A STRANGE NON-LOCAL MONOTONE OPERATOR ARISING IN THE HOMOGENIZATION OF A DIFFUSION EQUATION WITH DYNAMIC NONLINEAR BOUNDARY CONDITIONS ON PARTICLES OF CRITICAL SIZE AND ARBITRARY SHAPE

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ABSTRACT. We characterize the homogenization limit of the solution of a Poisson equation in a bounded domain, either periodically perforated or containing a set of asymmetric periodical small particles and on the boundaries of these particles a nonlinear dynamic boundary condition holds involving a Hölder nonlinear $\sigma(u)$. We consider the case in which the diameter of the perforations (or the diameter of particles) is critical in terms of the period of the structure. As in many other cases concerning critical size, a "strange" nonlinear term arises in the homogenized equation. For this case of asymmetric critical particles we prove that the effective equation is a semilinear elliptic equation in which the time arises as a parameter and the nonlinear expression is given in terms of a nonlocal operator H which is monotone and Lipschitz continuous on $L^2(0, T)$, independently of the regularity of σ.

1. INTRODUCTION

The main goal of this article is to extend previous papers in the literature dealing with the homogenization of a Poisson equation in a bounded domain, which we can assume either periodically perforated or containing a set of asymmetric periodical small particles, and on the internal boundaries a nonlinear dynamic boundary condition holds involving a Hölder continuous nonlinearity and some small parameters. In contrast to the case in which the diameter of the perforation (or the diameter of the particles) is equal to the period of the structure (see, e.g., [1, 29]) when the involved parameters are in a suitable balance with a " critical value" of the diameters then a new (and thus "strange" in the spirit of [8, 22, 23]) nonlocal term arises in the homogenized equation. That was shown in our previous paper [15] but merely for the case of a linear boundary condition and for symmetrical balls as perforations or particles (see also [33] for the case of Lipschitz nonlinear terms and nonhomogeneous boundary conditions). The more general case (Hölder continuous nonlinear terms and, which is more important, cavities, or particles, of arbitrary shape) leads to new difficulties which require a different framework: the "strange term" is now given by a nonlocal monotone operator which, curiously enough,

regularizes the nonlinearity (for instance, even if the nonlinearity involved in the dynamic boundary condition is merely Hölder continuous the strange operator is a $L^2(0,T)$-Lipschitz continuous operator. Our results also extend the treatment made in [11] for asymmetric particles but with Robin type boundary conditions (a problem which can be understood as the associate stationary problem associated with the evolution problem considered in the present paper). We point out that, in some sense, the assumption on critical values in the relation between size and distance, giving rise to a different reaction behavior is typical of many processes in Nanotechnology. New materials, in particular the so-called "Mechanical meta-materials", are built as artificial structures which have mechanical properties defined by their geometric structure rather than their chemical composition. The occurrence of "strange terms" in the homogenized equation can be understood as a similar process to the design of new materials having properties outside the scope found in Nature.

2. Problem statement

Let Ω be a bounded domain in R^n, $n \geq 3$, with Lipchsiz boundary $\partial\Omega$. In the cube $Y = (-1/2, 1/2)^n$ consider a subdomain G_0, $\overline{G_0} \subset Y$, which, for simplicity, we assume that is star shaped with respect to a ball $T_\rho^0 \subset G_0$ of radius ρ with the center at the origin. Out treatment remains valid if G_0 has a finite number of disjoint connect components satisfying the same geometric property (see Remark 5). Let $\delta B = \{x : \delta^{-1} x \in B\}$, $\delta > 0$. For $\varepsilon > 0$ let

$$\tilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\}.$$

Denote by Z^n the set of all vectors $j = (j_1, \ldots, j_n)$ with integer coordinates j_i, $i = 1, \ldots, n$. Consider the set

$$G_\varepsilon = \cup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \cup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in Z^n : \overline{G_\varepsilon^j} \subset Y_\varepsilon^j = \varepsilon Y + \varepsilon j, G_\varepsilon^j \cap \tilde{\overline{\Omega}}_\varepsilon \neq \emptyset\}$. We assume that

$$a_\varepsilon = C_0 \varepsilon^\gamma, \quad \text{for some } \gamma > 1 \text{ and } C_0 > 0. \tag{2.1}$$

It is easy to see that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = \text{const} > 0$. Note that

$$\overline{G_\varepsilon^j} \subset T_{Ca_\varepsilon}^j \subset T_{\varepsilon/4}^j \subset Y_\varepsilon^j,$$

where T_r^j is the ball in R^n of radius r with the center at $P_\varepsilon^j = \varepsilon j$ (the center of the cell Y_ε^j), C is a positive constant independent on ε. We introduce the sets

