2$. If we choose v^- as a test function in the integral identity for the problem (40) we arrive at

$$\int\limits_{(\mathbb{R}^n)^+} |\nabla v^-|^2 dx + C_0 \int\limits_{G_0} \left(\sigma(u_1 - \widehat{w}(u_1, y)) - \sigma(u_2 - \widehat{w}(u_2, y))\right) v^- ds = 0$$

The second integral in the obtained expression can be nonzero only if $v < 0$, i.e. $\widehat{w}(u_1, y) - \widehat{w}(u_2, y) < 0$. By combining this inequality with the condition $u_1 > u_2$ we get $u_1 - \widehat{w}(u_1, y) > u_2 - \widehat{w}(u_2, y)$. This inequality and monotonicity of the function σ imply that second integral is non-negative. Hence, two integrals must be equal to zero, so $v^- = 0$ $\mathcal{L}^{n-1}$-a.e. in G_0 and $v^- = c$ in $(\mathbb{R}^n)^+$. But we have $v \to 0$ as $|y| \to \infty$, hence, $c = 0$, i.e. $\widehat{w}(u_1, y) - \widehat{w}(u_2, y) \geq 0$.

One can construct function $\varphi(r) \in C_0^\infty(\mathbb{R})$ such that $\varphi = 0$ if $|r| > 1$ and $\varphi = 1$ if $|r| < 0.5$. We take $(u_1 - u_2 - v)^- \varphi(\rho(x, G_0)/R)$ as a test function in the integral identity for the problem (40) and obtain

$$\begin{aligned} -\int\limits_{(\mathbb{R}^n)^+} \nabla v \nabla (u_1 - u_2 - v)^- \varphi(\rho(x, G_0)/R) dx - \\ - \int\limits_{(\mathbb{R}^n)^+} \frac{\varphi'(\rho(x, G_0)/R)}{R} (u_1 - u_2 - v)^- \nabla v \nabla \rho dx + \\ + C_0 \int\limits_{G_0} \left(\sigma(u_1 - \widehat{w}(u_1, y)) - \sigma(u_2 - \widehat{w}(u_2, y))\right) (u_1 - u_2 - v)^- ds \\ = I_{1,R} + I_{2,R} + I_3 = 0. \end{aligned}$$

Since σ is monotone $I_3 \leq 0$.
For the first integral we have

$$I_{1,R} \leq -\int\limits_{\mathcal{D}_{1,R}} |\nabla (u_1 - u_2 - v)^-|^2 dx \leq 0,$$

where $\mathcal{D}_{1,R} = ((\mathbb{R}^n)^+) \bigcap \{x \in \mathbb{R}^n : \rho(x, G_0) < R\}$. We have that

$$I_1 = -\int\limits_{(\mathbb{R}^n)^+} |\nabla (u_1 - u_2 - v)^-|^2 dx = \lim_{R \to \infty} I_{1,R} \leq 0$$

For the second integral we derive estimation

$$\begin{aligned} |I_{2,R}| &\leq K_1 \int\limits_{\mathcal{D}_{2,R}} \frac{|v|}{R} |\nabla v| dx \leq \frac{K_1}{R} \|v\|_{L_2(\mathcal{D}_{2,R})} \|\nabla v\|_{L_2((\mathbb{R}^n)^+)} \\ &\leq \frac{K_2}{R^{\frac{n-2}{2}}} \|\nabla v\|_{L_2((\mathbb{R}^n)^+)} \to 0, \quad \text{as } R \to \infty \end{aligned}$$

where $\mathcal{D}_{2,R} = ((\mathbb{R}^n)^+)\bigcap\{x \in \mathbb{R}^n : R/2 < \rho(x, G_0) < R\}$. Thereby, as $R \to \infty$ we have $I_1 + I_3 = 0, I_1 \leq 0, I_3 \leq 0$, and so

$$I_1 = 0, \quad I_3 = 0.$$

Taking into account that $v \to 0$ as $|y| \to \infty$ we derive from the last corollary that $(u_1 - u_2 - v)^- \equiv 0$, i.e. $0 \leq v < u_1 - u_2$ in $(\mathbb{R}^n)^+$. Moreover, we have that $v < u_1 - u_2$ $\mathcal{L}^{n-1}$-a.e. in G_0.

The case $u_1 < u_2$ is analogous to that above, so we have $u_1 - u_2 \leq v \leq 0$ in $(\mathbb{R}^n)^+$ and $\mathcal{L}^{n-1}$-a.e. in G_0. This concludes the proof. □

The use of the comparison principle leads to an additional conclusion:

Lemma 2.4. *Let $u_1 > u_2$ then*

$$0 \leq \widehat{w}(u_1, y) - \widehat{w}(u_2, y) \leq (u_1 - u_2)\widehat{\kappa}(y). \tag{41}$$

Proof. The functions $\varphi_1(y) = \widehat{w}(u_1, y) - \widehat{w}(u_2, y)$ and $\varphi_2(y) = (u_1 - u_2)\widehat{\kappa}(y)$ can be extended (by symmetry) as harmonic functions to $\mathbb{R}^n \setminus \overline{G_0}$ such that $\lim_{|y|\to\infty}(\varphi_2 - \varphi_1) = 0$ and $\varphi_2 - \varphi_1 \geq 0$ on G_0. The comparison principles proves the result. □

A more regular dependence with respect to u can be also proved under additional regularity on function σ:

Lemma 2.5 (Differentiable dependence of solutions)**.** *Let $\sigma \in \mathcal{C}^1$ and $\sigma' \geq k_1 > 0$. Then, the map $u \in \mathbb{R} \mapsto \widehat{w}(u, \cdot) \in L^2_{loc}(K)$ is differentiable, for every smooth bounded set K such that $G_0 \subset K \subset \overline{(\mathbb{R}^n)^+}$. Furthermore, if we define*

$$\widehat{W}(u, y) = \frac{\partial \widehat{w}(y; u)}{\partial u} \tag{42}$$

then

$$\int\limits_{(\mathbb{R}^n)^+} \nabla \widehat{W}(u, y) \nabla \varphi dy = C_0 \int\limits_{G_0} \sigma'(u - \widehat{w}(u, y)) \left(1 - \widehat{W}(u, y)\right) \varphi(y) dS_y \tag{43}$$

for $\varphi \in C_c^\infty(\overline{(\mathbb{R}^n)^+})$, $\nabla \widehat{W} \in L^2(\Omega)^n$ and $0 \leq \widehat{W}(u, y) \leq \widehat{\kappa}(y)$. In particular,

$$0 \leq \frac{\partial \widehat{w}}{\partial u} \leq \widehat{\kappa}(y). \tag{44}$$

Proof. Considering the difference of two solutions

$$\int\limits_{(\mathbb{R}^n)^+} \nabla \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \nabla \varphi dy = C_0 \int\limits_{G_0} \frac{\sigma(u + h - \widehat{w}(u+h, y)) - \sigma(u - \widehat{w}(u, y))}{h} \varphi dS_y$$

$$= C_0 \int_{G_0} \sigma'(\xi_h(y)) \left(1 - \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h}\right) \varphi(y) dS_y$$

for some ξ_h in between $u+h-\widehat{w}(u+h,y)$ and $u-\widehat{w}(u,y)$. From this

$$\int\limits_{(\mathbb{R}^n)^+} \nabla \frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h} \nabla \varphi dy + C_0 \int_{G_0} \sigma'(\xi_h(y)) \frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h} \varphi(y) dS_y$$

$$= C_0 \int_{G_0} \sigma'(\xi_h(y)) \varphi(y) dS_y. \tag{45}$$

Taking $\varphi = \frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h}$, and using the fact that $\widehat{w}(u,y)$ can be bounded and σ' is continuous

$$\left\| \nabla \left(\frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h} \right) \right\|^2_{L^2(\Omega)^n} + k_1 C_0 \left\| \frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h} \right\|^2_{L^2(G_0)} \le C. \tag{46}$$

for h small. Thus, $\frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h}$ admits a weak limit as $h \to 0$ in $H^1(K)$, let it be $\widehat{W}(u,y)$. Thus, up to a subsequence, it admits a pointwise limit and strong limit in $L^2(K)$. It is clear that

$$\xi_h(y) \to u - \widehat{w}(u,y), \qquad \text{pointwise as } h \to 0.$$

By passing to the limit for $\varphi \in \mathcal{C}_c^\infty(\overline{(\mathbb{R}^n)^+})$ fixed, we characterize (43). From (41) we deduce that, for $h > 0$

$$0 \le \frac{\widehat{w}(u+h,y)-\widehat{w}(u,y)}{h} \le \kappa(y).$$

As $h \to 0$ we deduce the result $0 \le \widehat{W}(u,y) \le \kappa(y)$. □

Remark 2.1. Notice that $\widehat{W}$ is the unique solution of

$$\begin{cases} -\Delta \widehat{W} = 0 & y \in (\mathbb{R}^n)^+, \\ \partial_\nu \widehat{W} + C_0 \sigma'(u-\widehat{w})\widehat{W} = C_0 \sigma'(u-\widehat{w}) u & y \in G_0, \\ \partial_\nu \widehat{W} = 0 & y \notin G_0, y_1 = 0, \\ \widehat{W} \to 0 & |y| \to +\infty. \end{cases} \tag{47}$$

Remark 2.2. Assume $\sigma(u) = \mu u$. Then $\sigma'(u-\widehat{w}) \equiv \mu$, and $\widehat{W}$ does not depend on μ. Therefore, $\widehat{w}(x,u) = u\widehat{W}(x)$. Furthermore

$$H(u) = \lambda_\mu u \tag{48}$$

2.3 On the regularity of function H

Lemma 2.6. *Let σ be a maximal monotone graph. The function $H_{G_0}(u)$ defined by* (16) *is Lipschitz continuous nondecreasing of constant λ_{G_0} given by* (17)*, i.e. if $u_1 > u_2$*

$$0 \le H(u_1) - H(u_2) \le \lambda_{G_0}(u_1 - u_2), \tag{49}$$

i.e., in the notation of weak derivatives

$$0 \le H'(u) \le \lambda_{G_0} \quad \text{for a.e. } u \in \mathbb{R}. \tag{50}$$

Remark 2.3. Notice that $\widehat{\kappa}$ (and thus λ_{G_0}) does not depend on σ, but only on G_0.

Proof. First, let σ be smooth and $\sigma' \geq k_1 > 0$. Again, let $\widehat{W} = \frac{\partial \widehat{w}}{\partial u}$. Taking derivatives in (16) we have that

$$H'(u) = C_0 \int_{G_0} \sigma'(u - \widehat{w})(1 - \widehat{W}) dy'. \tag{51}$$

Since $\widehat{W} \leq 1$ we have that $H' \geq 0$. Using $\widehat{\kappa}$ as a test function in (43) we obtain that

$$\begin{aligned} H'(u) = C_0 \int_{G_0} \sigma'(u - \widehat{w})(1 - \widehat{W}) dy' &= \int_{G_0} \Big(\sigma'(u - \widehat{w})(1 - \widehat{W})\Big) \widehat{\kappa} dy' \\ = \int_{(\mathbb{R}^n)^+} \nabla \widehat{W} \nabla \widehat{\kappa} dy &= \int_{G_0} \widehat{W} (\partial_\nu \widehat{\kappa}) dy' \leq \int_{G_0} (\partial_\nu \widehat{\kappa}) dy' = \lambda \end{aligned}$$

using the facts that $\partial_\nu \widehat{\kappa} \geq 0$ and $0 \leq \widehat{W} \leq 1$.
If σ is a general maximal monotone graph, estimate (49) is maintained by approximation by a smooth sequence of function σ_k. ☐

3 Convergence of the boundary integrals where $u_\varepsilon \geq 0$

3.1 On the auxiliary function w_ε^j

We introduce a function $w_\varepsilon^j(u, x)$ as a solution of the boundary value problem

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+, \\ \partial_\nu w_\varepsilon^j = \varepsilon^{-k} \sigma(u - w_\varepsilon^j), & x \in G_\varepsilon^j, \\ \partial_\nu w_\varepsilon^j = 0 & x \in (T_{\varepsilon/4}^j)^0 \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 0, & x \in (\partial T_{\varepsilon/4}^j)^+, \end{cases} \tag{52}$$

where $u \in \mathbb{R}$ is a parameter. We will compare this auxiliary functions with the functions

$$\widehat{w}_\varepsilon^j(u, x) = \widehat{w}\left(u, \frac{x - P_\varepsilon^j}{a_\varepsilon}\right) \tag{53}$$

The function $w_\varepsilon^j \in H_0^1(T_{\varepsilon/4}^j)$ is a weak solution of the problem (52) if it satisfies the integral identity

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla \varphi dx - \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) \varphi dx' = 0,$$

for an arbitrary function $\varphi \in H_0^1(T_{\varepsilon/4}^j)$.

