Linearized stability for degenerate and singular semilinear and quasilinear parabolic problems: the linearized singular equations

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Abstract

We study some linear eigenvalue problems for the Laplacian operator with singular absorption or/and source coefficients arising in the linearization around positive solutions to some quasilinear degenerate parabolic equations and singular semilinear parabolic problems as well. We show that the linearization process applies even if the coefficients behave singularly with the distance to the boundary to the exponent two. This improves previous references in the literature. Applications to the above mentioned nonlinear problems are also presented.

Dedicated to Ioan I. Vrabie: a great mathematician and a great person

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1 Introduction

In this paper we study some linear eigenvalue problems with singular coefficients arising in the linearization around positive solutions to some quasilinear degenerate parabolic equations and singular semilinear parabolic problems as well.

More precisely, we consider problems of the form

$$\begin{cases} -\Delta w + c(x)w = \lambda b(x)w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $b(x)$ and $c(x)$ are unbounded coefficients going to infinity close to the boundary. As we will explain in what follows, the interesting examples arising when linearizing singular problems are (modulo positive constants)

$$(P_\pm) \equiv \begin{cases} -\Delta w \pm \frac{kw}{d(x)^\beta} = \frac{\lambda w}{d(x)^\gamma} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $0 \leq \beta, \gamma \leq 2$, $k > 0$ and $d(x) = d(x, \partial\Omega)$. In fact, the exact value of the coefficient k is not too relevant except for the limit case $\gamma = 2$, so in the other cases we shall assume $k = 1$.

These problems were studied by many authors in the last thirty years and many references will be indicated below. In particular, the motivation to study problem (1), in the paper by Bertsch and Rostamian [10], was to obtain linearized stability results for positive solutions to the degenerate quasilinear parabolic problem

$$\begin{cases} \beta(u)_t - \Delta u = f(u) & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega. \end{cases} \tag{3}$$

Here $\beta(s)$ is smooth with $\beta(s) \geq 0$ for $s > 0$, $\beta(0) = 0$, $\beta'(0) = +\infty$ and $\beta'(s) > 0$ for $s > 0$. Moreover $f(s)$ can be either a smooth function, with $f(0) = 0$ and such that $f \circ \beta^{-1}$ is locally Lipschitz continuous for $s \geq 0$, as for instance $\beta(s) = s^{1/m}$, $m > 1$ and $f(s) = s^{p/m}$ with $1 < p < m$ already considered in [10], or a singular function as for instance $\beta(s) = s^{1/m}$, $m > 1$ and $f(s) = s^{p/m}$ with $-m < p < m$ already considered in the literature (see references in Remark 4.1 below). We point out that the results in [10] are obtained for classical solutions (at least $C^{2,\delta}(\overline{\Omega})$, $0 < \delta < 1$) such that not only $u > 0$ in Ω but also

$$\frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega, \tag{4}$$

where n denotes the outward normal unit vector (i.e., that are interior points of the positive cone in $C_0^1(\overline{\Omega})$). Solutions $u > 0$ in Ω such that

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{5}$$

or with compact support in Ω raise interesting problems (see section 5 in [10] and below). Among the many improvements of the results of [10] we mention specially the papers by Brezis and Marcus [14] and Brezis, Marcus and Shafrir [15].

In what follows, we shall call "flat solution" to any solution of the corresponding partial differential equation such that $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ and $u > 0$ in Ω.

If $\overline{u} > 0$ is a stationary solution to (3), the corresponding linearized parabolic problem can be rewritten as

$$\beta'(\overline{u})w_t - \Delta w - f'(\overline{u})w = 0$$

or equivalently as

$$w_t - \frac{1}{\beta'(\overline{u})}(\Delta w + f'(\overline{u})w) = 0$$

since $\beta' > 0$. The associated linear eigenvalue problem is

$$\begin{cases} -\Delta w - f'(\overline{u})w = \lambda\beta'(\overline{u})w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

which is a problem of type (1).

In [10] the authors obtained under suitable assumptions some interesting results concerning existence and properties of eigenvalues for (1) by working in the usual Sobolev space $H_0^1(\Omega)$ and the weighted Sobolev space $H_0^1(\Omega, b)$. Then they are applied in order to prove (in a nontrivial way) linearized stability for positive stationary solutions to (3) in the sense that the sign of the first eigenvalue gives the asymptotic stability (or instability) of the solution.

Linear eigenvalue problems as (1) also arise when studying linearized stability for positive solutions to the semilinear singular equation

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{7}$$

corresponding to $\beta(s) = s$ but where now $f : (0, \infty) \to \mathbb{R}$ is a smooth function such that $f(s) \underset{s \searrow 0}{\to} +\infty$. Two model problems are $f(s) = s^{-\alpha}$ and $f(s) = -s^{-\alpha}$ with $\alpha > 0$ (see [58], [62], [63]). In the case of $f(s) = -s^{-\alpha}$ with $\alpha > 0$ it may arise solutions with compact support and then the equation is only well-defined by replacing $f(s) = -s^{-\alpha}$ by $f(s) = -s^{-\alpha}\chi_{\{s>0\}}$, see [76], [29], [24] and their references. Moreover, since there is global quenching in finite time the stability question we shall consider in this paper is only relevant for perturbations of the form $f(s) = -s^{-\alpha}\chi_{\{s>0\}} + \gamma s^{\theta}$ for some $\theta \in \mathbb{R}$, see, e.g., [28], [29], [63], [59] and [24]. One of the main goals of this paper is to see if the linearization process is well defined for problems in which there are compact support solutions or when the linearization is applied near a *flat solution*.

Problem (1) was studied in [63] for a much more general class of problems including second order linear differential operators not necessarily in divergence form and rather general nonlinear terms $f(x, u)$ but this time not in the framework of Sobolev spaces but in Hölder continuous function spaces and $C_0^1(\overline{\Omega})$. Most of the well-known theorems for continuous (on $\overline{\Omega}$) coefficients b and c were extended to this more general situation and then it was proved that linearized stability implies stability in the sense of Lyapunov (something that we shall consider in a companion but separate paper [39]). Applications to a variety of singular problems were given in [64] (see also [62]). All results in [63] are restricted to the case $0 < \alpha < 1$ for the above model example and to solutions $u > 0$ satisfying (4) as well. This means that the case $\alpha \geq 1$ (where stationary solutions to (7) are not in $C_0^1(\overline{\Omega})$ but only in $C^{\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$) is excluded. We point out that this low regularity of the gradient of solutions occurs in a large class of nonlinear partial differential equations (see, e.g., [22], [45] and [8]). But it is also useful to have linearized stability results for $0 < \alpha < 1$ (and even $-1 < \alpha < 1$) in the Hilbert space $H_0^1(\Omega)$, now we have a sequence of eigenvalues. This is useful if we want to show that

$\lambda = 0$ is not an eigenvalue of the linearized operator, which allows to apply the Implicit Function Theorem in [63] to functions at the interior of the positive cone in $C_0^1(\overline{\Omega})$ (see [40], [41]). The variational characterization is a useful tool when applying this kind of results. Moreover, we emphasize that results in [10] are obtained for stationary solutions $\overline{u} \in C^{2,\gamma}(\overline{\Omega})$ with $\gamma \in [0,1)$, a condition which is *never* satisfied for stationary solutions to (7) when $f(u)$ is singular. In this sense, we improve *all* results in [10].

An interesting application of the linearization procedure along a singular solution of an ODE associated to some singular BVP can be found in [13]. A nice application of the results in [63] was considered in [30] in order to study the existence and smoothness of the solution branch to some singular problems with super exponential growth in $\mathbb{R}^2$ by bifurcation arguments.

Most of the results on the resulting linear problem after linearization in this paper have been extended to the analogous quasilinear problem for the p-Laplacian in [50]. But the linearization process, such as it is presented here, is not directly applicable to the linearization of p-Laplacian type quasilinear equations since the diffusion coefficients are extremely singular.

The general idea of linearization, or linear approximation, plays a fundamental role in all what concerns differential calculus and in many more places in mathematics. In the field of ordinary differential equations the basic results by Poincaré and Lyapunov are, together with the use of Lyapunov functions, the main tool in order to study stability in the finite-dimensional case. These ideas were extended to the infinite-dimensional situation not only for nonlinear parabolic equations ([78], [73]) but also for other relevant nonlinear evolution equations as, e.g., Navier-Stokes system [79], [72], and the classical bifurcation problems of Bénard and Taylor in Fluid Mechanics [67], or very relevant problems arising in magneto-hydrodynamics [70]. A general theory was elaborated by Henry [61], actually the results in [63] (see [64]) are obtained as an application of [61]. We also mention that the linearization process was also applied to several problems in combustion theory (see, e.g., the many references presented in the monograph [52]). See [2], [65] for related results concerning linearization of some other sublinear problems.