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega,$$
$$Q_\varepsilon^T = \Omega_\varepsilon \times (0, T), \quad S_\varepsilon^T = S_\varepsilon \times (0, T), \quad \Gamma^T = \partial\Omega \times (0, T).$$

The main problem considered in this paper deals with the following case of nonlinear dynamic boundary conditions

$$
\begin{aligned}
-\Delta_x u_\varepsilon &= f(x, t), \quad (x, t) \in Q_\varepsilon^T, \\
\varepsilon^{-\gamma} \partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} \sigma(u_\varepsilon) &= \varepsilon^{-\gamma} g(x, t), \quad (x, t) \in S_\varepsilon^T, \\
u_\varepsilon(x, t) &= 0, \quad (x, t) \in \Gamma^T, \\
u_\varepsilon(x, 0) &= 0, \quad x \in S_\varepsilon,
\end{aligned}
\tag{2.2}
$$

where

$$\gamma = \frac{n}{n-2}, \tag{2.3}$$

ν is the unit outward normal to the boundary S_ε^T, $\partial_\nu u_\varepsilon$ is the normal derivative of u_ε, and, for simplicity in the exposition we assume

$$f \in H^1(0, T; L^2(\Omega)), \tag{2.4}$$

$$g \in L^2(0, T; C(\overline{\Omega})). \tag{2.5}$$

Problem (2.2) arises in many different contexts (see, e.g., the exposition made in [3, 15, 20] and their many references). The nonlinear term $\sigma(u_\varepsilon)$ represents, in some models, the chemical reaction on the boundary of the particles. It is well known that a relevant choice of this term is the function given by $\sigma(u) = \sigma_0 u^\alpha$, for some positive constant σ_0 and where $\alpha \in [0, 1]$ represents the "order of the chemical reaction" (see, e.g. [10]). Motivated by this, we assume that σ is a Hölder continuous function (with a Lipschiz behavior for large values of u), $\sigma : \mathbb{R} \to \mathbb{R}$ is nondecreasing, $\sigma(0) = 0$, and

$$|\sigma(s) - \sigma(t)| \leq K_1 |s - t|^\alpha + K_2 |s - t| \quad \forall s, t \in \mathbb{R} \text{ and for some } 0 < \alpha \leq 1, \tag{2.6}$$

Here K_1, K_2 are positive constants.

We recall that a function $u_\varepsilon \in C([0, T] : L^2(S_\varepsilon))$ is a *strong solution of* (2.2) if $u_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega))$, $\partial_t u_\varepsilon \in L^2(0, T; L^2(S_\varepsilon))$ and $\sigma(u_\varepsilon) \in L^2(0, T; L^2(S_\varepsilon))$, such that

$$\varepsilon^{-\gamma} \int_{S_\varepsilon^T} \partial_t u_\varepsilon v \, ds \, dt + \int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla v \, dx \, dt + \varepsilon^{-\gamma} \int_{S_\varepsilon^T} \sigma(u_\varepsilon) v \, ds \, dt$$
$$= \varepsilon^{-\gamma} \int_{S_\varepsilon^T} gv \, ds \, dt + \int_{Q_\varepsilon^T} fv \, dx \, dt, \tag{2.7}$$

for all $v \in L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega))$, and the initial condition $u_\varepsilon(x, 0) = 0$ holds for $x \in S_\varepsilon$. Here $H^1(\Omega_\varepsilon, \partial\Omega)$ denotes the Hilbert space obtained as the closure, with the norm $H^1(\Omega_\varepsilon)$, of the set of all $\phi \in C^\infty(\overline{\Omega})$ such that $\phi = 0$ in a neighborhood of $\partial\Omega$. Notice that if we denote by $H^{1/2}(S_\varepsilon, \partial\Omega)$ to the space of traces on S_ε of a function from $H^1(\Omega_\varepsilon, \partial\Omega)$, with the norm

$$\|v\|_{H^{1/2}(S_\varepsilon, \partial\Omega)} = \inf_{w \in H^1(\Omega_\varepsilon, \partial\Omega)} \{\|w\|_{H^1(\Omega_\varepsilon, \partial\Omega)} : w|_{S_\varepsilon} = v\},$$

then strong solutions are more regular than other type of weak solutions satisfying merely that $\partial_t u_\varepsilon \in H^{-1/2}(S_\varepsilon, \partial\Omega)$, where $H^{-1/2}(S_\varepsilon, \partial\Omega)$ is the dual of the space $H^{1/2}(S_\varepsilon, \partial\Omega)$.

Our first result concerns the existence and uniqueness of strong solutions of problem (2.2) under the assumption of Hölder continuity on σ.

