Unexpected regionally negative solutions of the homogenized problem associated with the Poisson equation and dynamic unilateral boundary conditions: the case of symmetric particles of critical size *

J.I. Díaz, A.V. Podol'skii, T.A. Shaposhnikova

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Abstract

We obtain the homogenized problem associate to the Poisson equation in a domain perforated by "tiny" balls (or in a domain defined as the exterior to a periodic set of very small particles) of radius $a_\varepsilon = C_0\varepsilon^\gamma$ with $\gamma = \frac{n}{n-2}$, $C_0 > 0$ (the, so-called, "critical case"). On the boundary of these balls we assume a dynamic unilateral boundary condition (the so-called "Signorini boundary condition"). We prove that the homogenized problem consists of an elliptic equation coupled with an ordinary differential unilateral problem: in contrast with the case of "big perforations" (or "big particles") for which the equation is a parabolic unilateral problem. In particular, we prove that the solution of the homogenized problem may become regionally negative (at least in the interior of some subset of $Q^T = \Omega \times (0,T)$ on which $f(x,t)$ is negative). Nothing similar may happen for the case of big particles since the corresponding homogenized problem implies that the solution is always non-negative, even for very negative data $f(x,t)$

1 Introduction

We study the asymptotic behavior as $\varepsilon \to 0$ of the solution u_ε of the Poisson equation, in a domain Ω of $\mathbb{R}^n$, $n \geq 3$, perforated by balls G_ε (or defined as the exterior to a set of particles G_ε given by balls) of radius $a_\varepsilon \ll \varepsilon$, when on the boundary of these inclusions (or of these particles) we assume that an unilateral dynamic Signorini boundary condition takes place containing a suitably large coefficient at the time derivative. Although we will present the technical details on the domain in the next section, we outline now that the problem under consideration can be simply formulated in the following terms:

$$\begin{cases} -\Delta_x u_\varepsilon = f(x,t), & (x,t) \in Q_\varepsilon^T, \\ u_\varepsilon(x,t) = 0, & (x,t) \in \Gamma^T, \\ \left.\begin{array}{l} u_\varepsilon \geq 0, \quad \varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon \geq 0, \\ u_\varepsilon\big(\varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon\big) = 0, \end{array}\right\} & (x,t) \in S_\varepsilon^T, \\ u_\varepsilon(x,0) = u^0(x), & x \in S_\varepsilon, \end{cases} \tag{1}$$

for some given data $f \in H^1(0,T;L^2(\Omega))$ and, for simplicity, we assume, at least $u^0 \in H_0^1(\Omega)$ (and thus its trace over S_ε, here labeled in the same way, satisfies $u^0 \in H^{1/2}(S_\varepsilon)$) with $u^0 \geq 0$ on Ω (see Remark 1). Here $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$, $Q_\varepsilon^T = \Omega_\varepsilon \times (0,T)$, $S_\varepsilon = \partial G_\varepsilon$, $S_\varepsilon^T = S_\varepsilon \times (0,T)$ and $\Gamma^T = \partial\Omega \times (0,T)$. As usual H^s denotes the Sobolev spaces with values in L^2, of order s, on the respective domains.

Dynamic boundary conditions arise in various physical and chemical processes (see, for instance, [6], [20], and [4]). The homogenization of such problems attracted a wide research interest in the last years

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([30], [17], [1], [19] and [2] for a brief overview). On the other hand, the homogenization of a large class of problems with unilateral constraints on the boundary were studied in many papers: here we refer only some of them [3], [11], [23], [28], [29], [13], [22], and [14]. Our main interest in this paper deals with the so called "critical case" in which the radius of the balls a_ε, the exponent in the intertia term γ and the dimension of the space n are linked by the conditions

$$a_\varepsilon = C_0\varepsilon^\gamma,\ \gamma = \frac{n}{n-2},\ C_0 > 0. \tag{2}$$

(see a general exposition on this type of problems in the monograph [16]).

We point out that in the case of "big particles" ($a_\varepsilon = C_0\varepsilon^\gamma$, with $\gamma = 1$), some easy modifications of the results of [30], [1], by using the arguments given in [10] (see also [16]), allow to see that $u_\varepsilon \to u$ (in a certain sense) and that the homogenized problem can be formulated in terms of the parabolic obstacle variational inequality for a modified diffusion operator

$$\begin{cases} \left.\begin{array}{l} C_n\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} q_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_,} - \widehat{C}_n f(x,t) \geq 0,\ u \geq 0, \\ u\left(C_n\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} q_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_,} - \widehat{C}_n f(x,t)\right) = 0 \end{array}\right\} & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u^0(x), & x \in \Omega, \end{cases} \tag{3}$$

for some suitable positive constants $C_n, \widehat{C}_n$ and a positive definite matrix $(q_{i,j})$.

By the contrary, when the radius of the balls is critical as indicated in (2), we will show that the homogenized problem is characterized in a completely different manner: we arrive to a system given by an elliptic equation (with t as a parameter) coupled with an ordinary differential obstacle problem (with x as a $n-$dimensional parameter) satisfied by a new nonlocal term called, usually, as an "strange term" in the literature (see [9], [16])

$$\begin{cases} -\Delta u + \mathcal{A}_n(u - H_{u,u^0}) = f(x,t), & (x,t) \in Q^T, \\ u = 0, & (x,t) \in \Gamma^T, \\ \left.\begin{array}{l} H_{u,u^0} \geq 0, \quad \partial_t H_{u,u^0} + \mathcal{B}_n H_{u,u^0} \geq \mathcal{B}_n u, \\ H_{u,u^0}\big(\partial_t H_{u,u^0} + \mathcal{B}_n(H_{u,u^0} - u)\big) = 0, \end{array}\right\} & (x,t) \in Q^T, \\ H_{u,u^0}(x,0) = u^0(x), & x \in \Omega, \end{cases} \tag{4}$$

where

$$\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n,\ \omega_n = |\partial G_0|,\ \mathcal{B}_n = (n-2)C_0^{-1}.$$

We will obtain several qualitative properties on this term H_{u,u^0}. We will use the general method of "oscillating test functions", suggested (to some standard formulations) by L. Tartar, but requiring quite complicated adaptations to each problem under consideration (see the general exposition, and references, presented in the monograph [16])).

We also should mention the difference of the limit problem (4) with the one obtained in the case of standard (non-dynamic) unilateral boundary conditions of Robin type (see, [3], [11], [24] and [14]). In that case, the homogenized problem also contains a "strange term" H which involves the negative part of the limit solution u, but in this case H is a local function and this changes dramatically the approach.

One of the most important consequences of the different limit problem, in the critical case, is that, in contrast with the case of big particles (in which the solution is always non-negative, even for very negative data $f(x,t)$), as we will prove (see Theorem 3 below) the solution of (4) may become negative in the interior of some subset of $Q^T = \Omega \times (0,T)$ on which $f(x,t)$ is negative. This is an unexpected property which can be useful for many different purposes (in the same spirit that the searching of new materials in Nanotechnology: see some references in the monograph [16]) and that, as far as we know, was not indicated in other papers dealing with the homogenization under unilateral boundary conditions (see the mention made above) although this property also holds for non-dynamic boundary conditions.

Another "strange term" arising in the limit problem (4) concerns the implicit initial value $u(x,0)$ of its solution. We will show that $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and thus $u(x,0)$ is given as the unique weak solution of the stationary problem

$$\begin{cases} -\Delta u(x,0) + \mathcal{A}_n u(x,0) = f(x,0) + \mathcal{A}_n u^0(x), & x \in \Omega, \\ u(x,0) = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

So that, the sequence of solutions u_ε, being non-negative on infinite points of Ω at $t = 0$, converges to the function u, solution of (4), which satisfies initially the problem (5) and which may become negative if $f(x,0)$ is suitably negative on some parts of Ω.

In fact, this paper forms part of an attempt to extend previous homogenization results to the more general class of nonlinear terms on the boundary conditions on the boundary of the particles. The case of Robin type boundary conditions given by maximal monotone graphs (which include the case of Signorini boundary conditions as a special case) was carried out in the paper [14]. For the case of dynamic boundary conditions, as far as we know, the more general result in the available literature assumes a Hölder continuity on the monotone function (see [19]). The present paper goes beyond this regularity by considering one of the most relevant maximal monotone graphs which are not Hölder continuous: the case of dynamic Signorini unilateral boundary conditions. Concerning the geometry of the particles, we assume here the simpler case of radially symmetric particles. A separate study by the authors will be presented concerning particles of arbitrary shape in a paper in preparation.