From the uniqueness of problem (52) and the method of sub and supersolutions, we have the following:

Lemma 3.1. *The function w_ε^j satisfies following estimations*

1. *if* $u \geq 0$ *then* $0 \leq w_\varepsilon^j(u,x) \leq \widehat{w}_\varepsilon^j(u,x) \leq u$,
2. *if* $u \leq 0$ *then* $u \leq \widehat{w}_\varepsilon^j(u,x) \leq w_\varepsilon^j(u,x) \leq 0$.

Remark 3.1. Since σ is monotone, from the previous result,

$$|\sigma(u - w_\varepsilon^j)| \leq |\sigma(u)| \tag{54}$$

We define the function

$$W_\varepsilon = \begin{cases} w_\varepsilon^j(u,x), & x \in (T_{\varepsilon/4}^j)^+,\ j \in \Upsilon_\varepsilon \\ 0, & x \in (\mathbb{R}^n)^+ \setminus \bigcup\limits_{j\in\Upsilon_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}. \end{cases} \tag{55}$$

Notice that $W_\varepsilon \in H^1(\Omega, \Gamma_2)$ for all $u \in \mathbb{R}$. The following Lemma proposes estimate of the introduced function and its gradient

Lemma 3.2. *The following estimations for the function W_ε, that was defined in* (55), *are valid*

$$\begin{aligned} \|\nabla W_\varepsilon\|^2_{L_2(\Omega)} &\leq K|u||\sigma(u)|, \\ \|W_\varepsilon\|^2_{L_2(\Omega)} &\leq K\varepsilon^2|u||\sigma(u)|, \end{aligned} \tag{56}$$

Proof. From the weak formulation of problem (52) for $j \in \Upsilon_\varepsilon$ we know

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla\varphi dx - \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j)\varphi dx' = 0.$$

We take $\varphi = w_\varepsilon^j$ as a test function in this expression and obtain

$$\int\limits_{(T_{\varepsilon/4}^j)^+} |\nabla w_\varepsilon^j|^2 dx - \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) w_\varepsilon^j dx' = 0.$$

Then we can transform the obtained relation to the following expression

$$\int\limits_{(T_{\varepsilon/4}^j)^+} |\nabla w_\varepsilon^j|^2 dx + \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j)(u - w_\varepsilon^j) dx' = \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) u dx'.$$

By using the monotonicity of $\sigma(u)$ we derive the following inequality

$$\int\limits_{(T_{\varepsilon/4}^j)^+} |\nabla w_\varepsilon^j|^2 dx + \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j)(u - w_\varepsilon^j) dx' \leq \varepsilon^{-k} \int\limits_{G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)||u| dx'.$$

Due to the monotonicity σ and (54) we have that

$$\|\nabla w_\varepsilon^j\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le \varepsilon^{-k} \int\limits_{G^j_\varepsilon} |\sigma(u-w^j_\varepsilon)||u|dx' \le k_2|u||\sigma(u)|\varepsilon^{-k}|G^j_\varepsilon|$$

Hence, the following estimate is valid

$$\|\nabla w_\varepsilon^j\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le K|u||\sigma(u)|\varepsilon^{n-1}.$$

Adding over all cells we get

$$\|\nabla W_\varepsilon\|^2_{L_2(\Omega)} \le K|u||\sigma(u)|.$$

Friedrich's inequality implies

$$\|w_\varepsilon^j\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le K\varepsilon^2\|\nabla w_\varepsilon^j\|^2_{L_2((T^j_{\varepsilon/4})^+)}.$$

Summing over all cells and using obtained estimations we derive

$$\|W_\varepsilon\|^2_{L_2(\Omega)} \le K\varepsilon^2\|\nabla W_\varepsilon\|^2_{L_2(\Omega)} \le K\varepsilon^2|u||\sigma(u)|,$$

which concludes the proof. □

Hence, as $\varepsilon \to 0$ we have

$$\begin{aligned} W_\varepsilon &\rightharpoonup 0 \text{ weakly in } H^1(\Omega),\\ W_\varepsilon &\to 0 \text{ strongly in } L_2(\Omega). \end{aligned} \tag{57}$$

3.2 The comparison between w^j_ε and $\widehat{w}^j_\varepsilon$

As an immediate consequence of Lemma 3.1 we have that

Lemma 3.3. *For all $u \in \mathbb{R}$ and a.e. $x \in (T^j_{\varepsilon/4})^+$ we have*

$$|w^j_\varepsilon(u,x)| \le \left|\widehat{w}^j_\varepsilon(u,x)\right| \tag{58}$$

The following Lemma gives an estimate of proximity of functions w^j_ε and $\widehat{w}$.

Lemma 3.4. *For the introduced functions $w^j_\varepsilon(u,x)$ and $\widehat{w}(y;u)$ following estimations hold*

$$\|\nabla(v^j_\varepsilon(u,x))\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le K|u|^2\varepsilon^n, \tag{59}$$

$$\|v^j_\varepsilon(u,x)\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le K|u|^2\varepsilon^{n+2}, \tag{60}$$

where $v^j_\varepsilon(u,x) = w^j_\varepsilon(u,x) - \widehat{w}^j_\varepsilon(u,x)$.

Proof. The function v_ε^j is solution to the following boundary value problem

$$\begin{cases} \Delta v_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{G_\varepsilon^j}, \\ \partial_\nu v_\varepsilon^j = \varepsilon^{-k} \left(\sigma(u - \widehat{w}_\varepsilon^j) - \sigma(u - w_\varepsilon^j)\right), & x \in G_\varepsilon^j, \\ \partial_\nu v_\varepsilon^j = 0 & x \in (T_{\varepsilon/4}^j)^0 \setminus \overline{G_\varepsilon^j}, \\ v_\varepsilon^j = -\widehat{w}_\varepsilon^j, & x \in (\partial T_{\varepsilon/4}^j)^+. \end{cases}$$

Applying the comparison principle we have $|v_\varepsilon^j(u,x)| \leq |\widehat{w}_\varepsilon^j(u,x)|$ a.e. in $(T_{\varepsilon/4}^j)^+$. We take v_ε^j as a test function in the corresponding weak solution integral expression for the above problem

$$\|\nabla v_\varepsilon^j\|^2_{L_2((T_{\varepsilon/4}^j)^+)} + \varepsilon^{-k} \int\limits_{G_\varepsilon^j} \Big(\sigma(u - w_\varepsilon^j) - \sigma(u - \widehat{w}_\varepsilon^j)\Big) v_\varepsilon^j dx' = - \int\limits_{(\partial T_{\varepsilon/4}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds. \quad (61)$$

We transform the right-hand side expression of the inequality in the following way:

$$- \int\limits_{(\partial T_{\varepsilon/4}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds = - \int\limits_{(T_{\varepsilon/4}^j)^+ \setminus \overline{(T_{\varepsilon/8}^j)^+}} \nabla v_\varepsilon^j \nabla \widehat{w}_\varepsilon^j dx + \int\limits_{(\partial T_{\varepsilon/8}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds.$$

Let us estimate the obtained terms. We can extend v_ε^j by symmetry $v_\varepsilon^j(x,u) = v_j^\varepsilon(-x,u)$ for $x \in (T_{\varepsilon/4}^j)^-$ which is harmonic in $T_{\varepsilon/4}^j \setminus \overline{G_0}$. By using some estimates on the derivatives of harmonic functions and the maximum principle, for $\widetilde{x} \in \partial T_{\varepsilon/8}^j$, we get

$$|\partial_{x_i} v_\varepsilon^j(\widetilde{x})| \leq \frac{1}{|T_{\varepsilon/16}(\widetilde{x})|} \Bigg| \int\limits_{T_{\varepsilon/16}(\widetilde{x})} \frac{\partial v_\varepsilon^j}{\partial x_i} dx \Bigg| =$$

$$= \frac{K}{\varepsilon^n} \Bigg| \int\limits_{\partial T_{\varepsilon/16}(\widetilde{x})} v_\varepsilon^j \nu_i dx \Bigg| \leq K|u|,$$

since, on $\partial T_{\varepsilon/16}(\widetilde{x})$, from the maximum principle we have

$$|v_\varepsilon^j| \leq |\widehat{w}_\varepsilon^j| \leq \frac{K|u|}{\Big|(x - P_\varepsilon^j)/a_\varepsilon\Big|^{n-2}} = K|u|\varepsilon^{2-n} a_\varepsilon^{n-2} = K|u|\varepsilon^{2-n}\varepsilon^{n-1} = K|u|\varepsilon. \quad (62)$$

This last estimate implies that $|\nabla v_\varepsilon^j(\widetilde{x})| \leq K$ for $\widetilde{x} \in \partial T_{\varepsilon/8}^j$. Therefore, we get an estimate of the second term

$$\Bigg| \int\limits_{(\partial T_{\varepsilon/8}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds \Bigg| \leq K|u| \max_{\partial T_{\varepsilon/8}^j} |\widehat{w}_\varepsilon^j| |\partial T_{\varepsilon/8}^j| \leq K|u|^2 \varepsilon^{n.}$$

Then we estimate the first term

$$\left| \int\limits_{(T^j_{\varepsilon/4})^+ \setminus \overline{(T^j_{\varepsilon/8})^+}} \nabla v^j_\varepsilon \nabla \widehat{w} dx \right| \leq K|u|^2 \varepsilon a_\varepsilon^{n-2} = K|u|^2 \varepsilon^n$$

Combining the obtained estimates and using the properties of function σ we get

$$\|\nabla v^j_\varepsilon\|^2_{L_2((T^j_{\varepsilon/4})^+)} \leq K|u|^2 \varepsilon^n$$

Friedrichs' inequality implies that

$$\|v^j_\varepsilon\|^2_{L_2((T^j_{\varepsilon/4})^+)} \leq K|u|^2 \varepsilon^{n+2}$$

This concludes the proof. □

Lemma 3.5. *We have that*

$$\frac{1}{|G^j_\varepsilon|} \int_{G^j_\varepsilon} \left| v^j_\varepsilon(u, x') \right|^2 dx' \leq \varepsilon. \tag{63}$$

[1]

Proof. From (61), using (4), we deduce that

$$k_1 a_\varepsilon^{-1} \int_{G^j_\varepsilon} |v^j_\varepsilon|^2 \leq K|u|^2 \varepsilon^n. \tag{64}$$

Thus

$$\frac{1}{|G^j_\varepsilon|} \int_{G^j_\varepsilon} |v^j_\varepsilon(u, x')|^2 dx' \leq K a_\varepsilon^{1-n} \int_{G^j_\varepsilon} |v^j_\varepsilon(u, x')|^2 dx' \tag{65}$$

$$\leq K a_\varepsilon^{2-n} a_\varepsilon^{-1} \int_{G^j_\varepsilon} |v^j_\varepsilon(u, x')|^2 dx' \tag{66}$$

$$\leq K|u|^2 a_\varepsilon^{2-n} \varepsilon^n = K|u|^2 \varepsilon. \tag{67}$$

This completes the proof. □

3.3 Convergence to the "strange term"

The following result plays a crucial rol in the proof of Theorem 1.

