Energy and large time estimates for nonlinear porous medium flow with nonlocal pressure in $\mathbb{R}^N$

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Abstract

We study the general family of nonlinear evolution equations of fractional diffusive type $\partial_t u - \operatorname{div}\left(|u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u]\right) = f$. Such type of nonlocal equations are related to the porous medium equations with a fractional Laplacian pressure. Our study concerns the case in which the flow takes place in the whole space. We consider $m_1, m_2 > 0$, and $s \in (0,1)$, and prove existence of weak solutions. Moreover, when $f \equiv 0$ we obtain the L^p-L^∞ decay estimates of solutions, for $p \geq 1$. Besides, we also investigate the finite time extinction of solution. Our results improve the recent papers in the literature.

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1 Introduction

The main purpose of this paper is to study the following evolution equation of diffusive type with nonlocal effects

$$\begin{cases} \partial_t u - \operatorname{div}\left(|u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u]\right) = f & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

with $m_1, m_2 > 0$, $s \in (0,1)$, and space dimension $N \geq 2$. The symbol $(-\Delta)^{-s}$ denotes by the inverse of the fractional Laplacian operator as usual (see, e.g. [28]).

Equation (1.1) corresponds to the well-known Porous Medium Equation $\partial_t u = \operatorname{div}(u^{m_1}\nabla u)$, when one consider $s = 0$, and $m_2 = 1$. This model arises, for instance, from considering a compressible fluid, with a density distribution $u(x,t)$ and with a Darcy's law leading to the equation

$$u_t - \operatorname{div}(u\nabla p) = 0,$$

where p denotes the pressure. Many other different relations between the density, the velocity and the pressure arise in the applications. For example, the model, proposed by Leibenzon and Muskat states a law in that $p = g(u)$, where g is a nondecreasing scalar function (see more examples in [31]). Such an equation of this type has been studied by many authors (see, e.g. [3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 21, 26, 29, 32]). There are many questions, addressed to the equation of this type, which are being the object of active researches, such as the existence and uniqueness, the regularity, the behaviour of solution in short time and in large time, the finite and infinite speed of propagation, and so on.

Here, we would like to mention specially the recent results, being close to the ones in our paper. It is known that equation (1.1) with $m_1 = m_2 = 1$ reads as: $u_t = \operatorname{div}(u\nabla(-\Delta)^{-s}u)$ was first introduced by Caffarelli and Vázquez, [11]. In [3], Biler et al. studied a particular case of equation (1.1):

$$\partial_t u - \operatorname{div}\left(|u|\nabla^{\alpha-1}(|u|^{m-2}u)\right) = 0,$$

where $\alpha = 2(1-s) \in (0,2), m_1 = 1, m = m_2+1$, and $f = 0$. The authors constructed non-negative self-similar solutions, the so called Barenblatt-Pattle-Zeldovich solutions. Furthermore, they proved the existence of weak solutions for $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and the decay estimate L^1-L^p (see Theorem 6.1) as follows.

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{N(1-\frac{1}{p})}{N(m-1)+\alpha}} \|u_0\|_{L^1}^{\frac{N(m-1)/p+\alpha}{N(m-1)+\alpha}}. \tag{1.2}$$

Thanks to this decay, they also obtained an existence of solution for $u_0 \in L^1(\mathbb{R}^N)$, under the assumptions

$$\begin{cases} m > 1 + \frac{1-\alpha}{N}, \text{ if } \alpha \in (0,1), \\ m > 3 - \frac{2}{\alpha}, \text{ if } \alpha \in [1,2). \end{cases}$$

Equation (1.1) with $s \in (0,1), m_2 = 1, m_1 = m - 1 > 0$, and $f = 0$ was investigated by Stan et al. in [27]. The authors studied the existence of nonnegative weak solutions for all integrable initial data u_0. In addition, they obtained the smoothing effect L^p-L^∞, for $p \geq 1$:

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{N(m-1)+2p(1-s)}} \|u_0\|_{L^p}^{\frac{2p(1-s)}{N(m-1)+2p(1-s)}}, \tag{1.3}$$

with $C = C(N, s, m, p) > 0$. In particular, by considering the case of $p = 1$, (1.3) allows them to obtain the existence result for initial data with bounded measure. Moreover, the finite and infinite speed of propagation have been also studied by the same authors, see [26] (see also [4] for a different equation of this type).

It is also interesting to note that the mean field equation

$$u_t = \text{div}(u\nabla(-\Delta)^{-1}u), \tag{1.4}$$

could be considered as a limit of (1.1) with $m_1 = m_2 = 1$, and $f = 0$, as $s \to 1^-$. In fact, Serfaty and Vázquez [24] proved an existence of solution of (1.4) for all integrable initial data, even for data measure. A uniqueness result was also given in the class of bounded solutions. Furthermore, the solution, constructed in [24] satisfies a universal bound

$$\|u(t)\|_{L^\infty} \leq \frac{C}{t},$$

with $C = C(N) > 0$.
Very recently, Nguyen and Vázquez [21] proved existence of weak solutions of (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N$, with the homogeneous Dirichlet boundary condition. Besides, they also obtained a universal bound

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{1}{m_1+m_2-1}},$$

with $m_1 + m_2 > 1$ and $C = C(N, s, |\Omega|, m_1 + m_2)$.

The main goal of this paper is to carry out a qualitative study of weak solutions of (1.1). We first prove the existence of weak solutions with data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $f \in L^1(Q_T) \cap L^\infty(Q_T)$, where $Q_T = \mathbb{R}^N \times (0, T)$. Moreover, when $f = 0$ we show L^p-L^∞ decay estimates of solutions, for all $p \geq 1$, see Theorem 2 below. We also emphasize that our decay results below holds for $m_1 + m_2 > 1 - \frac{2p(1-s)}{N}$. Thus, we improved the previous range of $m = m_1+m_2 > 1$, described in (1.2) and (1.3). For the case $m_1+m_2 < 1-\frac{2p(1-s)}{N}$ and $f = 0$, we show that every weak solution vanishes in a finite time, see Theorem 3 below. In addition, we also obtain the regularity of

$$\operatorname{div}\left(|u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u]\right) \in L^2\left(0,T,H^{-1}(B_R)\right),$$

for any $R > 0$, if provided that either $s \in [\frac{1}{2}, 1)$, or $m_2 > m_1$. And

$$\operatorname{div}\left(|u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u]\right) \in L^p\left(0,T,W^{-2,p}(\mathbb{R}^N)\right)$$

if provided $s \in (0, \frac{1}{2})$, see Propositions 8 and 9 below. The ones improves the regularity $\operatorname{div}\Theta(u) \in L^1(0,T,(W_0^{2,\infty}(B_R))')$ of Nguyen and Vázquez [21].

Our proof is self contained, and it is merely based on the Fourier analysis and the fundamental estimates of the Riesz potential. This enables us to avoid using the spectral theory approach, which is useful in study the equation of this type on a bounded domain with the homogeneous boundary condition (see e.g. [6, 21, 27]), or avoid using the characterization of Besov and Triebel-Lizorkin space in order to obtain some estimates involving the fractional Sobolev spaces $W^{s,p}$, see e.g. [3].

Definition and main results

Let us put $\Theta(u) = |u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u]$. Now, we introduce first the definition of a weak solution that we are going to use in this paper.

Definition 1. *Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f \in L^1(Q_T) \cap L^\infty(Q_T)$. We say that u is a weak solution of problem (1.1) if $u \in L^1(Q_T) \cap L^\infty(Q_T)$ satisfies $\operatorname{div}\Theta(u) \in L^2(0,T,Y(B_R))$, and*

$$\int_0^T\int_{\mathbb{R}^N} (-u\varphi_t + \Theta(u).\nabla\varphi - f\varphi)\,dxdt = 0, \quad \forall\varphi \in \mathcal{C}_c^\infty(Q_T),$$

where

$$Y(B_R) = \begin{cases} H^{-1}(B_R), & \text{if } s \in [\frac{1}{2}, 1), \\ W^{-2,p}(B_R), & \text{if } s \in (0, \frac{1}{2}). \end{cases}$$

Note that $H^{-1}(B_R)$ (resp. $W^{-2,p}(B_R)$) is the dual space of $H_0^1(B_R)$ $\left(\text{resp. } W_0^{2,p}(B_R)\right)$, and B_R is the ball in $\mathbb{R}^N$, with center at 0 and radius R.

Remark 1. *It follows from the Definition 1 that $u \in \mathcal{C}([0,T];Y(B_R))$, for any $R > 0$. Thus, $u(t)$ possesses an initial trace u_0 in this sense. Particularly, if either $s \in [\frac{1}{2}, 1)$ or $m_2 > m_1$, then $u \in \mathcal{C}([0,T];H^{-1}(B_R))$ for every $R > 0$.*

Under this framework, our existence result is as follows.

Theorem 1. *Let $m_1, m_2 > 0$ and $s \in (0,1)$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f \in L^1(Q_T) \cap L^\infty(Q_T)$. Then, there exists a weak solution u of (1.1). Moreover, u satisfies the following properties:*
i) L^q-estimate: *For any $1 \le q \le \infty$, we have*

$$\|u(t)\|_{L^q} \le \|u_0\|_{L^q} + t^{\frac{q-1}{q}} \|f\|_{L^q(Q_t)}, \quad \text{for a.e. } t \in (0,T). \tag{1.5}$$

Here, we denote $\frac{q-1}{q} = 1$ if $q = \infty$.
ii) Energy estimates:
If $m_2 > m_1$, then there is a constant $C = C(u_0, f, m_1, m_2) > 0$ such that

$$\|(-\Delta)^{\frac{1-s}{2}}[|u|^{m_2-1}u]\|_{L^2(Q_T)} \le C. \tag{1.6}$$

If $m_2 = m_1$, then there is a constant $C = C(u_0, f, m_2) > 0$ such that

$$\|(-\Delta)^{\frac{1-s}{2}}[|u|^{m_2 p_0-1}u]\|_{L^2(Q_T)} \le C, \tag{1.7}$$

with $p_0 = \frac{N+2(1-s)}{N+2(1-2s)}$.
If $m_2 < m_1$, then there is a constant $C = C(u_0, f, m_1, m_2) > 0$ such that

$$\|(-\Delta)^{\frac{1-s}{2}}[|u|^{m_1-1}u]\|_{L^2(Q_T)} \le C. \tag{1.8}$$

Next, we provide a sharper decay result of solution of (1.1) for the case $f = 0$.

Theorem 2. *Let $p \ge 1$, and $s \in (0,1)$. Let $m_1, m_2 > 0$ be such that $m_1 + m_2 > 1 - \frac{2p(1-s)}{N}$. Assume that $f = 0$ and $u_0 \in L^p(\mathbb{R}^N)$. Then, there exists a constant $C = C(N, s, m_1 + m_2, p) > 0$ such that*

$$\|u(t)\|_{L^\infty} \le C t^{-\frac{1}{p(1-\alpha_0)+\beta_0}} \|u_0\|_{L^p}^{\frac{p(1-\alpha_0)}{p(1-\alpha_0)+\beta_0}}, \tag{1.9}$$

with $\alpha_0 = \frac{N-2(1-s)}{N}$, and $\beta_0 = m_1 + m_2 - 1$.

Remark 2. *We emphasize that (1.9) holds for the case $m_1 + m_2 > 1 - \frac{2p(1-s)}{N}$. Thus, we improve the decay result of the authors in [3, 27], where the authors assumed $m = m_1 + m_2 > 1$.*

Finally, we study the finite time extinction of solution.

Theorem 3. *Let $s \in (0,1)$, and $m_1, m_2 > 0$ be such that $m_1 + m_2 < \alpha_0$. Assume that $f = 0$ and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, there is a finite time $\tau_0 > 0$ such that*

$$u(x,t) = 0, \quad \text{for } (x,t) \in \mathbb{R}^N \times (\tau_0, \infty).$$

Our paper is organized as follows: The next section is devoted to review the fractional Sobolev spaces, and the approximation of the fractional Laplacian $(-\Delta)^s$. Moreover, we prove some functional inequalities, which will be useful later. In Section 3, we would like to study the existence of solution to a regularized equation to (1.1), and we justify the passing to the limit in order to obtain existence of solution of (1.1). The last section is devoted to investigate some decay estimates, and the extinction in a finite time of weak solutions.

Through this paper, the constant C may change step by step. Moreover, $C = C(\alpha, \beta, \gamma)$ means that the constant C merely depends on the parameters α, β, γ.

We denote $\|.\|_{X(\mathbb{R}^N)} = \|.\|_X$, and $\int_{\mathbb{R}^N} f(x)dx = \int f(x)dx$ for short. Finally, the notation $A \lesssim B$ means that there exists a positive constant $c > 0$, being independent of data such that $A \leq cB$.

2 Functional setting

Let $p \geq 1$, and $s \in (0, 1)$. For a given domain $\Omega \subset \mathbb{R}^N$, we define the fractional Sobolev space

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < \infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p}.$$

Moreover, we also denote the homogeneous fractional Sobolev space by $\dot{W}^{s,p}(\Omega)$, endowed with the seminorm

$$\|u\|_{\dot{W}^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p}.$$

In particular, we denote $W^{s,2}(\mathbb{R}^N)$ by $H^s(\mathbb{R}^N)$, which turns out to be a Hilbert space. It is well-known that we have the equivalent characterization

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int (1 + |\xi|^{2s}) |\mathcal{F}\{u\}(\xi)|^2 d\xi < \infty \right\},$$

where $\mathcal{F}$ denotes the Fourier transform, and that we have

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int (1 + |\xi|^{2s}) |\mathcal{F}\{u\}(\xi)|^2 d\xi \right)^{1/2}.$$

In addition, for $u \in H^s(\mathbb{R}^N)$, the fractional Laplacian is defined by

$$(2.1) \qquad (-\Delta)^s u(x) = C(N, s) P.V. \int \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = \mathcal{F}^{-1}\{|\xi|^{2s} \mathcal{F}(u)(\xi)\}.$$

Then,

$$\|u\|_{H^s(\mathbb{R}^N)}^2 = \|u\|_{L^2}^2 + C\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2.$$

We emphasize that if $s > 0$, then $(-\Delta)^{-s} = \mathcal{I}_{2s}$, the Riesz potential, (see, e.g. [28]). Moreover, the fractional gradient ∇^s can be written as $\nabla\mathcal{I}_{1-s}$. And for any smooth bounded function $v : \mathbb{R}^N \to \mathbb{R}$, we have

$$\nabla^s v = C(N,s)\int_{\mathbb{R}^N} (v(x) - v(x+z))\,\frac{z}{|z|^{N+1+s}}dz,$$

with a suitable constant $C(N,s)$, see [3].

Approximation of the fractional Laplacian $(-\Delta)^s$

For fixed $s \in (0,1)$, and each $\varepsilon > 0$, let us define the operator

$$\mathcal{L}_\varepsilon^s[f](x) := C(N,s)\int \frac{f(x) - f(y)}{(|x-y|^2 + \varepsilon^2)^{\frac{N+2s}{2}}}dy, \tag{2.2}$$

for $x \in \mathbb{R}^N$, and for $f \in \mathcal{S}(\mathbb{R}^N)$ (the Schwartz space). It is known that the operator $\mathcal{L}_\varepsilon^s$ can be considered as a regularization of the fractional Laplacian $(-\Delta)^s$, see [9].
Next, we recall some properties of the operator $\mathcal{L}_\varepsilon^s$.

- **Square root:** By the symmetry, we observe that

$$\langle \mathcal{L}_\varepsilon^s[f], f\rangle_{L^2} = \frac{C}{2}\int\int \frac{|f(x) - f(y)|^2}{(|x-y|^2 + \varepsilon^2)^{\frac{N+2s}{2}}}dxdy.$$

Then, we denote $\mathcal{L}_\varepsilon^{\frac{s}{2}}[f]$ as a square root of $\mathcal{L}_\varepsilon^s[f]$ in the Fourier transform sense, and

$$\|\mathcal{L}_\varepsilon^{\frac{s}{2}}[f]\|_{L^2}^2 = \langle \mathcal{L}_\varepsilon^s[f], f\rangle_{L^2}.$$

Lemma 1. *Let $f \in H^1(\mathbb{R}^N)$, and $s \in (\frac{1}{2}, 1)$. Then, there holds*

$$\sup_{\varepsilon>0} \|(-\Delta)^{-\frac{1}{2}}\mathcal{L}_\varepsilon^s[f]\|_{L^2} \le C\|f\|_{H^1}, \tag{2.3}$$

where the constant $C = C(N,s) > 0$.