In this paper we shall only consider the linearization process in a formal way, paying special attention to the singular linear problems originated in such process. So, we shall use the expression that a stationary solution $\overline{u}(x)$ of a nonlinear parabolic equation containing nonlinear terms as $\beta(u(x,t))_t$ and $f(u(x,t))$ is *linearly stable* if the *first eigenvalue* λ_1 of the associated linear problem (containing now terms of the form $\beta'(\overline{u})$ and $f'(\overline{u})$) [as explained by means of (6)] is positive. The techniques needed to prove that any *linearly stable* solution $\overline{u}(x)$ is stable in the Lyapunov sense have a very different nature and will be the object of a separate paper by the authors [39]. See also [40], [41], [42] for other stability results concerning stationary *ground solutions*.

In section 2 we study the linear problem (1) giving a more complete and unified version of the results in [10]. Even the simple model example given above for singular problems ($f(s) = s^{-\alpha}$ with $0 < \alpha < 1$) does not fall under the scope of [10]. Our results allow to deal with the case not considered before $\alpha > 1$. We devote some attention to the "critical case" $\beta = 2$ and / or $\gamma = 2$ below corresponding to $\alpha > 1$. The results concerning the boundary behaviour of positive eigenfunctions that are based in recent work by the first author ([31], [32]: see also [35], [34] and [36]) are new. Section 3 deals with applications to semilinear singular equations studied in [58], [62], [63], [64]). Section 4 is devoted to applications to stationary solutions to degenerate quasilinear parabolic equations studied in [10] and some linearized stability results are improved. Some remarks on the cases of "flat" positive solutions and compact support solutions are developed. Finally, some variants of methods used in section 2 are given in an Appendix at the end in order to show the flexibility of this kind of arguments. In particular, we use a version of the Hardy's inequality in [66] to improve an argument used in [1].

2 The singular linearized eigenvalue problem

In this section we study the linear eigenvalue problem (1)

$$\begin{cases} -\Delta w - c(x)w = \lambda b(x)w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. This problem was studied in [10], where the existence of an infinite sequence of eigenvalues was proved under the assumptions

$$b, c \in L^\infty_{loc}(\Omega),\ b(x) \geq b_0 > 0, \tag{9}$$

$$|c(x)| \leq kb(x),\ \ k > 0 \tag{10}$$

where

$$d(x) := d(x, \partial\Omega). \tag{11}$$

a function which plays an important role in all this theory. As a matter of fact, in some results of [10] they assume the additional condition

$$b(x)d(x)^2 \underset{d(x)\to 0}{\longrightarrow} 0, \tag{12}$$

but, as we shall indicate below, this assumption is not needed in some cases. We point out that our results could be also stated for the general formulation by assumming conditions of the type

$$\begin{cases} 0 < \liminf |c(x)|\, d(x)^\beta \leq \limsup |c(x)|\, d(x)^\beta < +\infty \\ 0 < \liminf b(x)d(x)^\gamma \leq \limsup b(x)d(x)^\gamma < +\infty \end{cases} \tag{13}$$

for some $0 \leq \beta, \gamma \leq 2$, but we shall not follow this presentation. Concerning the constant k in (10) we shall see that the exact value of the coefficient k is not too relevant except for the limit case $\gamma = 2$ (see subsection 2.3), so in the other cases we shall assume $k = 1$.

It is claimed in [10] that the exponent 2 "is the critical growth condition for b and c". We can see immediately that assumption (10) is only satisfied if $\beta < \gamma$. This means that $\gamma < \beta$, which is precisely the condition arising in our intended applications (in the model example $\beta = 1 + \alpha > 0$, $0 < \alpha < 1$, and $\gamma = 0$) is not included in [10].

2.1 Case $0 \leq \beta \leq 2$ and $\gamma = 0$

In order to illustrate the method of proof we start by showing that the case $\beta = 2$ in (P_+) is not actually "critical". We deal first with the case $\gamma = 0$.

We first prove some auxiliary results. In all which follows we use the notation

$$\|u\| = \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \right)^{1/2}.$$

Lemma 2.1 . *For any $h \in H^{-1}(\Omega)$, there is a unique solution $w \in H_0^1(\Omega)$ of the linear problem*

$$\begin{cases} -\Delta w + \frac{w}{d(x)^\beta} = h \in H^{-1}(\Omega) \\ \qquad w = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{14}$$

for $0 < \beta \leq 2$. Moreover, if $h \geq 0$, then $w \geq 0$.

Proof. For $\beta = 2$ the associated bilinear form in $H_0^1(\Omega)$ is well-defined, continuous and coercive. Indeed, for

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega \frac{uv}{d(x)^2}$$

we have, by using Hardy's inequality (see, e.g., [75])

$$a(u,v) \leq \|u\| \, \|v\| + \int_\Omega \left| \frac{u}{d(x)} \right| \left| \frac{v}{d(x)} \right| \leq (1+C) \, \|u\| \, \|v\| \,,$$

for some $C > 0$. Moreover, from the weak maximum principle it follows that $w \geq 0$ if $h \geq 0$. The proof is similar if $0 < \beta < 2$.■

Lemma 2.2 . *The solution operator $P : H^{-1}(\Omega) \to H_0^1(\Omega)$ defined by $w = Ph$ is continuous, and $Ph \geq 0$ if $h \geq 0$. Then the linear operator $T = i \circ P \circ j$, where $j : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is the standard embedding and $i : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact injection, is a self-adjoint compact linear operator $T : L^2(\Omega) \to L^2(\Omega)$.*

Proof. The first part is contained in Lemma 2.1. The second one follows from the continuity of j and Rellich's theorem. It is very easy to show that T is self-adjoint.■

Theorem 2.1 . *If $0 \leq \beta \leq 2$ and $\gamma = 0$ there is an infinite sequence $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$ of eigenvalues to (P_+) such that $\lim \lambda_n = +\infty$, with eigenfunctions $\varphi_n \in H_0^1(\Omega)$. The first eigenvalue $\lambda_1 > 0$ has an associated eigenfunction $\varphi_1 \geq 0$.*

Proof. It is clear that λ is an eigenvalue with eigenfunction u if and only if $u = \lambda Tu$. The existence of the infinite sequence λ_n of eigenvalues such that $\lim_{n\to+\infty} \lambda_n = +\infty$ follows from the well-known spectral theory for compact self-adjoint linear operators in Hilbert spaces. From the variational characterization

$$\lambda_1 = \inf_{w\neq 0} \frac{\int_\Omega |\nabla w|^2 + \int_\Omega \frac{w^2}{d(x)^\beta}}{\int_\Omega w^2},$$

for $0 \leq \beta \leq 2$, it follows $\lambda_1 > 0$. If u_1 is an associated minimizing function for λ_1, $|u_1|$ is also suitable and hence $u_1 \geq 0$.$\blacksquare$

Remark 2.1 *Much more general existence results can be obtained outside of the energy space $H_0^1(\Omega)$ when, for instance, it is merely assumed that $h \in L^1(\Omega, d)$ even for $\beta > 2$ (see e.g. [35], [34] and [36] and its many references).*

2.2 Case $0 \leq \beta \leq 2$ and $0 \leq \gamma < 2$

Next we deal with the case $\gamma > 0$, more precisely we study the case $0 < \gamma < \beta \leq 2$. The case $0 < \beta < \gamma < 2$ follows in a completely similar way, and the "critical" case $\gamma = 2$ will be considered at the end of the section.

We study the eigenvalue problem (corresponding to the problem (P_+) in the formulation $(P_\pm)$)

$$\begin{cases} -\Delta w + \frac{w}{d(x)^\beta} = \frac{\lambda w}{d(x)^\gamma} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $0 < \gamma < \beta < 2$. The case $0 < \gamma < \beta = 2$ is, once again, very similar. The same for $0 < \beta < \gamma < 2$.