Lemma 3.6. *Let H be the function defined by* (16)*, and let φ an arbitrary function in $C^\infty(\Omega)$. Then for any test function $h \in H^1(\Omega, \Gamma_2)$ we have*

$$\left| \sum_{j \in \Upsilon_\varepsilon} \int\limits_{(\partial T^j_{\varepsilon/4})^+} \partial_{\nu_x} \widehat{w}^j_\varepsilon(\varphi(\widetilde{P}^j_\varepsilon), s) h(s) ds + C_0^{n-2} \int\limits_{\Gamma_1} H(\varphi(x')) h(x') dx' \right| \to 0, \tag{68}$$

as $\varepsilon \to 0$, where $\widetilde{P}^j_\varepsilon \in G^j_\varepsilon$ and $\nu = (-1, 0, \cdots, 0)$ is the unit outward normal to $(\partial T^j_{\varepsilon/4})^+$.

[1]NOTE: Function v^j_ε in this section depends on u.

Proof. Consider the cylinder

$$Q^j_\varepsilon = \left\{ x \in \mathbb{R}^n \quad : \quad 0 < x_1 < \varepsilon, \quad -\frac{\varepsilon}{2} < x_i - (\widetilde{P}^j_\varepsilon)_i < \frac{\varepsilon}{2}, \quad i = 2, \cdots, n \right\}$$

We define the auxiliary function θ^j_ε as the unique solution to the following boundary value problem

$$\begin{cases} \Delta \theta^j_\varepsilon = 0, & Y^j_\varepsilon = Q^j_\varepsilon \setminus \overline{(T^j_{\varepsilon/4})^+}, \\ \partial_\nu \theta^j_\varepsilon = -\partial_\nu \widehat{w}^j_\varepsilon(\varphi(\widetilde{P}^j_\varepsilon), x), & x \in (\partial T^j_{\varepsilon/4})^+, \\ \dfrac{\partial \theta^j_\varepsilon}{\partial x_1} = \mu^j_\varepsilon, & x \in \gamma^j_\varepsilon = \partial Q^j_\varepsilon \cap \{x : \, x_1 = \varepsilon\}, \\ \dfrac{\partial \theta^j_\varepsilon}{\partial x_1} = 0, & \text{on the rest of the boundary } \partial Q^j_\varepsilon, \\ \langle \theta^j_\varepsilon \rangle_{Y^j_\varepsilon} = 0. \end{cases} \tag{69}$$

The constant μ^j_ε is defined from the solvability condition for the problem (69)

$$\mu^j_\varepsilon = -C_0^{n-2} H(\varphi(\widetilde{P}^j_\varepsilon)). \tag{70}$$

We take θ^j_ε as a test function in the integral identity associated to the problem (69) and obtain

$$\int\limits_{Y^j_\varepsilon} |\nabla \theta^j_\varepsilon|^2 dx = -\mu^j_\varepsilon \int\limits_{\gamma^j_\varepsilon} \theta^j_\varepsilon dx' + \int\limits_{S^{j,+}_{\varepsilon/4}} \partial_\nu \widehat{w} \theta^j_\varepsilon ds. \tag{71}$$

Using the embedding theorems we obtain the following estimate

$$\int\limits_{\gamma^j_\varepsilon} |\theta^j_\varepsilon| dx' \le K \varepsilon^{(n-1)/2} \|\theta^j_\varepsilon\|_{L_2(\gamma^j_\varepsilon)} \le K \varepsilon^{n/2} \|\nabla \theta^j_\varepsilon\|_{L_2(Y^j_\varepsilon)}. \tag{72}$$

Taking into account that

$$\max_{(\partial T^j_{\varepsilon/4})^+} |\partial_\nu \widehat{w}^j_\varepsilon(\varphi(\widetilde{P}^j_\varepsilon), x)| \le K \frac{a_\varepsilon^{-1}}{\left| \frac{x - P^j_\varepsilon}{a_\varepsilon} \right|^{n-1}} = K a_\varepsilon^{n-2} \varepsilon^{1-n} \le K$$

and using some estimates proved in [31] we derive

$$\begin{aligned} \int\limits_{(\partial T^j_{\varepsilon/4})^+} |(\partial_\nu \widehat{w}^j_\varepsilon(\varphi(\widetilde{P}^j_\varepsilon), s)) \theta^j_\varepsilon| ds \le K \int\limits_{(\partial T^j_{\varepsilon/4})^+} |\theta^j_\varepsilon| ds \le K \varepsilon^{(n-1)/2} \|\theta^j_\varepsilon\|_{L_2((\partial T^j_{\varepsilon/4})^+)} \\ \le K \varepsilon^{(n-1)/2} \{ \varepsilon^{-1/2} \|\theta^j_\varepsilon\|_{L_2(Y^j_\varepsilon)} + \sqrt{\varepsilon} \|\nabla \theta^j_\varepsilon\|_{L_2(Y^j_\varepsilon)} \} \\ \le K \varepsilon^{n/2} \|\nabla \theta^j_\varepsilon\|_{L_2(Y^j_\varepsilon)} \end{aligned} \tag{73}$$

From the above estimates (72) and (73) we get

$$\|\nabla\theta_\varepsilon^j\|^2_{L_2(Y_\varepsilon^j)} \le K\varepsilon^n \tag{74}$$

From the estimate (74) it follows that

$$\sum_{j\in\Upsilon_\varepsilon} \|\theta_\varepsilon^j\|^2_{L_2(Y_\varepsilon^j)} \le K\varepsilon.$$

Adding all the above integral identities for problems (69) we derive that for $h\in H^1(\Omega)$ the following inequality holds

$$\begin{aligned}
&\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(\partial T^j_{\varepsilon/4})^+}\partial_\nu \widehat{w}_\varepsilon^j(\varphi(\widetilde{P}_\varepsilon^j),s)h\,ds + C_0^{n-2}\int\limits_{\Gamma_1} H(\varphi(x'))h\,dx'\Big|\\
&\quad\le \Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\widehat{Y}_\varepsilon^j}\nabla\theta_\varepsilon^j\nabla h\,dx\Big| + \Big|\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon^j\int\limits_{\gamma_\varepsilon^j}h\,dx' + C_0^{n-2}\int\limits_{\Gamma_1}H(\varphi(x'))h\,dx'\Big|.
\end{aligned}\tag{75}$$

Let us estimate the terms in the right-hand side of the inequality (75). By using the estimate (74), we get the following inequality on the first term

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j}\nabla\theta_\varepsilon^j\nabla h\,dx\Big| \le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega).} \tag{76}$$

Denote

$$\widehat{\gamma}_\varepsilon^j = (Q_\varepsilon^j)^0, \qquad \Gamma_1^\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}\widehat{\gamma}_\varepsilon^j. \tag{77}$$

Then we have

$$\begin{aligned}
&\left|\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon^j\int\limits_{\gamma_\varepsilon^j}h\,dx' + C_0^{n-2}\int\limits_{\Gamma_1}H(\varphi(x'))h\,dx'\right|\\
&\quad\le \left|C_0^{n-2}\sum_{j\in\Upsilon_\varepsilon}\left(\int\limits_{\gamma^-_{j,\varepsilon}}H(\varphi(\widetilde{P}_\varepsilon^j))h\,dx' - \int\limits_{\widehat{\gamma}_\varepsilon^j}H(\varphi(x'))h\,dx'\right)\right|\\
&\quad\le C_0^{n-2}\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\widehat{\gamma}_\varepsilon^j}(H(\varphi(\widetilde{P}_\varepsilon^j)) - H(\varphi(x')))h\,dx'\Big|\\
&\qquad + C_0^{n-2}\Big|\sum_{j\in\Upsilon_\varepsilon}\left(\int\limits_{\gamma_\varepsilon^j}H(\varphi(\widetilde{P}_\varepsilon^j))h\,dx' - \int\limits_{\widehat{\gamma}_\varepsilon^j}H(\varphi(\widetilde{P}_\varepsilon^j))h\,dx'\right)\Big|.
\end{aligned}$$

Let us estimate the terms in the right-hand side of the obtained inequality. For the first term we have

$$\begin{aligned}\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\widehat{\gamma}_\varepsilon^j}(H(\varphi(\widetilde{P}_\varepsilon^j))-H(\varphi(x')))h dx'\Big| &\le K\|h\|_{L_2(\Gamma_1^\varepsilon)}\max_{\substack{j\in\Upsilon_\varepsilon\\ x'\in\widehat{\gamma}_\varepsilon^j}}\Big|H(\varphi(\widetilde{P}_\varepsilon^j))-H(\varphi(x'))\Big| \\ &\le K\|h\|_{L_2(\Gamma_1^\varepsilon)}\max_{\substack{j\in\Upsilon_\varepsilon\\ x'\in\widehat{\gamma}_\varepsilon^j}}\Big|\varphi(\widetilde{P}_\varepsilon^j)-\varphi(x')\Big| \\ &\le K\varepsilon\|h\|_{H^1(\Omega)}.\end{aligned}$$

By using continuity in L^2-norm on the hyperplanes of the functions from $H^1(\Omega)$ we estimate the second term

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\left(\int\limits_{\gamma_\varepsilon^j}H(\varphi(\widetilde{P}_\varepsilon^j))h dx'-\int\limits_{\widehat{\gamma}_\varepsilon^j}H(\varphi(\widetilde{P}_\varepsilon^j))h dx'\right)\Big|\le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega)}.$$

Hence, we have

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon^j\int\limits_{\gamma_\varepsilon^j}h dx'+C_0^{n-2}\int\limits_{\Gamma_1}H(\varphi(x'))h dx'\Big|\le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega)}. \tag{78}$$

Combining estimates (76) and (78) we conclude the proof. □

4 Convergence of the boundary integrals where $u_\varepsilon \le 0$

4.1 The auxiliary function κ_ε^j

We introduce the function κ_ε^j as the unique solution of the following problem

$$\begin{cases}\Delta\kappa_\varepsilon^j=0, & x\in(T_{\varepsilon/4}^j)^+\setminus\overline{G_\varepsilon^j},\\ \kappa_\varepsilon^j=1, & x\in G_\varepsilon^j,\\ \partial_\nu\kappa_\varepsilon^j=0 & x\in(\partial T_{\varepsilon/4}^j)^0\setminus\overline{G_\varepsilon^j},\\ \kappa_\varepsilon^j=0, & x\in\partial T_{\varepsilon/4}^j,\end{cases} \tag{79}$$

and then we define

$$\kappa_\varepsilon=\begin{cases}\kappa_\varepsilon^j(x), & x\in(T_{\varepsilon/4}^j)^+,\ j\in\Upsilon_\varepsilon\\ 0, & x\in\mathbb{R}^n\setminus\overline{\bigcup\limits_{j\in\Upsilon_\varepsilon}(T_{\varepsilon/4}^j)^+}.\end{cases} \tag{80}$$

It is easy to see that $\kappa_\varepsilon\in H^1(\Omega)$ and

$$\kappa_\varepsilon\rightharpoonup 0 \text{ weakly in } H^1(\Omega) \text{ as } \varepsilon\to 0. \tag{81}$$

4.2 Estimate of the difference between κ_ε^j and $\widehat{\kappa}_\varepsilon^j$

Lemma 4.1. *Let* κ_ε^j *and* $\widehat{\kappa}$ *as above. Then*

$$\sum_{j\in\Upsilon_\varepsilon} \|\kappa_\varepsilon^j - \widehat{\kappa}_\varepsilon^j\|^2_{H^1((T^j_{\varepsilon/4})^+)} \leq K\varepsilon. \tag{82}$$