Theorem 2.1. *Assume σ Hölder continuos as well the rest of the above assumptions. Then for any $\varepsilon > 0$, problem (2.2) has a unique strong solution, satisfying the estimates*

$$\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))}^2 + \varepsilon^{-\gamma}\|u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))}^2$$
$$\leq K(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2),$$
$$\varepsilon^{-\gamma}\|\partial_t u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))}^2 \leq K\Big(\|f\|_{H^1(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2$$
$$+ \|f\|_{L^2(0,T;L^2(\Omega))}^{2\alpha} + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^{2\alpha}\Big), \tag{2.8}$$

where K is a positive constant independent on ε, f and g.

The proof of this Theorem will be given below by means of Galerkin expansion arguments. We recall that, by [24], it is well-known the existence of a linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon, \partial\Omega) \to H^1_0(\Omega)$, such that

$$\|\nabla(P_\varepsilon u)\|_{L^2(\Omega)} \leq K\|\nabla u\|_{L^2(\Omega_\varepsilon)}, \quad \|P_\varepsilon u\|_{H^1_0(\Omega)} \leq K\|u\|_{H^1(\Omega_\varepsilon)},$$

where $K > 0$ is independent of ε. Then, by using the estimate from Theorem 2.1, we conclude that

$$\|P_\varepsilon u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} \leq K,$$

as $\varepsilon \to 0$. Therefore, for some subsequence (still denoted by u_ε), we have

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0, \quad \text{weakly in } L^2(0,T;H^1_0(\Omega)). \tag{2.9}$$

The main goal of our research is to characterize the function u_0 in terms of a homogenized boundary value problem on Ω for nonlinear terms σ assumed to be in the same class of the existence of solutions. To do that we will argue without passing by a regularizing of σ followed by a passing to the limit in such a process. This is in contrast with many other papers for related problems (see, e.g. [11] and its references).

3. Statement of main results

In the case of pure Robin type boundary conditions, $f(x,t) = f(x)$, and when the diameter of the perforation (or the diameter of the particles) is equal to the period of the structure ($\gamma = 1$ in (2.3)), it can be proved (see [10]) that u_0 satisfies a semilinear equation of the type $-Lu_0 + C\sigma(u_0) = f(x)$ in Ω, for a suitable second order elliptic linear operator L and a suitable constant $C > 0$. As mentioned before we are interested in the characterization of u_0 when the critical size condition (2.3) holds in presence of dynamic nonlinear boundary conditions. A "strange term" (replacing the above nonlinear term $C\sigma(x, u_0)$) arises in the homogenized semilinear equation satisfied by u_0, in the spirit of the series of previous works for different type of boundary conditions (see, e.g., [8, 9, 19, 22, 23] and the monograph [14]). When G_0 is a symmetric ball, it was shown in [15, 33] that this strange term is nonlocal in time and contains some memory expressions. In this paper we will consider the general asymmetric case on G_0 and we will prove that the correspondent strange term is now given by a monotone operator $H : L^2(0,T) \to L^2(0,T)$. As a matter of facts, we will apply this operator to functions which are also $x-$dependent, where $x \in \Omega$, but concerning the operator H the variable x plays the role of a parameter, so that we can also understand that the above mentioned monotone operator can be understood as H $: L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$. To be more precise, for any function $\phi \in L^2(0,T;L^2(\Omega))$, we define the value of H$[\phi]$ by its pointwise application, for a.e. $(x,t) \in \Omega \times (0,T)$:

$$\text{H}[\phi](x,t) = \phi(x,t)\lambda_{G_0} - \int_{\partial G_0} \partial_\nu \widetilde{w}_\phi(x,y,t)ds_y, \tag{3.1}$$

where $\widetilde{w}_\phi(x,y,t)$ satisfies the G_0-capacity type exterior nonlinear auxiliary problem

$$\Delta_y \widetilde{w}_\phi = 0, \quad \mathbb{R}^n \setminus \overline{G_0} \times (0,T),$$
$$C_0 \partial_t \widetilde{w}_\phi + \partial_\nu \widetilde{w}_\phi + C_0 \sigma(\widetilde{w}_\phi) = C_0 g(x,t) + \phi(x,t)\partial_\nu \kappa(y), \quad \partial G_0 \times (0,T),$$
$$\widetilde{w}_\phi(x,y,0) = 0, \quad y \in \partial G_0,$$
$$\widetilde{w}_\phi(x,y,t) \to 0, \quad |y| \to +\infty, \; t \in (0,T),$$