Now we should use the weighted $L^2(\Omega, b)$ space of functions u such that

$$\int_\Omega u^2(x) b(x) dx < +\infty.$$

We need an auxiliary result in [10], namely

Lemma 2.3 . *If $b(x) = \frac{1}{d(x)^\delta}$ with $0 < \delta < 2$, then the embedding $i : H_0^1(\Omega) \hookrightarrow L^2(\Omega, b)$ is compact.*$\blacksquare$

Now we "factorize" the operator T in a similar way:

$$L^2(\Omega, b) \xrightarrow{F} H^{-1}(\Omega) \xrightarrow{P} H_0^1(\Omega) \xrightarrow{i} L^2(\Omega, b)$$

where $b(x) = \frac{1}{d(x)^\delta}$ for some $0 < \delta < 2$ and $F(w) = w/d(x)^\gamma$.

First we prove the

Lemma 2.4 . *The mapping* $F : L^2(\Omega, b) \to H^{-1}(\Omega)$, *where* $b(x) = \frac{1}{d(x)^\gamma}$, *defined as* $F(w) = \frac{w}{d(x)^\gamma}$, *is linear continuous for any* $0 < \gamma < 2$.

Proof. We should show first that F is well defined, i.e., that if $w \in L^2(\Omega, b)$ then $\frac{w}{d(x)^\gamma} \in H^{-1}(\Omega)$. Indeed, if $z \in H_0^1(\Omega)$ we have

$$\left| \left\langle \frac{w}{d(x)^\gamma}, z \right\rangle \right| = \left| \int_\Omega \frac{wz}{d(x)^\gamma} \right| \leq \int_\Omega \left| \frac{z}{d(x)} \right| \left| wd(x)^{1-\gamma} \right|$$

and since we have

$$\left\| wd(x)^{1-\gamma} \right\|_{L^2(\Omega)}^2 = \int_\Omega w^2 d(x)^{2(1-\gamma)} = \int_\Omega \frac{w^2}{d(x)^\gamma} d(x)^{2-\gamma} \leq C \left\| w \right\|_{L^2(\Omega,b)}^2$$

for some $C > 0$, by Hardy inequality

$$\left| \left\langle \frac{w}{d(x)^\gamma}, z \right\rangle \right| \leq C \left\| w \right\|_{L^2(\Omega,b)} \left\| z \right\|,$$

which gives the result.■

As above, the linear operator $T : L^2(\Omega, b) \to L^2(\Omega, b)$ is compact (by Lemma 2.3) and self-adjoint and we reason as for Theorem 2.1. We have then proved the following

Theorem 2.2 *If* $0 \leq \gamma < 2$ *and* $\beta \in [0, 2]$, *there is an infinite sequence* $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$ *of eigenvalues to* (P_+) *such that* $\lim \lambda_n = +\infty$, *with eigenfunctions* $\varphi_n \in H_0^1(\Omega)$. *The first eigenvalue* $\lambda_1 > 0$ *has an associated eigenfunction* $\varphi_1 \geq 0$.■

It is well known that if both the domain and the coefficients b and c are smooth enough then the first eigenvalue λ_1 is simple and has an eigenfunction $\varphi_1 > 0$ with $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$; moreover, λ_1 is the only eigenvalue with this property. These results follow in some cases from the classical version of the Krein-Rutman theorem applied to the positive cone in $C_0^1(\overline{\Omega})$ by invoking the Strong Maximum Principle, now the eigenfunction φ_1 belongs to the interior of this cone. When the positive cone of the corresponding space has an empty interior (as for $L^p(\Omega)$, $1 < p < +\infty$) an alternative version of the Krein-Rutman theorem holds (see [23]) and can be applied: in this case φ_1 is a quasi-interior point of the cone, i.e. $\varphi_1 > 0$ a.e. in Ω (see [55] for an application of this idea when $b, c \in L^r(\Omega)$, $r > N/2$).

In [10] the authors prove that if $b, c \in C^\delta(\Omega)$ for some $0 < \delta < 1, \gamma = 1$, $\beta < 2$, then $\varphi_n \in C^{2,\delta}(\Omega) \cap C^1(\overline{\Omega})$. Again, by using an extension of the Strong Maximum Principle, they obtain that $\varphi_1 > 0$ in Ω, $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$. These questions remain open in [10] not only for the "critical" case $\beta = 2$ but also for $1 < \gamma \leq 2$. We greatly improve all these results here.

The problem was settled for $0 \le \beta, \gamma < 2$ in [63] in the framework of classical solutions by showing that $\varphi_1 \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$ for some $0 < \delta < 1$, $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$ and λ_1 is a simple eigenvalue by applying an extension of the classical Strong Maximum Principle (see also [80]) and Krein-Rutman theorem.

We can try to apply both versions of the Krein-Rutman theorem in our case if (some suitable version of) the theorem holds (see [21] for this kind of results). However, we prefer to follow a different approach. First we state that $\varphi_1 \in L^\infty(\Omega)$ and this will allow to show that $\varphi_1 > 0$ and its interior regularity.

Theorem 2.3 *The eigenfunction φ_1 to $(P_\pm)$ (corresponding to the first eigenvalue λ_1) is bounded for any $0 \le \gamma < 2$ and $\beta \in [0, 2]$. Moreover, for $\beta = 2$ and $0 \le \gamma < 2$, any eigenfunction φ_n to (P_+) is a flat solution of the equation.*

Proof. The proof consists in a variant of the general iterative technique presented in [51] (see also [49] for another application of these arguments). Actually this is a particular version of more general results, namely Theorem 2.3 in [50] for the p-Laplacian. That the eigenfunctions φ_n to (P_+) are flat solutions was shown in [32] for $\beta = 2$ and $\gamma = 0$ but the same method of proof applies if $0 \le \gamma < 2$.■

Remark 2.2 *If $\gamma = 2$ the eigenfunction φ_1 to $(P_\pm)$ is unbounded (see [26]).*

Corollary 2.4 *Under the conditions of Theorem 2.2, if $\varphi_1 \ge 0$ is an eigenfunction to $(P_\pm)$ corresponding to λ_1, then $\varphi_1 > 0$ and $\varphi_1 \in W^{2,p}_{loc}(\Omega)$ for any $p \in (1, \infty)$ and $\varphi_1 \in C^{1,\delta}_{loc}(\Omega)$ for any $0 < \delta < 1$.*

Proof. Since φ_1 is bounded we can apply the interior L^p regularity (see [60]) and then the conclusions follow from well-known embedding theorems and Bony's Maximum Principle [11].■

Theorem 2.5 *Under the conditions of Theorem 2.2, the first eigenvalue λ_1 to $(P_\pm)$ is simple and it is the only eigenvalue having positive eigenfunction.*

Proof. The second part follows immediately from the fact that eigenfunctions corresponding to different eigenvalues of a self-adjoint operator are orthogonal. For the first one can reason as in [5] following the ideas in [56]. We sketch the proof for the reader's convenience. The first eigenvalue is given by

$$\lambda_1 = \inf_{\int_\Omega \frac{w^2}{d(x)^\gamma} = 1} \int_\Omega |\nabla w|^2 + \int_\Omega \frac{w^2}{d(x)^\beta}.$$

Assume that $u, v > 0$ are eigenfunctions associated to λ_1 such that $\int_\Omega \frac{u^2}{d(x)^\gamma} = \int_\Omega \frac{v^2}{d(x)^\gamma} = 1$. Consider the function $w = \eta^{1/2}$, where $\eta = (u^2 + v^2)/2$. Now w is a test function since

$$\int_\Omega \frac{w^2}{d(x)^\gamma} = \frac{1}{2}\left(\int_\Omega \frac{u^2}{d(x)^\gamma} + \int_\Omega \frac{v^2}{d(x)^\gamma}\right) = 1.$$

We have, by convexity,

$$|\nabla w|^2 = \eta^{-1}\left|\tfrac{1}{2}(u\nabla v + v\nabla u)\right|^2$$
$$= \eta\left|t(x)\tfrac{\nabla u}{u} + (1-t(x)\tfrac{\nabla v}{v}\right|^2 \le \eta\left(t(x)\left|\tfrac{\nabla u}{u}\right|^2 + (1-t(x)\left|\tfrac{\nabla v}{v}\right|^2)\right)$$
$$= \tfrac{1}{2}\left(u^2\left|\tfrac{\nabla u}{u}\right|^2 + v^2\left|\tfrac{\nabla v}{v}\right|^2\right) = \tfrac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right),$$

where

$$t(x) = \frac{u^2}{u^2+v^2}.$$

Hence

$$\int_\Omega |\nabla w|^2 \le \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2\right)$$

and equality should follow since u and v are solutions. Then $\frac{\nabla u}{u} = \frac{\nabla v}{v}$ and $u = Cv$ for some $C > 0$.■