2 Statement of the problem and estimates of solution

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with Lipschitz boundary $\partial\Omega$. We denote by $Y = (-1/2, 1/2)^n$ the unit cube and by $G_0 = \{x : |x| < 1\}$ the unit ball. Define $\delta B = \{x : \delta^{-1}x \in B\}$, $\delta > 0$. For $\varepsilon > 0$, let

$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \ \rho(x, \partial\Omega) > 2\varepsilon\}.$$

We denote by $\mathbb{Z}^n$ the set of all vectors $j = (j_1, \dots, j_n)$ with an integer coordinates j_i, $i = 1, \dots, n$. We consider the set

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in Z^n : \ \overline{G_\varepsilon^j} \subset Y_\varepsilon^j = \varepsilon Y + \varepsilon j, \ G_\varepsilon^j \cap \overline{\widetilde{\Omega}_\varepsilon} \neq \emptyset\}$. Note that $|\Upsilon_\varepsilon| = d\varepsilon^{-n}$, d is a positive constant.

We consider the critical balance between the parameters (see [16])

$$a_\varepsilon = C_0 \varepsilon^\gamma, \ \gamma = \frac{n}{n-2}, \ C_0 > 0. \tag{6}$$

It is easy to see that $\overline{G_\varepsilon^j} \subset T_{\varepsilon/4}^j \subset Y_\varepsilon^j$, where T_r^j is a ball of $\mathbb{R}^n$ of the radius r centered at the center of the cell Y_ε^j which we denote by $P_\varepsilon^j = \varepsilon j$. We introduce the sets

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \qquad S_\varepsilon = \partial G_\varepsilon, \qquad \partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega,$$
$$Q_\varepsilon^T = \Omega_\varepsilon \times (0,T), \quad S_\varepsilon^T = S_\varepsilon \times (0,T), \quad \Gamma^T = \partial\Omega \times (0,T).$$

We consider the convex closed set

$$K_\varepsilon = \{v \in H^1(\Omega_\varepsilon, \partial\Omega) : \ v \geq 0 \text{ a.e. } x \in S_\varepsilon\}.$$

As usual, we denote by $H^1(\Omega_\varepsilon, \partial\Omega)$ the completion with respect to the norm in $H^1(\Omega_\varepsilon)$ of the set of infinitely differentiable functions on $\overline{\Omega_\varepsilon}$ vanishing in a neighborhood of the boundary $\partial\Omega$. Along with K_ε, we consider the temporary convex set

$$\mathcal{K}_\varepsilon = \{v \in L^2\big(0,T; H^1(\Omega_\varepsilon, \partial\Omega)\big) : \ v(\cdot,t) \in K_\varepsilon \text{ for a.e. } t \in (0,T)\}.$$

The starting problem consists in searching $u_\varepsilon \in \mathcal{K}_\varepsilon \cap C\big([0,T]; L^2(S_\varepsilon)\big)$ with $\partial_t u_\varepsilon \in L^2\big(0,T; L^2(S_\varepsilon)\big)$ and $u_\varepsilon(x,0) = u^0(x)$, such that u_ε satisfies the variational inequality

$$\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \partial_t u_\varepsilon (\phi - u_\varepsilon) ds dt + \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\phi - u_\varepsilon) dx dt \geq \int_0^T \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx dt, \tag{7}$$

for any $\phi \in \mathcal{K}_\varepsilon$. Here, $f \in H^1(0,T; L^2(\Omega))$ and $u^0 \in H^1_0(\Omega)$, $u^0 \geq 0$. Note that (7) corresponds to the weak formulation of the Signorini problem with a dynamic boundary condition on S^T_ε

$$\begin{cases} -\Delta_x u_\varepsilon = f(x,t), & (x,t) \in Q^T_\varepsilon, \\ \left.\begin{array}{l} u_\varepsilon \geq 0, \quad \varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon \geq 0, \\ u_\varepsilon\big(\varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon\big) = 0, \end{array}\right\} & (x,t) \in S^T_\varepsilon, \\ u_\varepsilon(x,t) = 0, & (x,t) \in \Gamma^T, \\ u_\varepsilon(x,0) = u^0(x), & x \in S_\varepsilon. \end{cases} \tag{8}$$

The existence and uniqueness of solutions to (7) can be obtained through different techniques.

Remark 1 The existence and uniqueness of solutions to (7) can be obtained under very general assumptions: for instance, by applying semigroups theory (see, e.g. [6]) the initial datum u^0 can be taken merely in $H^1(\Omega)$ instead $H^1_0(\Omega)$. Nevertheless, here we will present a different proof (which requires $u^0 \in H^1_0(\Omega)$) but which allows to get some *a priori* estimates which will be very useful for different purposes.

Theorem 1 *Given $f \in H^1\big(0,T; L^2(\Omega)\big)$ and $u^0 \in H^1_0(\Omega)$, $u^0 \geq 0$, for any $\varepsilon > 0$ the problem (8) has a unique solution u_ε and the following estimates hold*

$$\begin{cases} \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \varepsilon^{-\gamma/2}\|u_\varepsilon\|_{C([0,T];L^2(S_\varepsilon))} \leq K(\|f\|_{L^2(Q^T)} + \big\|u^0\big\|_{H^1(\Omega)}), \\ \varepsilon^{-\gamma}\|\partial_t u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))} + \max_{t\in[0,T]}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon))} \leq K\bigg(\|f\|_{H^1(0,T;L^2(\Omega))} + \big\|u^0\big\|_{H^1(\Omega)}\bigg), \end{cases} \tag{9}$$

where K is a positive constant independent of ε, f and u^0.

Proof. We start by considering the approximate problem corresponding to the application of the penalization method (see [25] for a general exposition). Given any positive number δ we consider the auxiliary problem

$$\begin{cases} -\Delta_x u^\delta_\varepsilon = f(x,t), & (x,t) \in Q^T_\varepsilon, \\ \varepsilon^{-\gamma}\partial_t u^\delta_\varepsilon + \partial_\nu u^\delta_\varepsilon + \delta^{-1}(u^\delta_\varepsilon)^- = 0, & (x,t) \in S^T_\varepsilon, \\ u^\delta_\varepsilon(x,t) = 0, & (x,t) \in \Gamma^T, \\ u^\delta_\varepsilon(x,0) = u^0(x), & x \in S_\varepsilon, \end{cases} \tag{10}$$

where $u^+ = sup(0,u)$, $u^- = u - u^+$. Note that $\sigma(u) = u^-$ is a monotone Lipschitz continuous function

$$|u^- - v^-| \leq |u - v| \quad \forall u, v \in \mathbb{R}.$$

We say that a function $u^\delta_\varepsilon \in C\big([0,T]; L^2(S_\varepsilon)\big)$ is a solution to the problem (10) if $u^\delta_\varepsilon \in L^2\big(0,T; H^1(\Omega_\varepsilon, \partial\Omega)\big)$, $\partial_t u^\delta_\varepsilon \in L^2\big(0,T; L^2(S_\varepsilon)\big)$, we have

$$\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \partial_t u^\delta_\varepsilon v ds dt + \int_0^T \int_{\Omega_\varepsilon} \nabla u^\delta_\varepsilon \nabla v dx dt + \delta^{-1} \int_0^T \int_{S_\varepsilon} (u^\delta_\varepsilon)^- v ds dt = \int_{Q^T_\varepsilon} f v dx dt, \tag{11}$$

for any $v \in L^2\big(0,T; H^1(\Omega_\varepsilon, \partial\Omega)\big)$, and the initial condition $u^\delta_\varepsilon(x,0) = u^0(x)$ holds for $x \in S_\varepsilon$.
After some easy modifications, we can apply, for instance, the results from [19] (there stated for $u^0 \equiv 0$) to conclude that for any $\delta > 0$ the problem (10) has a unique solution and that the following estimates hold

$$\begin{cases} \|u^\delta_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \varepsilon^{-\gamma/2}\|u^\delta_\varepsilon\|_{C([0,T];L^2(S_\varepsilon))} \leq K\big(\|f\|_{L^2(Q^T)} + \big\|u^0\big\|_{H^1(\Omega)}\big), \\ \varepsilon^{-\gamma/2}\|(u^\delta_\varepsilon)^-\|_{L^2(0,T;L^2(S_\varepsilon))} \leq K\sqrt{\delta}\big(\|f\|_{L^2(Q^T)} + \big\|u^0\big\|_{H^{1/2}(S_\varepsilon)}\big), \\ \varepsilon^{-\gamma/2}\|\partial_t u^\delta_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))} + \max_{t\in[0,T]}\|\nabla u^\delta_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq K(\|f\|_{H^1(0,T;L^2(\Omega))} + \big\|u^0\big\|_{H^1(\Omega)}). \end{cases} \tag{12}$$