(3.2)

with $\kappa(y)$ the standard G_0-capacity exterior problem

$$\Delta_y \kappa(y) = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0},$$
$$\kappa(y) = 1, \quad y \in \partial G_0,$$
$$\kappa(y) \to 0, \quad |y| \to +\infty,$$

(3.3)

and where λ_{G_0} is the capacity of the set G_0 given by

$$\lambda_{G_0} = \int_{\partial G_0} \partial_\nu \kappa(y) ds_y.$$

Notice that we are denoting by y to the variable in the domain $\mathbb{R}^n \setminus \overline{G_0}$ and that function $\widetilde{w}_\phi(x,y,t)$ depends, in a crucial way, not only of the values of $\phi(t,\cdot)$ but also of the data G_0, σ and g, so that, a more significative notation for this operator $H[\phi]$ would be

$$H[\phi] = H_{G_0,\sigma,g}[\phi].$$

Nevertheless, for the sake of simplicity in the notation, we avoid such a sophisticated notation. Notice that in the definition of $H[\phi]$ the variable $x \in \Omega$ plays the role of a parameter (since the partial differential problems (3.2) and (3.3) are formulated in terms of the $y \in \mathbb{R}^n \setminus \overline{G_0}$ variable).

We point out that although $\kappa(y) = 1$ on ∂G_0 (and then there is no direct influence of κ on the definition of operator H), the y−decay of $\kappa(y)$ allows a better manageability of the function $\widetilde{w}_\phi(x,y,t)$. We shall prove in later that the problem (3.2) has a unique weak solution and that $H[\phi] \in L^2(0,T;L^2(\Omega))$ for any $\phi \in L^2(0,T;L^2(\Omega))$. As a matter of fact, we shall prove that H is Lipschitz continuous operator, independently of the regularity of σ.

Remark 3.1. When G_0 is a ball, $G_0 = \{|y| < 1\}$ (which we will also denote as $G_0 = T_1^0$), then $\kappa(y) = |y|^{2-n}$ and the solution of problem (3.2) is explicitly given as

$$\widetilde{w}_\phi(x,y,t) = \frac{\mathcal{H}_\phi(x,t)}{|y|^{n-2}},$$

where $\mathcal{H}_\phi(x,t)$ for any $x \in \Omega$ is the unique solution of nonlinear Cauchy problem

$$\partial_t \mathcal{H}_\phi + \frac{n-2}{C_0}\mathcal{H}_\phi + \sigma(\mathcal{H}_\phi) = \frac{n-2}{C_0}\phi(x,t) + g(x,t),$$
$$\mathcal{H}_\phi(x,0) = 0.$$

(3.4)

where $\mathcal{H}_\phi$ is the unique solution (according to [5]) of the Cauchy problem associated with the maximal monotone graph given through function σ. A similar expression can be also found for the case $\sigma = \sigma(x,\phi)$, when $x \in \Omega$ is taken as a parameter. In the linear case $\sigma(\phi) = \lambda\phi$, for some $\lambda > 0$, we obtain

$$\mathcal{H}_\phi(x,t) = \int_0^t \Big(\frac{n-2}{C_0}\phi(x,s) + g(x,s)\Big) \exp\big\{-(\lambda + \frac{n-2}{C_0})(t-s)\big\}ds.$$

Therefore, $H[\phi](x,t) = (n-2)\omega_n(\phi(x,t) - \mathcal{H}_\phi(x,t))$, with $\omega_n = |\partial T_1^0|$, the area of the unit sphere.

In Section 7 we shall prove some properties of the "strange operator" H: it is a monotone operator (see Theorem 9), in the sense that for any $\phi_1, \phi_2 \in L^2(0,T)$,

$$\int_0^T (H[\phi_1] - H[\phi_2])(\phi_1 - \phi_2)dt \geq 0. \tag{3.5}$$

We will prove also that H satisfies the growth relation (Theorem 6.5)

$$\|H[\phi]\|_{L^2(0,T)} \leq K(\|\phi\|_{L^2(0,T)} + \|g\|_{L^2(0,T)}), \tag{3.6}$$

for any $\phi \in L^2(0,T)$ and that, in addition, H is a Lipschiz continuous operator on $L^2(0,T)$, in the sense that

$$\|H[\phi_1] - H[\phi_2]\|_{L^2(0,T)} \leq K\|\phi_1 - \phi_2\|_{L^2(0,T)}. \tag{3.7}$$

for any $\phi_1, \phi_2 \in L^2(0,T)$ and for a suitable constant $K > 0$. In addition, H is a Lipschiz continuous operator on $L^2(0,T)$ with respect to g , i.e.

$$\|H[\phi_{g_1}] - H[\phi_{g_2}]\|_{L^2(0,T)} \leq K\|g_1 - g_2\|_{L^2(0,T)}. \tag{3.8}$$