Remark 2.3 *The results in [5] and [56] are actually valid for the p-Laplacian operator with $p > 1$: see [17] (for $p = 2$) and [46] for related ideas. The same argument is applicable to "sublinear" problems for the p-Laplacian operator giving a simple proof of the uniqueness of positive solutions: see [49] for previous results using the L^∞ estimate as in Theorem 2.5.*

Corollary 2.4 yields the best regularity of φ_1 if $0 < \beta, \gamma < 2$. Indeed, since λ_1 is the only eigenvalue having an eigenfunction $\varphi_1 > 0$, it should coincide with the principal eigenvalue obtained in [63] (see also [63] for more details on these points). The results in [63] are only valid if $\beta < 2$. We have thus proved the

Corollary 2.6 *Under the conditions of Theorem 2.2, if λ_1 is the first eigenvalue for $(P_\pm)$ with eigenfunction $\varphi_1 > 0$ for $0 < \beta, \gamma < 2$, then $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$ and $\varphi_1 \in C^{2,\delta}(\Omega) \cap C^{1,\delta}(\overline{\Omega})$ for some $0 < \delta < 1$.*

Now we study problem (P_-), namely

$$\begin{cases} -\Delta w - \frac{w}{d(x)^\beta} = \frac{\lambda w}{d(x)^\gamma} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

We recall the well-known variational characterization of the above first eigenvalue for (P_+)

$$\lambda_1 = \inf_{w\neq 0 \; w\in L^2(\Omega,b)} \frac{\int_\Omega |\nabla w|^2 + \int_\Omega \frac{w^2}{d(x)^\beta}}{\int_\Omega \frac{w^2}{d(x)^\gamma}}.$$

We consider the case $0 < \gamma, \beta < 2$. We use a fixed point argument and apply the above result for (P_+). To any $\lambda \in \mathbb{R}$ fixed, with $\lambda > 0$, we associate the eigenvalue problem

$$\begin{cases} -\Delta w = \mu(\frac{w}{d(x)^\beta} + \frac{\lambda w}{d(x)^\gamma}) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

i) We look for positive eigenvalues, $\lambda > 0$ of (P_-). We study (17) where λ is a fixed coefficient and μ plays the role of a eigenvalue parameter. By applying Theorem 2.5 to (17) working in the "bigger" space $L^2(\Omega, b)$, where b depends on β and γ (it is very easy to see that $\beta > \gamma$ implies $L^2(\Omega, d^{-\beta}) \subset L^2(\Omega, d^{-\gamma})$), we find a first eigenvalue $\mu_1 = r(\lambda) > 0$ of (17) with a positive eigenfunction $\psi > 0$ having the variational characterization

$$r(\lambda) = \inf_{w \neq 0 \ w \in L^2(\Omega,b)} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega \frac{w^2}{d(x)^\beta} + \lambda \int_\Omega \frac{w^2}{d(x)^\gamma}}.$$

We know (and see) that $r(\lambda)$ is a continuous and monotone (decreasing) function of λ. Moreover we have

$$r(\lambda) \leq \frac{1}{\lambda} \inf_{w \neq 0 \ w \in L^2(\Omega,b)} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega \frac{w^2}{d(x)^\beta}}$$

which implies $r(\lambda) \underset{\lambda \to +\infty}{\to} 0$.

Hence the existence of a positive eigenvalue λ to (16) is equivalent to the existence of $\overline{\lambda} > 0$ such that $r(\overline{\lambda}) = 1$, and in turn this is equivalent to

$$r(0) = \inf_{w \neq 0 \ w \in L^2(\Omega,b)} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega \frac{w^2}{d(x)^\beta}} > 1.$$

ii) Next, we treat the case $\lambda < 0$ with the change of variable $\lambda \to -\lambda$ and now we have as associated eigenvalue problem

$$\begin{cases} -\Delta w + \frac{\lambda w}{d(x)^\gamma} = \nu \frac{w}{d(x)^\beta} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

where again λ is a fixed coefficient and ν is an eigenvalue parameter, and look for positive values $\overline{\lambda}$ such that $\nu(\overline{\lambda}) = 1$, where $\nu(\lambda) > 0$ is the first positive eigenvalue to (18) provided by Theorem 2.5. Now the variational characterization is

$$\nu(\lambda) = \inf_{w \neq 0 \ w \in L^2(\Omega,b)} \frac{\int_\Omega |\nabla w|^2 + \lambda \int_\Omega \frac{w^2}{d(x)^\gamma}}{\int_\Omega \frac{w^2}{d(x)^\beta}},$$

and $\nu(\lambda)$ is continuous and increasing in λ. Moreover $\nu(\lambda) \underset{\lambda \to +\infty}{\to} +\infty$. Hence there exists $\overline{\lambda} > 0$ such that $\nu(\overline{\lambda}) = 1$ if and only if

$$\nu(0) = \inf_{w \neq 0 \ w \in L^2(\Omega,b)} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega \frac{w^2}{d(x)^\beta}} < 1.$$

Notice that $\nu(0) = r(0)$.

We have then proved the following

Theorem 2.7 *The problem (P_{-}) for $0 < \gamma, \beta < 2$ has a first positive (resp. negative) eigenvalue $\lambda_1 > 0$ (resp. $\lambda_1 < 0$) with an associated positive eigenfunction if and only if $r(0) > 1$ (resp. $r(0) < 1$). If φ_1 is the associated eigenfunction, $\varphi_1 > 0$.*

At the best of our knowledge this result is completely new.

Remark 2.4 *Problem (P_{-}) for $\beta = 0$ and $\gamma = 2$ has received a great deal of attention in the recent years: see, e.g. [26], [27], [52]) and their references. In that case, if $c(x) = \frac{\mu}{d(x)^2}$ for some $\mu \leq 1/4$ and assuming for instance that Ω is convex, the similar statement to Lemma 2.1 requires the definition of a special Hilbert space $\mathcal{H}$ of norm*

$$\|u\|_{\mathcal{H}}^2 = \int_\Omega (|\nabla u|^2 \, dx - \frac{\mu u^2}{d(x)^2} + M u^2) dx$$

for some suitable $M > 0$. The case of $\mu \leq 0$ was considered in [31], [32], [35], [34] and [36] (see also their many references).

2.3 Case $\gamma = 2$

It remains to deal with the critical case $\gamma = 2$, with $0 \leq \beta \leq 2$ and both signs in (2). It is illustrative to start by recalling the results for the case $\gamma = 2$ and $\beta = 2$ which then (by an obvious change of notation in λ) reduces the problem to the study of nontrivial solutions of

$$\begin{cases} -\Delta w = \lambda \frac{w}{d(x)^2} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

That problem was considered previously by many authors in connection with the study of the best constant in the Hardy's inequality (see, e.g., [74], [14], [26], [27] and the exposition made in [52]). It is well-known (see e.g. [74]) that if Ω is convex then if we define

$$\mu(\Omega) := \inf_{w \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega \frac{w^2}{d(x)^2}} \tag{20}$$

then $\mu(\Omega) = 1/4$, $\lambda = 1/4$ is the infimum of the essential spectrum and problem (20) has no minimizer. Nevertheless, if $\mu(\Omega) < 1/4$ then there exists a $\lambda_{\mu(\Omega)} \in (0, 1/4)$ which is the first eigenvalue of the problem (19), and so there is a positive solution w of such problem (see Remark 3.2 of [14]).

Remark 2.5 *In the one-dimensional case $\Omega = (0,1)$, and by replacing $d(x)^2$ by $|x|^2$ in problem (19), it is easy to see ([10]) that $w(x) = -\sqrt{x} \lg x$ if $\lambda = 1/4$ and $w(x) = \sqrt{x} \sin(\omega_\lambda \lg x)$ with $\omega_\lambda = \sqrt{\lambda - 1/4}$ if $\lambda > 1/4$ are explicit solutions of the problem.*

These functions are not in $H_0^1(0,1)$ (they are bounded but their derivatives are not in $L^2(0,1)$) and so they were called "generalized eigenfunctions" in [10] (see also a related one-dimensional problem in [9]) . Nevertheless, it can be proved that by working with the notion of very weak solution such functions are well defined and belong to the weigthed space $H_0^1((0,1),d(x))$. See for this matter the study of L^1-eigenvalues made in [18].

From the above mentioned results, if $\gamma = 2$ we cannot always expect the existence of countably many eigenvalues λ_n of problem $(P_\pm)$ with $\lambda_n \to +\infty$ as $n \to +\infty$. Nevertheless, the existence of an eigenvalue λ can be proved under suitable additional conditions: as observed in [16], "lower order terms can reverse the situation".