Proof. It follows from the Plancherel theorem that

$$\begin{aligned}\|(-\Delta)^{-\frac{1}{2}}\mathcal{L}_\varepsilon^s[f]\|_{L^2}^2 &= \|\mathcal{F}\{(-\Delta)^{-\frac{1}{2}}\mathcal{L}_\varepsilon^s[f]\}\|_{L^2}^2\\ &= \|\mathcal{F}\{(-\Delta)^{-\frac{1}{2}}\}\mathcal{F}\{\mathcal{L}_\varepsilon^s\}\mathcal{F}\{f\}\|_{L^2}^2.\end{aligned}$$

On the other hand, we have

$$0 \leq \mathcal{F}\{\mathcal{L}_\varepsilon^s\} \leq \mathcal{F}\{(-\Delta)^s\} = C(N,s)|\xi|^{2s}.$$

We skip the proof of this inequality, and refer to Lemma 10. Thus, we obtain

$$\|(-\Delta)^{-\frac{1}{2}}\mathcal{L}_\varepsilon^s[f]\|_{L^2}^2 \leq C\int |\xi|^{2(2s-1)}|\hat{f}(\xi)|^2 d\xi. \tag{2.4}$$

By Hölder's inequality, we have

$$\int |\xi|^{2(2s-1)}|\hat{f}(\xi)|^2 d\xi \leq \left(\int |\xi|^2|\hat{f}(\xi)|^2 d\xi\right)^{2s-1}\left(\int |\hat{f}(\xi)|^2 d\xi\right)^{2-2s} \leq \|f\|_{H^1}^2. \tag{2.5}$$

From (2.4) and (2.5), we get the result. □

Lemma 2. *Let $\{f_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^2(\mathbb{R}^N)$ such that $f_\varepsilon \to f$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Then, for any $s \in (0,1)$, there holds*

$$\|(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f_\varepsilon] - f\|_{L^2} \to 0. \tag{2.6}$$

Proof. From the triangle inequality, we have

$$\begin{aligned}\|(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f_\varepsilon] - f\|_{L^2}^2 &\leq \|(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f_\varepsilon - f]\|_{L^2} + \|(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f] - f\|_{L^2} \\ &= \|\mathcal{F}\{(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f_\varepsilon - f]\}\|_{L^2}^2 + \|\mathcal{F}\left\{(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f] - f\right\}\|_{L^2}.\end{aligned} \tag{2.7}$$

By applying Lemma 10 in the Appendix, we have

$$|\mathcal{F}\{\mathcal{L}_\varepsilon^s\}(\xi)| \leq |\mathcal{F}\{(-\Delta)^s\}(\xi)| = C|\xi|^{2s}.$$

Then, we obtain

$$\begin{aligned}\|\mathcal{F}\{(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f_\varepsilon - f]\}\|_{L^2}^2 &= \|\mathcal{F}\{(-\Delta)^{-s}\}\mathcal{F}\{\mathcal{L}_\varepsilon^s\}\mathcal{F}\{f_\varepsilon - f\}\|_{L^2}^2 \\ &\leq \|\hat{f}_\varepsilon - \hat{f}\|_{L^2}^2.\end{aligned} \tag{2.8}$$

Similarly, we also get

$$\left|\mathcal{F}\left\{(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f] - f\right\}\right|^2 \leq \left(1 + |\mathcal{F}\{(-\Delta)^{-s}\}\mathcal{F}\{\mathcal{L}_\varepsilon^s\}|\right)^2 |\hat{f}|^2 \leq 4|\hat{f}|^2.$$

Moreover, we observe that $\mathcal{F}\left\{(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f] - f\right\}(\xi) \to 0$, for every $\xi \in \mathbb{R}^N$. Thanks to the Dominated Convergence Theorem, we conclude

$$\|\mathcal{F}\left\{(-\Delta)^{-s}\mathcal{L}_\varepsilon^s[f] - f\right\}\|_{L^2} \to 0, \tag{2.9}$$

as $\varepsilon \to 0$.
A combination of (2.7), (2.8) and (2.9) yields the proof of Lemma 2. □

Next, we prove a generalized version of Stroock-Varopoulos's inequality.

Lemma 3 (Generalized Stroock-Varopoulos Inequality for $\mathcal{L}_\varepsilon^s$)**.** *Let $s \in (0,1)$, and let $\psi, \phi \in \mathcal{C}^1(\mathbb{R})$ be such that $\psi', \phi' \geq 0$. Then,*

$$\int \psi(f)\mathcal{L}_\varepsilon^s\left[\phi(f)\right]dx \geq 0. \tag{2.10}$$

If we take $\psi(f) = f$, then we obtain

$$\int f\mathcal{L}_\varepsilon^s\left[\phi(f)\right]dx \geq \int |\mathcal{L}_\varepsilon^{\frac{s}{2}}\Phi(f)|^2 dx, \tag{2.11}$$

where $\phi' = (\Phi')^2$.

Proof. We have

$$\begin{aligned}\int \psi(f)\mathcal{L}_\varepsilon^s\left[\phi(f)\right]dx &= C_{N,s}\int\int \psi(f(x))\frac{\phi(f(x)) - \phi(f(y))}{(|x-y|^2+\varepsilon^2)^{\frac{N+2s}{2}}}dxdy \\ &= \frac{C}{2}\int\int \frac{\left[\psi(f(x)) - \psi(f(y))\right]\left[\phi(f(x)) - \phi(f(y))\right]}{(|x-y|^2+\varepsilon^2)^{\frac{N+2s}{2}}}dxdy.\end{aligned}$$

Since $\psi', \phi' \geq 0$, we have

$$\left[\psi(f(x)) - \psi(f(y))\right]\left[\phi(f(x)) - \phi(f(y))\right] \geq 0.$$

Hence, we get (2.10).
Finally, (2.11) is proved in Theorem 3.2, [27]. □

To end this part, we point out some well-known fundamental inequalities, used several times in this paper.

Lemma 4. *For any $\alpha > 0$ and $\beta \in (0,1)$, there holds*

$$\left||a|^{\alpha\beta-1}a - |b|^{\alpha\beta-1}b\right| \leq 2^{1-\beta}\left||a|^{\alpha-1}a - |b|^{\alpha-1}b\right|^\beta, \quad \forall a,b \in \mathbb{R}.$$

Lemma 5. *Let $\alpha, \beta > 0$, and $\theta = \frac{\alpha+\beta}{2}$. Then, there is a constant $C > 0$ such that*

$$\left||a|^{\theta-1}a - |b|^{\theta-1}b\right|^2 \leq C\left||a|^{\alpha-1}a - |b|^{\alpha-1}b\right|\left||a|^{\beta-1}a - |b|^{\beta-1}b\right|, \quad \forall a,b \in \mathbb{R}. \tag{2.12}$$

3 A regularized problem

In this section, we study the solutions of the following problem.

$$
\begin{cases} \partial_t u - \delta_1 \Delta u + \delta_2 \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)] - \operatorname{div} \Theta_{\varepsilon,\nu}(u) = f, & \text{in } \mathbb{R}^N \times (0,T), \\ u(0) = u_0, \quad \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}
$$

where $s_0 = (1-2s)_+$, $\Theta_{\varepsilon,\nu}(u) = H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]$, and

$$
H_\nu(u) = \frac{|u|^{m_1+2}}{\nu^2+u^2}, \ G_\nu(u) = \frac{|u|^{m_2+1}u}{\nu^2+u^2}, \ J_\kappa(u) = \frac{|u|^{m_0+1}u}{u^2+\kappa^2},
$$

with $m_0 = \frac{1}{2}\min\{m_1, \dfrac{m_2(N-2s_0)}{N}\}$, and for every $\delta_1, \delta_2, \varepsilon, \kappa, \nu \in (0,1)$. Note that (3.1) is a regularization of (1.1). We shall prove the existence of solutions of (3.1) in a suitable functional space by using the fixed-point theorem, and derive some energy estimates. Let us put

$$
X = L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
$$

The associated norm $\|.\|_X$ is $\|.\|_{L^1(\mathbb{R}^N)} + \|.\|_{L^\infty(\mathbb{R}^N)}$. Then, we have

Theorem 4. *Let $u_0 \in X$ and $f \in L^1(Q_T) \cap L^\infty(Q_T)$. Then, there exists a weak solution $u \in \mathcal{C}([0,T];X)$ satisfying problem (3.1) in the weak sense, i.e:*

$$
\int_0^T \int (-u\varphi_t + \delta_1 \nabla u.\nabla\varphi + \delta_2 \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\varphi - \Theta_{\varepsilon,\nu}(u).\nabla\varphi - f\varphi)\, dxdt = 0,
$$

for all $\varphi \in \mathcal{C}_c^\infty(Q_T)$.

Proof. To prove Theorem 4, we first look for a mild solution $u \in \mathcal{C}([0,T];X)$ as a fixed point of the map

$$
\mathcal{T} : u \mapsto e^{t\delta_1\Delta}u_0 + \int_0^t \nabla e^{(t-\tau)\delta_1\Delta}\Theta_{\varepsilon,\nu}(u)d\tau + \int_0^t e^{(t-\tau)\delta_1\Delta}\left(-\delta_2\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)] + f(x,\tau)\right)d\tau,
$$

where $e^{t\Delta}$ is the semigroup corresponding to the heat kernel $(4\pi t)^{-\frac{N}{2}}\exp(-\frac{|x|^2}{4t})$. Furthermore, we have a fundamental estimate for the heat semigroup $e^{t\Delta}$ (see Proposition 1.2, Ch. 15, [30]).

Proposition 1. *For every $1 \le q \le r \le \infty$, there holds*

$$
\|\nabla^k e^{t\delta\Delta}u\|_{L^r} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{k}{2}}\|u\|_{L^q}, \quad \forall t > 0,
$$

where the constant $C > 0$ depends on the parameters involved.

Proof. The proof of Proposition 1 is quite easy. It follows from Young's inequality for convolution, so we skips the detail and leave it to the reader. $\square$

Next, the following lemma shows that the operator $\mathcal{T}$ is a local contraction:

Lemma 6. *For any $T \in (0,1)$, the operator $\mathcal{T}$ maps $\mathcal{C}([0,T];X)$ into itself. Moreover, there is a real number $\gamma \in (0,1)$ such that for all $u, v \in \overline{B}(0,R) \subset \mathcal{C}([0,T];X)$,*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{\mathcal{C}([0,T];X)} \leq C(R) T^{\gamma} \|u - v\|_{\mathcal{C}([0,T];X)}, \tag{3.2}$$

where $C(R)$ depends on R and the parameters involved.

Proof of Lemma 6. Let us drop the dependence on the parameters ε, ν, κ of the terms $\Theta_{\varepsilon,\nu}, H_\nu, G_\nu, J_\kappa$ for short. Then, we have

$$\mathcal{T}(u) - \mathcal{T}(v) = \int_0^t \nabla e^{(t-\tau)\delta_1\Delta} \left(\Theta(u) - \Theta(v)\right) d\tau + \delta_2 \int_0^t e^{(t-\tau)\delta_1\Delta} (\mathcal{L}_\varepsilon^{s_0}[J(u) - J(v)]) d\tau. \tag{3.3}$$

By applying Proposition (1), we obtain

$$\begin{aligned} \|\mathcal{T}(u)(t) - \mathcal{T}(v)(t)\|_{L^r} \leq & C \int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|\Theta(u) - \Theta(v)\|_{L^q} d\tau \\ & + C \int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|\mathcal{L}_\varepsilon^{s_0}[J(u) - J(v)]\|_{L^q} d\tau. \end{aligned} \tag{3.4}$$

Obviously, we will consider $r = 1$ and $r = \infty$ alternatively in the following.
We now consider the first term on the right hand side of (3.4). Let us write

$$\begin{aligned} A = \Theta(u) - \Theta(v) = & \left(H(u) - H(v)\right) \nabla(-\Delta)^{-1} \mathcal{L}_\varepsilon^{1-s}[G(u)] \\ & + H(v) \nabla(-\Delta)^{-1} \mathcal{L}_\varepsilon^{1-s}\left[G(u) - G(v)\right]. \end{aligned}$$

Let us fix $q > \frac{N}{N-1}$, and put $q' = \frac{q}{q-1}$, $q^\star = \frac{Nq}{N+q}$. Then,

$$\begin{aligned} \|A\|_{L^1} &\leq \|H(u) - H(v)\|_{L^{q'}} \|\mathcal{I}_1[\mathcal{L}_\varepsilon^{1-s}[G(u)]]\|_{L^q} + \|H(v)\|_{L^{q'}} \|\mathcal{I}_1[\mathcal{L}_\varepsilon^{1-s}[G(u) - G(v)]]\|_{L^q} \\ &\lesssim \sup_{|z| \leq 2R} \{|H'(z)|\} \|u - v\|_{L^{q'}} \|\mathcal{L}_\varepsilon^{1-s}[G(u)]\|_{L^{q^\star}} + \sup_{|z| \leq R} \{|H'(z)|\} \|v\|_{L^{q'}} \|\mathcal{L}_\varepsilon^{1-s}[G(u) - G(v)]\|_{L^{q^\star}} \\ &\lesssim \sup_{|z| \leq 2R} \{|H'(z)|\} \|u - v\|_{L^{q'}} \|G(u)\|_{L^{q^\star}} + \sup_{|z| \leq R} \{|H'(z)|\} \|v\|_{L^{q'}} \|G(u) - G(v)\|_{L^{q^\star}} \\ &\lesssim \sup_{|z| \leq 2R} \{|H'(z) G'(z)|\} \|u - v\|_X \|u\|_X + \sup_{|z| \leq 2R} \{|H'(z) G'(z)|\} \|v\|_X \|u - v\|_X. \end{aligned}$$

Thus,

$$\|A\|_{L^1} \leq C(R, \varepsilon) \|u - v\|_X. \tag{3.5}$$

Note that the second inequality is obtained by using the well known property of Riesz potential $\mathcal{I}_1$, and the fourth inequality follows from the interpolation inequality that

$$\|u\|_{L^r} \le \|u\|_X, \quad \text{for } r \ge 1.$$

Similarly, we also get

$$\begin{aligned}
\|\mathcal{L}_\varepsilon^{s_0}[J(u)-J(v)]\|_{L^1} &\le C\|J(u)-J(v)\|_{L^1} \\
&\le C \sup_{|z|\le 2R}\{|J'(z)|\}\|u-v\|_{L^1} \\
&\le C_1(R)\|u-v\|_X. \qquad (3.6)
\end{aligned}$$

By choosing $r=q=1$ in (3.4), and by (3.5), (3.6), we obtain

$$\begin{aligned}
\|\mathcal{T}(u)-\mathcal{T}(v)\|_{L^1} &\le C(R,\varepsilon)\int_0^t (t-\tau)^{-\frac{1}{2}}\|u-v\|_X d\tau + C_1(R)\int_0^t \|u-v\|_X d\tau \\
&\le C_2(R,\varepsilon)\sqrt{T}\|u-v\|_{\mathcal{C}([0,T];X)}, \qquad (3.7)
\end{aligned}$$

for any $t \in (0,T)$, with $C_2(R,\varepsilon) = \max\{C_1(R), C(R,\varepsilon)\}$.

Next, we estimate $\|A\|_{L^q}$ for every $q > N$. In a similar way to the proof of (3.5), we have

$$\begin{aligned}
\|A\|_{L^q} &\le \|H(u)-H(v)\|_{L^\infty}\|\mathcal{I}_1\mathcal{L}_\varepsilon^{1-s}[G(u)]\|_{L^q} + \|H(v)\|_{L^\infty}\|\mathcal{I}_1\mathcal{L}_\varepsilon^{1-s}[G(u)-G(v)]\|_{L^q} \\
&\lesssim \sup_{|z|\le 2R}\{|H'(z)|\}\|u-v\|_{L^\infty}\|G(u)\|_{L^{q^\star}} + \sup_{|z|\le R}\{|H'(z)|\}\|v\|_{L^\infty}\|G(u)-G(v)\|_{L^{q^\star}} \\
&\lesssim \sup_{|z|\le 2R}\{|H'(z)G'(z)|\}\|u-v\|_X\|u\|_X + \sup_{|z|\le 2R}\{|H'(z)G'(z)|\}\|v\|_X\|u-v\|_X \\
&\le C_3(R,\varepsilon)\|u-v\|_X. \qquad (3.8)
\end{aligned}$$

By the same argument as in (3.6), we also obtain

$$\|\mathcal{L}_\varepsilon^{s_0}[J(u)-J(v)]\|_{L^q} \le C_4(R)\|u-v\|_X. \qquad (3.9)$$

Now, let us take $r=\infty$ in (3.4). By (3.8) and (3.9), we obtain

$$\begin{aligned}
\|\mathcal{T}(u)-\mathcal{T}(v)\|_{L^\infty} &\le C_3(R,\varepsilon)\int_0^t (t-\tau)^{-\frac{N}{2q}-\frac{1}{2}}\|u-v\|_X d\tau \\
&\quad + C_4(R)\int_0^t (t-\tau)^{-\frac{N}{2q}}\|u-v\|_X d\tau.
\end{aligned}$$

Thus,

$$\|\mathcal{T}(u)(t)-\mathcal{T}(v)(t)\|_{L^\infty} \le C_5(R,\varepsilon)T^{\frac{1}{2}-\frac{N}{2q}}\|u-v\|_{\mathcal{C}([0,T];X)}. \qquad (3.10)$$

From (3.10) and (3.7), we get (3.2) with $\gamma = \frac{1}{2} - \frac{N}{2q}$.

Finally, it remains to show that $\mathcal{T}$ maps $\overline{B}(0,R)$ into $\overline{B}(0,R)$, with

$$R = 2C(\delta_1)\left(\|u_0\|_X + \|f\|_{L^\infty(Q_T)} + \|f\|_{L^1(Q_T)}\right).$$

Indeed, let us take $v = 0$ in (3.2). Then,

$$\|\mathcal{T}(u)\|_{\mathcal{C}([0,T];X)} \le \|\mathcal{T}(0)\|_{\mathcal{C}([0,T];X)} + C_6(R,\varepsilon)T^\gamma \|u\|_{\mathcal{C}([0,T];X)}, \tag{3.11}$$

with

$$\mathcal{T}(0)(t) = e^{t\delta_1\Delta}u_0 + \int_0^t e^{(t-\tau)\delta_1\Delta} f(.,\tau)d\tau.$$

Now, for every $\|u\|_{\mathcal{C}([0,T];X)} < R$, let $T \in (0,1)$ be small enough such that $C_6(R,\varepsilon)T^\gamma < \frac{1}{2}$. Therefore, (3.11) implies

$$\|\mathcal{T}(u)\|_{\mathcal{C}([0,T];X)} \le \|\mathcal{T}(0)\|_{\mathcal{C}([0,T];X)} + \frac{R}{2}. \tag{3.12}$$

On the other hand, it is not difficult to show that

$$\|\mathcal{T}(0)(t)\|_{L^1} \le C(\delta_1)\left(\|u_0\|_{L^1} + \|f\|_{L^1(Q_T)}\right). \tag{3.13}$$

And

$$\|\mathcal{T}(0)(t)\|_{L^\infty} \le C(\delta_1)\left(\|u_0\|_{L^\infty} + t\|f\|_{L^\infty(Q_T)}\right). \tag{3.14}$$

A combination of (3.13) and (3.14) implies

$$\|\mathcal{T}(0)\|_{\mathcal{C}([0,T];X)} \le C(\delta_1)\left(\|u_0\|_X + \|f\|_{L^1(Q_T)} + T\|f\|_{L^\infty(Q_T)}\right) \le \frac{R}{2}.$$

This inequality and (3.12) implies that $\mathcal{T}$ maps $\overline{B}(0,R)$ into $\overline{B}(0,R)$. Thus, we obtain Lemma 6. □

Now, by applying Lemma 6, there is a unique mild solution $u_{\varepsilon,\nu,\kappa} \in \mathcal{C}([0,T];X)$ (denoted as u for short) satisfying the equation $\mathcal{T}(u) = u$. This yields Theorem 4. □

Remark 3. *By the standard regularity, if u_0 and f are smooth then so is u. Thanks to this point, in what follows, we can use a smoothing effect to the data by assuming that $u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and $f \in \mathcal{C}_c^\infty(Q_T)$.*

Next, we derive some estimates for solution u of (3.1). The first estimate is the L^q-estimate.