It will be very useful to get a stronger regularity on the operator $H[\phi]$ under some more regularity on the datum g on the boundary:

$$g \in L^2(0,T; W^{1,\infty}(\Omega)), \tag{3.9}$$

i.e, there exists $L \in L^2(0,T)$ with $L(t) > 0$ such that

$$|g(x_1,t) - g(x_2,t)| \leq L(t)|x_1 - x_2|, \quad \forall x_1, x_2 \in \overline{\Omega}, \text{ and for a.e. } t \in (0,T). \tag{3.10}$$

Remember that by Rademacher's theorem $W^{1,\infty}(\Omega) = \text{Lip}(\Omega)$. This will be used as an intermediate step of the proof of the main result concerning of the function u_0 defined in (2.9).

Theorem 3.2. *Let $n \geq 3$, $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$, and assume (2.6). Let u_ε be the strong solution of problem (2.2). Then the function $u_0 \in L^2(0,T; H_0^1(\Omega))$ is the unique weak solution of the family of problems (depending on the parameter $t \in (0,T)$),*

$$\begin{aligned} -\Delta_x u_0(x,t) + C_0^{n-2}H[u_0](x,t) &= f(x,t), \quad x \in \Omega, t \in (0,T), \\ u_0(x,t) &= 0, \quad x \in \partial\Omega, t \in (0,T), \end{aligned} \tag{3.11}$$

with $H[u_0](x,t)$ defined by the nonlocal operator (3.1) for a.e. $x \in \Omega$.

The proof of this theorem follows the master lines of the so called "Tartar's oscillating test functions method" (see, e.g., [28]), nevertheless many quite sophisticated variants must be introduced for its application to the problem under consideration. We send the reader to the general exposition made in the monograph [14] for several different problems.

4. Proof of Theorem 3.2

As usual, we can prove the uniqueness of the strong solution of (2.2) using the integral identity in the definition of solution and then applying the monotonicity of the function σ.

For proving the existence, we consider the auxiliary problem

$$
\begin{aligned}
\delta \partial_t u_\varepsilon^\delta - \Delta u_\varepsilon^\delta &= f(x,t), \quad (x,t) \in Q_\varepsilon^T, \\
\varepsilon^{-\gamma} \partial_t u_\varepsilon^\delta + \partial_\nu u_\varepsilon^\delta + \varepsilon^{-\gamma} \sigma(u_\varepsilon^\delta) &= \varepsilon^{-\gamma} g(x,t), \quad (x,t) \in S_\varepsilon^T, \\
u_\varepsilon^\delta(x,t) &= 0, \quad (x,t) \in \Gamma^T, \\
u_\varepsilon^\delta(x,0) &= 0, \quad x \in \Omega_\varepsilon, \\
u_\varepsilon^\delta(x,0) &= 0, \quad x \in S_\varepsilon,
\end{aligned}
\tag{4.1}
$$

where $\delta > 0$ is a small parameter. We use the Galerkin method to prove the existence of weak solution of this problem. By a weak solution we mean a function u_ε^δ in $L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, with $\partial_t u_\varepsilon^\delta \in L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))$, $\partial_t u_\varepsilon^\delta \in L^2(0,T;H^{-1/2}(S_\varepsilon,\partial\Omega))$, $u_\varepsilon^\delta(x,0) = 0$ for $x \in \Omega_\varepsilon$, $u_\varepsilon^\delta(x,0) = 0$, $x \in S_\varepsilon$, and such that the following integral identity holds for any test function $v \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$,

$$
\begin{aligned}
&\delta \int_0^T \langle \partial_t u_\varepsilon^\delta, v \rangle_{\Omega_\varepsilon} dt + \varepsilon^{-\gamma} \int_0^T \langle \partial_t u_\varepsilon^\delta, v \rangle_{S_\varepsilon} dt + \int_{Q_\varepsilon^T} \nabla u_\varepsilon^\delta \nabla v \, dx \, dt \\
&+ \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \sigma(u_\varepsilon^\delta) v \, ds \, dt \\
&= \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} g(x,t) v(x,t) \, ds \, dt + \int_{Q_\varepsilon^T} f v \, dx \, dt.
\end{aligned}
\tag{4.2}
$$

Here, we have denoted by $\langle \cdot, \cdot \rangle_{\Omega_\varepsilon}$ and $\langle \cdot, \cdot \rangle_{S_\varepsilon}$ the duality relations between the spaces $H^1(\Omega_\varepsilon,\partial\Omega)$ and $H^{-1}(\Omega_\varepsilon,\partial\Omega)$, and $H^{1/2}(S_\varepsilon,\partial\Omega)$ and $H^{-1/2}(S_\varepsilon,\partial\Omega)$, respectively.