Let us start by considering the limit case of problem (P_-), i.e.

$$\begin{cases} -\Delta u - \frac{ku}{d(x)^\beta} = \frac{\lambda u}{d(x)^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

for $k > 0$ and $\beta \in [0,2)$. That problem was considered in the papers by Brezis and Marcus [14] and Brezis, Marcus and Shafrir [15] in connection with the so called "improved Hardy inequality". Then the question of the best constant in such inequality becomes related to the consideration of the quantity

$$J_k^\Omega = \inf_{w\neq 0 \ w\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla w|^2 - k\int_\Omega \frac{w^2}{d(x)^\beta}}{\int_\Omega \frac{w^2}{d(x)^2}}, \tag{22}$$

for any $k \in \mathbb{R}$. Notice that if $b(x) = \frac{1}{d(x)^2}$ and we define $H_0^1(\Omega, b)$ as the Hilbert space with norm

$$\|u\|^2_{H_0^1(\Omega,b)} = \int_\Omega |\nabla u|^2\, dx + \int_\Omega u^2 b(x) dx.$$

then, using once again Hardy's inequality as above, we conclude that $\|u\|_{H_0^1(\Omega,b)}$ and $\|u\|_{H_0^1(\Omega)}$ are equivalent for the space $H_0^1(\Omega,b) = H_0^1(\Omega)$. In [14] it is shown that there exists a $k^* = k^*(\Omega)$ such that

i) $J_k^\Omega = 1/4$ for any $k \leq k^*$

ii) $J_k^\Omega < 1/4$ for any $k > k^*$

iii) if $k > k^*$ the infimum in (22) is achieved (by a positive function $w \in H_0^1(\Omega)$). The main argument of their proof is, as in [10], the method introduced in [16] to overcome the lack of compactness.

iv) if $k < k^*$ then the infimum in (22) is not achieved.

The study of the borderline case $k = k^*$ was the main goal of the paper [15]. Their main result can be particularized to our formulation and shows that, for any $\beta \in [0,2)$, the infimum in (22) is not achieved. Moreover, as a consequence of the estimates obtained in [14] and [15] the positive solution u of (21) is not a flat solution since $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$.

Remark 2.6 *There are some important generalizations of most of the results in this section concerning the case in which function $d(x)$ is replaced by*

$$d(x) = d(x, \Sigma_k)$$

where $\Sigma_k \subset \overline{\Omega}$ is a smooth compact manifold of co-dimension k, $0 \le k \le N-1$ (Σ_0 corresponds to a single point and an example of Σ_{N-1} is $\partial\Omega$). see. e.g. [52], [27], [53] and their references.

3 Applications to nonlinear problems: I. Linearized stability for singular semilinear parabolic problems

In this section we apply the results in the precedent one to obtain linearized stability results for positive solutions (actually for positive solutions satisfying condition (4) as well) of some semilinear elliptic singular problems. This problem was studied in [63] working in the space $C_0^1(\overline{\Omega})$, where the classical version of the Krein-Rutman theorem was used to prove the existence of a first principal eigenvalue. Then it was proved that linearized stability implies stability in the sense of Lyapunov (something which will be considered in [39]). Applications were given also in [64] and [62].

But these results were applicable to the model problem example

$$\begin{cases} -\Delta u = \frac{1}{u^\alpha} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

only for $0 < \alpha < 1$. And even in this case it is sometimes useful to have an infinite sequence of eigenvalues and the well-known variational characterization of the eigenvalues involving the Rayleigh quotient. For example, this is very useful when applying the Implicit Function Theorem at the interior of positive cone in $C_0^1(\overline{\Omega})$ in [63] to show the existence of smooth curves of solutions. See also [64], [62], [12], [40], [41]. We have the following results for (23): see, e.g., [58], [62], [63], [22], and [8].

Theorem 3.1 *If $0 < \alpha < 1$, there exists a unique solution $u \in C^2(\Omega) \cap C_0^{1,1-\alpha}(\overline{\Omega})$ such that $u > 0$ in Ω, $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. Moreover*

$$c_1 d(x) \le u(x) \le c_2 d(x) \tag{24}$$

for some $c_1, c_2 > 0$;

Theorem 3.2 *i) if $\alpha > 1$, there exists a unique solution $u \in C^2(\Omega) \cap C_0^{0,\frac{2}{1+\alpha}}(\overline{\Omega})$ such that $u > 0$ in Ω and*

$$c_1 d(x)^{\frac{2}{1+\alpha}} \le u(x) \le c_2 d(x)^{\frac{2}{1+\alpha}} \tag{25}$$

for some $c_1, c_2 > 0$; ii) if $\alpha = 1$, there exists a unique solution $u \in C^2(\Omega) \cap C_0^{0,\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$.

In what follows we give some applications of the previous theorems in section 2. Similar results working in $C_0^1(\overline{\Omega})$ were obtained in [64] and the above references.

We only deal with part i) in Theorem 3.2. Part ii) can be treated by using similar arguments.

Example 1. We consider the model example (23). The eigenvalue problem for the linearized operator at u can be written as

$$\begin{cases} -\Delta w + \frac{\alpha w}{u^{1+\alpha}} = \lambda w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

From (24) it follows immediately from the comparison results arising from the variational characterization that for some $k_1 > 0$

$$\lambda_1(-\Delta + \frac{k_1}{d^{1+\alpha}}) \leq \lambda_1(-\Delta + \frac{\alpha}{u^{1+\alpha}}).$$

It is enough to show that $\lambda_1(-\Delta + \frac{k_1}{d^{1+\alpha}}) > 0$, a particular case in Theorem 2.1 for $\beta = 1 + \alpha < 2$.We have thus proved

Theorem 3.3 *The unique solution $u > 0$ for (23) with $0 < \alpha < 1$ is linearly stable.*∎

If $\alpha > 1$ we use (25) and reduce in the same way the problem to show that

$$\lambda_1(-\Delta + \frac{k_2}{d^2}) > 0$$

where $k_2 > 0$. We apply again Theorem 2.1, this time with $\beta = 2$. We have thus proved

Theorem 3.4 *The unique solution $u > 0$ for (23) with $\alpha > 1$ is linearly stable.*∎

Remark 3.1 *The spectrum of the operator $-\Delta + \frac{k_1}{d^2}$ is very relevant for the study of Schroedinger solution with singular potentials (see [31], [41], [35], [34]).*

Example 2. We study now positive solutions of the problem

$$\begin{cases} -\Delta u + \frac{1}{u^\alpha} = \frac{\lambda}{u^\beta}\chi_{\{u>0\}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

where $0 < \alpha < \beta < 1$ (notice that due to the positivity of solutions the singular terms, as e.g. $\frac{\lambda}{u^\beta}$, do not need to be written otherwise, as$\frac{\lambda}{u^\beta}\chi_{\{u>0\}}$, which is needed for solutions with compact support). It was proved in [64] that there is a unique positive solution to (27) for any $\lambda > 0$. The linearized eigenvalue problem is

$$\begin{cases} -\Delta w + \frac{\lambda\beta w}{u^{\beta+1}} - \frac{\alpha w}{u^{1+\alpha}} = \mu w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

or

$$\begin{cases} -\Delta w + (\frac{\lambda\beta}{u^{\beta+1}} - \frac{\alpha}{u^{1+\alpha}})w = \mu w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

It is not difficult to show, by using the estimate $0 < u \leq \lambda^{\frac{1}{\beta-\alpha}}$ ([64]), that for the coefficient

$$(\frac{\lambda\beta}{u^{\beta+1}} - \frac{\alpha}{u^{1+\alpha}}) = \frac{\lambda\beta - \alpha u^{\beta-\alpha}}{u^{\beta+1}} > 0$$

and we apply Theorem 2.1. We have thus proved

Theorem 3.5 *The unique solution $u > 0$ for (27) with $0 < \alpha < \beta < 1$ is linearly stable.* ■

In these examples there exists a unique positive solution which is linearly stable. A more flexible way of using the above ideas which can be interesting when dealing with multiple positive solutions is the following.