Proposition 2. *Let u be a solution of (3.1) in Q_T. Then, for every $q \in [1, \infty)$ we have*

$$\|u(t)\|_{L^q(\mathbb{R}^N)} \le \|u_0\|_{L^q(\mathbb{R}^N)} + t^{\frac{q-1}{q}} \|f\|_{L^q(Q_t)}, \quad \forall t \in (0,T). \tag{3.15}$$

In particular, if $q = \infty$ then

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le t\|f\|_{L^\infty(Q_t)} + \|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{3.16}$$

Moreover, there is a positive constant C, depending only T, u_0, f such that

$$\delta_1 \|u\|^2_{L^2((0,T);H^1(\mathbb{R}^N))} \le C. \tag{3.17}$$

Proof. For every $q > 1$ and for $t \in (0,T)$, we use $|u|^{q-2}u$ as a test function to (3.1) and integrate on $\mathbb{R}^N$ in order to obtain

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\int |u(t)|^q dx + (q-1)\int |u|^{q-2} H_\nu(u) \nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)].\nabla u \; dx \\ &+ \delta_1(q-1)\int |u|^{q-2}|\nabla u|^2 \; dx + \delta_2 \int \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]|u|^{q-2}u \; dx \\ &= \int f(x,t)|u|^{q-2}u \; dx. \end{aligned} \tag{3.18}$$

Thanks to Lemma 3, we get

$$\int |u|^{q-2}u\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)] \; dx \ge 0, \tag{3.19}$$

with $\psi(u) = |u|^{q-2}u$, and $\phi(u) = J_\kappa(u)$.
On the other hand, we observe that

$$\begin{aligned} &\int |u|^{q-2} H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)].\nabla u \; dx \\ &= \int \nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)].\nabla \tilde{H}_\nu(u) \; dx \\ &= \int \tilde{H}_\nu(u)(-\Delta)(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)] \; dx \\ &= \int \tilde{H}_\nu(u)\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)] \; dx \ge 0, \end{aligned} \tag{3.20}$$

with

$$\tilde{H}_\nu(u) = \int_0^u |s|^{q-2} H_\nu(s) ds.$$

Note that the inequality in (3.20) is also obtained by applying Lemma 3.
A combination of (3.18), (3.19) and (3.20) implies

$$\frac{1}{q}\frac{d}{dt}\int |u(t)|^q dx \le \int f(x,t)|u|^{q-2}u \; dx.$$

Using Hölder's inequality yields

$$\frac{1}{q}\frac{d}{dt}\int |u(t)|^q dx \leq \left(\int |f(t)|^q dx\right)^{1/q}\left(\int |u(t)|^q dx\right)^{(q-1)/q}.$$

This leads to

$$\frac{1}{q}[y(t)]^{\frac{(1-q)}{q}} y'(t) \leq \|f(t)\|_{L^q},$$

with $y(t) = \int |u(t)|^q dx$. By solving the above OD inequality, we obtain

$$[y(t)]^{1/q} \leq [y(0)]^{1/q} + \int_0^t \|f(t)\|_{L^q(\mathbb{R}^N)}.$$

Again, applying Hölder's inequality yields (3.15).
Passing to the limit as $q \to \infty$, we deduce (3.16).

Next, we prove L^1-estimate for u.
For any $\eta > 0$, let us put

$$\chi_\eta(r) = \begin{cases} sign(r), & \text{if } |r| > \eta, \\ \frac{1}{\eta}r, & \text{if } |r| \leq \eta, \end{cases}$$

By multiplying (3.1) with $\chi_\eta(u)$, and integrating on $\mathbb{R}^N$, we get

$$\int (u_t\chi_\eta(u) + \delta_1\nabla u.\nabla\chi_\eta(u) + \delta_2\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\chi_\eta(u) + \Theta(u).\nabla\chi_\eta(u))\, dx = \int f\chi_\eta(u)dx. \tag{3.21}$$

Since $\chi'_\eta(u) \geq 0$, it is clear that

$$\int \nabla u.\nabla\chi_\eta(u)dx = \int |\nabla u|^2\chi'_\eta(u)dx \geq 0,$$

and

$$\int \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\chi_\eta(u)dx, \int \Theta(u).\nabla\chi_\eta(u)dx \geq 0$$

by using Lemma 3.
Thus, it follows from (3.21) after integrating on $(0,t)$ that

$$\int S_\eta(u(t))dx \leq \int S_\eta(u_0)dx + \|f\|_{L^1(Q_t)},$$

with

$$S_\eta(u) = \int_0^u \chi_\eta(r)dr = \frac{u^2}{2\eta}\chi_{\{|u|<\eta\}} + (|u| - \frac{\eta}{2})\chi_{\{|u|\geq\eta\}}.$$

Note that χ_A is the characteristic function of the set A.
It is not difficult to verify that

$$\lim_{\eta\to 0}\int S_\eta(u(t))dx = \int |u(t)|dx.$$

So, (3.15) follows with $q=1$.

It remains to prove (3.17). By the same above argument, we obtain from (3.18) with $q=2$

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \delta_1\int_0^t\int |\nabla u|^2 dxds \le \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^t\int fudxds.$$

Applying Hölder's inequality yields

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \delta_1\int_0^t\int |\nabla u|^2 dxds \le \frac{1}{2}\|u_0\|_{L^2}^2 + \|f\|_{L^2(Q_t)}\|u\|_{L^2(Q_t)}. \tag{3.22}$$

Moreover, we have from (3.15) with $q=2$

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + \sqrt{t}\|f\|_{L^2(Q_t)}. \tag{3.23}$$

A combination of (3.22) and (3.23) implies (3.17). Then, we obtain Proposition 2. □

Proposition 3. *Let u as in Proposition 2. Then, there is a constant $C=C(m_0,u_0,f)>0$ such that for every $\kappa,\varepsilon>0$*

$$\delta_2\|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)]\|_{L^2(Q_T)} \le C. \tag{3.24}$$

Proof. Let us denote $u=u_\varepsilon$ for short. Now, by using $J_\kappa(u)$ as a test function to equation (3.1) and integrating both sides on Q_T, we obtain

$$\begin{aligned}\int \tilde{J}_\kappa(u(T))dx &+ \delta_1\int_0^T\int \left(J_\kappa'(u)|\nabla u|^2 + H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]J_\kappa'(u).\nabla u\right)dxdt\\ &+\delta_2\int_0^T\int \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]J_\kappa(u)dxdt = \int \tilde{J}_\kappa(u_0)dx + \int_0^T\int fJ_\kappa(u)dxdt,\end{aligned} \tag{3.25}$$

with

$$\tilde{J}_\kappa(u) = \int_0^u J_\kappa(s)ds.$$

By a simple calculation, we have

$$0\le \tilde{J}_\kappa(s) \le \frac{|s|^{m_0+1}}{m_0+1}, \quad \forall s\in\mathbb{R}. \tag{3.26}$$

Note that $J'(s) \geq 0$, so

$$\int_0^T \int J'_\kappa(u)|\nabla u|^2 dxdt \geq 0, \tag{3.27}$$

and

$$\begin{aligned} &\int_0^T \int H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]J'_\kappa(u).\nabla u dxdt \\ &= \int_0^T \int W(u)\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]dxdt \geq 0, \end{aligned} \tag{3.28}$$

with

$$W(u) = \int_0^u H_\nu(s)J'_\kappa(s)ds.$$

We observe that $W'(s) \geq 0$, so inequality (3.28) is obtained by using the Stroock-Varopoulos's inequality as (3.20).
Thus, it follows from (3.25), (3.26), (3.27) and (3.28) that

$$\delta_2\|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u)]\|^2_{L^2(Q_T)} \leq \frac{1}{m_0+1}\int |u_0|^{m_0+1}dx + \|f\|_{L^1(Q_T)}\|J_\kappa(u)\|_{L^\infty(Q_T)}. \tag{3.29}$$

Furthermore, thanks to Proposition 2, we have

$$\|J_\kappa(u)\|_{L^\infty(Q_T)} \leq \|u\|^{m_0}_{L^\infty(Q_T)} \leq C(u_0, f, m_0). \tag{3.30}$$

Combining (3.29) and (3.30) yields (3.24). This completes the proof of Proposition 3. □

Remark 4. *As a consequence of* (3.16), *the norm $\|u(t)\|_{L^\infty(\mathbb{R}^N)}$ cannot be explosive for $t < T$. Furthermore, we get the global existence of solution u provided that $f \in L^\infty(Q_\infty) \cap L^1(Q_\infty)$. In particular, if $f \equiv 0$ the norm $\|u(t)\|_{L^q(\mathbb{R}^N)}$ is nonincreasing with respect to t for any $q \geq 1$.*

Remark 5. *We emphasize that for any given $\delta_1, \delta_2 > 0$ the right hand side of the estimates in Proposition 2 and Proposition 3 are independent of ε, ν, κ. Moreover, the two perturbation terms $-\delta_1\Delta u$ and $\delta_2(-\Delta)^{s_0}[|u|^{m_0-1}u]$ are positive and play a role in absorbing $\operatorname{div}(|u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u])$. This observation will enable us to pass to the limit as $\varepsilon, \nu, \kappa \to 0$ in the following.*

3.1 Limit as $\varepsilon \to 0$

Next, we shall pass to the limit as $\varepsilon \to 0$.

Proposition 4. *Let u_ε be the solution of problem* (3.1)*, obtained from Lemma 6. Then, there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ (still denoted as $\{u_\varepsilon\}_{\varepsilon>0}$) such that for any $R>0$*

$$u_\varepsilon \to u, \text{ in } L^2\left(B_R \times (0,T)\right).$$

Furthermore, $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap L^2(0,T;H^1(\mathbb{R}^N))$ is a solution of the following problem

$$u_t - \delta_1 \Delta u - \operatorname{div}\left(H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]\right) + \delta_2(-\Delta)^{s_0} J_\kappa(u) = f, \quad \text{in } Q_T. \tag{3.31}$$

In addition, there exists a positive constant $C = C(u_0, f_0, m_0)$ such that

$$\delta_2 \|(-\Delta)^{\frac{s_0}{2}} J_\kappa(u)\|^2_{L^2(Q_T)} \le C. \tag{3.32}$$

Proof. The main idea of the proof is to pass to the limit as $\varepsilon \to 0$ in the equation satisfied by u_ε

$$\int_0^T \int \left(-u_\varepsilon \varphi_t + \delta_1 \nabla u_\varepsilon . \nabla \varphi + \delta_2 \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]\varphi + \Theta_{\varepsilon,\nu}(u_\varepsilon)\cdot \nabla\varphi - f\varphi\right) dxdt = 0, \tag{3.33}$$

for all $\varphi \in \mathcal{C}_c^\infty(Q_T)$. Here, we denote

$$\Theta_{\varepsilon,\nu}(u_\varepsilon) = H_\nu(u_\varepsilon)\nabla(-\Delta)^{-s}[G_\nu(u_\varepsilon)].$$

At the beginning, let us fix a test function $\varphi \in \mathcal{C}_c^\infty(Q_T)$ such that $Supp(\varphi) \subset B_R$, for $R>0$. Now, we recall a compactness result of Simon, [25], used several times in the following.

Lemma 7. *Assume that the spaces $V_1 \subset V_2 \subset V_3$ with compact embedding $V_1 \subset V_2$. Let $\{u_n\}_{n\ge1}$ be a bounded sequence in $L^p(0,T;V_1)$ and let $\{\partial_t u_n\}_{n\ge1}$ be bounded in $L^1(0,T;V_3)$. Then $\{u_n\}_{n\ge1}$ is relatively compact in $L^p(0,T;V_2)$.*

Next, we have the following uniform estimates:

Lemma 8. $\operatorname{div}(\Theta_{\varepsilon,\nu}(u_\varepsilon))$ *and $\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]$ are uniformly bounded in $L^2(0,T;H^{-1}(B_R))$ with respect to $\varepsilon,\kappa>0$, where $H^{-1}(B_R)$ is the dual space of $H_0^1(B_R)$.*

Proof of Lemma 8. In fact, we have

$$\begin{aligned}
\|\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]\|_{H^{-1}(B_R)} &= \sup_{\{\|\psi\|_{H_0^1(B_R)}\le 1\}} \left| \int_{B_R} \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]\psi(x)dx \right| \\
&= \sup_{\{\|\psi\|_{H_0^1(B_R)}\le 1\}} \left| \int_{\mathbb{R}^N} \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]\psi(x)dx \right| \\
&= \sup_{\{\|\psi\|_{H_0^1(B_R)}\le 1\}} \left| \int_{\mathbb{R}^N} \mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)]\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[\psi](x)dx \right| \qquad (3.34)
\end{aligned}$$

The equality in (3.34) is obtained by using the Plancherel theorem. By Hölder's inequality, we get

$$\|\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]\|_{H^{-1}(B_R)} \leq \sup_{\{\|\psi\|_{H_0^1(B_R)}\leq 1\}} \|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)]\|_{L^2(\mathbb{R}^N)}\|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[\psi]\|_{L^2(\mathbb{R}^N)}. \tag{3.35}$$

Moreover, applying Hölder's inequality and Young's inequality yields

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[\psi]\|_{L^2(\mathbb{R}^N)} &\lesssim \left(\int_{\mathbb{R}^N} |\xi|^{2s_0}|\hat{\psi}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{\mathbb{R}^N} |\xi|^{2}|\hat{\psi}(\xi)|^2 d\xi\right)^{\frac{s_0}{2}} \left(\int_{\mathbb{R}^N} |\hat{\psi}(\xi)|^2 d\xi\right)^{\frac{1-s_0}{2}} \\ &\lesssim \|\psi\|_{H_0^1(B_R)}. \end{aligned} \tag{3.36}$$

A combination of (3.35) and (3.36) deduces

$$\int_0^T \|\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]\|^2_{H^{-1}(B_R)}dt \lesssim \int_0^T \|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)]\|^2_{L^2(\mathbb{R}^N)}dt.$$

It follows from the last inequality and (3.24) that $\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)]$ is bounded in $L^2(0,T;H^{-1}(B_R))$ by a constant, not depending on ε, κ.

Next, we claim that $\|\operatorname{div}\Theta_{\varepsilon,\nu}(u_\varepsilon)\|_{L^2(0,T;H^{-1}(B_R))}$ is bounded by a constant, being independent of ε. Due to some technical reasons, we divide our proof into the two following cases:

i) If $\frac{1}{2} \leq s < 1$, for any $t > 0$ we apply the Plancherel theorem and Hölder's inequality in order to obtain

$$\begin{aligned} &\|\operatorname{div}\Theta_{\varepsilon,\nu}(u_\varepsilon,\nu)(t)\|_{H^{-1}(B_R)} \\ &= \sup_{\{\|\psi\|_{H_0^1(B_R)}\leq 1\}} \left|\int_{\mathbb{R}^N} H_\nu(u_\varepsilon(t))\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon(t))]\nabla\psi(x)dx\right| \\ &\leq \sup_{\{\|\psi\|_{H_0^1(B_R)}\leq 1\}} \|H_\nu(u_\varepsilon(t))\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)}\|\nabla\psi\|_{L^{q_s'}(\mathbb{R}^N)} \\ &\leq \sup_{\{\|\psi\|_{H_0^1(B_R)}\leq 1\}} \|H_\nu(u_\varepsilon)\|_{L^\infty(Q_T)}\|\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)}\|\nabla\psi\|_{L^2(B_R)}|B_R|^{1-\frac{q_s'}{2}} \\ &\leq |B_R|^{1-\frac{q_s'}{2}}\|u_\varepsilon\|^{m_1}_{L^\infty(Q_T)}\|\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)}, \end{aligned} \tag{3.37}$$

where $q_s = \frac{2N}{N-2(2s-1)}$, and $q_s' = \frac{q_s}{q_s-1}$. Note that $q_s \geq 2$.
According to Proposition 2, $\|u_\varepsilon\|^{m_1}_{L^\infty(Q_T)}$ is bounded by a constant, not depending on ε.