In fact, we will prove that the solution of (4.1) is more regular. Namely, u_ε^δ is a strong solution in the sense that $u_\varepsilon^\delta \in C([0,T];L^2(\Omega_\varepsilon))$, $u_\varepsilon^\delta \in C([0,T];L^2(S_\varepsilon))$, $u_\varepsilon^\delta \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, $\partial_t u_\varepsilon^\delta \in L^2(0,T;L^2(\Omega_\varepsilon))$, $\partial_t u_\varepsilon^\delta \in L^2(0,T;L^2(S_\varepsilon))$, $u_\varepsilon^\delta(x,0) = 0$, if $x \in \Omega_\varepsilon \cup S_\varepsilon$ and the integral identity

$$
\begin{aligned}
&\delta \int_{Q_\varepsilon^T} \partial_t u_\varepsilon^\delta v \, dx \, dt + \int_{Q_\varepsilon^T} \nabla u_\varepsilon^\delta \nabla v \, dx \, dt + \varepsilon^{-\gamma} \int_{S_\varepsilon^T} \partial_t u_\varepsilon^\delta v \, ds \, dt \\
&+ \varepsilon^{-\gamma} \int_{S_\varepsilon^T} \sigma(u_\varepsilon^\delta) v \, ds \, dt \\
&= \varepsilon^{-\gamma} \int_{S_\varepsilon^T} g(x,t) v(x,t) \, ds \, dt + \int_{Q_\varepsilon^T} f v \, dx \, dt
\end{aligned}
\tag{4.3}
$$

holds for any function $v \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$.

We introduce the space $H_\varepsilon^\delta = L^2(\Omega_\varepsilon) \times L^2(S_\varepsilon)$ with the scalar product

$$
((u,\widetilde{u}),(v,\widetilde{v}))_{H_\varepsilon^\delta} = \delta(u,v)_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}(\widetilde{u},\widetilde{v})_{L^2(S_\varepsilon)}.
$$

Let $V_\varepsilon = \{(u,u|_{S_\varepsilon}) : u \in H^1(\Omega_\varepsilon,\partial\Omega)\}$, where $u|_{S_\varepsilon}$ is the trace of function $u \in H^1(\Omega_\varepsilon,\partial\Omega)$ on S_ε. On V_ε we introduce the norm

$$
\|(v,v|_{S_\varepsilon})\|_{V_\varepsilon}^2 = \|v\|_{H^1(\Omega_\varepsilon,\partial\Omega)}^2 + \|v|_{S_\varepsilon}\|_{H^{1/2}(S_\varepsilon,\partial\Omega)}^2.
$$

It is easy to see that V_ε is a reflexive separable Banach space dense in the space H_ε^δ, so that the linear span of the basis is dense in V_ε. In $H^{1/2}(S_\varepsilon, \partial\Omega)$ we use the following norm $\|v\|_{H^{1/2}(S_\varepsilon, \partial\Omega)} = \inf\{\|g\|_{H^1(\Omega_\varepsilon, \partial\Omega)} : g|_{S_\varepsilon} = v\}$. We denote by $\{(w_\varepsilon^m, w_\varepsilon^m|_{S_\varepsilon})\}_{m=1}^\infty$ the orthonormal basis for the space H_ε^δ, so that the linear span of the this basis is dense in V_ε.

Let us apply the Galerkin method to prove the existence of a weak solution to the problem (2.2). Let us start by showing the existence of $(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon}) \in C([0,T]; V_\varepsilon)$, such that $\partial_t(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon}) \in L^2(0,T; H_\varepsilon^\delta)$ and

$$
\begin{aligned}
&(\partial_t(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon}), (w_\varepsilon^l, w_\varepsilon^l|_{S_\varepsilon}))_{H_\varepsilon^\delta} + (\nabla u_\varepsilon^{\delta,m}, \nabla w_\varepsilon^l)_{L^2(\Omega_\varepsilon, \partial\Omega)} \\
&+ \varepsilon^{-\gamma}(\sigma(u_\varepsilon^{\delta,m}|_{S_\varepsilon}), w_\varepsilon^l|_{S_\varepsilon})_{L^2(S_\varepsilon)} \\
&= \varepsilon^{-\gamma}(g, w_\varepsilon^l|_{S_\varepsilon})_{L^2(S_\varepsilon)} + (f, w_\varepsilon^l)_{L^2(\Omega_\varepsilon)},
\end{aligned}
\tag{4.4}
$$

where $l = 1, \dots, m$, and

$$
(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon})(x,0) = 0.
\tag{4.5}
$$