Assume that we consider the general semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{30}$$

with f smooth and let $\overline{u} > 0$ (with $\frac{\partial \overline{u}}{\partial n} < 0$ on $\partial\Omega$) be a solution. The associated linearized problem is

$$\begin{cases} -\Delta w - f'(\overline{u})w = \mu w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

Assume that this linearized problem has a first eigenvalue μ_1 with positive (smooth) eigenfunction $\psi_1 > 0$, thus

$$\begin{cases} -\Delta \psi_1 - f'(\overline{u})\psi_1 = \mu_1 \psi_1 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

Multiplying (30) by ψ_1, (32) by $\overline{u}$ and integrating by parts on Ω using Green's formula gives

$$\int_\Omega \nabla\overline{u}\cdot\nabla\psi_1 - \int_\Omega f(\overline{u})\psi_1 = 0 = \int_\Omega \nabla\overline{u}\cdot\nabla\psi_1 - \int_\Omega f'(\overline{u})\overline{u}\psi_1 - \mu_1 \int_\Omega \overline{u}\psi_1,$$

and finally

$$\mu_1 = \frac{\int_\Omega \left[f(\overline{u}) - f'(\overline{u})\overline{u}\right]\psi_1}{\int_\Omega \overline{u}\psi_1}.$$

If $H(\overline{u}) = [f(\overline{u}) - f'(\overline{u})u] > 0$, $\mu_1 > 0$ and $\overline{u}$ is linearly stable. For Example 1, $H(u) = (1+\alpha)u^{-\alpha} > 0$. For Example 2, $H(u) = u^{-\beta}[\lambda(1+\beta) - (1+\alpha)u^{\beta-\alpha}] > 0$, using that $0 < u < \lambda^{1/(\beta-\alpha)}$ for any solution.

Notice that $H(u) > 0$ is just the assumption in the uniqueness theorem in [64]. The above computations are justified using (24) and (25).

Example 3. We consider again (27) but this time with $0 < \beta < \alpha < 1$. We have

$$H(u) = u^{-\beta}[\lambda(1+\beta)u^{\alpha-\beta} - (1+\alpha)] \leq 0$$

if $0 < u \leq \left(\frac{1+\alpha}{\lambda(1+\beta)}\right)^{\frac{1}{\alpha-\beta}}$. This means that solutions satisfying this estimate are linearly unstable (if they exist !). Indeed, the situation is now more delicate. In the one-dimensional case it was proved in [43] (extending the previous paper [37]: see also [44]) the existence of an upper branch of positive solutions $u_\lambda > 0$ with $\frac{\partial u_\lambda}{\partial n} < 0$ on $\partial\Omega$. We shall prove in [39] that the solutions of this branch are Lyapunov stable for $\lambda > \lambda^* > 0$ for some $\lambda^* > 0$ and for $\lambda \in (\lambda^*, \lambda^{**})$ there is a lower branch v_λ with $\frac{\partial v_\lambda}{\partial n} < 0$ on $\partial\Omega$, $v_\lambda < u_\lambda$ possibly unstable, which prolongates in continua of compact support solutions. But we do not know if our result can be applied to v_λ. We recall that they do <u>not</u> apply to the solution $v_{\lambda^{**}}$ such that $\frac{\partial v_{\lambda^{**}}}{\partial n} = 0$ on $\partial\Omega$ nor to compact support solutions For this problem linearization is not the only way of obtaining stability results. This has been done in [40], [41] and [42] by using variational arguments. The situation is again more complicated for a general domain if $N > 1$. Existence of a positive solution was proved in [64] by using a continuation argument and some multiplicity results were obtained in [42] by combining variational and continuation methods. For the case $\beta > \alpha > 0$ see [3] (see also [6],[7] concerning the multivalued case $\beta = 0$ and $\alpha = -1$). The pseudo-linearization process introduced in [20] can be applied to the multivalued case $\alpha = 0$ and $\beta \in (-1, 0)$). The existence of solutions for the parabolic and elliptic equations were given in [47], [48], [81] and [43] respectively. The study of the nonlinear eigenvalue type problems for variational inequalities (such as it corresponds when we assume $\alpha = 0$) is already quite classical in the literature (see, e.g. [68] and [71]).

Example 4. If $f(u) = \lambda u - u^\alpha$, with $0 < \alpha < 1$, we have $H(u) = f(u) - f'(u)u = (\alpha-1)u^\alpha < 0$. This implies the *linearized instability* for solutions in the interior of the positive cone.

It was proved in [77], [40] and [33] that for any $\lambda > \lambda_1$ there exists a non-negative solution $u_\lambda \in H_0^1(\Omega)$ for the problem

$$\begin{cases} -\Delta u + u^\alpha = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Here λ_1 (and in what follows λ_2) is the first (repectively, the second) eigenvalue of the problem $-\Delta w = \lambda w$ in Ω, $w = 0$ on $\partial\Omega$. Moreover, bifurcation at infinity arises from λ_1 (see [38], [40]) and this means that solutions with a large norm u_λ close to λ_1 are such that $u_\lambda > 0$ with $\frac{\partial u_\lambda}{\partial n} < 0$ on $\partial\Omega$. Now the results in [63] allow to apply the Implicit Function Theorem at the interior of the positive cone if the linearized operator is an isomorphism. Since we have $\mu_1(-\Delta - \alpha u_\lambda^{\alpha-1} - \lambda) < 0$ the result would follow clearly from $0 < \mu_2(-\Delta - \alpha u_\lambda^{\alpha-1} - \lambda)$.

From the usual variational characterization

$$\mu_2 = \inf_{w \in [\varphi_1]^\perp} \frac{\int_\Omega \left(|\nabla w|^2 + \frac{\alpha w^2}{u_\lambda^{1-\alpha}} - \lambda w^2\right) dx}{\int_\Omega w^2 dx}$$

we obtain the estimate

$$\mu_2 > \lambda_2 - \lambda + \inf_{w \in [\varphi_1]^\perp} \frac{\int_\Omega \frac{\alpha w^2}{u_\lambda^{1-\alpha}} dx}{\int_\Omega w^2 dx}.$$

Using that $u_\lambda \leq c_1 d(x)$, for some $c_1(\lambda) > 0$, we get

$$\int_\Omega \frac{w^2}{u_\lambda^{1-\alpha}} dx \geq \int_\Omega \frac{w^2}{c_1^{1-\alpha} d(x)^{1-\alpha}} dx \geq \frac{1}{D^{1-\alpha} c_1^{1-\alpha}} \int_\Omega w^2 dx$$

where $D > 0$ is such that $d(x) \leq D$ for any $x \in \Omega$. Hence

$$\mu_2 > \lambda_2 - \lambda + \frac{1}{D^{1-\alpha} c_1^{1-\alpha}}$$

and the condition $\mu_2 > 0$ is satisfied for some $\lambda > \lambda_1$ close to λ_1, and in particular for $\lambda_1 < \lambda < \lambda_2$.

4 Applications to nonlinear problems: II. Linearized stability for degenerate quasilinear parabolic problems

We study in this section the quasilinear degenerate parabolic problem (3)

$$\begin{cases} \beta(u)_t - \Delta u = f(u) & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{34}$$

where $\Omega \subset \mathbb{R}^N$ is again a smooth bounded domain, $\beta(s)$ is smooth for $s > 0$, $\beta(s) \geq 0$ for $s > 0$, $\beta(0) = 0$, $\beta'(0) = +\infty$ and $\beta'(s) > 0$ for $s > 0$. Moreover $f(s)$ is smooth for $s > 0$ with $f(0) = 0$ and, in principle, as in [10], $f \circ \beta^{-1}$ is locally Lipschitz continuous for $s \geq 0$ (this includes $s = 0$!). Under these assumptions it is proved in [10] that (3) has a unique weak solution and an associate comparison principle which is useful in order to prove results concerning asymptotic behaviour of solutions.