Thus, it suffices to prove that $\|\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)}$ is uniformly bounded for all $\varepsilon > 0$, and for $t > 0$. Indeed, it follows from the Riesz potential estimate that

$$\begin{aligned}
\|\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)} &= \|\nabla^{1-2s}(-\Delta)^{-(1-s)}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)} \\
&= \|\mathcal{I}_{2s-1}(-\Delta)^{-(1-s)}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^{q_s}(\mathbb{R}^N)} \\
&\lesssim \|(-\Delta)^{-(1-s)}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon(t))]\|_{L^2(\mathbb{R}^N)} \\
&\lesssim \|G_\nu(u_\varepsilon(t))\|_{L^2(\mathbb{R}^N)}. \qquad (3.38)
\end{aligned}$$

Morever, it is clear that

$$\|G_\nu(u_\varepsilon(t))\|_{L^2(\mathbb{R}^N)} \leq \frac{1}{\nu^2}\|u_\varepsilon^{m_2+2}(t)\|_{L^2(\mathbb{R}^N)}.$$

It follows from Proposition 2, and the Interpolation theorem that $\|u_\varepsilon^{m_2+2}(t)\|_{L^2(\mathbb{R}^N)}$ is uniformly bounded for all $t > 0$, and all $\varepsilon > 0$.
From (3.37), (3.38), and the last inequality, we get the claim for the case $s \in [\frac{1}{2}, 1)$.
In the following, we remove the dependence on time t of the terms in our estimates for brief if no confusion.

ii) If $0 < s < \frac{1}{2}$, we then have from the Plancherel theorem and Hölder's inequality that

$$\begin{aligned}
\|\operatorname{div}\Theta_\varepsilon(u_\varepsilon,\nu)\|_{H^{-1}(B_R)} &= \sup_{\{\|\psi\|_{H_0^1(B_R)}\leq 1\}} \left|\int_{\mathbb{R}^N} H_\nu(u_\varepsilon)\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon)]\nabla\psi(x)dx\right| \\
&\leq \sup_{\{\|\psi\|_{H_0^1(B_R)}\leq 1\}} \|H_\nu(u_\varepsilon)\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon)]\|_{L^2(\mathbb{R}^N)}\|\nabla\psi\|_{L^2(\mathbb{R}^N)} \\
&\leq \|u_\varepsilon\|^{m_1}_{L^\infty(Q_T)}\|\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon)]\|_{L^2(\mathbb{R}^N)} \qquad (3.39)
\end{aligned}$$

On the other hand, using the Plancherel theorem yields

$$\begin{aligned}
\|\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon)]\|^2_{L^2(\mathbb{R}^N)} &= \|\mathcal{F}\left\{\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon)]\right\}\|^2_{L^2(\mathbb{R}^N)} \\
&\lesssim \int |\xi|^{2(1-2s)}|\mathcal{F}\left\{G_\nu(u_\varepsilon)\right\}(\xi)|^2 d\xi. \qquad (3.40)
\end{aligned}$$

Since $0 < s < \frac{1}{2}$, we can apply Hölder's inequality in order to obtain

$$\begin{aligned}
\int |\xi|^{2(1-2s)}|\mathcal{F}\left\{G_\nu(u_\varepsilon)\right\}|^2 d\xi &\leq \left(\int |\xi|^2|\mathcal{F}\left\{G_\nu(u_\varepsilon)\right\}|^2 d\xi\right)^{1-2s}\left(\int |\mathcal{F}\left\{G_\nu(u_\varepsilon)\right\}|^2 d\xi\right)^{2s} \\
&\leq \|G_\nu(u_\varepsilon)\|^2_{H^1(\mathbb{R}^N)}. \qquad (3.41)
\end{aligned}$$

Combining (3.40) and (3.41) yields

$$\int_0^T \|\nabla(-\Delta)^{-1}\mathcal{L}_{\varepsilon}^{1-s}[G_\nu(u_\varepsilon)]\|^2_{L^2(\mathbb{R}^N)}dt \lesssim \int_0^T \|G_\nu(u_\varepsilon)\|^2_{H^1(\mathbb{R}^N)}dt. \qquad (3.42)$$

Since G_ν is a Lipschitz function with $G_\nu(0) = 0$, and by (3.17), there exists a constant $C > 0$, being independent of ε such that

$$\|G_\nu(u_\varepsilon)\|^2_{L^2(0,T;H^1(\mathbb{R}^N))} \leq C. \tag{3.43}$$

Thus, the claim follows from (3.42) and (3.43).
This puts an end to the proof of Lemma 8. □

Now, thanks to Lemma 8, $\partial_t u_\varepsilon$ is bounded in $L^2(0,T;H^{-1}(B_R))$ by a constant not depending on ε. Moreover, it follows from Proposition 2 that u_ε is bounded in $L^2(0,T;H^1_0(B_R))$. Thus, applying Lemma 7 implies that there is a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ (still denoted as $\{u_\varepsilon\}_{\varepsilon>0}$) such that

$$u_\varepsilon \to u, \quad \text{in} \;\; L^2\left(B_R \times (0,T)\right). \tag{3.44}$$

Thanks to Proposition 2, we deduce

$$u_\varepsilon \to u, \quad \text{in} \;\; L^p\left(B_R \times (0,T)\right), \text{ for } 1 \leq p < \infty,$$

and

$$u \in L^\infty(Q_T).$$

By (3.17), ∇u_ε converges weakly to ∇u in $L^2(B_R \times (0,T))$ up to a subsequence. Thus, we get

$$\int_0^T \int \left(-u_\varepsilon \varphi_t + \delta_1 \nabla u_\varepsilon . \nabla \varphi\right) dxdt \to \int_0^T \int \left(-u\varphi_t + \delta_1 \nabla u . \nabla \varphi\right) dxdt. \tag{3.45}$$

Next, we consider the difference between the two integrals as follows

$$\begin{aligned}
&\int_0^T \int \left(\mathcal{L}^{s_0}_\varepsilon[J_\kappa(u_\varepsilon)] - (-\Delta)^{s_0}[J_\kappa(u)]\right) \varphi \; dxdt \\
&= \int_0^T \int \mathcal{F}\{\mathcal{L}^{s_0}_\varepsilon[J_\kappa(u_\varepsilon)] - (-\Delta)^{s_0}[J_\kappa(u)]\}\mathcal{F}\{\varphi(t)\}(\xi) \; d\xi dt \\
&= \int_0^T \int \left(\mathcal{F}\{\mathcal{L}^{s_0}_\varepsilon\}\mathcal{F}\{J_\kappa(u_\varepsilon)\} - |\xi|^{2s_0}\mathcal{F}\{J_\kappa(u)\}\right) \mathcal{F}\{\varphi(t)\}(\xi) \; d\xi dt.
\end{aligned} \tag{3.46}$$

We claim that

$$\int_{Q_T} |A_\varepsilon(\xi)| d\xi dt \to 0, \tag{3.47}$$

as $\varepsilon \to 0$, with

$$A_\varepsilon = \left(\mathcal{F}\{\mathcal{L}^{s_0}_\varepsilon\}\mathcal{F}\{J_\kappa(u_\varepsilon)\} - |\xi|^{2s_0}\mathcal{F}\{J_\kappa(u)\}\right) \mathcal{F}\{\varphi(t)\}(\xi).$$

In fact, it is obvious that $A_\varepsilon(\xi) \to 0$ a.e in Q_T.
Moreover, we have

$$\int |A_\varepsilon(\xi)| d\xi \lesssim \int |\mathcal{F}\{\mathcal{L}_\varepsilon^{s_0}\}||\mathcal{F}\{J_\kappa(u_\varepsilon)\}| + |\xi|^{2s_0}|\mathcal{F}\{J_\kappa(u)\}||\mathcal{F}\{\varphi(t)\}|\ d\xi dt$$
$$\lesssim \int |\xi|^{2s_0} \left(|\mathcal{F}\{J_\kappa(u_\varepsilon)\}| + |\mathcal{F}\{J_\kappa(u)\}|\right) |\mathcal{F}\{\varphi(t)\}(\xi)| d\xi dt.$$

The last inequality is obtain by using the fact $|\mathcal{F}\{\mathcal{L}_\varepsilon^s\}(\xi)| \leq |\mathcal{F}\{(-\Delta)^s(\xi)| = C|\xi|^{2s}$, for every $\xi \in \mathbb{R}^N$, and for $s \in (0,1)$.
Furthermore, using the standard property of Fourier transform yields

$$|\mathcal{F}\{J_\kappa(u)\}(\xi)| \leq \|J_\kappa(u)\|_{L^1} \leq \frac{1}{\kappa^2} \int |u(t)|^{m_0+2} dx \leq C(u_0, f, m_0, \kappa),$$

by (3.15). Similarly, we also obtain

$$|\mathcal{F}\{J_\kappa(u_\varepsilon)\}(\xi)| \leq C(u_0, f, m_0, \kappa).$$

Thus,

$$\int |A_\varepsilon(\xi)|\ d\xi dt \leq C \int |\xi|^{2s_0} |\mathcal{F}\{\varphi(t)\}(\xi)| d\xi.$$

Since $\varphi(t) \in \mathcal{S}(\mathbb{R}^N)$, so is $\mathcal{F}\{\varphi(t)\}$. This fact implies that $|\xi|^{2s_0}|\mathcal{F}\{\varphi(t)\}(\xi)|$ is integrable on Q_T. Thanks to the Dominated Convergence Theorem, we obtain (3.47).
This leads to

$$\int_0^T \int \left(\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u_\varepsilon)] - (-\Delta)^{s_0}[J_\kappa(u)]\right) \varphi\ dxdt \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.48}$$

It remains to prove that

$$\int_0^T \int \left(\operatorname{div} \Theta_{\varepsilon,\nu}(u_\varepsilon) - \operatorname{div}\left(H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]\right)\right) \varphi\, dxdt \to 0, \tag{3.49}$$

as $\varepsilon \to 0$. By technical reasons, we divide our proof into the two following cases:

- If $\frac{1}{2} \leq s < 1$, we rewrite

$$\int_0^T \int \left(\operatorname{div} \Theta_{\varepsilon,\nu}(u_\varepsilon) - \operatorname{div}\left(H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]\right)\right) \varphi dxdt$$
$$= \int_0^T \int \left(H_\nu(u_\varepsilon)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]\right) .\nabla\varphi dxdt$$

Put

$$A_1 = \int_0^T \int |H_\nu(u_\varepsilon) - H_\nu(u)| \left|\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]\right| |\nabla\varphi| dxdt,$$

and

$$A_2 = \int_0^T \int |H_\nu(u)| \left|\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - \nabla(-\Delta)^{-s}[G_\nu(u)]\right| |\nabla\varphi| dxdt.$$

To obtain (3.49), it suffices to show that $A_1, A_2 \to 0$, as $\varepsilon \to 0$.

$$\begin{aligned} A_1 &= \int_0^T \int |H_\nu(u_\varepsilon) - H_\nu(u)| \left|\nabla^{1-2s}(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]\right| |\nabla\varphi| dxdt \\ &= \int_0^T \int |H_\nu(u_\varepsilon) - H_\nu(u)| \left|\mathcal{I}_{2s-1}\left[(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]\right]\right| |\nabla\varphi| dxdt. \end{aligned}$$

By the fundamental estimate for the Riesz potential and the Plancherel theorem, we get

$$\begin{aligned} \|\mathcal{I}_{2s-1}\left[(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]\right]\|_{L^{q_s}} &\lesssim \|(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]\|_{L^2} \\ &\lesssim \|G_\nu(u_\varepsilon)\|_{L^2}, \end{aligned}$$

with $q_s = \frac{2N}{N-2(2s-1)} \geq 2$.
Again, we observe that $\|G_\nu(u_\varepsilon)\|_{L^2}$ is bounded by a constant not depending on ε. This implies that the term $\mathcal{I}_{2s-1}\left[(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]\right]$ is also bounded in L^{q_s}. Moreover, it is not difficult to prove that $H_\nu(u_\varepsilon) \to H_\nu(u)$ in $L^p(B_R \times (0,T))$, for any $p \in [1,\infty)$. Thus, $A_1 \to 0$ as $\varepsilon \to 0$.

Similarly, we also have

$$\|\mathcal{I}_{2s-1}\left[(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\right]\|_{L^{q_s}} \lesssim \|(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\|_{L^2}.$$

Applying Lemma 2 yields

$$\|(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\|_{L^2} \to 0,$$

as $\varepsilon \to 0$. Thus

$$\|\mathcal{I}_{2s-1}\left[(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\right]\|_{L^{q_s}} \to 0.$$

This implies $A_2 \to 0$.

- If $s \in (0, \frac{1}{2})$, we write

$$\int_0^T \int \operatorname{div}(\Theta_{\varepsilon,\nu}(u_\varepsilon))\varphi dxdt = \int_0^T \int \operatorname{div}\left(H_\nu(u_\varepsilon)\nabla\varphi\right)(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]dxdt. \tag{3.50}$$

We will show that

$$\int_0^T \int \operatorname{div}\left(H_\nu(u_\varepsilon)\nabla\varphi\right)(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)]dxdt \to$$

$$\int_0^T \int \operatorname{div}\left(H_\nu(u)\nabla\varphi\right)(-\Delta)^{-s}[G_\nu(u)]dxdt$$

$$= \int_0^T \int \operatorname{div}\Big(H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]\Big)\varphi dxdt. \tag{3.51}$$

One hand, we see that

$$\|\operatorname{div}\left(H_\nu(u_\varepsilon)\nabla\varphi\right)\|_{L^2} \le \|H'_\nu(u_\varepsilon)\nabla u_\varepsilon.\nabla\varphi\|_{L^2} + \|H_\nu(u_\varepsilon)\Delta\varphi\|_{L^2}. \tag{3.52}$$

It follows from Proposition 2 that $H'_\nu(u_\varepsilon)$ and $H_\nu(u_\varepsilon)$ are bounded by a constant, being independent of ε. This implies that the right hand side of (3.52) is also bounded, so is $\|\operatorname{div}\left(H_\nu(u_\varepsilon)\nabla\varphi\right)\|_{L^2}$. Other hand, it is not difficult to verify that

$$\operatorname{div}\left(H_\nu(u_\varepsilon)\nabla\varphi\right) \to \operatorname{div}\left(H_\nu(u)\nabla\varphi\right), \quad \text{in } \mathcal{D}'(Q_T).$$

Then, $\operatorname{div}\left(H_\nu(u_\varepsilon)\nabla\varphi\right)$ converges weakly to $\operatorname{div}\left(H_\nu(u)\nabla\varphi\right)$ in $L^2(B_R)$ (up to a subsequence).
Therefore, it is sufficient to prove that

$$(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] \to (-\Delta)^{-s}[G_\nu(u)], \quad \text{in } L^2_{loc}(Q_T). \tag{3.53}$$

Indeed, we have

$$\begin{aligned}
&\|(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - (-\Delta)^{-s}[G_\nu(u)]\|_{L^{q_s^\star}(\mathbb{R}^N)} \\
&= \|(-\Delta)^{-s}\left((-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\right)\|_{L^{q_s^\star}(\mathbb{R}^N)} \\
&= \|\mathcal{I}_{2s}\left((-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\right)\|_{L^{q_s^\star}(\mathbb{R}^N)} \\
&\lesssim \|(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\|_{L^2(\mathbb{R}^N)},
\end{aligned} \tag{3.54}$$

with $q_s^\star = \frac{2N}{N-4s} > 2$.
Since $G_\nu(u_\varepsilon) \to G_\nu(u)$ strongly in $L^2(Q_T)$, then a modification of Lemma 2 implies

$$\left\|(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\right\|_{L^2(Q_T)} \to 0. \tag{3.55}$$

By applying Hölder's inequality and by (3.55), we obtain

$$\begin{aligned}
&\int_0^T \|(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - (-\Delta)^{-s}[G_\nu(u)]\|^2_{L^2(B_R)}dt \\
&\le \int_0^T \|(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - (-\Delta)^{-s}[G_\nu(u)]\|^2_{L^{q_s^\star}(B_R)}|B_R|^{2(1-\frac{2}{q_s^\star})}dt \\
&\le |B_R|^{2(1-\frac{2}{q_s^\star})}\int_0^T \left\|(-\Delta)^{-(1-s)}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u_\varepsilon)] - G_\nu(u)\right\|^2_{L^2(\mathbb{R}^N)} dt \to 0.
\end{aligned}$$

This implies (3.53).
As a consequence, we obtain (3.51) and (3.49) alternatively.
A combination of (3.45), (3.48) and (3.49) ensures that u is a weak solution of (3.1).

Next, we prove (3.32). Indeed, we can mimic the proof of (3.47) to obtain

$$\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)] \to (-\Delta)^{\frac{s_0}{2}}[J_\kappa(u)], \quad \text{in the sense of distribution } \mathcal{D}'(Q_T),$$

as $\varepsilon \to 0$. Furthermore, it follows from (3.24) that

$$\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)] \to (-\Delta)^{\frac{s_0}{2}}[J_\kappa(u)], \quad \text{weakly in } L^2(Q_T).$$

Then,

$$\liminf_{\varepsilon\to 0} \|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)]\|_{L^2(Q_T)} \geq \|(-\Delta)^{\frac{s_0}{2}}[J_\kappa(u)]\|_{L^2(Q_T)},$$

which implies (3.32).

To complete the proof of Proposition 4, it remains to show that $u \in \mathcal{C}([0,T]; L^p(\mathbb{R}^N))$, for all $p \geq 1$. By (3.17), we observe that u_ε is bounded in $L^2(0,T; H_0^1(B_R))$ by a constant, not depending on ε. Moreover, $\partial_t u_\varepsilon$ is also bounded in $L^1(Q_T) + L^2(0,T; H^{-1}(B_R))$, for any $R > 0$. Thanks to Theorem 1.1, [22], we obtain

$$u \in \mathcal{C}\left([0,T]; L^2_{loc}(\mathbb{R}^N)\right).$$

From this fact, we can mimic the argument, given in page 21 of [15] in order to get

$$u \in \mathcal{C}\left([0,T]; L^1(\mathbb{R}^N)\right).$$

By the boundedness of u_ε in Q_T, we have $u \in \mathcal{C}\left([0,T]; L^p(\mathbb{R}^N)\right)$, for every $p \in [1,\infty)$.
This ends to the proof of Proposition 4. □

Remark 6. *It is not difficult to verify that the solution u, obtained by passing to the limit as $\varepsilon \to 0$ also satisfies Proposition 2. Moreover, the estimates in this part are independent of κ. This observation will allow us to pass to the limit as $\kappa \to 0$ in the following.*

3.2 Limit as $\kappa \to 0$

In this part, we shall pass to the limit as $\kappa \to 0$.

Proposition 5. *Let u_κ be the solution of problem* (3.31)*, obtained in Proposition 2. Then, there holds for any $R > 0$*

$$u_\kappa \to u, \text{ in } L^2\left(B_R \times (0,T)\right)$$

up to a subsequence.
Furthermore, $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap L^2(0,T;H^1(\mathbb{R}^N))$ *is a solution of the following problem*

$$u_t - \delta_1 \Delta u - \operatorname{div} \Theta_\nu(u) + \delta_2 (-\Delta)^{s_0}(|u|^{m_0-1}u) = f, \text{ in } Q_T, \tag{3.56}$$

where we use the notation $\Theta_\nu(u) = H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]$.
In addition, we have

$$\delta_2 \|(-\Delta)^{\frac{s_0}{2}}(|u|^{m_0-1}u)\|^2_{L^2(Q_T)} \leq C, \tag{3.57}$$

where $C > 0$ *depends only on* m_0, u_0, f.