Proof. The function $v_\varepsilon^j = \kappa_\varepsilon^j - \widehat{\kappa}$ satisfies the following problem

$$\begin{cases} \Delta v_\varepsilon^j = 0 & x \in (T^j_{\varepsilon/4})^+, \\ v_\varepsilon^j = 0 & x \in G_\varepsilon^j, \\ \partial_\nu v_\varepsilon^j = 0 & x \in (T_\varepsilon^j)^0 \setminus \overline{G_\varepsilon^j}, \\ v_\varepsilon^j = -\widehat{\kappa}_\varepsilon^j & x \in (\partial T^j_{\varepsilon/4})^+. \end{cases}$$

We take v_ε^j as a test function in an integral identity for the above problem

$$\|\nabla v_\varepsilon^j\|_{L_2((T^j_{\varepsilon/4})^+)} = -\int\limits_{\partial(T^j_{\varepsilon/4})^+} \partial_\nu v_\varepsilon^j \widehat{\kappa} ds.$$

We transform the right-hand side expression of the identity in the following way:

$$-\int\limits_{\partial(T^j_{\varepsilon/4})^+} \partial_\nu v_\varepsilon^j \widehat{\kappa} ds = -\int\limits_{(T^j_{\varepsilon/4})^+\setminus\overline{T^j_{\varepsilon/8}}} \nabla v_\varepsilon^j \nabla\widehat{\kappa} dx + \int\limits_{\partial T^j_{\varepsilon/8}} \partial_\nu v_\varepsilon^j \widehat{\kappa} ds.$$

For an arbitrary point $x_0 \in \partial T^j_{\varepsilon/8}$ we have

$$\begin{aligned} |\partial_{x_i} v_\varepsilon^j(x_0)| &\leq \frac{1}{|T_{\varepsilon/16}(x_0)|}\Bigg|\int\limits_{T_{\varepsilon/16}(x_0)} \frac{\partial v_\varepsilon^j}{\partial x_i} dx\Bigg| \\ &= \frac{K}{\varepsilon^n}\Bigg|\int\limits_{\partial T_{\varepsilon/16}(x_0)} v_\varepsilon^j \nu_i dx\Bigg| \leq K. \end{aligned}$$

Last estimate implies that $|\nabla v_\varepsilon^j(\widetilde{x})| \leq K$ for $\widetilde{x} \in \partial T^j_{\varepsilon/8}$. Therefore, we can estimate the second term in the following way

$$\Bigg|\int\limits_{\partial T^j_{\varepsilon/8}} \partial_\nu v_\varepsilon^j \widehat{\kappa} ds\Bigg| \leq K \max_{\partial T^j_{\varepsilon/8}} |\widehat{\kappa}||\partial T^j_{\varepsilon/8}| \leq K\varepsilon^n.$$

Then we estimate the first term

$$\Bigg|\int\limits_{(T^j_{\varepsilon/4})^+\setminus\overline{T^j_{\varepsilon/8}}} \nabla v_\varepsilon^j \nabla\widehat{\kappa} dx\Bigg| \leq K\varepsilon a_\varepsilon^{n-2} = K\varepsilon^n$$

By combining acquired estimations we derive

$$\|\nabla v_\varepsilon^j\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le K\varepsilon^n$$

Friedrichs' inequality imply

$$\|v_\varepsilon^j\|^2_{L_2((T^j_{\varepsilon/4})^+)} \le K\varepsilon^{n+2}$$

This concludes the proof. □

4.3 Convergence to the strange term

Lemma 4.2. *Let λ_{G_0} be given by* (17)*. Then for all functions $h \in H^1(\Omega, \Gamma_2)$ we have*

$$\Big| \sum_{j\in\Upsilon_\varepsilon} \int\limits_{(\partial T^j_{\varepsilon/4})^+\cap\Omega} \partial_{\nu_x}\widehat{\kappa}^j_\varepsilon(s)h(s)ds + C_0^{n-2}\lambda_{G_0}\int\limits_{\Gamma_1} hdx' \Big| \to 0, \tag{83}$$

as $\varepsilon \to 0$, where ν is the unit outward normal to $\partial T^j_{\varepsilon/4} \cap \Omega$.

Proof. By analogy with the proof of Lemma 3.6, we define the function θ^j_ε as a solution to the following boundary value problem

$$\begin{cases} \Delta\theta^j_\varepsilon = 0, & Y^j_\varepsilon, \\ \partial_\nu\theta^j_\varepsilon = -\partial_\nu\widehat{\kappa}^j_\varepsilon, & x \in (\partial T^j_{\varepsilon/4})^+, \\ \dfrac{\partial\theta^j_\varepsilon}{\partial x_1} = \mu, & x \in \gamma^j_\varepsilon, \\ \dfrac{\partial\theta^j_\varepsilon}{\partial x_1} = 0, & \partial Q^j_\varepsilon \setminus ((\partial T^j_{\varepsilon/4})^+ \cup \gamma^j_\varepsilon), \\ \langle\theta^j_\varepsilon\rangle_{Y^j_\varepsilon} = 0, & \end{cases} \tag{84}$$

The constant μ is defined from the solvability condition for the problem (84)

$$\mu = -C_0^{n-2}\lambda. \tag{85}$$

By using the same technique as in the proof of the Lemma 68 we have

$$\|\nabla\theta^j_\varepsilon\|^2_{L_2(Y^j_\varepsilon)} \le K\varepsilon^n, \qquad \sum_{j\in\Upsilon_\varepsilon} \|\theta^j_\varepsilon\|^2_{L_2(Y^j_\varepsilon)} \le K\varepsilon. \tag{86}$$

Summing up all integral identities for the problems (84) we derive that for the arbitrary function from $H^1(\Omega)$ the following inequality is true

$$\begin{aligned} &\Big| \sum_{j\in\Upsilon_\varepsilon} \int\limits_{(\partial T^j_{\varepsilon/4})^+} (\partial_\nu\widehat{\kappa}^j_\varepsilon) hds + C_0^{n-2}\lambda\int\limits_{\Gamma_1} hdx' \Big| \\ &\le \Big| \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\widehat{Y}^j_\varepsilon} \nabla\theta^j_\varepsilon\nabla hdx \Big| + \Big| \sum_{j\in\Upsilon_\varepsilon} \mu\int\limits_{\gamma^j_\varepsilon} hdx' + C_0^{n-2}\lambda\int\limits_{\Gamma_1} hdx' \Big|, \end{aligned} \tag{87}$$

Let us estimate terms in the right-hand side of the inequality (87). By using the estimate (74), we get following estimation of the first term

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j}\nabla\theta_\varepsilon^j\nabla h dx\Big| \le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega)}. \tag{88}$$

Then we have

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\mu\int\limits_{\gamma_\varepsilon^j} hdx' + C_0^{n-2}\lambda\int\limits_{\Gamma_1} hdx'\Big| \le \Big|C_0^{n-2}\lambda\sum_{j\in\Upsilon_\varepsilon}\left(\int\limits_{\gamma_{j,\varepsilon}^-} hdx' - \int\limits_{\widehat{\gamma}_\varepsilon^j} hdx'\right)\Big|$$

By using continuity in L_2-norm on the hyperplanes of the functions from $H^1(\Omega)$ we estimate the second term

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\left(\int\limits_{\gamma_\varepsilon^j} hdx' - \int\limits_{\widehat{\gamma}_\varepsilon^j} hdx'\right)\Big| \le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega)}$$

Hence, we have

$$\Big|\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon\int\limits_{\gamma_\varepsilon^j} hdx' + C_0^{n-2}\int\limits_{\Gamma_1} hdx'\Big| \le K\sqrt{\varepsilon}\|h\|_{H^1(\Omega)} \tag{89}$$

Combining estimations (88) and (89) we conclude the proof. □

5 Proof of Theorem 1

For different reasons it is convenient to introduce some new notation: instead to use the decomposition $u_0 = u_{0,+} - u_{0,-}$ mentioned at the introduction (see the statement of Theorem 1) we shall use the alternative decomposition $u_0 = u_0^+ + u_0^-$ (i.e. $u_0^+ = u_{0,+}$ but $u_0^- = -u_{0,-}$).

Proof. Let $\varphi(x)$ be an arbitrary function from $C_0^\infty(\Omega)$. We choose point $\widehat{P}_\varepsilon^j \in \overline{G_\varepsilon^j}$ such that

$$\min_{x\in\overline{G_\varepsilon^j}}\varphi^+(x) = \varphi^+(\widehat{P}_\varepsilon^j),$$

where $\varphi^+ = \max\{0,\varphi(x)\}$ and $\varphi^-(x) = \varphi(x) - \varphi^+(x)$. Define the function

$$\mathcal{W}_\varepsilon(\varphi^+,x) = \begin{cases} w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x), & x\in T_\varepsilon^j, \quad j\in\Upsilon_\varepsilon \\ 0, & x\in\mathbb{R}^n\setminus\bigcup\limits_{j\in\Upsilon_\varepsilon}\overline{T_\varepsilon^j}. \end{cases} \tag{90}$$

From estimates (56) we conclude that

$$\mathcal{W}_\varepsilon(\varphi^+, x) \rightharpoonup 0 \tag{91}$$

in $H^1(\Omega, \Gamma_2)$ as $\varepsilon \to 0$. We set

$$v = \varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^-$$

as a test function in integral inequality (11) where φ is an arbitrary function from $C_0^\infty(\Omega)$. Notice that $v \in K_\varepsilon$. Indeed, according to the Lemma 3.1 and using that $\kappa_\varepsilon \equiv 1$ in G_ε we have for all $x \in G_\varepsilon^j$ that

$$v = \varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^- \geq \varphi^+(\widehat{P}_\varepsilon^j) - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j), x) \geq 0. \tag{92}$$

Hence, we get

$$\int\limits_\Omega \Big\{ \nabla\Big(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^-\Big) \tag{93}$$

$$\cdot \nabla\Big(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^- - u_\varepsilon\Big)\Big\} dx + \tag{94}$$

$$+ \varepsilon^{-k} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{G_\varepsilon^j} \sigma(\varphi^+ - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j), x))(\varphi^+ - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j), x) - u_\varepsilon) dx' \geq \tag{95}$$

$$\geq \int\limits_\Omega f(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^- - u_\varepsilon) dx. \tag{96}$$

Considering the first integral of the right-hand side of the inequality above we have:

$$\begin{aligned}
&\int\limits_\Omega \nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon\varphi^-)\nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon\varphi^- - u_\varepsilon) dx \\
&\quad = \int\limits_\Omega \nabla\varphi\nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+(\widehat{P}_\varepsilon^j), x) - \kappa_\varepsilon\varphi^- - u_\varepsilon) dx - \\
&\qquad - \int\limits_\Omega \nabla\mathcal{W}_\varepsilon(\varphi^+, x)\nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon\varphi^- - u_\varepsilon) dx \\
&\qquad - \int\limits_\Omega \nabla(\kappa_\varepsilon\varphi^-)\nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon\varphi^- - u_\varepsilon) dx \\
&\quad = \sum_{i=1}^3 J_\varepsilon^i.
\end{aligned} \tag{97}$$

By using (81) and (91) we have

$$\lim_{\varepsilon \to 0} J_\varepsilon^1 = \int\limits_\Omega \nabla\varphi\nabla(\varphi - u_0) dx. \tag{98}$$

Then we proceed by transforming J_ε^2 in the following way

$$
\begin{aligned}
J_\varepsilon^2 = &-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(T^j_{\varepsilon/4})^+}\nabla w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\cdot\nabla(\varphi - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)-\kappa_\varepsilon\varphi^- - u_\varepsilon)dx\\
= &-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(T^j_{\varepsilon/4})^+}\Big\{\nabla\Big(w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)-\widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\Big)\\
&\qquad\cdot\nabla\Big(\varphi - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)-\kappa_\varepsilon\varphi^- - u_\varepsilon\Big)\Big\}dx-\\
&-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(T^j_{\varepsilon/4})^+}\nabla \widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\nabla(\varphi - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)-\kappa_\varepsilon\varphi^- - u_\varepsilon)dx\\
= &\, I_\varepsilon^1 + I_\varepsilon^2. \qquad (99)
\end{aligned}
$$