As it was shown above, the formally associated linearized problem around the positive stationary solution $\overline{u} > 0$ to (34) is

$$\begin{cases} \beta'(\overline{u}) w_t - \Delta w - f'(\overline{u}) w = 0 & \text{in } \Omega \times (0, +\infty), \\ w = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{35}$$

which leads to the linear eigenvalue problem (6)

$$\begin{cases} -\Delta w - f'(\overline{u}) w = \lambda \beta'(\overline{u}) w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{36}$$

We will consider problem (34) with

$$\beta(s) = s^{1/m}, \ f(s) = s^{p/m}$$

under the conditions

$$-1 < \frac{p}{m} < 1 \text{ and } m > 1. \tag{37}$$

This corresponds to reaction-diffusion phenomena with a balance between suitable slow diffusion and strong forcing (sometimes called as exothermic) reaction terms. Notice that $f \circ \beta^{-1}(s) = s^p$ and that it is locally Lipschitz continuous if $p \in [1, m)$ although it is not the case for the function $f(s)$ when $-m < p < 1$. The associated stationary problem

$$\begin{cases} -\Delta u = u^{\frac{p}{m}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{38}$$

has a unique solution $\overline{u} > 0$ to which the above results may be applied (the case $p \in (-m, 0)$ corresponds to Theorem 3.1) and the case $p \in (0, m)$ is well-known in the literature: see, e.g., the survey [62])). Problem (6) can be written as

$$\begin{cases} -\Delta w - \frac{pw}{m\overline{u}^{\frac{m-p}{m}}} = \frac{\lambda w}{m\overline{u}^{\frac{m-1}{m}}} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{39}$$

Recalling that $\overline{u} \sim d(x)$ near $\partial\Omega$ (see Theorem 3.1) if $-1 < \frac{p}{m} < 0$ and applying the strong maximum principle if $0 < \frac{p}{m} < 1$) the problem could be reduced (modulo some positive constants) to problem (16) with

$$\beta = \frac{m-p}{m} = 1 - \frac{p}{m} < 2, \ \gamma = \frac{m-1}{m} < 1$$

and the results in section 2 apply.

If λ_1 is the first eigenvalue given by Theorem 2.1 and $\psi_1 > 0$ the associated eigenfunction, we have

$$\begin{cases} -\Delta\psi_1 - f'(\overline{u})\psi_1 = \lambda\beta'(\overline{u})\psi_1 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

The same computation as above yields

$$\lambda_1 = \frac{\int_\Omega [f(\overline{u}) - f'(\overline{u})\overline{u}]\, \psi_1}{\int_\Omega \beta'(\overline{u})\psi_1} > 0$$

since $f(\overline{u}) - f'(\overline{u})\overline{u} > 0$. Then, the application of Theorem 2.7 and the above analysis leads to the following conclusion:

Theorem 4.1 *Assume that (37) holds. Then the quasilinear problem (34) has a unique stationary strictly positive solution $\overline{u} > 0$, $\overline{u}$ satisfies (38) and $\overline{u}$ is linearly stable.*$\blacksquare$

As noticed in the introduction, we claim that linearized stability in this context implies Lyapunov stability: we plan to settle this question in [39]. This was proved in the framework of $C_0^{1,\gamma}(\overline{\Omega})$, with $\gamma \in (0,1)$, in [63] (see also [20] for the case of a delayed parabolic problem).

Remark 4.1 *This problem was studied in [10] under the assumption* $1 \leq p < m$. *If* $p \in (0,1)$ *the uniqueness of solutions of the associate parabolic equation was proved in [19] for suitable initial data. For the case* $p \in (-m, 0)$, *under suitable additional conditions, the corresponding parabolic problem has a unique solution for smooth positive initial data (this is a simple variation of the results of [25] [54]; see also [57] for the case of* $\mathbb{R}^N$*).*

On the other side, if we assume now

$$p \in (-\infty - m), \text{ and } m > 1 \tag{41}$$

there is still a unique positive solution to (38) but now its behaviour is $\overline{u} \sim d(x)^{\frac{2m}{m-p}}$ and the behaviour of the coefficients in the linearized equation is

$$\overline{u}^{\frac{m-p}{m}} \sim d(x)^2$$

and

$$\overline{u}^{\frac{m-1}{m}} \sim d(x)^{\frac{2(m-1)}{m-p}},$$

with $\frac{2(m-1)}{m-p} < 2$.

Again, we use the above results, now with $\beta = 2$, and $\gamma < 2$ and prove

Theorem 4.2 *Assume that (41) holds. Then the unique positive solution* $\overline{u} > 0$ *of (38) is linearly stable.*■

Next we study the case of compactly supported solutions $\overline{u}$, where the above results for the linearized problem cannot be applied in a strict sense but we can analyze the stability of flat solutions of the associate stationary problem. The following special case was studied in [4]: consider the degenerate problem

$$\begin{cases} u_t - (u^m)_{xx} = f(u) = u(1-u)(u-\alpha) & \text{in } (-L, L) \times (0, +\infty), \\ u^m(\pm L, t) = 0 & \text{on } (0, +\infty), \\ u(x, 0) = u_0(x) & \text{on } (-L, L), \end{cases} \tag{42}$$

with $m > 1$ and some $0 < \alpha < 1$. It was shown in [4] that depending on the parameter $L > 0$ the associated stationary problem

$$\begin{cases} -(u^m)_{xx} = u(1-u)(u-\alpha) & \text{in } (-L, L), \\ u^m(\pm L) = 0 \end{cases} \tag{43}$$

may have compact support solutions such that $0 \leq \overline{u} \leq 1$ (this corresponds to the case L large enough). Assume that $0 \leq \overline{u} \leq 1$ is a solution satisfying

$$\begin{cases} -(\overline{u}^m)_{xx} = \overline{u}(1-\overline{u})(\overline{u}-\alpha) & \text{in } (-L, L), \\ \overline{u}^m(\pm L) = 0 \end{cases} \tag{44}$$

with $\overline{u} > 0$ on (a, b), $-L < a < b < L$ and $\overline{u} \equiv 0$ on $[-L, a] \cup [b, L]$. This corresponds to formulation (34) with

$$\beta(s) = s^{1/m}, \;\; f(s) = s^{1/m}(1 - s^{1/m})(s^{1/m} - \alpha).$$

With the change of variable $\overline{v} = \overline{u}^m$ we get

$$\begin{cases} -\overline{v}_{xx} = \overline{v}^{1/m}(1-\overline{v}^{1/m})(\overline{v}^{1/m}-\alpha) & \text{in } (-L, L), \\ \overline{v}(\pm L) = 0. \end{cases} \tag{45}$$

We know (by applying the study for more general one-dimensional semilinear equations made in [31]) that for $\overline{v} \sim 0$ then

$$\overline{v} \sim d(x)^{\frac{2}{1-\frac{1}{m}}} = d(x)^{\frac{2m}{m-1}}$$

where $\frac{2m}{m-1} > 2$ for any $m > 0$. Hence

$$\overline{u} \sim \overline{v}^{\frac{1}{m}} \sim d(x)^{\frac{2}{m-1}},$$

and several boundary behaviours are possible:

$$\begin{aligned} \frac{2}{m-1} > 1 \Longleftrightarrow m < 3 \; &\Longrightarrow \overline{u}'(a) = \overline{u}'(b) = 0, \\ \frac{2}{m-1} \leq 1 \Longleftrightarrow m \geq 3 \; &\Longrightarrow \overline{u} \notin C^1([a, b]) \text{ (but } \overline{v} \in C^1([a, b])). \end{aligned}$$

For the coefficients of the "linearized problem" on (a, b) we obtain

$$\beta'(\overline{v}) \sim \overline{v}^{\frac{1}{m}-1} = \frac{1}{\overline{v}^{\frac{m-1}{m}}} \sim \frac{1}{d(x)^{\frac{2m}{m-1}\frac{m-1}{m}}} = \frac{1}{d(x)^2}$$

and in the same way

$$f'(\overline{v}) \sim \frac{-1}{d(x)^2} \text{ near } x = a \text{ and } x = b.$$

Again, we can apply the results in subsection 2.3 to the associate problem (36) corresponding to $\gamma = 2$, as in [10]. In any case, the linearized instability of the stationary solution $\overline{u}$, follows by the arguments used in [10]. Indeed, once that $m > 1$ we get that $\overline{v} \in C^2([a, b])$ and that $\overline{v}'(a) = \overline{v}'(b) = 0$. Thus $z := \overline{v}' \in H_0^1(a, b)$ and since

$$-z'' = f'(\overline{v})z \text{ in } (a, b),$$

(notice that there is a misprint in the corresponding formula of page 398 in [10]) we get that

$$\lambda_1 = \inf_{w\neq 0\ w\in H_0^1(a,b)} \frac{\int_a^b |w'|^2 - \int_a^b f'(\overline{u}^m)w}{\int_a^b \beta'(\overline{u}^m)w^2} \leq 0.$$

Moreover, as indicated in [10], $\lambda_1 \neq 0$ since, otherwise, the eigenfunction is changing sign. So $\overline{u}$ is unstable (in fact, that was already proved in [4] by means of the comparison principle and the use of suitable auxiliary super and subsolutions). See also Lemma 5.3 and the following remarks of [10].