Proof. We first note that $\operatorname{div} \Theta_\nu(u_\kappa)$ and $(-\Delta)^{s_0}[J_\kappa(u_\kappa)]$ are bounded in $L^2(0,T;H^{-1}(B_R))$ by a constant not depending on κ, see Remark 6. Thanks to the compactness result in Lemma 7, there is a subsequence of $\{u_\kappa\}_{\kappa>0}$ such that

$$u_\kappa \to u, \quad \text{in } \ L^2\left(B_R \times (0,T)\right),$$

as $\kappa \to 0$. It follows from Proposition 2 that

$$u_\kappa \to u, \quad \text{in } \ L^p\left(B_R \times (0,T)\right), \text{ for } 1 \leq p < \infty,$$

and

$$u \in L^\infty(Q_T).$$

Now, it suffices to show that u satisfies equation (3.56) in the weak sense.
Indeed, it is not difficult to verify that

$$\begin{aligned} &\int_0^T \int \left(-u_\kappa \varphi_t + \delta_1 \nabla u_\kappa . \nabla \varphi - \Theta_\nu(u_\kappa) \cdot \nabla \varphi\right) dxdt \to \\ &\qquad \int_0^T \int \left(-u \varphi_t + \delta_1 \nabla u . \nabla \varphi - \Theta_\nu(u) \cdot \nabla \varphi\right) dxdt. \end{aligned} \tag{3.58}$$

Thus, it remains to demonstrate that

$$\int_0^T \int \left((-\Delta)^{s_0}[J_\kappa(u_\kappa)] - (-\Delta)^{s_0}[|u|^{m_0-1}u]\right) \varphi dxdt \to 0. \tag{3.59}$$

Indeed, we have from the Plancherel's theorem

$$\begin{aligned} &\left|\int_0^T \int \left((-\Delta)^{s_0}[J_\kappa(u_\kappa)] - (-\Delta)^{s_0}[|u|^{m_0-1}u]\right) \varphi dxdt\right| \\ &= \left|\int_0^T \int \left(J_\kappa(u_\kappa) - |u|^{m_0-1}u\right) (-\Delta)^{s_0} \varphi dxdt\right| \\ &\leq \int_0^T \int \left|J_\kappa(u_\kappa) - |u|^{m_0-1}u\right| |(-\Delta)^{s_0}\varphi| dxdt. \end{aligned}$$

By (3.16), we have

$$\begin{aligned}\left|\left(J_\kappa(u_\kappa) - |u|^{m_0-1}u\right)(x,t)\right| &\leq |J_\kappa(u_\kappa)(x,t)| + |u(x,t)|^{m_0} \\ &\leq |u_\kappa(x,t)|^{m_0} + |u(x,t)|^{m_0} \\ &\leq 2\left(\|u_0\|_{L^\infty} + T\|f\|_{L^\infty(Q_T)}\right)^{m_0}, \ \forall (x,t) \in Q_T.\end{aligned}$$

Moreover, it is clear that $J_\kappa(u_\kappa) \to |u|^{m_0-1}u$ as $\kappa \to 0$, for a.e $(x,t) \in Q_T$.
Thus, applying the Dominated Convergence Theorem yields

$$\int_0^T \int \left|J_\kappa(u_\kappa) - |u|^{m_0-1}u\right| |(-\Delta)^{s_0}\varphi| dxdt \to 0,$$

when $\kappa \to 0$. This implies (3.59).
In conclusion, u is a weak solution of problem (3.56).
Finally, (3.57) follows from (3.32), and we obtain the proof of Proposition 5. □

3.3 Limit as $\nu \to 0$

Proposition 6. *Let u_ν be the solution, obtained in Proposition 5. Then, there exists a subsequence of $\{u_\nu\}_{\nu>0}$ converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$. Moreover, $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap L^2(0,T; H^1(\mathbb{R}^N))$ is a solution of the equation*

$$u_t - \delta_1 \Delta u - \operatorname{div} \Theta(u) + \delta_2(-\Delta)^{s_0}(|u|^{m_0-1}u) = f, \quad \text{in } Q_T. \tag{3.60}$$

Recall here that $\Theta(u) = H(u)\nabla(-\Delta)^{-s}[G(u)]$, with $H(u) = |u|^{m_1}$ and $G(u) = |u|^{m_2-1}u$.

Proof. By (3.57) and the same argument as in Lemma 8, we obtain $\delta_2(-\Delta)^{s_0}(|u_\nu|^{m_0-1}u_\nu)$ is bounded in $L^2(0,T;H^{-1}(B_R))$ by a constant not depending on ν.
Now, we show that

$$\|\operatorname{div}\left(H_\nu(u_\nu)\nabla(-\Delta)^{-s}[G_\nu(u_\nu)]\right)\|_{L^2(0,T;H^{-1}(B_R))} \leq C, \tag{3.61}$$

with $C > 0$ is independent of ν.
The idea of the proof of (3.61) is most likely to the one of Lemma 8, but we need to derive the estimates, not depending on the parameter ν. To do that, we have to use some properties of the term $(-\Delta)^{s_0}(|u_\nu|^{m_0-1}u_\nu)$. We divide our proof into the two following cases:

i) If $s \in [\frac{1}{2}, 1)$, we mimic the proof of (3.37) and (3.38) to obtain

$$\|\operatorname{div}\left(H_\nu(u_\nu)\nabla(-\Delta)^{-s}[G_\nu(u_\nu)]\right)\|_{H^{-1}(B_R)} \leq \|u_\nu\|^{m_1}_{L^\infty(Q_T)}\|\mathcal{I}_{2s-1}[G_\nu(u_\nu)]\|_{L^2(B_R)}, \tag{3.62}$$

and

$$\|\mathcal{I}_{2s-1}[G_\nu(u_\nu)]\|_{L^{q_s}} \lesssim \|G_\nu(u_\nu)\|_{L^2} \leq \left(\int |u_\nu(x)|^{2m_2} dx\right)^{\frac{1}{2}}. \tag{3.63}$$

From the Sobolev embedding, we have

$$\||u_\nu|^{m_0-1}u_\nu\|_{L^{q^\star_{s_0}}} \lesssim \|(-\Delta)^{\frac{s_0}{2}}(|u_\nu|^{m_0-1}u_\nu)\|_{L^2},$$

with $q^\star_{s_0} = \frac{2N}{N-2s_0}$. It follows from (3.57) that there exists a constant $C > 0$ (not depending on ν) such that

$$\int |u_\nu(x)|^{m_0 q^\star_{s_0}} dx \leq C. \tag{3.64}$$

Since $m_0 \leq \frac{m_2(N-2s_0)}{2N}$, then we have from (3.64)

$$\int |u_\nu(x)|^{2m_2} dx \leq \|u_\nu\|^{2m_2 - m_0 q^\star_{s_0}}_{L^\infty(Q_T)} \int |u_\nu(x)|^{m_0 q^\star_{s_0}} dx \leq C, \tag{3.65}$$

We also remind here that $\|u_\nu\|_{L^\infty(Q_T)}$ is bounded by a constant $C = C(u_0, f)$.
A combination of (3.62), (3.63) and (3.65) deduces

$$\|\operatorname{div}\left(H_\nu(u_\nu)\nabla(-\Delta)^{-s}[G_\nu(u_\nu)]\right)\|_{H^{-1}(B_R)} \leq C.$$

Or, we obtain (3.61).

ii) If $s \in (0, \frac{1}{2})$ then

$$\begin{aligned}
&\|\operatorname{div}\left(H_\nu(u_\nu)\nabla(-\Delta)^{-s}[G_\nu(u_\nu)]\right)\|_{H^{-1}(B_R)} = \sup_{\|\psi\|_{H^1_0(B_R)}=1} \left|\int_{B_R} \Theta(u_\nu)\cdot\nabla\psi dx\right| \\
&= \sup_{\|\psi\|_{H^1_0(B_R)}=1} \left|\int_{\mathbb{R}^N} \operatorname{div}(H_\nu(u_\nu)\nabla\psi)(-\Delta)^{-s}[G_\nu(u_\nu)]dx\right| \\
&= \sup_{\|\psi\|_{H^1_0(B_R)}=1} \left|\int_{\mathbb{R}^N} (-\Delta)^{-\frac{1}{2}}\operatorname{div}(H_\nu(u_\nu)\nabla\psi)(-\Delta)^{\frac{1}{2}-s}[G_\nu(u_\nu)]dx\right| \\
&\leq \sup_{\|\psi\|_{H^1_0(B_R)}=1} \|(-\Delta)^{\frac{1}{2}-s}G_\nu(u_\nu)\|_{L^2(\mathbb{R}^N)}\|(-\Delta)^{-\frac{1}{2}}\operatorname{div}(H_\nu(u_\nu)\nabla\psi)\|_{L^2(\mathbb{R}^N)}. \qquad (3.66)
\end{aligned}$$

Thanks to Plancherel's theorem, there is a constant $C = C(N) > 0$ such that

$$\|(-\Delta)^{-\frac{1}{2}}\operatorname{div}(H_\nu(u_\nu)\nabla\psi)\|_{L^2(\mathbb{R}^N)} \leq C\|H_\nu(u_\nu)\nabla\psi\|_{L^2(\mathbb{R}^N)} \leq C\|\psi\|_{H^1_0(B_R)}. \tag{3.67}$$

By (3.66) and (3.67), we obtain

(3.68)
$$\int_0^T \| \operatorname{div}\left(H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]\right)\|^2_{H^{-1}(B_R)}dt \le C\int_0^T \|(-\Delta)^{\frac{1}{2}-s}G_\nu(u_\nu)\|^2_{L^2(\mathbb{R}^N)}dt.$$

On the other hand, we have

$$|G_\nu(a)-G_\nu(b)| \le C\left||a|^{m_0-1}a-|b|^{m_0-1}b\right|(|a|+|b|)^{m_2-m_0}, \quad \forall a,b\in\mathbb{R},$$

with $C=C(m_0,m_2)$, for all $\nu\in(0,1)$. Thus,

$$\begin{aligned}\|(-\Delta)^{\frac{1}{2}-s}G_\nu(u_\nu)\|^2_{L^2(Q_T)} &= \int_0^T\int\int \frac{|G_\nu(u_\nu(x))-G_\nu(u_\nu(y))|^2}{|x-y|^{N+2(1-2s)}}dxdydt \\ &\le C\|u_\nu\|^{m_2-m_0}_{L^\infty(Q_T)}\int_0^T\int\int \frac{\left||u_\nu(x)|^{m_0-1}u_\nu(x)-|u_\nu(y)|^{m_0-1}u_\nu(y)\right|^2}{|x-y|^{N+2(1-2s)}}dxdydt \\ &\le C\|u_\nu\|^{m_2-m_0}_{L^\infty(Q_T)}\|(-\Delta)^{\frac{s_0}{2}}\left(|u_\nu|^{m_0-1}u_\nu\right)\|^2_{L^2(Q_T)}. \end{aligned} \tag{3.69}$$

Note that $s_0=1-2s$ in this case. Thanks to (3.57) and Proposition 2, the right hand side of the last inequality is bounded by a constant, being independent of ν, so is $\|(-\Delta)^{\frac{1}{2}-s}G_\nu(u_\nu)\|^2_{L^2(Q_T)}$.

Thus, (3.61) follows from (3.68) and the boundedness of $\|(-\Delta)^{\frac{1}{2}-s}G_\nu(u_\nu)\|^2_{L^2(Q_T)}$. Thanks to Lemma 7, there is a subsequence of $\{u_\nu\}_{\nu>0}$, converging to u in $L^2(B_R\times(0,T))$ when $\nu\to 0$.

By Proposition 2, we deduce

$$u_\nu \to u, \text{ in } L^p(B_R\times(0,T)), \text{ for } 1\le p<\infty,$$

and

$$u\in L^\infty(Q_T).$$

Now, we shall show that u is a weak solution of problem (3.60).

We claim that

$$(-\Delta)^{-\frac{1}{2}}\operatorname{div}(H_\nu(u_\nu)\nabla\varphi) \to (-\Delta)^{-\frac{1}{2}}\operatorname{div}(|u|^{m_1}\nabla\varphi), \tag{3.70}$$

strongly in $L^2(Q_T)$.

It follows from Plancherel's theorem that

$$\left\|(-\Delta)^{-\frac{1}{2}}\operatorname{div}\left(\left(H_\nu(u_\nu)-|u|^{m_1}\right)\right)\nabla\varphi\right\|_{L^2(\mathbb{R}^N)} \lesssim \left\|\left(H_\nu(u_\nu)-|u|^{m_1}\right)\nabla\varphi\right\|_{L^2(\mathbb{R}^N)}.$$

Thus,

$$\left\|(-\Delta)^{-\frac{1}{2}}\operatorname{div}\left(\left(H_\nu(u_\nu)-|u|^{m_1}\right)\right)\nabla\varphi\right\|_{L^2(Q_T)} \lesssim \left\|\left(H_\nu(u_\nu)-|u|^{m_1}\right)\nabla\varphi\right\|_{L^2(Q_T)}.$$

Since $H_\nu(u_\nu(x,t)) \to |u(x,t)|^{m_1}$ for a.e. $(x,t) \in Q_T$ (up to a subsequence if necessary), then we have

$$(H_\nu(u_\nu) - |u|^{m_1})\,\nabla\varphi \to 0, \text{ for a.e. } (x,t) \in Q_T.$$

Furthermore, by Proposition 2 we have

$$|(H_\nu(u_\nu(x,t)) - |u(x,t)|^{m_1})\nabla\varphi| \le C(u_0, f, T)|\nabla\varphi|, \quad \forall (x,t) \in Q_T.$$

Thanks to the Dominated Convergence Theorem, we obtain (3.70).
Next, we deduce from (3.69)

$$(-\Delta)^{\frac{1}{2}-s} G_\nu(u_\nu) \to (-\Delta)^{\frac{1}{2}-s}(|u|^{m_2-1}u),$$

weakly in $L^2(Q_T)$ as $\nu \to 0$.
Thus,

$$\int \operatorname{div}\left(H_\nu(u_\nu)\nabla(-\Delta)^{-s}[G_\nu(u_\nu)]\right)\varphi dx \to \int \operatorname{div}\Theta(u)\varphi dx.$$

On the other hand, it is not difficult to show that

$$\begin{cases} \int_0^T \int u_\nu \varphi_t dxdt \to \int_0^T \int u\varphi_t dxdt, \\ \int_0^T \int \delta_1 \nabla u_\nu . \nabla\varphi dxdt \to \int_0^T \int \delta_1 \nabla u . \nabla\varphi dxdt, \\ \int_0^T \int \delta_2 (-\Delta)^{s_0}[|u_\nu|^{m_0-1}u_\nu]\varphi dxdt \to \int_0^T \int \delta_2(-\Delta)^{s_0}[|u|^{m_0-1}u]\varphi dxdt, \end{cases}$$

as $\nu \to 0$. Therefore, u is a weak solution of problem (3.60).
Or, we complete the proof of Proposition 6. □

3.4 Limit as $\delta_1, \delta_2 \to 0$

In this subsection, we will pass to the limit as $\delta_2, \delta_1 \to 0$ alternatively. Then, we have the following result.

Proposition 7. *Let u_{δ_2} be a solution of* (3.60) *above. Then, there exists a subsequence of $\{u_{\delta_2}\}_{\delta_2>0}$, converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$.*
Moreover, $u \in L^1(Q_T) \cap L^\infty(Q_T)$ is a weak solution of the following problem

$$u_t - \delta_1 \Delta u - \operatorname{div}\Theta(u) = f, \quad \text{in } Q_T, \tag{3.71}$$

Proof. We rewrite equation (3.60), satisfied by u_{δ_2} in the weak sense as follows

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} \big(&-u_{\delta_2}\varphi_t - \delta_1 u_{\delta_2}\Delta\varphi + H(u_{\delta_2})\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\cdot\nabla\varphi \\ &+\delta_2(-\Delta)^{s_0}[|u_{\delta_2}|^{m_0-1}u_{\delta_2}]\varphi - f\varphi\big)\,dxdt = 0, \quad \forall\varphi \in \mathcal{C}_c^\infty(Q_T). \end{aligned} \tag{3.72}$$

Our purpose is to pass to the limit as $\delta_2 \to 0$ in (3.72) in order to obtain

(3.73)
$$\int_0^T \int_{\mathbb{R}^N} \left(-u\varphi_t - \delta_1 u\Delta\varphi + H(u)\nabla(-\Delta)^{-s}[G(u)]\cdot\nabla\varphi - f\varphi\right) dxdt = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty(Q_T),$$

which says that u is a weak solution of equation (3.71).
First, we claim that

$$\delta_2(-\Delta)^{s_0}[|u_{\delta_2}|^{m_0-1}u_{\delta_2}] \to 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{3.74}$$

as $\delta_2 \to 0$.
Indeed, for any $\varphi \in \mathcal{D}(\mathbb{R}^N \times (0,T))$, we apply Hölder's inequality and (3.57) in order to obtain

$$\begin{aligned}\delta_2 \left|\int_0^T \int_{\mathbb{R}^N} (-\Delta)^{s_0}[|u_{\delta_2}|^{m_0-1}u_{\delta_2}]\varphi dxdt\right| &= \delta_2 \left|\int_0^T \int_{\mathbb{R}^N} (-\Delta)^{\frac{s_0}{2}}[|u_{\delta_2}|^{m_0-1}u_{\delta_2}](-\Delta)^{\frac{s_0}{2}}\varphi dxdt\right| \\ &\le \delta_2 \|(-\Delta)^{\frac{s_0}{2}}[|u_{\delta_2}|^{m_0-1}u_{\delta_2}]\|_{L^2(Q_T)}\|(-\Delta)^{\frac{s_0}{2}}\varphi\|_{L^2(Q_T)} \\ &\le C'\sqrt{\delta_2}\|(-\Delta)^{\frac{s_0}{2}}\varphi\|_{L^2(Q_T)}.\end{aligned}$$

This yields the claim.
Next, we prove that there is a subsequence of $\{u_{\delta_2}\}_{\delta_2>0}$, (still denoted as $\{u_{\delta_2}\}_{\delta_2>0}$) such that

$$u_{\delta_2} \to u, \quad \text{in } L^q\left(B_R \times (0,T)\right), \tag{3.75}$$

for any $R > 0$, and for any $q \in [1,\infty)$. Thus, up to a subsequence, we have

$$u_{\delta_2}(x,t) \to u(x,t), \quad \text{for a.e } (x,t) \in \mathbb{R}^N \times (0,T), \tag{3.76}$$

so

$$u \in L^\infty(Q_T).$$

In fact, using $|u|^{q-2}u$ as a test function to equation (3.60), we obtain as in (3.18)
(3.77)
$$\int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(G(u_{\delta_2}(x)) - G(u_{\delta_2}(y)))\,(|u_{\delta_2}|^{m_1+q-2}u_{\delta_2}(x) - |u_{\delta_2}|^{m_1+q-2}u_{\delta_2}(y))}{|x-y|^{N+2(1-s)}} dxdydt \le C,$$