Note that the last estimate can be also obtained through the application of Thèoreme 3.6 of [8] and Remark 3.1 of [6]. From (12), and some results of functional analysis (see, e.g., [25] and [6]) we derive that there exist a subsequence (still labeled as u_ε^δ) such that

$$
\begin{aligned}
&u_\varepsilon^\delta \rightharpoonup u_\varepsilon \text{ weakly in } L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega)),\\
&\partial_t u_\varepsilon^\delta \rightharpoonup \partial_t u_\varepsilon \text{ weakly in } L^2(0,T;L^2(S_\varepsilon)),\\
&u_\varepsilon^\delta \to u_\varepsilon \text{ strongly in } C([0,T];L^2(S_\varepsilon)),\\
&(u_\varepsilon^\delta)^- \to 0 \text{ strongly in } L^2(0,T;L^2(S_\varepsilon)),
\end{aligned}
$$

as $\delta \to 0$. This shows that u satisfies the estimates (9). Let us show that u is a solution of the variational inequality (7). Indeed, setting in (11) $v = \phi - u_\varepsilon^\delta$, where $\phi \in \mathcal{K}_\varepsilon$, we get

$$
\varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon^\delta(\phi - u_\varepsilon^\delta)dsdt + \int_0^T\int_{\Omega_\varepsilon}\nabla u_\varepsilon^\delta\nabla(\phi - u_\varepsilon^\delta)dxdt + +\delta^{-1}\int_0^T\int_{S_\varepsilon}(u_\varepsilon^\delta)^-(\phi - u_\varepsilon^\delta)dsdt
$$
$$
= \int_0^T\int_{\Omega_\varepsilon} f(\phi - u_\varepsilon^\delta)dxdt.
$$

Applying that

$$
\|\nabla u_\varepsilon\|_{L^2(Q_\varepsilon^T)} \le \lim_{\delta\to 0}\|\nabla u_\varepsilon^\delta\|_{L^2(Q_\varepsilon^T)},
$$

we obtain

$$
\lim_{\delta\to 0}\int_0^T\int_{\Omega_\varepsilon}\nabla u_\varepsilon^\delta\nabla(\phi - u_\varepsilon^\delta)dxdt \le \int_0^T\int_{\Omega}\nabla u_\varepsilon\nabla(\phi - u_\varepsilon)dxdt.
$$

Next, we have

$$
\|u_\varepsilon(x,t)\|^2_{L^2(S_\varepsilon)} \le \lim_{\delta\to 0}\|u_\varepsilon^\delta(x,t)\|^2_{L^2(S_\varepsilon)} \text{ for any } t \in [0,T],
$$

which implies

$$
\varepsilon^{-\gamma}\lim_{\delta\to 0}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon^\delta(\phi - u_\varepsilon^\delta)dsdt \le \varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon(\phi - u_\varepsilon)dsdt.
$$

Taking into account that $\phi \ge 0$ on S_ε a.e $t \in [0,T]$ we get

$$
\int_0^T\int_{S_\varepsilon}(u_\varepsilon^\delta)^-\phi dsdt - \int_0^T\int_{S_\varepsilon}|(u_\varepsilon^\delta)^-|^2 dsdt \le 0,
$$

and we conclude that u_ε satisfies the inequality (7) ■

We recall that by [26], it is well-known the existence of a linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon,\partial\Omega) \to H_0^1(\Omega)$, such that

$$
\|\nabla(P_\varepsilon u)\|_{L^2(\Omega)} \le K\|\nabla u\|_{L^2(\Omega_\varepsilon)}, \quad \|P_\varepsilon u\|_{H_0^1(\Omega)} \le K\|u\|_{H^1(\Omega_\varepsilon)},
$$

where $K > 0$ is a constant independent of ε. Then, by using the estimates in Theorem 1, we conclude that

$$
\|P_\varepsilon u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le K.
$$

Therefore, for some subsequence (that we still we denote by ε) we have that, as $\varepsilon \to 0$,

$$
P_\varepsilon u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{13}
$$

Remark 2 Under some additional conditions on f and u^0 it is possible to show that $u \in L^\infty\big(0,T;L^\infty(\Omega)\big)$, since u_ε is uniformly bounded (independently of ε) in the space $L^\infty\big(0,T;L^\infty(\Omega_\varepsilon)\big) = L^\infty\big(Q_\varepsilon^T\big)$. This is the case, for instance, if we assume $f \in W^{1,\infty}\big(0,T;L^\infty(\Omega)\big) \cap C^{0,\alpha}(Q^T)$ and $u^0 \in W^{1,\infty}(\Omega)$, $u^0 \geq 0$. Indeed, if we define $u_f(x,t)$ as the unique solution of the problem

$$\begin{cases} -\Delta u_f = f(x,t), & (x,t) \in Q^T, \\ u_f = 0, & (x,t) \in \Gamma^T, \end{cases}$$

then, by classical results, we know that $u_f \in W^{1,\infty}(Q^T)$. Thus, the function $U_\varepsilon^\delta(x,t) = u_\varepsilon^\delta(x,t) - u_f(x,t)$ satisfies the problem

$$\begin{cases} -\Delta_x U_\varepsilon^\delta = 0, & (x,t) \in Q_\varepsilon^T, \\ \partial_t U_\varepsilon^\delta + \varepsilon^\gamma \partial_\nu U_\varepsilon^\delta + \varepsilon^\gamma \delta^{-1}(U_\varepsilon^\delta + u_f)^- = -\partial_t u_f - \varepsilon^\gamma \partial_\nu u_f, & (x,t) \in S_\varepsilon^T, \\ U_\varepsilon^\delta(x,t) = 0, & (x,t) \in \Gamma^T, \\ U_\varepsilon^\delta(x,0) = u^0(x) - u_f(x,0), & x \in S_\varepsilon. \end{cases} \tag{14}$$

Then, the right hand side of the dynamic boundary condition is uniformly bounded (independently of ε and δ) and the nonlinear term is monotone non-decreasing in U_ε^δ. Thus we can apply the arguments of the papers [20] and [21] to get an uniform bound in $L^\infty\big(Q_\varepsilon^T\big)$. Clearly, this estimate remains valid also for u_ε, and since, by construction, the linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon, \partial\Omega) \to H_0^1(\Omega)$, is such that $P_\varepsilon\big(L^\infty(\Omega_\varepsilon)\big) \subset L^\infty(\Omega)$, then we get that $u \in L^\infty\big(0,T;L^\infty(\Omega)\big)$. In fact, it can be proved (by the arguments appearing in [18], [6]) that if $f(x,t) = f(x)$, problem (14) can be formulated as the Cauchy problem associated to the subdifferential of a convex function on $L^2(S_\varepsilon)$ and that this operator is also accretive in $L^\infty(S_\varepsilon)$.

One of the main results of this paper is to characterize the limit function u as the unique solution of a suitable homogenized problem.