Remark 4.2 *Notice that the behavior of function $f(s) = s^{1/m}(1-s^{1/m})(s^{1/m}-\alpha)$ is very similar (near $s = 0$) of the nonlinear function considered in the papers [41] and [42], $\widehat{f}(s) = \lambda u^{\beta} - u^{\alpha}$, with $0 < \alpha < \beta < 1$. In fact it seems possible to extend the results of the above mentioned papers to the case of the function $f(s) = s^{1/m}(1-s^{1/m})(s^{1/m}-\alpha)$. So, as shown in [41] and [42], the flat solutions are unstable in the one-dimensional (and also in two dimensional) space framework. But they are stable (as ground solutions) when Ω is a smooth bounded domain strictly star-shaped with respect to the origin in $\mathbb{R}^n$ and $0 < \alpha < \beta < 1$ are such that $2(1+\alpha)(1+\beta) - N(1-\alpha)(1-\beta) < 0$. Notice that if $N \geq 3$ this set of exponents (α, β) is not empty.*

If, as in Remark 5.7 in [10], we replace $f(s)$ by the non-Lipschitz function $f_p(s) = s^{p/m}(1-s^{p/m})(s^{p/m}-\alpha)$ with $1 < p < m$, then there are still compact support solutions to the associated stationary problem (36). The existence of local in time solutions for the parabolic problem is an easy consequence of the compactness of the associate semigroup (see [47], [48], [81]). Now some new facts arise since $\overline{v} \sim d(x)^{\frac{2m}{m-p}}$ and hence

$$\overline{u} \sim d(x)^{\frac{2}{m-p}},$$

and we have in this case

$$\frac{2}{m-p} > 1 \Longleftrightarrow m < 2+p \implies \overline{u}'(a) = \overline{u}'(b) = 0,$$
$$\frac{2}{m-1} \leq 1 \Longleftrightarrow m \geq 2+p \implies \overline{u} \notin C^1([a,b]) \text{ (but } \overline{v} \in C^1([a,b])).$$

Again, by the results of [31]) we get

$$f'(\overline{v}) \sim \overline{v}^{\frac{p}{m}-1} = \overline{v}^{\frac{p-m}{m}} \sim \frac{1}{d(x)^{\frac{2m}{m-p}\frac{m-p}{m}}} = \frac{1}{d(x)^2}$$

and on the other side

$$\beta'(\overline{v}) \sim \overline{v}^{\frac{1}{m}-1} \sim \frac{1}{\overline{v}^{\frac{m-1}{m}}} \sim \frac{1}{d(x)^{\frac{2m}{m-p}\frac{m-1}{m}}} = \frac{1}{d(x)^{\frac{2(m-1)}{m-p}}}.$$

We know (see, e.g., [15]) that in order to get existence of positive solutions of problem (36) the coefficient of the linear source term $\beta'(\overline{v})$ must be at most as $d(x)^{-2}$, so we get that for $p \in (1, m)$ there is not any positive solution of (36) since

$$\frac{2(m-1)}{m-p} > 2 \Longleftrightarrow p > 1.$$

Any solution of the parabolic problem blows-up in a finite time (see [57]). The situation radically changes if besides to assume $p \in (1, m)$ we perturb the parabolic equation with an absorption term of the form u^q and we assume $q < p < m$ (see [42]). We mention also here that the study of linear equations with a very singular absorption coefficient of the type $d(x)^{-\mu}$ with $\mu > 2$ was considered in the paper [35].

5 Appendix

As it was claimed in the introduction, we collect here several variants and different simple proofs of the existence of a first eigenvalue with positive eigenfunction. Some of them use a non-standard version of the Hardy-Sobolev inequality (see [66], [69] and [75]).

Example 1. We consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \frac{u}{d(x)^\tau} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{46}$$

with $0 \leq \tau < 1$. Here we use again a different Hardy-Sobolev inequality (see [66]): if $u \in W_0^{1,q}(\Omega)$ with $q > N, \frac{u}{d(x)^\tau} \in L^r(\Omega)$, where $r = \frac{q}{r}$ and we have

$$\left\| \frac{u}{d(x)^\tau} \right\|_{L^r(\Omega)} \leq C \left\| u \right\|_{W_0^{1,q}(\Omega)}.$$

Hence, if $u \in C_0^1(\overline{\Omega})$, $u \in W_0^{1,q}(\Omega)$ for any $q > N$ and then $\frac{u}{d(x)^\tau} \in L^r(\Omega)$ for any $r > N$. This implies that the linear equation

$$\begin{cases} -\Delta w = \lambda \frac{u}{d(x)^\tau} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{47}$$

has a unique solution $w = Tu \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for any $r > N$ and then $Tu \in C_0^{1,\beta}(\overline{\Omega})$ for all $\beta \in (0, 1)$, in particular $Tu \in C_0^1(\overline{\Omega})$. It follows from the Strong Maximum Principle that if $u \geq 0$, $u \neq 0$ then $w = Tu > 0$ in Ω and $\frac{\partial Tu}{\partial n} < 0$ on $\partial\Omega$. Moreover, $T : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is compact (the embedding $C_0^{1,\beta}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ is compact for any $\beta \in (0, 1)$) and the classical Krein-Rutman theorem gives the existence of a positive eigenvalue $\lambda_1 > 0$, which is simple with an eigenfunction $\varphi_1 > 0$ on Ω and $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$.

In order to apply this result to the problem

$$\begin{cases} -\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{48}$$

with $0 < q < 1$ (see [1]) we consider the corresponding linearized problem

$$\begin{cases} -\Delta w - \lambda q \overline{u}_\lambda^{q-1} w = \mu w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{49}$$

Reproducing the "fixed point" argument in section 2 it is possible to show that $\mu_1 > 0$ (resp. $\mu_1 < 0$) if and only if $r(0) < 1$ (resp. $r(0) > 1$), where

$$r(0) = \inf_{w \neq 0} \frac{\int_\Omega |\nabla w|^2 - \int_\Omega \lambda q \overline{u}_\lambda^{q-1} w^2}{\int_\Omega w^2}.$$

But we can reason more directly by using a comparison argument following from the variational characterization for the eigenvalues, namely

$$0 = \lambda_1(-\Delta - \lambda \overline{u}_\lambda^{q-1}) < \lambda_1(-\Delta - \lambda q \overline{u}_\lambda^{q-1}) = \mu_1,$$

and $\overline{u}_\lambda$ is linearly stable.

Example 2. The case $\beta = 2$, $\gamma = 0$ in section 2 can be treated with the different approach which follows. The cases $0 \leq \beta < 2$ and $0 < \gamma < 2$ are rather similar. We have the eigenvalue problem

$$\begin{cases} -\Delta w + \frac{w}{d(x)^2} = \mu w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

We define

$$\mu_1 = \inf_{\|w\|_{L^2}=1} \int_\Omega |\nabla w|^2 + \int_\Omega \frac{w^2}{d(x)^2}.$$

Let (w_n) be a minimizing sequence, then

$$\int_\Omega |\nabla w_n|^2 + \int_\Omega \frac{w_n^2}{d(x)^2} \to \mu_1, \text{ as } n \to +\infty, \ \|w_n\|_{L^2(\Omega)} = 1.$$

Then we have

$$\|w_n\|_{H_0^1(\Omega)} \leq C, \ \|w_n\|_{L^2(\Omega, \frac{1}{d(x)^2})} \leq C,$$

where C is independent of n. Then there exists a subsequence w_n such that

$$w_n \rightharpoonup \overline{w} \text{ weakly in } H_0^1(\Omega),$$

$$w_n \rightharpoonup \overline{w} \text{ weakly in } L^2(\Omega, \frac{1}{d(x)^2}),$$

$$w_n \to \overline{w} \text{ strongly in } L^2(\Omega).$$

Hence $\|\overline{w}\|_{L^2} = \lim \|w_n\|_{L^2} = 1$ and $\overline{w} \neq 0$. From the l.s.c. of norms it follows that

$$\int_\Omega |\nabla \overline{w}|^2 + \int_\Omega \frac{\overline{w}^2}{d(x)^2} \leq \liminf \left(\int_\Omega |\nabla w_n|^2 + \int_\Omega \frac{w_n^2}{d(x)^2} \right) = \mu_1.$$

We also have $w_n \to \overline{w}$ strongly in $H_0^1(\Omega)$. Indeed, if not

$$\int_\Omega |\nabla \overline{w}|^2 + \int_\Omega \frac{\overline{w}^2}{d(x)^2} < \liminf \left(\int_\Omega |\nabla w_n|^2 + \int_\Omega \frac{w_n^2}{d(x)^2} \right) = \mu_1$$

and since $\|w_n\|_{L^2} = 1$ we get a contradiction.

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