Let us fix $q > 1$ in (3.77) such that $\gamma = \frac{m_1+m_2+q-1}{2} \ge 1$. It follows from Lemma 5 and (3.77) that

$$\int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left||u_{\delta_2}(x)|^{\gamma-1}u_{\delta_2}(x) - |u_{\delta_2}(y)|^{\gamma-1}u_{\delta_2}(y)\right|^2}{|x-y|^{N+2(1-s)}} dxdydt \le C, \tag{3.78}$$

where $C > 0$ is independent of δ_2, δ_1.
This implies that $v_{\delta_2} = |u_{\delta_2}|^{\gamma-1}u_{\delta_2}$ is uniformly bounded in $L^2(0,T;H^{1-s}(\mathbb{R}^N))$ for all

$\delta_2 > 0$.
Moreover, since $\gamma \geq 1$, it follows then from Proposition 2 and the Interpolation theorem that

$$\lim_{|E|\to 0, E\subset(0,T)} \sup_{\delta_2>0} \int_E \int_{\mathbb{R}^N} |v_{\delta_2}(x,t)|^2 dxdt = 0,$$

and $\|v_{\delta_2}\|_{L^2(\mathbb{R}^N\times(0,T))}$ is bounded by a constant, being independent of δ_1, δ_2. Thus, there is a subsequence of $\{v_{\delta_2}\}_{\delta_2>0}$ (still denoted as $\{v_{\delta_2}\}_{\delta_2>0}$) such that

$$v_{\delta_2} \rightharpoonup v, \text{ weakly in } L^2\left(\mathbb{R}^N \times (0,T)\right).$$

Thanks to a result of Rakotoson and Temam, [23], we obtain for any $R > 0$

$$v_{\delta_2} \to v, \text{ in } L^2\left(B_R \times (0,T)\right). \tag{3.79}$$

Now, applying Lemma 4 and Hölder's inequality yields

$$\begin{aligned}\int_0^T \int_{B_R} |u_{\delta_2} - u|\, dxdt &\leq \int_0^T \int_{B_R} \left||u_{\delta_2}|^{\gamma-1}u_{\delta_2} - |u|^{\gamma-1}u\right|^{\frac{1}{\gamma}} dxdt \\ &\leq \left(\int_0^T \int_{B_R} \left||u_{\delta_2}|^{\gamma-1}u_{\delta_2} - |u|^{\gamma-1}u\right|^2 dxdt\right)^{\frac{1}{2\gamma}} (T|B_R|)^{1-\frac{1}{2\gamma}} \\ &= (T|B_R|)^{1-\frac{1}{2\gamma}} \|v_{\delta_2} - v\|^{\frac{1}{\gamma}}_{L^2(B_R\times(0,T))}.\end{aligned}$$

A combination of the last inequality, (3.79), and the uniform boundedness of u_{δ_2} implies (3.75).
It remains to show the convergence of $\nabla(-\Delta)^{-s}[G(u_{\delta_2})] \to \nabla(-\Delta)^{-s}[G(u)]$ in $\mathcal{D}'(Q_T)$. We divide our proof into the two following cases.

i) The case $s \in (0, \frac{1}{2})$. We show that

$$\nabla(-\Delta)^{-s}[G(u_{\delta_2})] \rightharpoonup \nabla(-\Delta)^{-s}[G(u)], \quad \text{in } L^p\left(0,T; W^{-1,p}(\mathbb{R}^N)\right) \tag{3.80}$$

up to a subsequence, for $p > 1$ such that $m_2 p' \geq 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $W^{-1,p}(\mathbb{R}^N)$ is the dual space of $W^{1,p}(\mathbb{R}^N)$.
We emphasize that it is enough to consider the case $0 < m_2 < 1$ in the following because the case $m_2 \geq 1$ is much easier.
By using Hölder's inequality and Plancherel's theorem, we get

$$\begin{aligned}\|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{W^{-1,p}(\mathbb{R}^N)} &= \sup_{\|\psi\|_{W^{1,p}(\mathbb{R}^N)}=1} \left|\int_{\mathbb{R}^N} \nabla(-\Delta)^{-s}[G(u_{\delta_2})]\psi dx\right| \\ &= \sup_{\|\psi\|_{W^{1,p}(\mathbb{R}^N)}=1} \left|\int_{\mathbb{R}^N} G(u_{\delta_2})\nabla(-\Delta)^{-s}\psi dx\right| \\ &\leq \sup_{\|\psi\|_{W^{1,p}(\mathbb{R}^N)}=1} \|G(u_{\delta_2})\|_{L^{p'}(\mathbb{R}^N)} \|\nabla^{1-2s}\psi\|_{L^p(\mathbb{R}^N)}.\end{aligned}$$

Now, we apply Lemma 11 in order to obtain

$$\|\nabla^{1-2s}\psi\|_{L^p(\mathbb{R}^N)} \leq C\|\nabla\psi\|_{L^p(\mathbb{R}^N)}^{1-2s}\|\psi\|_{L^p(\mathbb{R}^N)}^{2s}.$$

Combining the two last inequalities yields

$$\|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{W^{-1,p}(\mathbb{R}^N)} \lesssim \|G(u_{\delta_2})\|_{L^{p'}(\mathbb{R}^N)} = \|u_{\delta_2}\|_{L^{m_2p'}(\mathbb{R}^N)}^{m_2}. \tag{3.81}$$

Thanks to Proposition 2 and the fact $m_2p' \geq 1$, it follows from (3.81) that

$$\|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{L^p(0,T;W^{-1,p}(\mathbb{R}^N))} \leq C(u_0, f, p, m_2, T). \tag{3.82}$$

Thus, $\nabla(-\Delta)^{-s}[G(u_{\delta_2})]$ is uniformly bounded in $L^p\left(0,T;W^{-1,p}(\mathbb{R}^N)\right)$ for all $\delta_2 > 0$. Then, there exists a function w such that

$$\nabla(-\Delta)^{-s}[G(u_{\delta_2})] \rightharpoonup w, \text{ in } L^p\left(0,T;W^{-1,p}(\mathbb{R}^N)\right),$$

up to a subsequence.
On the other hands, we also have

$$G(u_{\delta_2}) \rightharpoonup G(u), \text{ in } L^{p'}(Q_T).$$

This implies (3.80).

ii) The case $s \in (\frac{1}{2}, 1)$. We prove that for any $R > 0$

$$\nabla(-\Delta)^{-s}[G(u_{\delta_2})] \rightharpoonup \nabla(-\Delta)^{-s}[G(u)], \text{ in } L^2(B_R \times (0,T)). \tag{3.83}$$

To do that, we first show that

$$\int_0^T \|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{L^2(B_R)}^2 dt \leq C, \tag{3.84}$$

where $C > 0$ is independent of δ_1, δ_2.
Let us fix $q > 1$ in (3.77) such that $\gamma \geq 1$ in (3.78). For any $\beta \in (s, \frac{1+s}{2})$, we apply Hölder's inequality and the Plancherel theorem to get

$$\begin{aligned}
\|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{L^r(\mathbb{R}^N)} &= \sup_{\|\psi\|_{L^{r'}(\mathbb{R}^N)}=1} \left|\int_{\mathbb{R}^N} (-\Delta)^{\beta-s}[G(u_{\delta_2})]\nabla(-\Delta)^{-\beta}\psi dx\right| \\
&\leq \sup_{\|\psi\|_{L^{r'}(\mathbb{R}^N)}=1} \|(-\Delta)^{\beta-s}[G(u_{\delta_2})]\|_{L^{p'}(\mathbb{R}^N)}\|\mathcal{I}_{2\beta-1}(\psi)\|_{L^p(\mathbb{R}^N)} \\
&\lesssim \sup_{\|\psi\|_{L^{r'}(\mathbb{R}^N)}=1} \|(-\Delta)^{\beta-s}[G(u_{\delta_2})]\|_{L^{p'}(\mathbb{R}^N)}\|\psi\|_{L^{\frac{Np}{N+p(2\beta-1)}}(\mathbb{R}^N)},
\end{aligned}$$

if provided that

$$\frac{Np}{N+p(2\beta-1)} > 1 \Leftrightarrow \frac{N}{p'} > (2\beta-1). \tag{3.85}$$

Now, we take $r' = \frac{Np}{N+p(2\beta-1)}$ in the last inequality in order to get

$$\|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{L^r(\mathbb{R}^N)} \lesssim \|(-\Delta)^{\beta-s}[G(u_{\delta_2})]\|_{L^{p'}(\mathbb{R}^N)}. \tag{3.86}$$

Next, let us put $\gamma_0 = \frac{m_2}{\gamma} \in (0,1)$, and let β be such that $\beta - s < \gamma_0(1-s)$.
Applying Lemma 12 with $v = |u_{\delta_2}|^{\gamma} sign(u_{\delta_2})$, and $\Gamma(v) = |v|^{\gamma_0} sign(v)$ yields

$$\|(-\Delta)^{\beta-s}[G(u_{\delta_2})]\|_{L^{p'}(\mathbb{R}^N)} \leq C\||u_{\delta_2}|^{\gamma-1}u_{\delta_2}\|_{\dot{H}^{1-s}(\mathbb{R}^N)},$$

with

$$\frac{\beta-s}{\gamma_0} + \frac{N}{2} = \frac{N}{p'} + 1 - s. \tag{3.87}$$

Since $N \geq 2$, $\gamma_0 \in (0,1)$, and $\beta \in (s, \frac{1+s}{2})$, then it is not difficult to verify that there exists a real number $p' \in (2, \frac{N}{2\beta-1})$ so that (3.85) and (3.87) hold.
Combining the last inequalities and (3.86) yields

$$\|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|_{L^r(\mathbb{R}^N)} \leq C\||u_{\delta_2}|^{\gamma-1}u_{\delta_2}\|_{\dot{H}^{1-s}(\mathbb{R}^N)}.$$

Here, we note that $r > 2$ since β is close enough to s. Then, for any ball B_R in $\mathbb{R}^N$, it follows from Hölder's inequality that

$$\begin{aligned}\int_0^T \|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|^2_{L^2(B_R)}dt &\leq |B_R|^{2(1-\frac{2}{r})}\int_0^T \|\nabla(-\Delta)^{-s}[G(u_{\delta_2})]\|^2_{L^r(B_R)}dt \\ &\lesssim |B_R|^{2(1-\frac{2}{r})}\int_0^T \||u_{\delta_2}|^{\gamma-1}u_{\delta_2}\|^2_{\dot{H}^{1-s}(\mathbb{R}^N)}dt.\end{aligned} \tag{3.88}$$

By (3.78) and (3.88), we obtain (3.84).
This implies that $\nabla(-\Delta)^{-s}[G(u_{\delta_2})]$ converges weakly to a function w in $L^2(B_R \times (0,T))$ as $\delta_2 \to 0$, up to a subsequence.
Moreover, since $G(u_{\delta_2}) \to G(u)$ for a.e $(x,t) \in \mathbb{R}^N \times (0,T)$, then we obtain (3.83).

iii) The case $s = \frac{1}{2}$. This case is quite simple.
Indeed, since $\nabla^0 = \nabla(-\Delta)^{-1/2} = \nabla\mathcal{I}_1 u$ (the Riesz transform of u), then we have

$$\|\nabla(-\Delta)^{-1/2}[G(u_{\delta_2})]\|^q_{L^q(Q_T)} \lesssim \|G(u_{\delta_2})\|^q_{L^q(Q_T)}$$

for any $q > 1$ such that $m_2 q \geq 1$. Thanks to Proposition 2, we obtain

$$\|G(u_{\delta_2})\|^q_{L^q(Q_T)} \leq C,$$

where $C > 0$ is independent of δ_1, δ_2. As a result, there is a subsequence of $\{\nabla(-\Delta)^{-1/2}[G(u_{\delta_2})]\}_{\delta_2>0}$ such that

$$\nabla(-\Delta)^{-1/2}[G(u_{\delta_2})] \rightharpoonup \nabla(-\Delta)^{-1/2}[G(u)], \text{ in } L^2(Q_T). \tag{3.89}$$

Thanks to (3.74), (3.75), (3.80), (3.83), and (3.89), we can pass to the limit as $\delta_2 \to 0$ in equation (3.72) in order to obtain equation (3.73). In other words, u is a weak solution of (3.71).
Hence, we get the proof of Proposition 7. □

Remark 7. *We emphasize that the estimates in the proof of Proposition 7 are also independent of* δ_1.

Next, we will pass to the limit as $\delta_1 \to 0$ in (3.71).

Proposition 8. *Let u_{δ_1} be a solution of* (3.71). *Then, there exists a subsequence of $\{u_{\delta_1}\}_{\delta_1>0}$, converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$.*
Furthermore, $u \in L^1(Q_T) \cap L^\infty(Q_T)$, which is a weak solution of equation (1.1).
In addition, we obtain the regularity of $\operatorname{div}(\Theta(u))$ *as follows:*

- *If $s \in [\frac{1}{2}, 1)$ then*

$$\operatorname{div}(\Theta(u)) \in L^2\left(0, T; H^{-1}(B_R)\right). \tag{3.90}$$

- *Otherwise, if $s \in (0, \frac{1}{2})$ then*

$$\operatorname{div}(\Theta(u)) \in L^p\left(0, T; W^{-2,p}(\mathbb{R}^N)\right), \tag{3.91}$$

  for $p > 1$ such that $\frac{m_2 p}{p-1} \geq 1$, and $W^{-2,p}(\mathbb{R}^N)$ is the dual space of $W^{2,p}(\mathbb{R}^N)$.

Proof. Thanks to Remark 7, we observe that the proof of Proposition 8 can be done by repeating the one of Proposition 7. Thus, it remains to prove $\delta_1 \Delta u_{\delta_1} \to 0$ in $\mathcal{D}'(Q_T)$, as $\delta_1 \to 0$, (3.90) and (3.91).
We first show that

$$\delta_1 \Delta u_{\delta_1} \rightharpoonup 0, \text{ in } L^2\left(0, T; H^{-1}(\mathbb{R}^N)\right) \tag{3.92}$$

as $\delta_1 \to 0$, .
Indeed, for any $\varphi \in L^2\left(0, T; H^1(\mathbb{R}^N)\right)$ we have from (3.17) and Hölder's inequality

$$\begin{aligned} \delta_1 \left| \int_0^T \int_{\mathbb{R}^N} \Delta u_{\delta_1} \varphi dx dt \right| &\leq \sqrt{\delta_1}\sqrt{\delta_1} \|\nabla u_{\delta_1}\|_{L^2(Q_T)} \|\nabla \varphi\|_{L^2(Q_T)} \\ &\leq \sqrt{\delta_1} C \|\nabla \varphi\|_{L^2(Q_T)}. \end{aligned}$$

Hence, (3.92) follows as $\delta_1 \to 0$.

Next, we prove (3.90). It follows from Hölder's inequality that

$$\begin{aligned}
\| \operatorname{div}(\Theta(u)) \|_{L^2(0,T;H^{-1}(B_R))} &= \sup_{\|\psi\|_{H_0^1(B_R)}=1} \left| \int_{B_R} \operatorname{div}(\Theta(u))\, \psi \, dx \right| \\
&= \sup_{\|\psi\|_{H_0^1(B_R)}=1} \left| \int_{B_R} \Theta(u) \nabla \psi \, dx \right| \\
&\le \sup_{\|\psi\|_{H_0^1(B_R)}=1} \|\Theta(u)\|_{L^2(B_R)} \|\nabla \psi\|_{L^2(B_R)} \\
&\le \|\Theta(u)\|_{L^2(B_R)}.
\end{aligned}$$

Then,

$$\int_0^T \| \operatorname{div}(\Theta(u)) \|^2_{L^2(0,T;H^{-1}(B_R))} dt \le \int_0^T \|\Theta(u)\|^2_{L^2(B_R)} dt.$$

It follows from (3.84) and (3.89) that

$$\int_0^T \|\Theta(u)\|^2_{L^2(B_R)} dt \le C.$$

Then, we obtain (3.90).

Next, for any $p > 1$ such that $m_2 p' \ge 1$, we have

$$\begin{aligned}
\| \operatorname{div}(\Theta(u)) \|_{W^{-2,p}(\mathbb{R}^N)} &= \sup_{\|\psi\|_{W^{2,p}(\mathbb{R}^N)}=1} \left| \int_{\mathbb{R}^N} \operatorname{div}(\Theta(u))\, \psi \, dx \right| \\
&= \sup_{\|\psi\|_{W^{2,p}(\mathbb{R}^N)}=1} \left| \int_{\mathbb{R}^N} \Theta(u) \nabla \psi \, dx \right| \\
&\le \sup_{\|\psi\|_{W^{2,p}(\mathbb{R}^N)}=1} \|\Theta(u)\|_{W^{-1,p}(\mathbb{R}^N)} \|\nabla \psi\|_{W^{1,p}(\mathbb{R}^N)} \\
&\lesssim \sup_{\|\psi\|_{W^{2,p}(\mathbb{R}^N)}=1} \|\Theta(u)\|_{W^{-1,p}(\mathbb{R}^N)} \|\psi\|_{W^{2,p}(\mathbb{R}^N)} \\
&\le \|\Theta(u)\|_{W^{-1,p}(\mathbb{R}^N)}.
\end{aligned}$$

A combination of the last inequality and (3.80) implies that

$$\int_0^T \| \operatorname{div}(\Theta(u)) \|^p_{W^{-2,p}(\mathbb{R}^N)} dt \lesssim \int_0^T \|\Theta(u)\|^p_{W^{-1,p}(\mathbb{R}^N)} dt \le C.$$

Thus, (3.91) follows.
Then, we complete the proof of Lemma 8. □

In the case $m_2 > m_1$, we prove that (3.90) holds for any $s \in (0,1)$.