We look for $(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon})$ in the form

$$
(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon}) = \sum_{l=1}^m C_\varepsilon^l(t)(w_\varepsilon^l, w_\varepsilon^l|_{S_\varepsilon}).
$$

The coefficients C_ε^l, $l = 1, \dots, m$, are found as solutions of the system (4.4),(4.5) which is a Cauchy problem for a system of nonlinear ordinary differential equations on C_ε^l. By well known results, this problem has a unique global solution defined on the whole time-interval $[0,T]$. Notice that although σ is merely Hölder continuous, the uniqueness of solution is consequence of the monotonicity assumption made on σ. From (4.4) we obtain

$$
\begin{aligned}
&\frac{\delta}{2}\|u_\varepsilon^{\delta,m}(x,\tau)\|_{L^2(\Omega_\varepsilon)}^2 + \frac{\varepsilon^{-\gamma}}{2}\|u_\varepsilon^{\delta,m}(x,\tau)\|_{L^2(S_\varepsilon)}^2 + \int_0^\tau \|\nabla u_\varepsilon^{\delta,m}\|_{L^2(\Omega_\varepsilon)}^2 dt \\
&+ \varepsilon^{-\gamma}\int_0^\tau \int_{S_\varepsilon} \sigma(u_\varepsilon^{\delta,m})u_\varepsilon^{\delta,m}\, ds\, dt \\
&= \varepsilon^{-\gamma}\int_0^\tau (g, u_\varepsilon^{\delta,m})_{L^2(S_\varepsilon)} dt + \int_0^\tau (f, u_\varepsilon^{\delta,m})_{L^2(\Omega_\varepsilon)} dt.
\end{aligned}
$$

Using that $\sigma(u)u \geq 0$, and applying Friedrich's inequality we obtain

$$
\begin{aligned}
&\delta\|u_\varepsilon^{\delta,m}(x,\tau)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^{-\gamma}\|u_\varepsilon^{\delta,m}(x,\tau)\|_{L^2(S_\varepsilon)}^2 + \int_0^\tau \|\nabla u_\varepsilon^{\delta,m}\|_{L^2(\Omega_\varepsilon)}^2 dt \\
&\leq K(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2) + \varepsilon^{-\gamma}\int_0^\tau \|u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt,
\end{aligned}
\tag{4.6}
$$

where the constant K depends only on the domain Ω. From Gronwall's lemma we deduce the explicit estimate

$$
\varepsilon^{-\gamma}\operatorname*{ess\,sup}_{[0,T]} \|u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 \leq K(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2).
\tag{4.7}
$$

Using (4.6) and (4.7) we have

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{[0,T]} \|(u_\varepsilon^{\delta,m}, u_\varepsilon^{\delta,m}|_{S_\varepsilon})\|_{H_\varepsilon^\delta}^2 + \int_0^T \|\nabla u_\varepsilon^{\delta,m}\|_{L^2(\Omega_\varepsilon)}^2 dt \\
&\leq K(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2).
\end{aligned}
\tag{4.8}
$$

Similarly, by multiplying by $\partial_t u_\varepsilon^{\delta,m}$ we obtain

$$
\begin{aligned}
&\delta \int_0^\tau \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(\Omega_\varepsilon)}^2 dt + \varepsilon^{-\gamma} \int_0^\tau \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt + \frac{1}{2}\|\nabla u_\varepsilon^{\delta,m}(x,\tau)\|_{L^2(\Omega_\varepsilon)}^2 \\
&+ \varepsilon^{-\gamma} \int_0^\tau \int_{S_\varepsilon} \sigma(u_\varepsilon^{\delta,m})\partial_t u_\varepsilon^{\delta,m}\, ds\, dt \\
&= \int_0^\tau \int_{\Omega_\varepsilon} f \partial_t u_\varepsilon^{\delta,m}\, dx\, dt + \varepsilon^{-\gamma} \int_0^\tau \int_{S_\varepsilon} g(x,t)\partial_t u_\varepsilon^{\delta,m}\, ds\, dt.
\end{aligned}
\tag{4.9}
$$

Thanks to (2.4) we have

$$
\int_0^\tau \int_{\Omega_\varepsilon} f \partial_t u_\varepsilon^{\delta,m}\, dx\, dt = -\int_0^\tau \int_{\Omega_\varepsilon} \partial_t f u_\varepsilon^{\delta,m}\, dx\, dt + \int_{\Omega_\varepsilon} f(x,\tau)u_\varepsilon^{\delta,m}(x,\tau)dx. \tag{4.10}
$$