3 Formulation of the main results and some properties on the strange term

We start by introducing a nonlocal operator which will play a crucial role in the rest of the paper. This corresponds to a term which it is called as "strange term" in the literature on homogenization with critical sizes (see the exposition made in [16]). Given $\phi \in L^2(0,T)$, $u^0 \geq 0$ and a positive constant $\mathcal{B}_n$, we introduce the function $H_{\phi,u^0} \in H^1(0,T)$ as the unique solution to the following evolution unilateral problem:

$$\begin{cases} \left.\begin{array}{l} \partial_t H_{\phi,u^0} + \mathcal{B}_n H_{\phi,u^0} \geq \mathcal{B}_n \phi, \;\; H_{\phi,u^0} \geq 0, \\ H_{\phi,u^0}\big(\partial_t H_{\phi,u^0} + \mathcal{B}_n H_{\phi,u^0} - \mathcal{B}_n \phi\big) = 0, \end{array}\right\} \quad t \in (0,T), \\ H_{\phi,u^0}(0) = u^0. \end{cases} \tag{15}$$

We will prove later that there is existence and uniqueness of solutions of this problem, but before to do it we are already in conditions to state our first main result:

Theorem 2 *Let $n \geq 3$, $f \in H^1\big(0,T;L^2(\Omega)\big)$, $u^0 \in H_0^1(\Omega)$, $u^0 \geq 0$, and let u_ε be the solution of* (8) *when $\gamma = \frac{n}{n-2}$. Then, the function u defined in* (13) *is the unique solution to the problem*

$$\begin{cases} -\Delta u + \mathcal{A}_n(u - H_{u,u^0}) = f(x,t), & (x,t) \in Q^T = \Omega \times (0,T), \\ u = 0, & (x,t) \in \Gamma^T, \\ \left.\begin{array}{l} H_{u,u^0} \geq 0, \quad \partial_t H_{u,u^0} + \mathcal{B}_n H_{u,u^0} \geq \mathcal{B}_n u, \\ H_{u,u^0}\big(\partial_t H_{u,u^0} + \mathcal{B}_n(H_{u,u^0} - u)\big) = 0, \end{array}\right\} & (x,t) \in Q^T, \\ H_{u,u^0}(x,0) = u^0(x), & x \in \Omega, \end{cases} \tag{16}$$

where $\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n$, $\omega_n = |\partial G_0|$, $\mathcal{B}_n = (n-2)C_0^{-1}$. *In addition,* $u \in H^1\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H_0^1(\Omega)\big)$ *and* $u(x,0)$ *is given as the unique solution of the stationary problem*

$$\begin{cases} -\Delta u(x,0) + \mathcal{A}_n u(x,0) = f(x,0) + \mathcal{A}_n u^0(x), & x \in \Omega, \\ u(x,0) = 0, & x \in \partial\Omega. \end{cases} \tag{17}$$

The second of our main results will prove that the above solution may become negative if $f(x,t)$ is suitably negative due to the structure of the homogenized problem under the critical size assumption. Something that, as indicated in the Introduction, can not occur if the balance between the sizes is not critical. More precisely, we have

Theorem 3 *Let* $f \in H^1\big(0,T;L^2(\Omega)\big)$, $u^0 \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $u^0 \geq 0$, *and assume that the unique solution of* (16) *is such that* $u \in L^\infty(Q^T)$. *Assume that there exists an interval* $[t_0,t_1] \subset [0,T]$, *such that the set* $Q_{f,\Lambda} := \{(x,t) \in \Omega \times [t_0,t_1] : f(x,t) \leq \Lambda < 0\}$ *is not empty for some* $\Lambda < 0$ *and* $\delta > 0$ *such that*

$$-\Lambda > \mathcal{A}_n\delta + \mathcal{A}_n\left(e^{-B_n t_0}\left\|u^0\right\|_{L^\infty(\Omega)} + \frac{\|u\|_{L^\infty(Q^T)}\left(e^{B_n t_1}-1\right)}{B_n}\right). \tag{18}$$

Then, $u(x,t) \leq -\delta$ *for a.e.* $(x,t) \in Q_{f,\Lambda}$ *and for any* $t \in [t_0,t_1]$ *such that* $d((x,t),\partial Q_{f,\Lambda}) \geq R$ *with*

$$\begin{cases} R = \left(\dfrac{\|u\|_{L^\infty(Q^T)}+\delta}{C}\right)^{\frac{1}{2}}, \\ C = \dfrac{-\Lambda - \mathcal{A}_n\left(\delta + e^{-B_n t_0}\left\|u^0\right\|_{L^\infty(\Omega)} + \frac{\|u\|_{L^\infty(Q^T)}\left(e^{B_n t_1}-1\right)}{B_n}\right)}{2n}. \end{cases} \tag{19}$$

Finally, let $t_0 = t_1 = 0$. *Assume* $f \in H^1(0,T;L^2(\Omega))$, *with* $f(.,0) \in L^\infty(\Omega)$, *and that the set* $\Omega_{f(.,0),\Lambda} := \{x \in \Omega : f(x,0) \leq \Lambda < 0\}$ *is not empty for some* $\Lambda < 0$ *and* $\delta > 0$ *such that*

$$-\Lambda > \mathcal{A}_n(\delta + \left\|u^0\right\|_{L^\infty(\Omega)}). \tag{20}$$

Then, $u(x,0) \leq -\delta$ *for a.e.* $x \in \Omega_{f(.,0),\Lambda}$ *such that* $d(x,\partial\Omega_{f(.,0),\Lambda}) \geq R$, *with*

$$\begin{cases} R = \left(\dfrac{\|f(\cdot,0)\|_{L^\infty(\Omega)} + \mathcal{A}_n^2\delta + \mathcal{A}_n\left\|u^0\right\|_{L^\infty(\Omega)}}{\mathcal{A}_n C}\right)^{\frac{1}{2}}, \\ C = \dfrac{-\Lambda - \mathcal{A}_n\left(\delta + \left\|u^0\right\|_{L^\infty(\Omega)}\right)}{2n}. \end{cases} \tag{21}$$

The proofs of the two main results will be given in the next section. We point out that some sufficient conditions on the data implying the boundedness of the unique solution of (16) were indicated in Remark 2.

Now, we shall study the nonlocal operator H_{u,u^0} since it will simplify the details of the proofs in the next section.

Proposition 1 *Given* $\phi \in L^2(0,T)$, $u^0 \geq 0$ *and a positive constant* $\mathcal{B}_n$, *there exists a unique solution* $H_{\phi,u^0} \in H^1(0,T)$ *of the problem* (15) *and the following estimates hold:*
i) We have

$$\|H_{\phi,u^0}\|_{C([0,T])} \leq u^0 + \mathcal{B}_n\|\phi\|_{L^2(0,T)}, \tag{22}$$

Moreover, if $\phi \in L^p(0,T)$, *for some* $p \in (2,+\infty]$, *then* $H_{\phi,u^0} \in W^{1,p}(0,T)$.

ii) Given $\phi_1, \phi_2 \in L^2(0,T)$ and $u_1^0,\ u_2^0 \geq 0$ we have

$$\| \left[H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0} \right]^{\pm} \|_{C([0,T])} \leq \left[u_1^0 - u_2^0 \right]^{\pm} + \mathcal{B}_n \| \left[\phi_1 - \phi_2 \right]^{\pm} \|_{L^2(0,T)}, \tag{23}$$

(i.e., the inequality holds for the positive and negative parts of the corresponding expressions). In particular

$$\| H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0} \|_{C([0,T])} \leq \left| u_1^0 - u_2^0 \right| + \mathcal{B}_n \| \phi_1 - \phi_2 \|_{L^2(0,T)}. \tag{24}$$

iii) Under the assumptions of ii) we also have

$$\left\{ \begin{array}{l} \dfrac{1}{2} \left| u_1^0 - u_2^0 \right|^2 + \mathcal{B}_n \displaystyle\int_0^T (H_{\phi_1,u_1^0}(t) - H_{\phi_2,u_2^0}(t))(\phi_1(t) - \phi_2(t)) dt \\ \quad \geq \dfrac{1}{2} \left| (H_{\phi_1,u_1^0}(T) - H_{\phi_2,u_2^0}(T)) \right|^2 + \mathcal{B}_n \displaystyle\int_0^T (H_{\phi_1,u_1^0}(t) - H_{\phi_2,u_2^0}(t))^2 dt. \end{array} \right. \tag{25}$$

Proof. There are several equivalent formulations to problem (15). For instance, it can be reformulated in terms of the following evolution variational inequality: find $H_{\phi,u^0} \in H^1(0,T)$, $H_{\phi,u^0} \geq 0$ on $[0,T]$, $H_{\phi,u^0}(0) = u^0$ (we recall that $H^1(0,T) \subset C([0,T])$) such that

$$\int_0^T (\partial_t H_{\phi,u^0} + \mathcal{B}_n H_{\phi,u^0} - \mathcal{B}_n \phi)(v - H_{\phi,u^0}) dt \geq 0, \tag{26}$$

for any $v \in L^2(0,T)$, $v \geq 0$ a.e. $t \in (0,T)$. Another equivalent formulation arises by using the notion of subdifferential of a convex function on a Hilbert space H (in our case simply H $= \mathbb{R}$). So, problem (15) can be also reformulated as the Cauchy problem for the maximal monotone operator $A : D(A) \to \mathcal{P}(\mathbb{R})$ defined by

$$\left\{ \begin{array}{l} D(A) = [0, +\infty) \\ A(u) = \left\{ \begin{array}{ll} \mathcal{B}_n u & \text{if } u > 0, \\ (-\infty, 0] & \text{if } u = 0. \end{array} \right. \end{array} \right.$$