Proposition 9. *Let $s \in (0,1)$, and let u be a weak solution of (1.1). Assume that $m_2 > m_1$. Then, for any ball B_R there holds*

$$\operatorname{div}(\Theta(u)) \in L^2\left(0,T; H^{-1}(B_R)\right). \tag{3.93}$$

Proof. By the same argument as in (3.77), we also obtain

$$\int_0^T \int\int \frac{(G(u(x)) - G(u(y)))\,(|u|^{m_1+q-2}u(x) - |u|^{m_1+q-2}u(y))}{|x-y|^{N+2(1-s)}} dxdydt \leq C, \tag{3.94}$$

for $q > 1$. Let us take $q = 1 + m_2 - m_1$. It follows from (3.94) that

$$0 \leq \int_0^T \int G(u)(-\Delta)^{1-s}[G(u)]dxdt \leq C = C(u_0, f, m_1, m_2).$$

Thus,

$$\|(-\Delta)^{\frac{1-s}{2}}[G(u)]\|^2_{L^2(Q_T)} \leq C. \tag{3.95}$$

Now, we have from the Plancherel theorem that

$$\begin{aligned}
\|\operatorname{div}\Theta(u)\|_{H^{-1}(B_R)} &= \sup_{\|\psi\|_{H_0^1(B_R)}=1} \left|\int_{B_R} \operatorname{div}(\Theta(u))\psi dx\right| \\
&= \sup_{\|\psi\|_{H_0^1(B_R)}=1} \left|\int_{\mathbb{R}^N} \operatorname{div}(H(u)\nabla\psi)\,(-\Delta)^{-s}[G(u)]dx\right| \\
&= \sup_{\|\psi\|_{H_0^1(B_R)}=1} \left|\int_{\mathbb{R}^N} (-\Delta)^{-\frac{1+s}{2}}\left[\operatorname{div}(H(u)\nabla\psi)\right](-\Delta)^{\frac{1-s}{2}}[G(u)]dx\right|.
\end{aligned} \tag{3.96}$$

Apply Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} (-\Delta)^{-\frac{1+s}{2}}\left[\operatorname{div}(H(u)\nabla\psi)\right](-\Delta)^{\frac{1-s}{2}}[G(u)]dx\right| \\
&\leq \|(-\Delta)^{-\frac{1+s}{2}}\left[\operatorname{div}(H(u)\nabla\psi)\right]\|_{L^2(\mathbb{R}^N)}\|(-\Delta)^{\frac{1-s}{2}}[G(u)]\|_{L^2(\mathbb{R}^N)}.
\end{aligned} \tag{3.97}$$

On the other hand, it follows from the Plancherel theorem that

$$\|(-\Delta)^{-\frac{1+s}{2}}\left[\operatorname{div}(H(u)\nabla\psi)\right]\|_{L^2(\mathbb{R}^N)} \leq C(N)\|(-\Delta)^{-\frac{s}{2}}\left[H(u)\nabla\psi\right]\|_{L^2(\mathbb{R}^N)}, \tag{3.98}$$

Moreover, we apply the Riesz potential estimate and Hölder's inequality to get

$$\begin{aligned}
\|(-\Delta)^{-\frac{s}{2}}\left[H(u)\nabla\psi\right]\|_{L^2(\mathbb{R}^N)} &= \|\mathcal{I}_s\left[H(u)\nabla\psi\right]\|_{L^2(\mathbb{R}^N)} \\
&\lesssim \|H(u)\nabla\psi\|_{L^{\frac{2N}{N+2s}}(\mathbb{R}^N)} \leq \|u_0\|^{m_1}_{L^\infty(\mathbb{R}^N)}\|\nabla\psi\|_{L^{\frac{2N}{N+2s}}(\mathbb{R}^N)} \\
&\leq \|u_0\|^{m_1}_{L^\infty(\mathbb{R}^N)}\|\nabla\psi\|_{L^2(B_R)}|B_R|^{\frac{2s}{N+2s}}.
\end{aligned} \tag{3.99}$$

Note that $\frac{2N}{N+2s} > 1$ since $s \in (0,1)$ and $N \geq 2$.
From (3.99) and (3.98), we obtain

$$\|(-\Delta)^{-\frac{1+s}{2}}\left[\operatorname{div}(H(u)\nabla\psi)\right]\|_{L^2(\mathbb{R}^N)} \lesssim |B_R|^{\frac{2s}{N+2s}}\|u_0\|^{m_1}_{L^\infty(\mathbb{R}^N)}\|\nabla\psi\|_{L^2(B_R)}. \tag{3.100}$$

A combination of (3.96), (3.97), and (3.100) yields

$$\|\operatorname{div}\Theta(u)\|_{H^{-1}(B_R)} \leq C\|(-\Delta)^{\frac{1-s}{2}}[G(u)]\|_{L^2(\mathbb{R}^N)},$$

where $C = C(R, u_0, N, s, m_1) > 0$. Then,

$$\int_0^T \|\operatorname{div}\Theta(u)\|^2_{H^{-1}(B_R)}dt \leq C\int_0^T \|(-\Delta)^{\frac{1-s}{2}}[G(u)]\|^2_{L^2(\mathbb{R}^N)}dt.$$

Thus, the conclusion follows from the last inequality and (3.95) . □

4 Decay estimates and the finite time extinction of solution

In this part, we study some decay estimates and the finite time extinction of solution u. We start with the case $f = 0$ by proving Theorem 2.

Proof of Theorem 2. By technical reasons, we consider first the case $p > 1$. It follows from (3.18) and after passing to the limit as $\varepsilon, \kappa, \nu, \delta_2 \to 0$ that

$$\frac{1}{p}\frac{d}{dt}\int |u(x,t)|^p dx + (p-1)\int\int \frac{(G(u(x)) - G(u(y)))\left(|u|^{m_1+p-2}u(x) - |u|^{m_1+p-2}u(y)\right)}{|x-y|^{N+2(1-s)}}dxdy \leq 0. \tag{4.1}$$

Thanks to Lemma 5, we obtain

$$\frac{1}{p}\frac{d}{dt}\int |u(x,t)|^p dx + (p-1)\int\int \frac{\left||u|^{\theta_0-1}u(x) - |u|^{\theta_0-1}u(y)\right|^2}{|x-y|^{N+2(1-s)}}dxdydt \leq C,$$

with $\theta_0 = \frac{m_1+m_2+p-1}{2} = \frac{\beta_0+p}{2}$,
By Sobolev embedding, we have

$$\||u(t)|^{\theta_0}\|_{L^{2^\star}} \leq C\||u(t)|^{\theta_0}\|_{\dot{H}^{1-s}},$$

with $C = C(N, s, 2)$, and $2^\star = \frac{2N}{N-2(1-s)} = \frac{2}{\alpha_0}$.
In order to use an iteration method, let us put $q_0 = p$. A combination of the two last inequalities leads to

$$\frac{d}{dt}\|u(t)\|^{q_0}_{L^{q_0}} + Cq_0(q_0-1)\|u(t)\|^{\alpha_0 q_1}_{L^{q_1}} \leq 0, \tag{4.2}$$

with $q_1 = 2^\star\theta_0$. Integrating both sides of (4.2) on $(s,t) \subset\subset (0,T)$ yields

$$\|u(t)\|_{L^{q_0}}^{q_0} + Cq_0(q_0-1)\int_s^t \|u(\tau)\|_{L^{q_1}}^{\alpha_0 q_1} d\tau \le \|u(s)\|_{L^{q_0}}^{q_0}. \tag{4.3}$$

It follows from (4.1) that $\|u(t)\|_{L^{q_0}}$ is nonincreasing with respect to t. Then, (4.3) deduces

$$\|u(t)\|_{L^{q_1}}^{\alpha_0 q_1} \le \frac{1}{Cq_0(q_0-1)}(t-s)^{-1}\|u(s)\|_{L^{q_0}}^{q_0}, \tag{4.4}$$

for every $0 < s < t < T$. It is important to note that the constant C in (4.4) will not change step by step, since we are going to use an iteration method starting from here. Now, let us set

$$t_n = t(1-2^{-n}), \quad q_{n+1} = 2^\star\theta_n, \quad \theta_n = \frac{\beta_0 + q_n}{2}, \quad n \ge 0,$$

and θ_0, q_0 are as above. Here, we note that the condition $m_1+m_2 > 1-\frac{2q_0(1-s)}{N}$ is equivalent to $q_1 > q_0$. Thus, by induction, we observe that the sequence $\{q_n\}_{n\ge 0}$ is increasing. Let us take $t = t_{n+1}, s = t_n$, and replace q_0 by q_n in (4.4). Then, we obtain

$$\|u(t_{n+1})\|_{L^{q_{n+1}}}^{\alpha_0 q_{n+1}} \le \frac{1}{Cq_n(q_n-1)}(t_{n+1}-t_n)^{-1}\|u(t_n)\|_{L^{q_n}}^{q_n}.$$

By induction, we get

$$\begin{aligned}\|u(t_{n+1})\|_{L^{q_{n+1}}} \le{}& \left[\frac{1}{Cq_n(q_n-1)}\right]^{\frac{1}{\alpha_0 q_{n+1}}}\left[\frac{1}{Cq_{n-1}(q_{n-1}-1)}\right]^{\frac{1}{\alpha_0^2 q_{n+1}}} \dots \left[\frac{1}{Cq_0(q_0-1)}\right]^{\frac{1}{\alpha_0^{n+1} q_{n+1}}}\\ &\times \left(t^{-1}2^{n+1}\right)^{\frac{1}{\alpha_0 q_{n+1}}}\left(t^{-1}2^{n}\right)^{\frac{1}{\alpha_0^2 q_{n+1}}} \dots \left(t^{-1}2\right)^{\frac{1}{\alpha_0^{n+1} q_{n+1}}}\\ &\times \|u_0\|_{L^{q_0}}^{\frac{q_0}{\alpha_0^{n+1}q_{n+1}}}.\end{aligned} \tag{4.5}$$

It is not difficult to verify that

$$\lim_{n\to\infty} \alpha_0^{n+1}q_{n+1} = q_0 + \frac{\beta_0}{1-\alpha_0}. \tag{4.6}$$

Next, by using (4.6), we obtain

$$\begin{aligned}\lim_{n\to\infty}\left(t^{-1}\right)^{\frac{1}{\alpha_0 q_{n+1}}}\left(t^{-1}\right)^{\frac{1}{\alpha_0^2 q_{n+1}}} \dots \left(t^{-1}\right)^{\frac{1}{\alpha_0^{n+1} q_{n+1}}} &= \lim_{n\to\infty} t^{-\frac{1}{\alpha_0 q_{n+1}}\sum_{j=0}^n (\frac{1}{\alpha_0})^j}\\ &= \lim_{n\to\infty} t^{-\frac{1}{\alpha_0 q_{n+1}}\frac{1-\alpha_0^{-(n+1)}}{1-\alpha_0^{-1}}}\\ &= t^{-\frac{1}{q_0(1-\alpha_0)+\beta_0}}.\end{aligned} \tag{4.7}$$

Similarly, we also have

$$\lim_{n\to\infty} C^{\frac{1}{\alpha_0 q_{n+1}}} C^{\frac{1}{\alpha_0^2 q_{n+1}}} ... C^{\frac{1}{\alpha_0^{n+1} q_{n+1}}} = C^{\frac{1}{q_0(1-\alpha_0)+\beta_0}}. \tag{4.8}$$

And also

$$\lim_{n\to\infty} (2^{n+1})^{\frac{1}{\alpha_0 q_{n+1}}} (2^n)^{\frac{1}{\alpha_0^2 q_{n+1}}} ... 2^{\frac{1}{\alpha_0^{n+1} q_{n+1}}} = 2^{\frac{1}{(1-\alpha_0)(q_0(1-\alpha_0)+\beta_0)}}. \tag{4.9}$$

After that, let us put

$$Z_n = q_n^{\frac{1}{\alpha_0 q_{n+1}}} q_{n-1}^{\frac{1}{\alpha_0^2 q_{n+1}}} ... q_0^{\frac{1}{\alpha_0^{n+1} q_{n+1}}}.$$

Let us show that Z_n is convergent as $n \to \infty$. Indeed, we consider the power series

$$S_n(s) = s^n \ln q_n + s^{n-1} \ln q_{n-1} + ... + s^1 \ln q_1 + \ln q_0. \tag{4.10}$$

Obviously, the radius of convergence of $S_n(s)$ is 1. Thus, $S_n(\alpha_0)$ converges absolutely to a real number λ_0 as $n \to \infty$.
On the other hand, we note that

$$\alpha_0^{n+1} q_{n+1} \ln Z_n = S(\alpha_0).$$

It follows then from (4.6) that

$$\lim_{n\to\infty} \ln Z_n = \frac{\lambda_0}{q_0 + \frac{\beta_0}{1-\alpha_0}}.$$

Then,

$$\lim_{n\to\infty} \frac{1}{Z_n} = \exp\left\{ -\frac{\lambda_0}{q_0 + \frac{\beta_0}{1-\alpha_0}} \right\}. \tag{4.11}$$

Similarly, there is a real positive number ζ_0 such that

$$\lim_{n\to\infty} \frac{1}{(q_n-1)^{\frac{1}{\alpha_0 q_{n+1}}} (q_{n-1}-1)^{\frac{1}{\alpha_0^2 q_{n+1}}} ... (q_0-1)^{\frac{1}{\alpha_0^{n+1} q_{n+1}}}} = \zeta_0. \tag{4.12}$$

A combination of (4.5), (4.6), (4.7), (4.8), (4.9), (4.11), and (4.12) implies that there is a constant $C = C(N, s, q_0, m_1, m_2) > 0$ such that

$$\|u(t)\|_{L^\infty} \le Ct^{-\frac{1}{q_0(1-\alpha_0)+\beta_0}} \|u_0\|_{L^{q_0}}^{\frac{q_0(1-\alpha_0)}{q_0(1-\alpha_0)+\beta_0}}. \tag{4.13}$$

This puts an end to the proof of Theorem 2 when $p > 1$.

Next, we prove L^1 decay estimate. To do that, we first prove an estimate of the decay L^1-L^q in the following lemma.

Lemma 9. *Let $s \in (0,1)$, and $m_1, m_2 > 0$ be such that $m_1 + m_2 > \alpha_0$. Assume that $u_0 \in L^1(\mathbb{R}^N)$. Then, for any $q > 1$ there holds*

$$\|u(t)\|_{L^q} \leq Ct^{-\frac{N(1-\frac{1}{q})}{(m_1+m_2-1)N+2(1-s)}} \|u_0\|_{L^1}^{\frac{N(m_1+m_2-1)+2(1-s)q}{[N(m_1+m_2-1)+2(1-s)]q}}, \tag{4.14}$$

with $C = C(N, s, m_1, m_2, q) > 0$.

Proof. Thanks to the Interpolation Inequality and the monotonicity of $\|u(t)\|_{L^1}$ for $t > 0$, it suffices to prove that (4.14) is true for any $q > 1$ large enough. We mimic the proof of Theorem 2 above by considering $q = q_0$. Let us recall here $q_1 = \frac{\beta_0+q_0}{\alpha_0}$. Note that $q_1 > q_0$ since q_0 is large enough.
Then, we apply the Interpolation Inequality to obtain

$$\|u(t)\|_{L^{q_0}} \leq \|u(t)\|_{L^1}^{\theta} \|u(t)\|_{L^{q_1}}^{1-\theta},$$

with $\theta = \frac{\frac{1}{q_0}-\frac{1}{q_1}}{1-\frac{1}{q_1}}$. Since $\|u(t)\|_{L^1}$ is nonincreasing for $t \geq 0$, we then get

$$\|u(t)\|_{L^{q_0}} \leq \|u_0\|_{L^1}^{\theta} \|u(t)\|_{L^{q_1}}^{1-\theta}.$$

It follows from (4.2) and the last inequality that

$$y'(t) + C\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} y(t)^{\frac{\alpha_0 q_1}{(1-\theta)q_0}} \leq 0, \tag{4.15}$$

with $C = C(N, s, q_0) > 0$, and $y(t) = \|u(t)\|_{L^{q_0}}^{q_0}$.
Note that $1 - \frac{\alpha_0 q_1}{(1-\theta)q_0} < 0$ since q_0 is large enough. Then, solving the OD inequality yields

$$y(t)^{1-\frac{\alpha_0 q_1}{(1-\theta)q_0}} \geq C\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} t,$$

with $C = C(N, s, q_0, m_1, m_2) > 0$.
Thus,

$$\|u(t)\|_{L^{q_0}} \leq C\|u_0\|_{L^1}^{\frac{\theta\alpha_0 q_1}{\alpha_0 q_1-(1-\theta)q_0}} t^{-\frac{(1-\theta)}{\alpha_0 q_1-(1-\theta)q_0}} = Ct^{-\frac{N(1-\frac{1}{q})}{(m_1+m_2-1)N+2(1-s)}} \|u_0\|_{L^1}^{\frac{N(m_1+m_2-1)+2(1-s)q}{[N(m_1+m_2-1)+2(1-s)]q}}.$$

This completes the proof of Lemma 9. □

Now, let us prove Theorem 2 for $p = 1$. In fact, we have from (4.13)

$$\|u(t)\|_{L^\infty} \leq C(t-\tau)^{-\frac{1}{q_0(1-\alpha_0)+\beta_0}} \|u(\tau)\|_{L^{q_0}}^{\frac{q_0(1-\alpha)}{q_0(1-\alpha_0)+\beta_0}}, \tag{4.16}$$

for any $\tau \in (0, t)$, and for any q_0 large enough.
Thanks to (4.14), we obtain

$$\|u(\tau)\|_{L^{q_0}} \leq C\tau^{-\frac{N(1-\frac{1}{q_0})}{(m_1+m_2-1)N+2(1-s)}} \|u_0\|_{L^1}^{\frac{N(m_1+m_2-1)+2(1-s)q_0}{[N(m_1+m_2-1)+2(1-s)]q_0}}. \tag{4.17}$$

Then, the conclusion follows from (4.16) and (4.17) with $\tau = \frac{t}{2}$. And the proof of Theorem 2 is complete. □

Finally, let us prove Theorem 3.