Lemma 3.4 implies that

$$I_\varepsilon^1 \to 0 \text{ as } \varepsilon\to 0. \qquad (100)$$

By using Green's formula we have the following decomposition of the second integral

$$
\begin{aligned}
I_\varepsilon^2 = &-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(\partial T^j_{\varepsilon/4})^+}\Big\{\partial_\nu\widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)(\varphi^+ - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)-u_\varepsilon)\Big\}ds\\
&-\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{G_\varepsilon^j}\Big\{\sigma\Big(\varphi^+(\widehat{P}_\varepsilon^j)-\widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\Big)\Big(\varphi^+ - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)-u_\varepsilon\Big)\Big\}dx'\\
= &\,\mathcal{I}_\varepsilon^1+\mathcal{I}_\varepsilon^2. \qquad (101)
\end{aligned}
$$

From Lemma 3.6 we have

$$\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon^1 = C_0^{n-2}\int\limits_\Omega H(\varphi^+(x))(\varphi^+ - u_0)dx. \qquad (102)$$

Combining (99)-(102) implies that

$$\lim_{\varepsilon\to0}J_\varepsilon^2 = C_0^{n-2}\int\limits_\Omega H(\varphi^+(x))(\varphi^+ - u_0)dx + \lim_{\varepsilon\to0}\mathcal{I}_\varepsilon^2. \qquad (103)$$

Now we consider third term of the identity (97). By using the fact that

$$
\begin{aligned}
\nabla(\kappa_\varepsilon^j\varphi^-)\cdot\nabla\rho_\varepsilon &= \Big(\nabla\kappa_\varepsilon^j\cdot\nabla\rho_\varepsilon\Big)\varphi^- + \kappa_\varepsilon^j\nabla\varphi^-\cdot\nabla\rho_\varepsilon\\
&= \nabla\kappa_\varepsilon^j\cdot\nabla(\varphi^-\rho_\varepsilon) - (\nabla\kappa_\varepsilon^j\cdot\nabla\varphi^-)\rho_\varepsilon + \kappa_\varepsilon^j\nabla\varphi^-\cdot\nabla\rho_\varepsilon\\
&= \nabla\widehat{\kappa}_\varepsilon^j\cdot\nabla(\varphi^-\rho_\varepsilon) - \nabla(\widehat{\kappa}_\varepsilon^j-\kappa_\varepsilon^j)\cdot\nabla(\varphi^-\rho_\varepsilon)\\
&\quad - (\nabla\kappa_\varepsilon^j\cdot\nabla\varphi^-)\rho_\varepsilon + \kappa_\varepsilon^j\nabla\varphi^-\cdot\nabla\rho_\varepsilon.
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
J_\varepsilon^3 = &- \sum_{j\in\Upsilon_\varepsilon} \int\limits_{(T^j_{\varepsilon/4})^+} \nabla\widehat{\kappa}^j_\varepsilon \nabla\Big(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)\Big)dx \\
&- \sum_{j\in\Upsilon_\varepsilon} \int\limits_{(T^j_{\varepsilon/4})^+} \nabla(\kappa^j_\varepsilon - \widehat{\kappa}^j_\varepsilon) \nabla\Big(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)\Big)dx \\
&+ \int\limits_\Omega \Big(\nabla\kappa_\varepsilon \cdot \nabla\varphi^-\Big)\Big(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)\Big)dx \\
&- \int\limits_\Omega \kappa_\varepsilon \nabla\varphi^- \cdot \nabla\Big(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)\Big)dx \\
= &\ \mathcal{Q}^1_\varepsilon + \mathcal{Q}^2_\varepsilon + \mathcal{Q}^3_\varepsilon + \mathcal{Q}^4_\varepsilon.
\end{aligned} \tag{104}
$$

Lemma 4.1 implies that $\mathcal{Q}^2_\varepsilon \to 0$ as $\varepsilon \to 0$. Then we use (9), (10), (81) and (91) to derive that $\mathcal{Q}^3_\varepsilon \to 0$ and $\mathcal{Q}^4_\varepsilon \to 0$ as $\varepsilon \to 0$. Then we transform $\mathcal{Q}^1_\varepsilon$ using the Green's formula

$$
\begin{aligned}
\mathcal{Q}^1_\varepsilon = &- \sum_{j\in\Upsilon_\varepsilon} \int\limits_{(\partial T^j_{\varepsilon/4})^+} (\partial_\nu \widehat{\kappa}^j_\varepsilon)\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)ds - \\
&- \sum_{j\in\Upsilon_\varepsilon} \int\limits_{G^j_\varepsilon} (\partial_\nu \widehat{\kappa}^j_\varepsilon)\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)dx'.
\end{aligned} \tag{105}
$$

By using the fact that $\partial_\nu\widehat{\kappa} \geq 0$ and $\varphi^-\mathcal{W}_\varepsilon(\varphi^+,x) \leq 0$, $\varphi^- u_\varepsilon \leq 0$ a.e. in G_ε we have that

$$
- \sum_{j\in\Upsilon_\varepsilon} \int\limits_{G^j_\varepsilon} (\partial_\nu \widehat{\kappa}^j_\varepsilon)\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+,x) - \kappa_\varepsilon\varphi^- - u_\varepsilon)dx' \leq 0. \tag{106}
$$

Hence, we conclude

$$
\mathcal{Q}^1_\varepsilon \leq - \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}\cap\Omega} (\partial_\nu \widehat{\kappa}^j_{\varepsilon}.)\varphi^-(\varphi - u_\varepsilon)ds \tag{107}
$$

Lemma 4.2 implies

$$
\lim_{\varepsilon\to 0} J^3_\varepsilon = \lim_{\varepsilon\to 0} \mathcal{Q}^1_\varepsilon \leq \lambda C_0^{n-2} \int\limits_\Omega \varphi^-(\varphi - u_\varepsilon)dx \tag{108}
$$

From (97), (98), (103), (108) we derive that

$$\int\limits_{\Omega} \nabla\varphi\nabla(\varphi-u_0)dx + C_0^{n-2}\int\limits_{\Gamma_1} H(\varphi^+)(\varphi-u_0)dx' + \lambda C_0^{n-2}\int\limits_{\Gamma_1}\varphi^-(\varphi-u_0)dx'$$

$$+\lim_{\varepsilon\to 0}\left\{\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{G_\varepsilon^j}\sigma\left(\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x)\right)(\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x)-u_\varepsilon)dx'\right.$$

$$\left.-\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{G_\varepsilon^j}\sigma\left(\varphi^+(\widehat{P}_\varepsilon^j)-\widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\right)(\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x)-u_\varepsilon)dx'\right\}$$

$$\geq \int\limits_{\Omega} f(\varphi-u_0)dx. \tag{109}$$

We first notice that $\varphi^+ - w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x) \geq \varphi^+(\widehat{P}_\varepsilon^j) - \widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j,x))$ and so

$$\left\{\sigma\left(\varphi^+-w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\right)-\sigma\left(\varphi^+(\widehat{P}_\varepsilon^j)-\widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\right)\right\}u_\varepsilon \geq 0.$$

Thus, we only need to study the term

$$\varepsilon^{-k}\int\limits_{G_\varepsilon^j}\left\{\sigma\left(\varphi^+-w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\right)-\sigma\left(\varphi^+(\widehat{P}_\varepsilon^j)-\widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\right)\right\}\left(\varphi^+-w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\right)dx'.$$

On the other hand,

$$|\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x)| \leq |\varphi^+| \leq K,$$
$$w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x) \leq \widehat{w}_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)| \leq |\varphi^+| \leq K.$$

Since σ is Hölder continuous, and $\varphi\in\mathcal{C}^1(\overline{\Omega})$, we have that, for a.e. $x\in G_\varepsilon^j$,

$$\begin{aligned}
&\Big|\sigma(\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x))-\sigma(\varphi^+(\widehat{P}_\varepsilon^j)-\mathcal{W}_\varepsilon(\varphi^+,x))\Big| \\
&\qquad\leq K\sum_{i=1}^{2}|\varphi^+(x)-\varphi^+(\widehat{P}_\varepsilon^j)|^{\rho_i} \leq K\sum_{i=1}^{2}|x-\widehat{P}_\varepsilon^j|^{\rho_i} \\
&\qquad= K\sum_{i=1}^{2}a_\varepsilon^{\rho_i}.
\end{aligned}$$

By using the same reasoning, estimate (56) implies that

$$\left|\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{G_\varepsilon^j}\Big(\sigma(\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x))-\sigma(\varphi^+(\widehat{P}_\varepsilon^j)-\mathcal{W}_\varepsilon(\varphi^+,x))\Big)\Big(\varphi^+-w_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)\Big)dx'\right|$$

$$\leq K_2\varepsilon^{-k}|G_\varepsilon|\sum_{i=1}^{2}a_\varepsilon^{\rho_i} \leq K\sum_{i=1}^{2}a_\varepsilon^{\rho_i}. \tag{110}$$

Then by Lemma 3.4 we have that

$$\varepsilon^{-k}\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{G_\varepsilon^j}\Big(\sigma(\varphi^+(\widehat{P}_\varepsilon^j)-\mathcal{W}_\varepsilon(\varphi^+,x))-\sigma(\varphi^+(\widehat{P}_\varepsilon^j)-\widehat{w}_\varepsilon^j)\Big)\Big(\varphi^+-\mathcal{W}_\varepsilon(\varphi^+,x)\Big)dx'\Big|$$
$$\leq K\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}\sum_{i=1}^{2}\int\limits_{G_\varepsilon^j}|v(\varphi^+(\widehat{P}_\varepsilon^j),x)|^{\rho_i}dx'$$
$$\leq K\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}|G_\varepsilon^j|\sum_{i=1}^{2}\frac{1}{|G_\varepsilon^j|}\int\limits_{G_\varepsilon^j}|v(\varphi^+(\widehat{P}_\varepsilon^j),x)|^{\rho_i}dx'$$

which, applying the $L^2(G_0)\to L^{\rho_i}(G_0)$ embedding for $0<\rho_i\leq 2$, can be estimated as

$$...\leq K\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}|G_\varepsilon^j|\sum_{i=1}^{2}\left(\frac{1}{|G_\varepsilon^j|}\int_{G_\varepsilon^j}|v_\varepsilon^j(\varphi^+(\widehat{P}_\varepsilon^j),x)|^2dx'\right)^{\frac{\rho_i}{2}}$$

and, by Lemma 3.5

$$...\leq K\varepsilon^{-k}\sum_{j\in\Upsilon_\varepsilon}|G_\varepsilon^j|\sum_{i=1}^{2}\varepsilon^{\frac{\rho_i}{2}}\leq K\varepsilon^{-k}|\Upsilon_\varepsilon||G_\varepsilon^j|\sum_{i=1}^{2}\varepsilon^{\frac{\rho_i}{2}}\leq K\varepsilon^{-k+1-n+k(n-1)}\sum_{i=1}^{2}\varepsilon^{\frac{\rho_i}{2}}$$
$$=K\sum_{i=1}^{2}\varepsilon^{\frac{\rho_i}{2}}\to 0, \tag{111}$$

by using that $0<\rho\leq 2$. Combining these estimates with (109) we derive, since $\rho>0$, that

$$\int\limits_\Omega\nabla\varphi\nabla(\varphi-u_0)dx+C_0^{n-2}\int\limits_{\Gamma_1}H(\varphi^+)(\varphi-u_0)dx'+\lambda C_0^{n-2}\int\limits_{\Gamma_1}\varphi^-(\varphi-u_0)$$
$$\geq\int\limits_\Omega f(\varphi-u_0)dx \tag{112}$$

holds for any $\varphi\in H^1(\Omega,\Gamma_2)$.