It is clear that $A = \partial j$, the subdifferential of the convex function $j : D(j) \to \mathbb{R}$ given by

$$\left\{ \begin{array}{l} D(j) = [0, +\infty) \\ j(u) = \left\{ \begin{array}{ll} \frac{\mathcal{B}_n u^2}{2} & \text{if } u > 0 \\ +\infty & \text{if } u = 0. \end{array} \right. \end{array} \right.$$

This is an easy modification of Example 2.8.1 of [8]. Thus, problem (15) becomes

$$\left\{ \begin{array}{l} \dfrac{d}{dt} H_{\phi,u^0}(t) + \partial j(H_{\phi,u^0}(t)) \ni \mathcal{B}_n \phi(t) \quad t \in (0,T), \\ H_{\phi,u^0}(0) = u^0. \end{array} \right.$$

Then, the proof of i) is a mere application of Théoreme 3.4 and Proposition 3.4 of [8]. Indeed, the operator A satisfies that $D(A)$ is closed and its minimal section $A^0(u)$ is given by

$$A^0(u) = \mathrm{Proj}_{A(u)} 0 = \left\{ \begin{array}{ll} \mathcal{B}_n u & \text{if } u > 0, \\ 0 & \text{if } u = 0, \end{array} \right.$$

where $\mathrm{Proj}_{A(u)} 0$ means the Projection of 0 on the set $A(u)$ (see, [8], page 26). In particular, $A^0(u)$ is bounded on the compact subsets of $D(A)$ and Proposition 3.4 of [8] can be applied.
The proof of ii) is a trivial consequence of the $T-$monotonicity of operator A (see, for instance, [8], [7] and [5]). In our case, the $T-$monotonicity of our operator A is a simple exercise since we have

$$(a-b)\,[u-v]^{+} \geq 0, \text{ for any } (a,b) \in A(u) \times A(v).$$

The final inequality is a simple consequence of the fact that $|h| = h^{+} + h^{-}$ and the previous inequalities of ii) (a direct proof of this estimate is given in Lemma 3.1 of [8]).
The proof of iii) is again a simple consequence of the monotonicity of the operator A (see, e.g. estimate (27) of Lemma 3.1 of [8]). ∎

Remark 3 It is not difficult to give a direct proof of Proposition 1 without passing by the abstract theory of maximal monotone operators. For instance, the existence of solutions can be also obtained by application of a penalization argument, i.e., by considering the approximate problem

$$\begin{cases} \partial_t H_{\phi,u^0} + \mathcal{B}_n H_{\phi,u^0} + \delta^{-1}\left[H_{\phi,u^0}\right]^{-} = \mathcal{B}_n \phi, \quad t \in (0,T), \\ H_{\phi,u^0}(0) = u^0. \end{cases} \tag{27}$$

In fact, the above approximate operator coincides with the so called Yosida approximation of the operator A (see, [8] Section 2.8). We also point out that when $u_1^0 = u_2^0$ then property (25) indicates the monotone dependence of the strange term H_{ϕ,u^0} with respect to the datum ϕ: something already found for a different family of problems with Robin type dynamic boundary conditions (see [19]). Here we used of the abstract theory for the proof of the above Proposition to point out that some similar results hold for much more general frameworks.

The qualitative properties of the operator H_{ϕ,u^0} indicated in Proposition 1 allow to get easily a proof of the existence and uniqueness of solution to the homogenized problem (16) indicated in the statement of Theorem 2. We say that the function $u \in L^2\big(0,T;H_0^1(\Omega)\big)$ is a weak solution to the problem (16) if $H_{u,u^0} \in H^1\big(0,T;L^2(\Omega)\big)$ and the following integral identity holds

$$\int\limits_{Q^T} \nabla u \nabla v dxdt + \mathcal{A}_n \int\limits_{Q^T} (u - H_{u,u^0}) v dxdt = \int\limits_{Q^T} f v dxdt \tag{28}$$

for any arbitrary function $v \in L^2(0,T;H_0^1(\Omega))$.

Proposition 2 *Let $n \geq 3$, $f \in H^1\big(0,T;L^2(\Omega)\big)$, $u^0 \in H_0^1(\Omega)$, $u^0 \geq 0$. Then there exits a unique $u \in L^2\big(0,T;H_0^1(\Omega)\big)$ weak solution to the problem (16). In addition, $u \in H^1\big(0,T;L^2(\Omega)\big)$ and $u(x,0)$ is given as the unique solution of the stationary problem (5).*

Proof. Given $u^0 \in H_0^1(\Omega)$, $u^0 \geq 0$, we consider the operator $\mathrm{A} : V{=}L^2\big(0,T;H_0^1(\Omega)\big) \to V'{=}L^2\big(0,T;H^{-1}(\Omega)\big)$ defined by the relation

$$\langle \mathrm{A}(u), v \rangle = \int\limits_0^T \int\limits_\Omega \nabla u \nabla v dxdt + \mathcal{A}_n \int\limits_0^T \int\limits_\Omega (u - H_{u,u^0}) v dxdt.$$

Let us show that this operator is coercive, continuous and monotone. First, by iii) of Proposition 1 we get the monotonicity of A since

$$\begin{aligned} \langle \mathrm{A}(u) - \mathrm{A}(v), u - v \rangle &= \|\nabla(u-v)\|^2_{L^2(Q^T)} + \mathcal{A}_n \int\limits_{Q^T} ((u-v) - (H_{u,u^0} - H_{v,u^0}))(u-v) dxdt \\ &= \|\nabla(u-v)\|^2_{L^2(Q^T)} + \mathcal{A}_n \|u-v\|^2_{L^2(Q^T)} - \mathcal{A}_n \int\limits_{Q^T} (H_{u,u^0} - H_{v,u^0})(u-v) dxdt \geq 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle \mathrm{A}(u), v \rangle &\leq \|\nabla u\|_{L^2(Q^T)} \|\nabla v\|_{L^2(Q^T)} + \mathcal{A}_n \|u - H_{u,u^0}\|_{L^2(Q^T)} \|v\|_{L^2(Q^T)} \\ &\leq K \|u\|_{L^2(0,T;H_0^1(\Omega))} \|v\|_{L^2(0,T;H_0^1(\Omega))}, \end{aligned}$$

for some positive constant K and thus the boundedness of the operator A follows. Moreover, we have

$$\begin{aligned}\langle \mathrm{A}(u)-\mathrm{A}(v),w\rangle &= \int\limits_{Q^T}\nabla(u-v)\nabla w dx+\mathcal{A}_n\int\limits_{Q^T}\big((u-v)-(H_{u,u^0}-H_{v,u^0})\big)wdxdt\\ &\leq \|\nabla(u-v)\|_{L^2(Q^T)}\|\nabla w\|_{L^2(Q^T)}+K\|u-v\|_{L^2(Q^T)}\|w\|_{L^2(Q^T)}\\ &\leq K\|u-v\|_{L^2(0,T;H_0^1(\Omega))}\|w\|_{L^2(0,T;H_0^1(\Omega))},\end{aligned}$$

for some positive constant K, which leads to the continuity of A.
Finally, the coerciveness of A follows from the estimate

$$\langle \mathrm{A}(u),u\rangle=\|\nabla u\|^2_{L^2(Q^T)}+\mathcal{A}_n\int\limits_{Q^T}(u-H_{u,u^0})udxdt\geq\|u\|^2_{L^2(0,T;H_0^1(\Omega))}.$$

Therefore, we can apply the theory of monotone operators (see [25]) and we get the existence of a weak solution to (16) for any arbitrary function $v\in L^2\big(0,T;H^{-1}(\Omega)\big)$. The uniqueness of solutions is a direct consequence of the monotonicity of the operator A. Indeed, if two solutions u_1, u_2 exists, then we take u_2 as a test function in the integral identity for u_1 and vice versa. Subtracting one identity from the other, we get

$$0=\int\limits_{Q^T}|\nabla(u_1-u_2)|^2dxdt+\mathcal{A}_n\int\limits_{Q^T}((u_1-u_2)-(H_{u_1,u^0}-H_{u_2,u^0}))(u_1-u_2)dxdt.$$