Proof of Theorem 3. We recall here q_0, q_1 as in the proof of Theorem 2. Then, we can mimic the proof of Lemma 9 in order to obtain as in (4.15)

$$y'(t) + C\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} y(t)^{\frac{\alpha_0 q_1}{(1-\theta)q_0}} \leq 0, \tag{4.18}$$

with $y(t) = \|u(t)\|_{L^{q_0}}^{q_0}$.
Now, we shall show that there exists a time $\tau_0 > 0$ such that $y(\tau_0) = 0$. If this is done, then the conclusion follows from the monotonicity of $y(t)$ by (4.18). Assume by contradiction that $y(t) > 0$, for any $t > 0$. Then, we solve the indicated ODE to obtain

$$y(t)^{1-\frac{\alpha_0 q_1}{(1-\theta)q_0}} - y(0)^{1-\frac{\alpha_0 q_1}{(1-\theta)q_0}} + C_1\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} t \leq 0.$$

Therefore,

$$y(0)^{1-\frac{\alpha_0 q_1}{(1-\theta)q_0}} \geq C_1\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} t. \tag{4.19}$$

By the fact $1 - \frac{\alpha_0 q_1}{(1-\theta)q_0} > 0$, (4.19) leads to a contradiction as $t \to \infty$.
Thus, we obtain the proof of Theorem 3. □

Remark 8. *By* (4.19)*, we can estimate the extinction time of u, denoted as τ_0 satisfying*

$$\tau_0 \leq C_1\|u_0\|_{L^1}^{\frac{\theta\alpha_0 q_1}{1-\theta}} \|u_0\|_{L^{q_0}}^{\frac{-\alpha_0 q_1 + (1-\theta)q_0}{1-\theta}}.$$

Remark 9. *Some of the results of this paper remain valid for the case of a bounded domain and homogeneous Dirichlet boundary conditions. Moreover function f may be given through a potential as in the case of nonlocal Schrödinger equation, such as $f(x,t) = -V(x)u(x,t)$, see [19]. To end this paper, we would like to refer to [1] for the study of the energy method in proving the complete quenching of solutions and the free boundary of solutions of nonlinear evolution equations.*

Now we shall consider the case of $f \neq 0$. In fact, the finite time extinction phenomenon also appears in problem (1.1) when $f \neq 0$ and f extincts in a finite time $T_f > 0$ (i.e: $f(x,t) = 0$ for $t \geq T_f$, and $x \in \mathbb{R}^N$). By assuming, for simplicity, that

$$f \in L^1(Q_T) \cap L^\infty(Q_T), \text{ for any } T > 0,$$

it is easy to adapt the proof of Theorem 3 to conclude that

$$y'(t) + K_1 y(t)^{\frac{\alpha_0 q_1}{(1-\theta)q_0}} \leq K_2 g(t), \tag{4.20}$$

with $y(t) = \|u(t)\|_{L^{q_0}}^{q_0}$ and

$$g(t) = \int |f(x,t)||u(x,t)|^{q-1}dx,$$

for some positive constants K_1 and K_2. Thus, we get the existence of a finite extinction time $\tau_0 \geq T_f$ for the solution u of problem (1.1) by repeating the arguments of Theorem 4 starting with the initial datum $u(x, T_f)$.

A less intuitive fact is that for certains source functions $f(t) \neq 0$, with a finite extinction time $T_f > 0$, the resulting extinction time τ_0 of the solution u let such that $\tau_0 = T_f$. Such type of behaviors was considered in the monograph [1] (see Theorem 2.1 of Chapter 2) for the case of local problems. As many other free boundary problems, this phenomenon requires a suitable balance between the domain (here the interval $(0, T_f)$) and the datum $\|u_0\|_{L^{q_0}}$, with a suitable decay of the right hand side (here given by the decay of $g(t)$ around $(t - T_f)_+$).

Proposition 10. *Let $s \in (0,1)$, and let $m_1, m_2 > 0$ be such that $m_1 + m_2 < \alpha_0 = \frac{N-2(1-s)}{N}$. Assume that $\|u_0\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^\infty(\mathbb{R}^N)}$ is small enough. Let ν_0 satisfy*

$$\max\{\alpha_0, 1 - (\alpha_0 - m_1 - m_2)\} < \nu_0 < 1.$$

Suppose that $f \in L^1(Q_T) \cap L^\infty(Q_T)$ and there exists a finite time $T_f > 0$ such that

$$\|f(t)\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon \left[1 - \frac{t}{T_f}\right]_+^{\frac{\nu_0}{1-\nu_0}}, \quad \text{for } t > 0, \tag{4.21}$$

and for some $\varepsilon > 0$ small enough. Then the finite extinction time of the solution u coincides with the extinction time of the source term f, i.e. $\tau_0 = T_f$.

Proof. Let us set

$$q_0 = \frac{1 - \nu_0 + (\alpha_0 - m_1 - m_2)}{1 - \nu_0}.$$

Note that $q_0 \geq 2$ since $\nu_0 > 1 + m_1 + m_2 - \alpha_0$. By a simple calculation, we have

$$\nu_0 = \frac{\alpha_0 q_1}{(1-\theta) q_0},$$

with $q_1 = \frac{\beta_0 + q_0}{\alpha_0}$, and $\theta = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{1 - \frac{1}{q_1}}$. We also emphasize that $q_1 > q_0$ since $\nu_0 > \alpha_0$.
In a similar way to the proof of (4.18), and thanks to the assumption (4.21), we observe that $y(t) = \|u(t)\|_{L^{q_0}}^{q_0}$ satisfies the following ordinary differential inequality:

$$y'(t) + C\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} y(t)^{\nu_0} \leq \varepsilon \left[1 - \frac{t}{T_f}\right]_+^{\frac{\nu_0}{1-\nu_0}} \|u_0\|_{L^{q_0-1}}^{q_0-1}, \quad y(0) = y_0 = \|u_0\|_{L^{q_0}}^{q_0}, \tag{4.22}$$

for some positive constant K. But, it is easy to see that the function

$$Y(t) = y_0 \left[1 - \frac{t}{T_f}\right]_+^{\frac{1}{1-\nu_0}}$$

is a supersolution of problem (4.22) once we assume the following condition on the data:

$$C\|u_0\|_{L^1}^{-\frac{\theta\alpha_0 q_1}{1-\theta}} {y_0}^{\nu_0} - \frac{y_0}{(1-\nu_0)T_f} > \varepsilon\|u_0\|_{L^{q_0-1}}^{q_0-1}. \tag{4.23}$$

We note that (4.23) occurs since u_0 is small and $\varepsilon > 0$ is also small enough, and it depends on u_0. Then, by applying the comparison principle for nonnegative solutions of the ODE associated to (4.22), we get

$$0 \leq y(t) \leq Y(t) \text{ for any } t \geq 0,$$

which implies that the extinction time of $y(t)$ coincides with T_f. □

Remark 10. *Notice that Theorem 3 extends to the case of the nonlocal problem* (1.1) *the result by Bénilan and Crandall [2] when we take $s = 0$ and $m := m_1 + m_2$.*

5 Appendix

Lemma 10. *Let $s \in (0, 1)$. For any $\varepsilon > 0$, there holds*

$$0 \leq \mathcal{F}\left\{\mathcal{L}_\varepsilon^s\right\} \leq \mathcal{F}\left\{(-\Delta)^s\right\}.$$

Proof. It is known that for any $u \in \mathcal{S}(\mathbb{R}^N)$ (the Schwartz space), $\mathcal{F}\{(-\Delta)^s\}$ can be considered as a multiplier of $\mathcal{F}\{(-\Delta)^s u\}$, i.e:

$$\mathcal{F}\left\{(-\Delta)^s u\right\}(\xi) = \mathcal{F}\{(-\Delta)^s\}\mathcal{F}\{u\}(\xi).$$

We have

$$(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+h) + u(x-h) - 2u(x)}{|h|^{N+2s}} dh.$$

Taking the Fourier transform yields

$$\begin{aligned}
\mathcal{F}\left\{(-\Delta)^s u\right\}(\xi) &= \frac{1}{2}\int_{\mathbb{R}^N} \frac{e^{i\xi\cdot h} + e^{-i\xi\cdot h} - 2}{|h|^{N+2s}} dh\, \mathcal{F}\{u\}(\xi) \\
&= \int_{\mathbb{R}^N} \frac{1 - \cos(\xi\cdot h)}{|h|^{N+2s}} dh\, \mathcal{F}\{u\}(\xi).
\end{aligned}$$

This implies that

$$\mathcal{F}\left\{(-\Delta)^s\right\}(\xi) = \int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot h)}{|h|^{N+2s}}dh. \tag{5.1}$$

Similarly, we also have

$$\begin{aligned}\mathcal{F}\left\{\mathcal{L}_\varepsilon^s u\right\}(\xi) &= \frac{1}{2}\int_{\mathbb{R}^N} \frac{e^{i\xi\cdot h}+e^{-i\xi\cdot h}-2}{(|h|^2+\varepsilon^2)^{\frac{N+2s}{2}}}dh\,\mathcal{F}\{u\}(\xi)\\ &= \int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot h)}{(|h|^2+\varepsilon^2)^{\frac{N+2s}{2}}}dh\,\mathcal{F}\{u\}(\xi).\end{aligned}$$

Therefore,

$$\mathcal{F}\left\{\mathcal{L}_\varepsilon^s\right\}(\xi) = \int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot h)}{(|h|^2+\varepsilon^2)^{\frac{N+2s}{2}}}dh. \tag{5.2}$$

Then, the conclusion of Lemma 10 follows from (5.1) and (5.2). □

Next, we have the following embedding results.

Lemma 11. *Let* $\alpha\in(0,1)$, $N\geq 1$, *and* $p\geq 1$. *Then, we have*

$$\|\nabla^\alpha v\|_{L^p(\mathbb{R}^N)} \leq C\|\nabla v\|_{L^p(\mathbb{R}^N)}^\alpha \|v\|_{L^p(\mathbb{R}^N)}^{1-\alpha}, \quad \forall v\in W^{1,p}(\mathbb{R}^N).$$

Proof. We have

$$\begin{aligned}\|\nabla^\alpha v\|_{L^p(\mathbb{R}^N)} &\leq C(N,\alpha)\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v(x+h)-v(x)|}{|h|^{N+\alpha}}dh\right)^p dx\right)^{1/p}\\ &\leq C(N,\alpha,p)\left[\left(\int_{\mathbb{R}^N}\left(\int_{|h|\leq\lambda}\frac{|v(x+h)-v(x)|}{|h|^{N+\alpha}}dh\right)^p dx\right)^{1/p}\right.\\ &\quad \left.+\left(\int_{\mathbb{R}^N}\left(\int_{|h|>\lambda}\frac{|v(x+h)-v(x)|}{|h|^{N+\alpha}}dh\right)^p dx\right)^{1/p}\right] := C(\mathbf{I}_1+\mathbf{I}_2).\end{aligned} \tag{5.3}$$

Now, we consider $\mathbf{I}_1$. Applying Young's inequality and Hölder's inequality yields

$$\begin{aligned}\mathbf{I}_1 &\leq \int_{|h|\leq\lambda}\left(\int_{\mathbb{R}^N}\left|\frac{|v(x+h)-v(x)|}{|h|^{N+\alpha}}\right|^p dx\right)^{1/p} dh\\ &\leq \int_{|h|\leq\lambda}\left(\int_{\mathbb{R}^N}\left(\int_0^1|\nabla v(t(x+h)+(1-t)x)|dt\right)^p dx\right)^{1/p}|h|^{-(N+\alpha-1)}dh\\ &\leq \int_{|h|\leq\lambda}\left(\int_0^1\int_{\mathbb{R}^N}|\nabla v(t(x+h)+(1-t)x)|^p dxdt\right)^{1/p}|h|^{-(N+\alpha-1)}dh\\ &\leq C(N,\alpha)\lambda^{1-\alpha}\|\nabla v\|_{L^p(\mathbb{R}^N)}.\end{aligned} \tag{5.4}$$

Next, we apply Young's inequality to get

$$\begin{aligned}\mathbf{I}_2 &\le \int_{|h|>\lambda}\left(\int_{\mathbb{R}^N}|v(x+h)-v(x)|^p dx\right)^{1/p}|h|^{-(N+\alpha)}dh\\ &\le 2\|v\|_{L^p(\mathbb{R}^N)}\int_{|h|>\lambda}|h|^{-(N+\alpha)}dh = C(N,\alpha)\lambda^{-\alpha}\|v\|_{L^p(\mathbb{R}^N)}. \qquad (5.5)\end{aligned}$$

A combination of (5.4) and (5.5) implies

$$\mathbf{I}_1+\mathbf{I}_2 \le C(N,\alpha)\left(\lambda^{1-\alpha}\|\nabla v\|_{L^p(\mathbb{R}^N)}+\lambda^{-\alpha}\|v\|_{L^p(\mathbb{R}^N)}\right).$$

The last inequality holds for any $\lambda>0$, then we obtain

$$\mathbf{I}_1+\mathbf{I}_2 \le C(N,\alpha)\|v\|^{1-\alpha}_{L^p(\mathbb{R}^N)}\|\nabla v\|^{\alpha}_{L^p(\mathbb{R}^N)}. \qquad (5.6)$$

By (5.3) and (5.6), we complete the proof of Lemma 11. □

Lemma 12. *Let $\theta\in(0,1)$, and $N\ge 1$. Let $\alpha_1,\alpha_2\in(0,1)$ be such that $\alpha_1<\alpha_2\theta$. Assume that Γ is a θ-Hölder continuous function on $\mathbb{R}$. Then, we have*

$$\|\nabla^{\alpha_1}\Gamma(v)\|_{L^r(\mathbb{R}^N)} \le C\|v\|_{\dot{H}^{\alpha_2}(\mathbb{R}^N)},$$

where

$$\frac{\alpha_1}{\theta}+\frac{N}{2}=\alpha_2+\frac{N}{r}. \qquad (5.7)$$

Remark 11. *Note that it follows from (5.7) that $r>2$.*

Proof. The proof of Lemma 12 can be found in [3, Lemma 6.6] . □

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References

[1] S. Antontsev, J. I. Díaz, S. Shmarev, *Energy methods for free boundary problems. Applications to nonlinear PDEs and Fluid Mechanics.* Series Progress in Nonlinear Differential Equations and Their Applications. No. 48, Birkhäuser, Boston, 2002.

[2] Ph. Bénilan and M. G. Crandall, *The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$.* Indiana Univ. Math. J. **30** (1981), 161-177.

[3] P. Biler, C. Imbert, G. Karch. *The nonlocal porous medium equation: Barenblatt profiles and other weak solutions.* Arch. Ration. Mech. Anal., **215** (2015), 497-529.

[4] M. Bonforte, A. Figalli, X. Ros-Otón. *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*, Comm. Pure Appl. Math. **70** (2017), no. 8, 1472-1508.

[5] M. Bonforte, A. Figalli, J. L. Vázquez. *Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains.* Analysis & PDE. **11** (2018), 945-982.

[6] M. Bonforte, Y. Sire, J. L. Vázquez. *Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains.* Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 5725-5767.

[7] M. Bonforte, J. L. Vázquez. *Quantitative Local and Global A Priori Estimates for Fractional Nonlinear Diffusion Equations.* Advances in Math. **250** (2014), 242-284.

[8] M. Bonforte, J. L. Vázquez. *A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains.* Arch. Ration. Mech. Anal. **218** (2015), 317-362.

[9] L. Caffarelli, L. Silvestre. *An extension problem related to the fractional Laplacian,* Comm. PDEs, **32** (2007), 1245-1260.

[10] L. A. Caffarelli, F. Soria, J. L. Vázquez. *Regularity of solutions of the fractional porous medium flow.* J. Eur. Math. Soc. **15** (2013), 1701-1746.

[11] L. A. Caffarelli, J. L. Vázquez. *Nonlinear porous medium flow with fractional potential pressure.* Arch. Rational Mech. Anal. **202** (2011), 537-565.

[12] L. A. Caffarelli, J. L. Vázquez. *Asymptotic behaviour of a porous medium equation with fractional diffusion*, Discrete Cont. Dyn. Systems A. **29** (2011), 1393-1404.

[13] L. Caffarelli, P. R. Stinga. *Fractional elliptic equations: Caccioppoli estimates and regularity*, Annales Inst. Henri Poincaré (C) Analysis Non Linéaire, (2016), 767-807.

[14] P. Constantin, M. Ignatova. *Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications.* Int. Math. Res. Notices (2016), 1-21.

[15] N. A. Dao, J. I. Díaz, H. V. Kha. *Complete quenching phenomenon and instantaneous shrinking of support of solutions of degenerate parabolic equations with nonlinear singular absorption*, Proceedings of the Royal Society of Edinburgh, **149** (2019), 1323-1346.

[16] E. B. Davies. *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math. **109** (1987), 319-333.

[17] A. De Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez. *A fractional porous medium equation*, Advances in Mathematics. **226** (2011), 1378-1409.

[18] A. De Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez. *A general fractional porous medium equation.* Comm. Pure Appl. Math. **65** (2012), 1242-1284.

[19] J. I. Díaz, D. Gómez-Castro, J. L. Vázquez. *The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach.* Nonlinear Analysis, **177** (2018), 325-360.

[20] J. Dolbeault, A. Zhang *Flows and functional inequalities for fractional operators.* Applicable Analysis **96** (9) (2017), 1547-1560.

[21] Q.-H. Nguyen, J. L. Vázquez. *Porous medium equation with nonlocal pressure in a bounded domain.* Comm. PDEs, **43** (2018), 1502-1539.

[22] A. Porretta. *Existence Results for Nonlinear Parabolic Equations via Strong Convergence of Truncations.* Annali di Matematica pura ed applicata. (IV), **CLXXVII** (1999), 143-172.

[23] J. M. Rakotoson, R. Temam, *An optimal compactness theorem and application to elliptic-parabolic systems*, Appl. Math. Lett. **14** (2001), 303-306.

[24] S. Serfaty, J. L. Vázquez. *A Mean Field Equation as Limit of Nonlinear Diffusion with Fractional Laplacian Operators.* Calc. Var. PDEs. **49** (2014), 1091-1120.

[25] J. Simon. *Compact sets in the space* $L^p(0,T;B)$. Ann. Mat. Pura Appl. **146** (1987), 65-96.

[26] D. Stan, F. del Teso, J. L. Vázquez. *Finite and infinite speed of propagation for porous medium equations with fractional pressure.* Journal Diff. Eqns. **260** (2016), 1154-1199.

[27] D. Stan, F. del Teso, J. L. Vázquez. *Existence of weak solutions for porous medium equations with nonlocal pressure.* Arch. Ration. Mech. Anal. **233** (2019), no. 1, 451-496.

[28] E. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, 1970.

[29] P. R. Stinga, J. L. Torrea. *Extension problem and Harnack inequality for some fractional operators*, Comm. PDE. **35** (2010), 2092-2122.

[30] M. E. Taylor. *Partial Differential Equations. III: Nonlinear Equations*, 2nd edn, xxii, p. 715. Applied Mathematical Sciences 117. Springer, New York, 2011.

[31] J. L. Vázquez. *The Porous Medium Equation. Mathematical Theory*, vol. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.

[32] J. L. Vázquez, A. de Pablo, F. Quirós, A. Rodríguez. *Classical solutions and higher regularity for nonlinear fractional diffusion equations*, J. Eur. Math. Soc. **19** (2017), 1949-1975.

[33] Q. S. Zhang. *The boundary behavior of heat kernels of Dirichlet Laplacians*, Jour. Diff. Equa. **182** (2002), 416-430.