Finally, given $\psi\in H^1(\Omega,\Gamma_2)$ we consider the test function $\varphi=u_0\pm\delta\psi$, $\delta>0$ in (112) and we pass to the limit as $\delta\to 0$. By doing so we get that u_0 satisfies the integral condition

$$\int\limits_\Omega\nabla u_0\nabla\psi dx+C_0^{n-2}\int\limits_{\Gamma_1}H(u_0^+)\psi dx'+\lambda C_0^{n-2}\int\limits_{\Gamma_1}u_0^-\psi dx'=\int\limits_\Omega f\psi dx \tag{113}$$

for any $\psi\in H^1(\Omega,\Gamma_2)$. This concludes the proof. $\square$

6 Possible extensions and comments

6.1 Extension to the case of σ as a maximal monotone graph

In [11] the authors showed that a similar problem, although restricted to the case of spherical particles distributed through the whole domain, could be treated in the general framework of maximal monotone graphs σ, which allow for a *common roof* between the Dirichlet, Neumann and Signorini boundary conditions and many more. We have restricted here to the case of Hölder continuous σ (see (4)) but this condition is only used at very end, in estimates (110) and (111) to compute the last term of (109). The superlinearity condition is only used to obtain Lemma 3.5. These seem to be only technical difficulties, and can probably be avoided. Let us introduce what results can be expected, if these problems could be circumvented.

Maximal monotone graph of $\mathbb{R}^2$ A monotone graph of $\mathbb{R}^2$ is a map (or operator) $\sigma : D(\sigma) \subset \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ such that

$$(\xi_1 - \xi_2)(x_1 - x_2) \geq 0, \qquad \forall x_i \in D(\sigma), \forall \xi_i \in \sigma(x_i). \tag{114}$$

The set $D(\sigma)$ is called domain of the multivalued operator σ. Some authors define maximal monotone graphs as maps $\sigma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and define $D(\sigma) = \{x \in \Omega : \sigma(x) \neq \emptyset\}$.

A monotone graph σ is extended by another monotone graph $\widetilde{\sigma}$ if $D(\sigma) \subset D(\widetilde{\sigma})$ and $\sigma(x) \subset \widetilde{\sigma}(x)$ for all $x \in D(\sigma)$. A monotone graph is called maximal if its admits no proper extension. For further reference see [2].

Definition of solution The solution u_ε is also well defined, although the set K_ε must now be written as

$$K_\varepsilon = \{v \in H^1(\Omega, \Gamma_2) : \forall x' \in G_\varepsilon, v(x') \in D(\sigma)\}.$$

We will have that the integral condition (11) turns into

$$\int_\Omega \nabla\varphi \nabla(\varphi - u_\varepsilon) dx + \varepsilon^{-k} \int_{G_\varepsilon} \xi(x)(\varphi - u_\varepsilon) dx' \geq \int_\Omega f(\varphi - u_\varepsilon) dx, \tag{115}$$

for all $\varphi \in K_\varepsilon$ and $\xi \in L^2(G_\varepsilon)$ such that $\xi(x') \in \sigma(\varphi(x'))$ for a.e. $x' \in G_\varepsilon$. Existence and uniqueness of this solutions follows as in [11] and the references therein.

The auxiliary functions The equation of $\widehat{w}$ is well-defined when σ is a maximal monotone graph. As we have proved in this paper, the estimate $0 \leq H' \leq \lambda_{G_0}$ is independent of σ, and so H is Lipschitz continuous for any maximal monotone graph σ.

Signorini boundary conditions This is the case under study in this paper. Nonetheless, let us study in the general setting. For this kind of boundary condition, we need to consider the following maximal monotone graph:

$$\widetilde{\sigma}(s) = \begin{cases} \sigma(s) & s > 0, \\ (-\infty, 0] & s = 0, \\ \emptyset & s < 0 \end{cases} \tag{116}$$

and $D(\sigma) = [0, +\infty)$. Let us compute $\widetilde{H}_{G_0}$ in this setting:

- For $u < 0$, we can see what happens explicitly. We have that $u \leq \widehat{w}(u, \cdot) \leq 0$. Thus $u - \widehat{w} \leq 0$. Since $D(\sigma) = [0, +\infty)$ we must have that $\widehat{w}(y; u) = u$ on G_0. But then $\widehat{w}(y; u) = u\widehat{\kappa}(y)$. Hence $\widetilde{H}_{G_0}(u) = \lambda_{G_0} u$ when $u < 0$.
- When $u > 0$, we have that $0 \leq u - \widehat{w}(u, \cdot)$. Thus, only the values of σ affect $H_{G_0}(u)$.

We conclude that

$$\widetilde{H}_{G_0}(u) = \begin{cases} H_{G_0}(u) & u > 0, \\ \lambda_{G_0} u & u \leq 0. \end{cases}$$

The computations with maximal monotone graphs yield precisely Theorem 1. Notice that the bound on $\widetilde{H}'$ given by (18) is sharp.

Dirichlet boundary conditions In this case, we would have $D(\sigma) = \{0\}$ and

$$\widetilde{\sigma}(0) = (-\infty, +\infty)$$

By the same reasoning, we have that

$$\widetilde{H}_{G_0}(u) = \lambda_{G_0} u,$$

for all $u \in \mathbb{R}$. In this case of Dirichlet boundary conditions the critical case generates a linear term in the homogenized equation. This type of phenomena was already notice by the authors of [6].

Cases of finite and infinite permeable coefficient The Signorini boundary condition imposed as a maximal monotone graph (116) is the extreme case of *infinite permeability*, aiming to represent the behaviour of very large *finite* permeability given by a reaction term of the form

$$\widetilde{\sigma}_\mu(u) = \begin{cases} \sigma(u) & u > 0, \\ \mu u & u \leq 0, \end{cases} \tag{117}$$

where μ is very large. As in Remark 2.2 it is easy to show that the corresponding kinetic will be of the form

$$\widetilde{H}_\mu(u) = \begin{cases} H(u) & u > 0, \\ \lambda_\mu u & u \leq 0. \end{cases} \tag{118}$$

Furthermore, since we have proven that the Signorini boundary condition is an extremal case (i.e. $\widetilde{H}'(u) \leq \lambda_{G_0}$) we have that $\lambda_\mu \leq \lambda_{G_0}$.

6.2 On the super-linearity condition

The condition

$$|\sigma(s) - \sigma(t)| \geq k_1 |s - t|$$

is only used in the proof of Lemma 3.5. However, it is our belief that this condition can be remove and still obtain the result. We provide here a proof for $n = 3$ and G_0 a ball.

We define the auxiliary function w_ε unique solution of

$$\begin{cases} -\Delta w_\varepsilon^j = 0 & T_\varepsilon^0, \\ w_\varepsilon = 0 & G_\varepsilon^0 \\ w_\varepsilon = 1 & \partial T_\varepsilon^0 \end{cases} \tag{119}$$

where T_ε^0 is given by

$$T_\varepsilon^0 = \left\{ x \in \mathbb{R}^3 : \frac{x_1^2}{1 - (a_\varepsilon \varepsilon^{-1})^2} + x_2^2 + x_3^2 < \varepsilon^2 \right\}. \tag{120}$$

Using prolate ellipsoidal coordinates we can give an explicit expression of w_ε. These coordinates are given by

$$x_1 = a_\varepsilon \sinh \psi \cos \theta_1, \tag{121}$$

$$x_2 = a_\varepsilon \cosh \psi \sin \theta_1 \cos \theta_2, \tag{122}$$

$$x_3 = a_\varepsilon \cosh \psi \sin \theta_1 \sin \theta_2, \tag{123}$$

where $0 \leq \psi < \infty$, $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$. Defining $\sigma = \sinh \psi$ it can be proven through symmetry that $w_\varepsilon(x) = V_\varepsilon(\sigma)$. Furthermore, V_ε is the unique solution of the one-dimensional problem

$$\begin{cases} \dfrac{d}{d\sigma}\left((1 + \sigma^2) \dfrac{dV}{d\sigma} \right) = 0 & \sigma \in \left(0, \sqrt{(a_\varepsilon^{-1}\varepsilon)^2 - 1} \right), \\ V(0) = 0, \\ V\left(\sqrt{(a_\varepsilon^{-1}\varepsilon)^2 - 1} \right) = 1. \end{cases} \tag{124}$$

By integrating this simple one dimensional boundary value problem

$$V(\sigma) = \frac{\arctan \sigma}{\arctan \left(\sinh \sqrt{(a_\varepsilon^{-1}\varepsilon)^2 - 1} \right)}. \tag{125}$$

Since we can recover from the change in variable

$$\sigma = \sinh \psi = \sqrt{\frac{|x|^2 - a_\varepsilon^2 + \sqrt{(a_\varepsilon^2 - |x|^2)^2 + 4x_1^2 a_\varepsilon^2}}{2a_\varepsilon^2}}. \tag{126}$$

Due to mirror symmetry it is clear that $\partial_{x_1} w_\varepsilon|_{\{x_1=0\}} = 0$. Thus, we have

$$\int_{(T_\varepsilon^0)^+} \nabla w_\varepsilon \nabla (u^2) dx = \int_{(\partial T_\varepsilon^0)^+} \partial_{x_1} w_\varepsilon u^2 dS - \int_{G_\varepsilon^0} \partial_{x_1} w_\varepsilon u^2 dS. \tag{127}$$

Using the explicit expression of w_ε we can compute that

$$\partial_\nu w_\varepsilon|_{(\partial T_\varepsilon^0)^+} \sim a_\varepsilon \varepsilon^{-2} \tag{128}$$

$$\partial_\nu w_\varepsilon|_{G_\varepsilon^0} \sim -\frac{1}{\sqrt{a_\varepsilon^2 - |x|^2}}. \tag{129}$$

Now let

$$T_\varepsilon^j = P_\varepsilon^j + T_\varepsilon^0, \tag{130}$$

$$T_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} T_\varepsilon^j, \tag{131}$$

$$W_\varepsilon(x) = w_\varepsilon(x - P_\varepsilon^j) \quad \text{ for } x \in T_\varepsilon^j. \tag{132}$$

Adding over Υ_ε we deduce that

$$\int_{(T_\varepsilon)^+} \nabla W_\varepsilon \nabla (u^2) ds = \sum_{j\in\Upsilon_\varepsilon} \int_{(\partial T_\varepsilon^j)^+} \partial_\nu w_\varepsilon^j u^2 ds - \int_{G_\varepsilon} \partial_{x_1} W_\varepsilon u^2 ds. \tag{133}$$

It is easy to prove that

$$\int_{(\partial T_\varepsilon^j)^+} \partial_\nu w_\varepsilon^j h^2 ds \le K \sum_{j\in\Upsilon_\varepsilon} \|h\|^2_{H^1((T_\varepsilon^j)^+)} \tag{134}$$

for any $h \in H^1(\Omega)$. We now apply that

$$\|u\|^2_{L^2(G_\varepsilon)} \le K \left(\varepsilon^{-1} \|u\|_{L^2(T_\varepsilon^+)} + \varepsilon \|\nabla u\|^2_{L^2(T_\varepsilon^+)} \right). \tag{135}$$

With this (133), (134), (135) we can prove Lemma 3.5 for $k_1 = 0$.