Taking into account the inequality contained in the formulation of (16) we conclude

$$\int\limits_0^T\int\limits_\Omega(u_1-u_2)^2dxdt\geq\int\limits_0^T\int\limits_\Omega(H_{u_1,u^0}-H_{u_2,u^0})(u_1-u_2)dxdt$$

which implies that $u_1=u_2$ in $L^2(0,T;H_0^1(\Omega))$.
Since $f\in H^1\big(0,T;L^2(\Omega)\big)$ and $H_{u,u^0}\in H^1\big(0,T;L^2(\Omega)\big)$ we deduce that $u\in H^1\big(0,T;L^2(\Omega)\big)$. Using that $H^1\big(0,T;L^2(\Omega)\big)\subset C\big([0,T];L^2(\Omega)\big)$ we conclude that $u(x,0)$ is, at least, a very weak solution (by integrating twice by parts, after multiplying by a test function) of the stationary problem (5). In fact, as $f(\cdot,0)+\mathcal{A}_nu^0(\cdot)\in L^2(\Omega)$, by the uniqueness of solutions, we deduce that $u(\cdot,0)\in H_0^1(\Omega)$ and thus $u(x,0)$ satisfies (5) in a standard weak sense. ∎

Remark 4 As mentioned in the Introduction, if we take $\beta\partial_tu_\varepsilon$, $\beta>0$ in the boundary condition for the problem (8), and, for simplicity we assume $u^0=0$, then we will have $\beta(H_{\phi,0})'$ in the problem (15). Passing to the limit as $\beta\to0$ we get the complementarity conditions for the stationary problem: $H_{\phi,0}\geq\phi$, $H_{\phi,0}\geq0$, $H_{\phi,0}(H_{\phi,0}-\phi)=0$. In that case, this problem has an obvious solution $H_{\phi,0}=\phi^+$, hence, the "strange" term was only dependent on u^+ (although it was added to a different term $\mathcal{A}_n[u]^-$), such as it was obtained in [3], [11], [13], [24], [14] and [15]. Here, in the case of dynamic unilateral boundary conditions, the expression of the strange term H_{u,u^0} involves the own unknown u (i.e., the positive and negative parts of u).

4 Proof of Theorems 2 and 3.

Proof of Theorem 2. First step. Let us assume, additionally that $u^0\in H_0^1(\Omega)\cap W^{1,\infty}(\Omega)$, $u^0\geq0$. From the variational inequality (7), we conclude that u_ε also satisfies the following integral inequality in a weaker form

$$\int\limits_0^T\int\limits_{\Omega_\varepsilon}\nabla v\nabla(v-u_\varepsilon)dxdt+\varepsilon^{-\gamma}\int\limits_0^T\int\limits_{S_\varepsilon}\partial_tv(v-u_\varepsilon)dsdt\geq\int\limits_0^T\int\limits_{\Omega_\varepsilon}f(v-u_\varepsilon)dxdt-\frac12\varepsilon^{-\gamma}\|u^0(.)-v(.,0)\|^2_{L^2(S_\varepsilon)},\tag{29}$$

where v is any arbitrary function in $\mathcal{K}_\varepsilon$ with $\partial_t v \in L^2(0,T;L^2(S_\varepsilon))$.
Let $\phi(x,t) = \psi(x)\eta(t)$, $\psi \in C_0^\infty(\Omega)$, $\eta \in C^1([0,T])$. We define the function $H^{\varepsilon,j}_{\phi,u^0}(t)$, $j \in \Upsilon_\varepsilon$ as the unique solution to the problem

$$\begin{cases} \left.\begin{array}{l} \partial_t H^{\varepsilon,j}_{\phi,u^0} + \mathcal{B}_n H^{\varepsilon,j}_{\phi,u^0} \geq \mathcal{B}_n \phi(P^j_\varepsilon, t),\ H^{\varepsilon,j}_{\phi,u^0} \geq 0, \\ H^{\varepsilon,j}_{\phi,u^0_\varepsilon}\big(\partial_t H^{\varepsilon,j}_{\phi,u^0_\varepsilon} + \mathcal{B}_n(H^{\varepsilon,j}_{\phi,u^0} - \phi(P^j_\varepsilon,t))\big) = 0, \end{array}\right\} \text{ on } (0,T), \\ H^{\varepsilon,j}_{\phi,u^0}(0) = u^0(P^j_\varepsilon). \end{cases} \tag{30}$$

Note that $H^{\varepsilon,j}_{\phi,u^0}(t) = H_{\phi,u^0}(P^j_\varepsilon,t)$, with $j \in \Upsilon_\varepsilon$, where, in general, $H_{\phi,u^0}(x,t)$ denotes the unique solution to the problem

$$\begin{cases} \left.\begin{array}{l} \partial_t H_{\phi,u^0}(x,t) + \mathcal{B}_n H_{\phi,u^0}(x,t) \geq \mathcal{B}_n \phi(x,t),\ H_{\phi,u^0}(x,t) \geq 0, \\ H_{\phi,u^0}(x,t)\big(\partial_t H_{\phi,u^0}(x,t) + \mathcal{B}_n(H_{\phi,u^0}(x,t) - \phi(x,t)\big) = 0, \end{array}\right\} \ t \in (0,T), \\ H_{\phi,u^0}(x,0) = u^0(x), \end{cases} \tag{31}$$

where $x \in \Omega$ is taken as a parameter.
We define an auxiliary function $w^j_\varepsilon(x)$, $j \in \Upsilon_\varepsilon$, as the unique solution to the *capacity* linear boundary value problem

$$\begin{cases} \Delta w^j_\varepsilon = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ w^j_\varepsilon(x) = 1, & x \in \partial G^j_\varepsilon, \\ w^j_\varepsilon(x) = 0, & x \in \partial T^j_{\varepsilon/4}. \end{cases} \tag{32}$$

Since the perforations (or particles) are balls, we can find the explicit form of the solution to this problem

$$w^j_\varepsilon(x) = \frac{|x - P^j_\varepsilon|^{2-n} - (\varepsilon/4)^{2-n}}{a_\varepsilon^{2-n} - (\varepsilon/4)^{2-n}}.$$

(this is the main reason of the assumption $n \geq 3$: the case $n = 2$ needs suitable ad hoc arguments and this result does not hold if $n = 1$ (see, e.g., the exposition made in Section 3.1.5 of [16] and its references)).
Now we introduce the auxiliary function

$$W_{\varepsilon,\phi}(x,t) = \begin{cases} w^j_\varepsilon(x)(\phi(x,t) - H^{\varepsilon,j}_{\phi,u^0}(t)), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\cup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}. \end{cases} \tag{33}$$

Using the properties of the functions $w^j_\varepsilon(x)$ and $H^{\varepsilon,j}_{\phi,u^0}(t)$, we conclude that $W_{\varepsilon,\phi} \in H^1(Q^T_\varepsilon)$. It is easy to see that $P_\varepsilon W_{\varepsilon,\phi} \rightharpoonup 0$ weakly in $L^2\big(0,T;H^1_0(\Omega)\big)$ and that $\partial_t(P_\varepsilon W_{\varepsilon,\phi}) \rightharpoonup 0$ weakly in $L^2(0,T;L^2(\Omega))$ as $\varepsilon \to 0$.
Finally we consider the "oscillating test function" $v = \phi(x,t) - W_{\varepsilon,\phi}$, in (29). Note also that this choice of v does not coincide with the test function taken for Robin type dynamic boundary conditions ([17], [19]). We have that $v \in \mathcal{K}_\varepsilon$. Indeed, for $x \in \partial G^j_\varepsilon$, $t \in [0,T]$, we have

$$v(x,t) = \phi(x,t) - \phi(x,t) + H^{\varepsilon,j}_{\phi,u^0}(t) = H^{\varepsilon,j}_{\phi,u^0}(t) \geq 0.$$