6.3 Connections to fractional operators

Let us consider a domain $\Omega = \Omega' \times (0, +\infty)$ where $\Omega' \subset \mathbb{R}^{n-1}$ is a smooth bounded domain. Then $\Gamma_1 = \Omega'$ and $\Gamma_2 = \partial\Omega \times (0, +\infty)$. The related problem

$$\begin{cases} -\Delta u_\varepsilon = 0, & (x, y) \in \Omega' \times (0, +\infty), \\ \partial_\nu u_\varepsilon + \varepsilon^{-k} \sigma(u_\varepsilon) = \varepsilon^{-k} g_\varepsilon, & x \in G_\varepsilon \\ \partial_\nu u_\varepsilon = 0, & x \in \Omega' \setminus \overline{G}_\varepsilon, \\ u_\varepsilon = 0, & (x, y) \in \partial\Omega' \times (0, +\infty), \\ u_\varepsilon \to 0 & |y| \to +\infty. \end{cases} \tag{136}$$

is very relevant because it can be linked with the study of the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$. In fact, the boundary conditions on Ω' can be written compactly as

$$\partial_\nu u_\varepsilon + \varepsilon^{-k}\chi_{G_\varepsilon}\sigma(u_\varepsilon) = \varepsilon^{-k}\chi_{G_\varepsilon}g_\varepsilon \qquad x \in \Omega' \tag{137}$$

where χ is the indicator function. This boundary condition can be written as an equation of Ω' not involving the interior part of the domain, $\Omega' \times (0, +\infty)$, by understanding the normal derivative of problem (136) as the fractional Laplace operator $(-\Delta)^{\frac{1}{2}}$ in Ω' (see [3], [15] and their references). Then (137) can be written as

$$(-\Delta)^{\frac{1}{2}}u_\varepsilon + \varepsilon^{-k}\chi_{G_\varepsilon}\sigma(u_\varepsilon) = \varepsilon^{-k}\chi_{G_\varepsilon}g_\varepsilon \qquad x \in \Omega' \tag{138}$$

Thus, the study of the limit of (136) will provide an homogenization result for (138). Applying similar techniques to this paper and previous results in the literature [12], the homogenized problem

$$(-\Delta)^{\frac{1}{2}}u_0 + CH(x, u_0) = Ch \qquad x \in \Omega' \tag{139}$$

is expected, where H and h will depend on σ and g_ε. This could provide some new results of critical size homogenization for the fractional Laplacian (in the spirit of the important work [3], where some random aspects on the net, and for a general fractional power of the Laplacian, are also considered).

Part II

The case $n = 2$

7 Proof of Theorem 2

It's well known that problem (20), (21) has a unique weak solution $u_\varepsilon \in H^1(\Omega, \Gamma_2)$. By using (21) and conditions (22), that was set on the function σ, we get the following estimates

$$\|\nabla u_\varepsilon\|_{L_2(\Omega)} \le K, \qquad e^{\frac{\alpha^2}{\varepsilon}}\|u_\varepsilon\|^2_{L_2(G_\varepsilon)} \le K_1, \tag{140}$$

here and below, constants K, K_1 are independent of ε.

Hence there exists subsequence (denote as the original sequence u_ε) such that as $\varepsilon \to 0$ we have

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \\ u_\varepsilon &\to u_0 \text{ strongly in } L_2(\Omega). \end{aligned} \tag{141}$$

We introduce auxiliary functions w_ε^j and q_ε^j as a weak solution to the following problems

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}}, \\ w_\varepsilon^j = 1, & x \in \partial T^j_{a_\varepsilon}, \\ w_\varepsilon^j = 0, & x \in \partial T^j_{\varepsilon/4}, \end{cases} \tag{142}$$

and

$$\begin{cases} \Delta q_\varepsilon^j = 0, & x \in T_\varepsilon^j \setminus \overline{l_\varepsilon^j}, \\ q_\varepsilon^j = 1, & x \in l_\varepsilon^j, \\ q_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j. \end{cases} \tag{143}$$

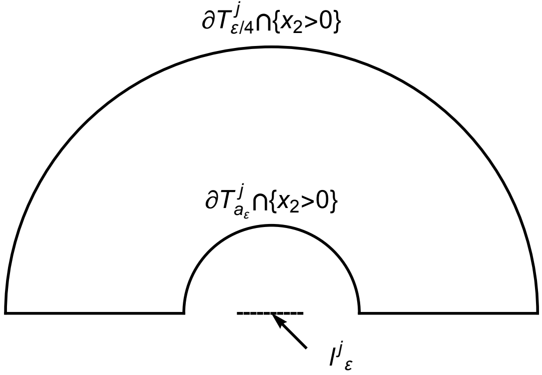

Figure 4: Domain $T_\varepsilon^{+,j} \setminus \overline{T_{a_\varepsilon}^j}$ and l_ε^j.

Note that w_ε^j and q_ε^j are also a solutions of the boundary value problems in the domains $(T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}$ and $(T_{\varepsilon/4}^j)^+$ respectively, where $(T_r^j)^+ = T_r^j \cap \{x_2 > 0\}$,

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ w_\varepsilon^j = 1, & x \in \partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} w_\varepsilon^j = 0, & x \in \{x_2 = 0\} \cap (\partial T_{\varepsilon/4}^j \setminus \overline{T_{a_\varepsilon}^j}), \end{cases} \tag{144}$$

$$\begin{cases} \Delta q_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+, \\ q_\varepsilon^j = 0 & x \in (\partial T_{\varepsilon/4}^j)^+ \\ q_\varepsilon^j = 1, & x \in l_\varepsilon^j, \\ \partial_{x_2} q_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j \cap \{x_2 = 0\}) \setminus \overline{l_\varepsilon^j}, \end{cases} \tag{145}$$

where $j \in \Upsilon_\varepsilon$, $l_\varepsilon^j = a_\varepsilon \widehat{l_0} + \varepsilon j$. Define

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, j \in \Upsilon_\varepsilon, \\ 1, & x \in \underline{(T_{a_\varepsilon}^j)^+}, \\ 0, & x \in \mathbb{R}_+^2 \setminus \bigcup_{j \in \Upsilon_\varepsilon} \overline{T_{\varepsilon/4}^j}, \end{cases} \tag{146}$$

where $\mathbb{R}_+^2 = \{x_2 > 0\}$,

$$Q_\varepsilon(x) = \begin{cases} q_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+,\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}_+^2 \setminus \bigcup_{j\in\Upsilon_\varepsilon} \overline{T_{\varepsilon/4}^j}^+. \end{cases} \tag{147}$$

We have $W_\varepsilon, Q_\varepsilon \in H_0^1(\Omega)$ and

$$W_\varepsilon \rightharpoonup 0, \quad \text{weakly in } \ H_0^1(\Omega),\ \varepsilon \to 0. \tag{148}$$

Lemma 7.1. *Let W_ε be a function defined by the formula* (146), *Q_ε be a function defined by the formula* (147). *Then*

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \le K\sqrt{\varepsilon}. \tag{149}$$

Proof. Note that for an arbitrary function $\psi \in H^1(T_{\varepsilon/4}^j)$ such that $\psi = 0$ on l_ε^j we have

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla q_\varepsilon^j \nabla \psi dx_1 dx_2 = 0. \tag{150}$$

We consider $\psi = w_\varepsilon^j - q_\varepsilon^j$ as a test function in the obtained equality and get

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla q_\varepsilon^j \nabla (w_\varepsilon^j - q_\varepsilon^j) dx_1 dx_2 = 0. \tag{151}$$

In addition, we have

$$\int\limits_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla (w_\varepsilon^j - q_\varepsilon^j) dx_1 dx_2 = \int\limits_{\partial T_{a_\varepsilon}^j \cap \{x_2>0\}} \partial_\nu w_\varepsilon^j (w_\varepsilon^j - q_\varepsilon^j) ds. \tag{152}$$

By subtracting (151) from (152) we derive

$$\int\limits_{(T_{\varepsilon/4}^j)^+} |\nabla (w_\varepsilon^j - q_\varepsilon^j)|^2 dx = \int\limits_{\partial T_{a_\varepsilon}^j \cap \{x_2>0\}} \partial_\nu w_\varepsilon^j (w_\varepsilon^j - q_\varepsilon^j) ds. \tag{153}$$

Note that $w_\varepsilon^j(x) = \frac{\ln(4r/\varepsilon)}{\ln(4a_\varepsilon/\varepsilon)}$ and $\partial_\nu w_\varepsilon^j = -\frac{1}{a_\varepsilon \ln(4a_\varepsilon/\varepsilon)}$. Hence, (153) implies that

$$\begin{aligned} \|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|^2_{L^2((T_{\varepsilon/4}^j)^+)} &\le \frac{1}{a_\varepsilon |\ln(4a_\varepsilon/\varepsilon)|} \int\limits_{\partial T_{a_\varepsilon}^j \cap \{x_2>0\}} |w_\varepsilon^j - q_\varepsilon^j| ds \\ &= \frac{1}{|\ln(4a_\varepsilon/\varepsilon)} | \int\limits_{\partial T_1^j \cap \{y_2>0\}} |w_\varepsilon^j - q_\varepsilon^j| ds_y \equiv J_\varepsilon. \end{aligned}$$

Given that $w_\varepsilon^j - q_\varepsilon^j = 0$ if $y \in \widehat{l_0}$ and using the embedding theorem, we get

$$J_\varepsilon \le \frac{K}{|\ln(4a_\varepsilon/\varepsilon)|}\Big(\int\limits_{(T_1^j)^+} |\nabla_y(w_\varepsilon^j - q_\varepsilon^j)|^2 dy\Big)^{1/2} \le K\varepsilon\|\nabla(w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T^j_{\varepsilon/4})^+)}. \tag{154}$$

From here we derive the estimate

$$\|\nabla(w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T^j_{\varepsilon/4})^+)} \le K\varepsilon.$$

From this estimation it follows that

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \le K\sqrt{\varepsilon}. \tag{155}$$

This concludes the proof. □

We introduce function $m(y) \in H^1((T_1^0)^+)$ as the weak solution to the following boundary value problem

$$\begin{cases} \Delta_y m = 0, & y \in T_1^0 \cap \{y_2 > 0\} = (T_1^0)^+, \\ \partial_{y_2} m = 1, & y \in \widehat{l_0}, \\ \partial_\nu m = \frac{2l_0}{\pi}, & y \in \partial T_1^0 \cap \{y_2 > 0\} = (\partial T_1^0)^+, \\ \partial_{y_2} m = 0, & y \in \partial (T_1^0)^+ \setminus \widehat{l_0} \cup (\partial T_1^0)^+. \end{cases} \tag{156}$$

Consider

$$m_\varepsilon^j(x) = \varepsilon m(\frac{x - P_\varepsilon^j}{a_\varepsilon}), \;\; x \in (T^j_{a_\varepsilon})^+. \tag{157}$$

The function $m_\varepsilon^j(x)$ verifies the problem

$$\begin{cases} \Delta_x m_\varepsilon^j = 0, & x \in (T^j_{a_\varepsilon})^+, \\ \partial_\nu m_\varepsilon^j = \frac{\varepsilon a_\varepsilon^{-1} 2l_0}{\pi}, & x \in (\partial T^j_{a_\varepsilon})^+, \\ \partial_{x_2} m_\varepsilon^j = \varepsilon a_\varepsilon^{-1}, & x \in \{x_2 = 0, |x - P_\varepsilon^j| \le a_\varepsilon l_0\} = l_\varepsilon^j, \\ \partial_{x_2} m_\varepsilon^j = 0, & \text{on the rest of the boundary.} \end{cases} \tag{158}$$

Lemma 7.2. *Let $n = 2$ and $h \in H^1(\Omega, \Gamma_2)$ then the following estimate holds:*

$$\Big|\frac{2l_0\varepsilon}{\pi a_\varepsilon}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(\partial T^j_{a_\varepsilon})^+} h ds - \frac{\varepsilon}{a_\varepsilon}\int\limits_{l_\varepsilon} h dx_1\Big| \le K\sqrt{\varepsilon}. \tag{159}$$

Proof. Denote $h_\varepsilon = H(\psi)(\psi - H(\psi) - u_\varepsilon)$. Then

$$\begin{aligned} &\Big|\frac{2l_0\varepsilon a_\varepsilon^{-1}}{\pi}\int\limits_{(\partial T^j_{a_\varepsilon})^+} h_\varepsilon ds - \varepsilon a_\varepsilon^{-1}\int\limits_{l_\varepsilon^j} h_\varepsilon dx_1\Big| = \\ &= \Big|\int\limits_{(T^j_{a_\varepsilon})^+} \nabla_x m_\varepsilon^j \nabla h_\varepsilon dx\Big| \le \|\nabla_x m_\varepsilon^j\|_{L^2((T^j_{a_\varepsilon})^+)}\|\nabla h_\varepsilon\|_{L^2((T^j_{a_\varepsilon})^+)}. \end{aligned} \tag{160}$$

Due to the fact that

$$\|\nabla_x m_\varepsilon^j\|^2_{L^2((T^j_{a_\varepsilon})^+)} = \varepsilon^2 \|\nabla_y m(y)\|^2_{L^2((T^0_1)^+)} \le K\varepsilon^2,$$

we have

$$\sum_{j\in\Upsilon_\varepsilon} \|\nabla_x m_\varepsilon^j\|^2_{L^2((T^j_{a_\varepsilon})^+)} \le K\varepsilon. \tag{161}$$

From (160), (161) we derive

$$\begin{gathered}\Big| e^{\alpha^2/\varepsilon}\frac{\pi}{2l_0}\int\limits_{l_\varepsilon} h_\varepsilon dx_1 - e^{\alpha^2/\varepsilon}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}\cap\{x_2>0\}} h_\varepsilon ds\Big| \le \\ \le \delta^{-1}\sum_{j\in\Upsilon_\varepsilon}\|\nabla_x m_\varepsilon^j\|^2_{L^2((T^j_{a_\varepsilon})^+)} + \delta\|\nabla h_\varepsilon\|^2_{L^2(\Omega)} \le K\sqrt{\varepsilon},\end{gathered} \tag{162}$$

if $\delta = \sqrt{\varepsilon}$. □

Proof of Theorem 2. First of all equation (25) has a unique solution $H(u)$ that is a Lipschitz continuous function in $\mathbb{R}$ and satisfies

$$\begin{gathered}(H(u)-H(v))(u-v) \ge \widetilde{k}_1|u-v|, \\ |H(u)| \le |u|,\end{gathered} \tag{163}$$

for all u, $v \in \mathbb{R}$ and a certain constant $\widetilde{k}_1 > 0$.

We take $v = \psi - Q_\varepsilon(H(\psi^+) + \psi^-)$ as a test function in (21), where $\psi \in C^\infty(\overline{\Omega})$, $\psi(x) = 0$ in the neighborhood of Γ_2, $H(u)$ satisfies the functional equation (25). Note that from (143), (147) and (163) we have $v \ge 0$ on l_ε so $v \in K_\varepsilon$. Hence we get

$$\begin{gathered}\int\limits_\Omega \nabla(\psi - Q_\varepsilon(H(\psi^+)+\psi^-))\nabla(\psi - Q_\varepsilon(H(\psi^+)+\psi^-) - u_\varepsilon)dx + \\ + e^{\frac{\alpha^2}{\varepsilon}}\int\limits_{l_\varepsilon}\sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon)dx_1 \ge \\ \ge \int\limits_\Omega f(\psi - Q_\varepsilon(H(\psi^+)+\psi^-) - u_\varepsilon)dx.\end{gathered} \tag{164}$$

We rewrite inequality (164) in the following way

$$\begin{gathered}\int\limits_\Omega \nabla(\psi - W_\varepsilon(H(\psi^+)+\psi^-))\nabla(\psi - Q_\varepsilon(H(\psi^+)+\psi^-) - u_\varepsilon)dx - \\ - \int\limits_\Omega \nabla((Q_\varepsilon - W_\varepsilon)(H(\psi^+)+\psi^-))\nabla(\psi - Q_\varepsilon(H(\psi^+)+\psi^-) - u_\varepsilon)dx + \\ + e^{\frac{\alpha^2}{\varepsilon}}\int\limits_{l_\varepsilon}\sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon)dx_1 \ge \\ \ge \int\limits_\Omega f(\psi - Q_\varepsilon(H(\psi^+)+\psi^-) - u_\varepsilon)dx.\end{gathered} \tag{165}$$

From the fact, that $Q_\varepsilon \rightharpoonup 0$ as $\varepsilon \to 0$ weakly in $H^1(\Omega, \Gamma_2)$, we have

$$\lim_{\varepsilon\to 0}\int\limits_\Omega f(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)dx = \int\limits_\Omega f(\psi - u_0)dx, \tag{166}$$

$$\lim_{\varepsilon\to 0} \nabla\psi\nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)dx = \int\limits_{\varepsilon\to 0} \nabla\psi\nabla(\psi - u_0)dx. \tag{167}$$

Lemma 7.1 implies that

$$\lim_{\varepsilon\to 0}\int\limits_\Omega \nabla(Q_\varepsilon - W_\varepsilon(H(\psi^+) + \psi^-))\nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)dx = 0. \tag{168}$$

Consider the remaining integrals in (165). Denote

$$\begin{aligned} I_\varepsilon \equiv &-\int\limits_\Omega \nabla(W_\varepsilon(H(\psi^+) + \psi^-))\nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)dx = \\ = &-\int\limits_\Omega \nabla W_\varepsilon \nabla\{(H(\psi^+) + \psi^-)(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)\}dx + \alpha_\varepsilon, \end{aligned}$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$.
It is easy to see that

$$\begin{aligned} I_\varepsilon = &-\int\limits_\Omega \nabla W_\varepsilon \nabla\{(H(\psi^+) + \psi^-)(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)\}dx \\ = &-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(T^j_{\varepsilon/4})^+\setminus\overline{T^j_{a_\varepsilon}}} \nabla w^j_\varepsilon \nabla\{(H(\psi^+) + \psi^-)(\psi - q^j_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon)\}dx + \widetilde{\alpha_\varepsilon} \\ = &-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}\cap\{x_2>0\}} \partial_\nu w^j_\varepsilon(H(\psi^+) + \psi^-)(\psi - u_\varepsilon)ds \\ &-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}\cap\{x_2>0\}} \partial_\nu w^j_\varepsilon(H(\psi^+) + \psi^-)(\psi^+ - H(\psi^+) - u_\varepsilon)ds + \widetilde{\alpha_\varepsilon}, \end{aligned} \tag{169}$$

where $\widetilde{\alpha_\varepsilon} \to 0$, $\varepsilon \to 0$.
Since $\partial_\nu w^j_\varepsilon\Big|_{\partial T^j_{\varepsilon/4}} = \frac{4}{\varepsilon\ln(4a_\varepsilon/\varepsilon)} = \frac{4}{-\alpha^2+\varepsilon\ln(4C_0)}$, using the results of [27], we derive

$$\begin{aligned} &-\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}\cap\{x_2>0\}} \partial_\nu w^j_\varepsilon(H(\psi^+) + \psi^-)(\psi - u_\varepsilon)ds \\ &= \lim_{\varepsilon\to 0}\frac{4}{\alpha^2 - \varepsilon\ln(4C_0)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}\cap\{x_2>0\}} (H(\psi^+) + \psi^-)(\psi - u_\varepsilon)ds \\ &= \frac{\pi}{\alpha^2}\int\limits_{\Gamma_2}(H(\psi^+) + \psi^-)(\psi - u_0)ds. \end{aligned} \tag{170}$$

Let us find the limit of the expression

$$
\begin{aligned}
&-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}\cap\{x_2>0\}} \partial_\nu w^j_\varepsilon (H(\psi^+)+\psi^-)(\psi^+-H(\psi^+)-u_\varepsilon)ds \\
&\qquad + e^{\frac{\alpha^2}{\varepsilon}}\int\limits_{l_\varepsilon}\sigma(\psi^+-H(\psi^+))(\psi^+-H(\psi^+)-u_\varepsilon)dx_1 \\
&\quad = \sum_{j\in\Upsilon_\varepsilon}\frac{(\alpha^2C_0)^{-1}e^{\alpha^2/\varepsilon}}{1-\varepsilon\alpha^{-2}\ln(4C_0)}\int\limits_{\partial T^j_{a_\varepsilon}\cap\{x_2>0\}}(H(\psi^+)+\psi^-)(\psi^+-H(\psi^+)-u_\varepsilon)ds \\
&\qquad + e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon}\sigma(\psi^+-H(\psi^+))(\psi^+-H(\psi^+)-u_\varepsilon)dx_1 \\
&\quad = e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon}\sigma(\psi^+-H(\psi^+))(\psi^+-H(\psi^+)-u_\varepsilon)dx_1 \\
&\qquad -\frac{e^{\alpha^2/\varepsilon}}{\alpha^2C_0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}\cap\{x_2>0\}}(H(\psi^+)+\psi^-)(\psi^+-H(\psi^+)-u_\varepsilon)ds+\widehat{\alpha}_\varepsilon \\
&\quad \equiv D_\varepsilon+\widehat{\alpha}_\varepsilon,
\end{aligned}
\tag{171}
$$

where $\widehat{\alpha}_\varepsilon \to 0$, $\varepsilon\to 0$.

To conclude the proof we will estimate the limit of D_ε. We have

$$
\begin{aligned}
D_\varepsilon = &\left\{\frac{\pi}{2l_0\alpha^2C_0}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon}(H(\psi^+)+\psi^-)(\psi^+-H(\psi)-u_\varepsilon)dx_1 - \right.\\
&\left. -\frac{e^{\alpha^2/\varepsilon}}{\alpha^2C_0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}\cap\{x_2>0\}}(H(\psi^+)+\psi^-)(\psi^+-H(\psi^+)-u_\varepsilon)ds\right\}+ \\
&+e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon}\{\sigma(\psi^+-H(\psi^+))-\frac{\pi}{2l_0\alpha^2C_0}H(\psi^+)\}(\psi^+-H(\psi^+)-u_\varepsilon)dx_1- \\
&-\frac{\pi}{2l_0\alpha^2C_0}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon}\psi^-(\psi^+-H(\psi)-u_\varepsilon)dx_1 = \mathcal{J}^1_\varepsilon+\mathcal{J}^2_\varepsilon+\mathcal{J}^3_\varepsilon
\end{aligned}
\tag{172}
$$

Lemma 7.2 implies that

$$|\mathcal{J}^1_\varepsilon| \le K\sqrt{\varepsilon}. \tag{173}$$

Then $\mathcal{J}^2_\varepsilon$ vanishes due to equation (25). By using that $u_\varepsilon\ge 0$, $\psi^-\le 0$ on l_ε and the fact that $\psi^-(\psi^+-H(\psi^+))\equiv 0$ we have

$$\mathcal{J}^3_\varepsilon \le 0. \tag{174}$$

Hence, we have that $\lim\limits_{\varepsilon\to 0} D_\varepsilon \le 0$ and

$$\lim_{\varepsilon\to 0}\left(e^{\frac{\alpha^2}{\varepsilon}}\int\limits_{l_\varepsilon}\sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon)dx_1 + I_\varepsilon\right) \le$$
$$\le \frac{\pi}{\alpha^2}\int\limits_{\Gamma_2}(H(\psi^+) + \psi^-)(\psi - u_0)ds. \tag{175}$$

Therefore, from (164)-(175) we conclude that $u_0 \in H^1(\Omega, \Gamma_2)$ satisfies the following inequality

$$\int\limits_\Omega \nabla\psi\nabla(\psi - u_0)dx + \frac{\pi}{\alpha^2}\int\limits_{\Gamma_1}(H(\psi^+) + \psi^-)(\psi - u_0)dx_1 \ge \int\limits_\Omega f(\psi - u_0)dx, \tag{176}$$

for any $\psi \in H^1(\Omega, \Gamma_2)$, where $H(u)$ satisfies the functional equation (25). This concludes the proof. □

Acknowledgments

The research of J.I. Díaz and D. Gómez-Castro was partially supported by the projects ref. MTM 2014-57113-P and MTM2017-85449-P of the DGISPI (Spain).

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