Thus, we get

$$\begin{aligned} &\varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_0^T \int_{S\varepsilon} \partial_t H^{\varepsilon,j}_{\phi,u^0}(t)(H^{\varepsilon,j}_{\phi,u^0}(t) - u_\varepsilon(s,t))dsdt \\ &+ \int_0^T \int_{\Omega\varepsilon} \nabla(\phi - W_{\varepsilon,\phi})\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt \geq \int_0^T \int_{\Omega_\varepsilon} f(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt. \end{aligned} \tag{34}$$

Using the properties of $W_{\varphi,\varepsilon}$, we derive

$$\lim_{\varepsilon\to0} \int_0^T \int_{\Omega_\varepsilon} f(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt = \int_{Q^T} f(\phi - u)dxdt, \tag{35}$$

and

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon}\nabla\phi\nabla(\phi-W_{\varepsilon,\phi}-u_\varepsilon)dxdt=\int_{Q^T}\nabla\phi\nabla(\phi-u)dxdt. \tag{36}$$

From the definition of $W_{\varepsilon,\phi}$, we get

$$-\int_0^T\int_{\Omega_\varepsilon}\nabla W_{\varepsilon,\phi}\nabla(\phi-W_{\varepsilon,\phi}-u_\varepsilon)dxdt=J_\varepsilon+\alpha_\varepsilon \tag{37}$$

where

$$J_\varepsilon\equiv-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla w^j_\varepsilon(x)\nabla((\phi(x,t)-H^{\varepsilon,j}_{\phi,u^0}(t))(\phi-W_{\varepsilon,\phi}-u_\varepsilon))dxdt$$

and $\alpha_\varepsilon\to 0$, as $\varepsilon\to 0$. Using Green's formula, we obtain

$$\begin{aligned}J_\varepsilon\ &=-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G^j_\varepsilon}\partial_\nu w^j_\varepsilon(\phi(s,t)-H^{\varepsilon,j}_{\phi,u^0}(t))(H^{\varepsilon,j}_{\phi,u^0}(t)-u_\varepsilon(s,t))dsdt-\\&-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon(s)(\phi(s,t)-H^{\varepsilon,j}_{\phi,u^0}(t))(\phi(s,t)-u_\varepsilon(s,t))dsdt.\end{aligned} \tag{38}$$

Taking into account that $\partial_\nu w^j_\varepsilon|_{\partial G^j_\varepsilon}=\mathcal{B}_n\varepsilon^{-\gamma}+\beta_\varepsilon$, where $\beta_\varepsilon\to 0$ as $\varepsilon\to 0$, we have

$$\begin{aligned}J_\varepsilon\ &=-\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G^j_\varepsilon}\mathcal{B}_n(\phi(s,t)-H^{\varepsilon,j}_{\phi,u^0}(t))(H^{\varepsilon,j}_{\phi,u^0}(t)-u_\varepsilon(s,t))dsdt-\\&-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon(s)(\phi(s,t)-H^{\varepsilon,j}_{\phi,u^0}(t))(\phi(s,t)-u_\varepsilon(s,t))dsdt+m_\varepsilon,\end{aligned} \tag{39}$$

where $m_\varepsilon\to 0$, as $\varepsilon\to 0$.
As in Remark 4.40 of [16], now the crucial argument consists in considering the total balance of the integrals over S_ε

$$\begin{aligned}&\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G^j_\varepsilon}((H^{\varepsilon,j}_{\phi,u^0})'+\mathcal{B}_n(H^{\varepsilon,j}_{\phi,u^0}(t)-\phi(P^j_\varepsilon,t))(H^{\varepsilon,j}_{\phi,u^0}(t)-u_\varepsilon(s,t))dsdt-\\&-\mathcal{B}_n\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G^j_\varepsilon}(\phi(s,t)-\phi(P^j_\varepsilon,t))(H^{\varepsilon,j}_{\phi,u^0}(t)-u_\varepsilon(s,t))dsdt\equiv K_{1,\varepsilon}+K_{2,\varepsilon}.\end{aligned} \tag{40}$$

Using that $H^{\varepsilon,j}_{\phi,u^0}(t)$ is a solution of the problem (30) and $u_\varepsilon\geq 0$ on S_ε, we conclude

$$K_{1,\varepsilon}=\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G^j_\varepsilon}((H^{\varepsilon,j}_{\phi,u^0})'+\mathcal{B}_n(H^{\varepsilon,j}_{\phi,u^0}-\phi(P^j_\varepsilon,t)))(H^{\varepsilon,j}_{\phi,u^0}-u_\varepsilon)dsdt\leq 0. \tag{41}$$

Additionally, we have

$$\lim_{\varepsilon\to 0}K_{2,\varepsilon}=0. \tag{42}$$

For the second term in (39), we can apply the "from surface to volume averaging convergence principle" (see Theorem 4.5 and the proof of Theorem 4.36 of [16]) and we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}} &\partial_\nu w^j_\varepsilon(s)(\phi(x,t)-H_{\phi,u^0}(s,t))(\phi(s,t)-u_\varepsilon(s,t))dsdt\\ &= \mathcal{A}_n\int_0^T\int_\Omega(\phi(x,t)-H_{\phi,u^0}(x,t))(\phi(x,t)-u(x,t))dxdt.\end{aligned} \tag{43}$$

Then, we have to prove that

$$I_\varepsilon=\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon(s)(H^{\varepsilon,j}_{\phi,u^0}(P^j_\varepsilon,t)-H_{\phi,u^0}(s,t))(\phi(s,t)-u_\varepsilon(s,t))dsdt\to 0.$$

Taking into account (24), since

$$\left|\partial_\nu w^j_\varepsilon|_{\partial T^j_{\varepsilon/4}}\right|\le C\varepsilon,$$

we conclude

$$|I_\varepsilon|\le C\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}}\left|u^0(P^j_\varepsilon)-u^0(s)\right||\phi(s,t)-u_\varepsilon(s,t)|\,dsdt+\kappa_\varepsilon$$

with $\kappa_\varepsilon\to 0$ as $\varepsilon\to 0$. Since u^0 is Lipschitz continuous (remeber that $u^0\in W^{1,\infty}(\Omega)$ in this first step)

$$\left|u^0(P^j_\varepsilon)-u^0(s)\right|\le K\left|P^j_\varepsilon-s\right|\le K\varepsilon.$$

Then

$$\begin{aligned}|I_\varepsilon|&\le K\varepsilon\,\|\phi\|_{L^\infty(\overline{Q^T})}+C\varepsilon^2\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}}|u_\varepsilon(s,t)|\,dsdt\\ &\le K\varepsilon\,\|\phi\|_{L^\infty(\overline{Q^T})}+C\varepsilon^2\left|\sum_{j\in\Upsilon_\varepsilon}\varepsilon^{n-1}\right|^{1/2}\left(\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T^j_{\varepsilon/4}}|u_\varepsilon(s,t)|^2\,dsdt\right)^{1/2}.\end{aligned}$$

Using now the a priori estimate

$$\|u_\varepsilon\|^2_{L^2\left((0,T)\times\left(\bigcup_{j\in\Upsilon_\varepsilon}\partial T^j_{\varepsilon/4}\right)\right)}\le K\varepsilon^{-1}$$

(see, Remark 1 of [27] making there $a_\varepsilon=C\varepsilon$) we get that

$$|I_\varepsilon|\le C\varepsilon.$$

Combining (34)-(43), and using a density argument we conclude that u satisfies the inequality

$$\int_{Q^T}\nabla\phi\nabla(\phi-u)dxdt+\mathcal{A}_n\int_{Q^T}(\phi-H_{\phi,u^0})(\phi-u)dxdt\ge\int_{Q^T}f(\phi-u)dxdt, \tag{44}$$

where ϕ is an arbitrary test function in $L^2\big(0,T;H^1_0(\Omega)\big)$. From here, using the hemicontinuity of the operator $\phi\to H_{\phi,u^0}$ (as in Théorem 2.2 of [25], and Proposition 2), we derive that u is the unique solution to the problem (16).

Second step. We assume now that $u^0\in H^1_0(\Omega)$, $u^0\ge 0$. Let $u^{0,m}\in H^1_0(\Omega)\cap W^{1,\infty}(\Omega)$, $u^{0,m}\ge 0$ such that $u^{0,m}\to u^0$ in $H^1_0(\Omega)$. Then, by the first step, we know that there exists a sequence $\{u^m\}$ of limit solutions of the corresponding sequence $\{u_\varepsilon^m\}$ such that u^m satisfies the limit problem (4) and then, equivalently, the variational inequality

$$\int_{Q^T} \nabla\phi\nabla(\phi - u^m)dxdt + \mathcal{A}_n \int_{Q^T} (\phi - H_{\phi,u^{0,m}})(\phi - u^m)dxdt \geq \int_{Q^T} f(\phi - u^m)dxdt, \tag{45}$$

for any test function $\phi \in L^2(0,T;H_0^1(\Omega))$. Since $u^{0,m}(x) \to u^0(x)$ a.e. $x \in \Omega$, by (24) we get that $H_{\phi,u^{0,m}} \to H_{\phi,u^0}$ in $C([0,T];L^2(\Omega))$.
On the other hand, from equation (4) for u^m, by multiplying by u^m, we get that $\{u^m\}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Then there exists $u \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $u^m \rightharpoonup u$ weakly in $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ with u satisfying (44) with the strange term H_{ϕ,u^0}, for any test function $\phi \in L^2(0,T;H_0^1(\Omega))$. In fact, we also know that u is the unique solution of (4) corresponding to the initial datum u^0.
Finally, let $\{u_{\varepsilon,m}\}$ be the sequence of solutions of problem (8) corresponding to the sequence of initial data $\{u^{0,m}\}$. Then, for any test function $\phi \in L^2(0,T;H_0^1(\Omega))$, we can use the following diagonal argument

$$\int_{Q^T} (P_\varepsilon u_\varepsilon - u)\phi dxdt = \int_{Q^T} \{(P_\varepsilon u_\varepsilon - P_\varepsilon u_{\varepsilon,m})\phi + (P_\varepsilon u_{\varepsilon,m} - u^m)\phi + (u^m - u)\phi\} dxdt.$$

Then, using the monotonicity of the nonlinear term in the dynamic boundary condition we get the weak convergence of $P_\varepsilon u_{\varepsilon,m} \rightharpoonup P_\varepsilon u_\varepsilon$, for any $\varepsilon > 0$, as $m \to +\infty$, since $u^{0,m} \to u^0$ in $H_0^1(\Omega)$. Moreover, by applying the first step 1 we get the weak convergence $P_\varepsilon u_{\varepsilon,m} \rightharpoonup u^m$ (which in fact is uniform in m: recall the estimates given in Theorem 1), and then we get that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$, with u solution of (4) and the proof of the result is complete.■

Proof of Theorem 3. Given $t \in [t_0,t_1]$ and $(x_0,t) \in Q_{f,\Lambda}$ we will use the local barrier function

$$\overline{u}(x;x_0) = C\left|x - x_0\right|^2 - \delta,$$

with $C > 0$ to be chosen later. We have (see, e.g., Remark 2.7 of [12]) that

$$-\Delta\overline{u} = -2nC.$$

On the other hand, we know that

$$0 \leq H_{u,u^0}(t) \leq H_{u^+,u^0}(t) \leq H_{\|u\|_{L^\infty(Q^T)},\|u^0\|_{L^\infty(\Omega)}}(t),$$

for any $t \in [0,T]$. Then, for any $t \in [t_0,t_1]$

$$\begin{aligned} H_{\|u\|_{L^\infty(Q^T)},\|u^0\|_{L^\infty(\Omega)}}(t) &= e^{-B_n t}\left\|u^0\right\|_{L^\infty(\Omega)} + \int_0^t e^{B_n(t-s)}\left\|u\right\|_{L^\infty(Q^T)} ds \\ &\leq e^{-B_n t_0}\left\|u^0\right\|_{L^\infty(\Omega)} + \frac{\left\|u\right\|_{L^\infty(Q^T)}(e^{B_n(t_1-t)} - 1)}{B_n}. \end{aligned}$$

Thus we have

$$\begin{aligned} -\Delta\overline{u} + \mathcal{A}_n(\overline{u} - H_{u,u^0}(t)) &\geq -2nC - \mathcal{A}_n\Bigg(\delta - e^{-B_n t_0}\left\|u^0\right\|_{L^\infty(\Omega)} \\ &\qquad - \frac{\left\|u\right\|_{L^\infty(Q^T)}(e^{B_n(t_1-t_0)} - 1)}{B_n}\Bigg) \\ &\geq \Lambda \geq f(x,t) \text{ on } Q_{f,\Lambda}, \end{aligned}$$

if we assume C given by (19), thanks to the assumption (18). Then, if $B_R(x_0) \times [t_0,t_1] \subset Q_{f,\Lambda}$ we get that $\overline{u}(x;x_0)$ will be a local supersolution assumed that

$$CR^2 - \delta \geq \left\|u\right\|_{L^\infty(Q^T)}.$$

This is satisfied once we take R given by (19). Then, by the comparison principle for the operator $u \to -\Delta u + \mathcal{A}_n u$ on $B_R(x_0)$, with Dirichlet conditions on $\partial B_R(x_0)$, we get that

$$u(x,t) \leq C\,|x - x_0|^2 - \delta \text{ a.e. } x \in B_R(x_0) \text{ and for any } t \in [t_0, t_1], \tag{46}$$

which implies the first part of the result.
Finally, if assume that $t_0 = t_1 = 0$, then we can use, directly that $u(x,0)$ is the weak solution of the stationary problem (5). Thus we repeat the above arguments but in a more direct way. We assume that the set $\Omega_{f(.,0),\Lambda} := \{x \in \Omega : f(x,0) \leq \Lambda < 0\}$ is not empty for some $\Lambda < 0$ small enough and consider again the local barrier function

$$\overline{u}(x;x_0) = C\,|x - x_0|^2 - \delta,$$

Thus we have

$$-\Delta\overline{u} + \mathcal{A}_n\overline{u} - \mathcal{A}_n u^0 \geq -2nC - \mathcal{A}_n\delta - \mathcal{A}_n \left\|u^0\right\|_{L^\infty(\Omega)} \geq \Lambda \geq f(x,t) \text{ on } Q_{f,\Lambda},$$

if we assume (20). Then, if $B_R(x_0) \subset \Omega_{f(.,0),\Lambda}$, since we have the estimate

$$\|u(.,0)\|_{L^\infty(\Omega)} \leq \frac{\|f(\cdot,0)\|_{L^\infty(\Omega)}}{\mathcal{A}_n} + \left\|u^0\right\|_{L^\infty(\Omega)},$$

we get that $\overline{u}(x;x_0)$ will be a local supersolution assumed that

$$CR^2 - \mathcal{A}_n\delta \geq \frac{\|f(\cdot,0)\|_{L^\infty(\Omega)}}{\mathcal{A}_n} + \left\|u^0\right\|_{L^\infty(\Omega)}.$$

This is satisfied once we take R given by (21). Then, by the comparison principle for the operator $u \to -\Delta u + \mathcal{A}_n u$ on $B_R(x_0)$, with Dirichlet conditions on $\partial B_R(x_0)$, we get that

$$u(x,0) \leq C\,|x - x_0|^2 - \delta \text{ a.e. } x \in B_R(x_0), \tag{47}$$

and thus, $u(x,0) \leq -\delta$ for a.e. $x \in \Omega_{f(.,0),\Lambda}$ such that $d(x, \partial\Omega_{f(.,0),\Lambda}) \geq R$, with R and C satisfying (21). ∎

Remark 5 It seems possible to extend the results of this paper in several directions: the case of $n = 2$ can be also considered, the diffusion operator can be replaced by the quasilinear degenerate p-Laplacian operator (see [2] and [16] for some related papers), it seems possible to add a nonhomogeneous forcing term $g(x,t)$ in the right hand side of the dynamic boundary condition (see, e.g. [17], [16] for some related studies) and to improve the regularity on the initial datum, etc. The more delicate new aspect would be the consideration of perforations (or particles) of arbitrary shape. The previous paper [19] was dealing with the case of Robin type dynamic boundary conditions and the extension to the case of Signorini nonlinear terms will require the correct definition of some auxiliary capacity problems. This will be the object of a separated work by the authors.

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Jesus Ildefonso Díaz
Instituto de Matematica Interdisciplinar, Universidad Complutense de Madrid, 28040 ´
Madrid, Spain
Email address: jidiaz@ucm.es

Alexander V. Podol'skii
Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
originalea@ya.ru

Tatiana A. Shaposhnikova
Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
Email address: shaposh.tan@mail.ru