Using assumption (2.6) on σ, and applying the Hölder inequality, we obtain

$$
\begin{aligned}
&\varepsilon^{-\gamma}\int_0^\tau \int_{S_\varepsilon} |\sigma(u_\varepsilon^{\delta,m})||\partial_t u_\varepsilon^{\delta,m}|\, ds\, dt \\
&\leq C_\beta \varepsilon^{-\gamma}\int_0^\tau \|\sigma(u_\varepsilon^{\delta,m})\|_{L^2(S_\varepsilon)}^2 dt + \beta \varepsilon^{-\gamma}\int_0^\tau \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt \\
&\leq C_{1,\beta}\varepsilon^{-\gamma}\int_0^\tau \int_{S_\varepsilon} |u_\varepsilon^{\delta,m}|^{2\alpha} ds dt + C_{2,\beta}\varepsilon^{-\gamma}\int_0^\tau \|u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt \\
&+ \beta \varepsilon^{-\gamma}\int_0^\tau \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt.
\end{aligned}
\tag{4.11}
$$

Then, applying again Hölder inequality, we have

$$
\begin{aligned}
\varepsilon^{-\gamma}\int_0^\tau \int_{S_\varepsilon} |u_\varepsilon^{\delta,m}|^{2\alpha}\, ds\, dt &\leq \varepsilon^{-\gamma}\Big(\int_0^\tau \int_{S_\varepsilon} |u_\varepsilon^{\delta,m}|^2\, dx\, dt\Big)^\alpha |S_\varepsilon|^{1-\alpha} \\
&\leq K\varepsilon^{\gamma(\alpha-1)}\Big(\varepsilon^{-\gamma}\int_0^\tau \int_{S_\varepsilon} |u_\varepsilon^{\delta,m}|^2\, ds\, dt\Big)^\alpha \varepsilon^{\gamma(1-\alpha)} \\
&\leq K(\|f\|_{L^2(0,T;L^2(\Omega))}^{2\alpha} + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^{2\alpha}),
\end{aligned}
\tag{4.12}
$$

where the constant K does not depend on f and g, on m, δ, ε.

From (4.8), (4.11) and (4.12) we conclude that

$$
\begin{aligned}
&\varepsilon^{-\gamma}\int_0^\tau \int_{S_\varepsilon} |\sigma(u_\varepsilon^{\delta,m})||\partial_t u_\varepsilon^{\delta,m}|\, ds\, dt \\
&\leq K_\beta\Big((\|f\|_{L^2(0,T;L^2(\Omega))}^{2\alpha} + \|f\|_{L^2(0,T;L^2(\Omega))}^2) + (\|g\|_{L^2(0,T;C(\overline{\Omega}))}^{2\alpha} \\
&+ \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2)\Big) + \beta \varepsilon^{-\gamma}\int_0^\tau \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt,
\end{aligned}
\tag{4.13}
$$

where β is an arbitrary positive constant. Setting $\beta = 1/2$, from (4.9), (4.10), (4.13), we obtain

$$
\begin{aligned}
&\delta \int_0^T \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(\Omega_\varepsilon)}^2 dt + \varepsilon^{-\gamma}\int_0^T \|\partial_t u_\varepsilon^{\delta,m}\|_{L^2(S_\varepsilon)}^2 dt + \operatorname{ess\,sup}_{[0,T]}\|\nabla u_\varepsilon^{\delta,m}\|_{L^2(\Omega_\varepsilon)}^2 \\
&\leq K\Big(\|f\|_{H^1(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^{2\alpha} + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^{2\alpha}\Big).
\end{aligned}
\tag{4.14}
$$

Consequently, from (4.8) and (4.14), we derive that there exists a subsequence such that as $m \to \infty$,

$$u_\varepsilon^{\delta,m} \rightharpoonup u_\varepsilon^\delta, \quad \text{weakly in } L^2(0,T;H^1(\Omega_\varepsilon, \partial\Omega)),$$
$$\partial_t u_\varepsilon^{\delta,m} \rightharpoonup \partial_t u_\varepsilon^\delta, \quad \text{weakly in } L^2(0,T;L^2(S_\varepsilon)),$$
$$\partial_t u_\varepsilon^{\delta,m} \rightharpoonup \partial_t u_\varepsilon^\delta, \quad \text{weakly in } L^2(0,T;L^2(\Omega_\varepsilon)),$$
$$u_\varepsilon^{\delta,m} \to u_\varepsilon^\delta, \quad \text{in } L^2(0,T;L^2(S_\varepsilon)).$$

From (4.4) we deduce that u_ε^δ satisfies the integral identity

$$\delta \int_0^T \int_{\Omega_\varepsilon} \partial_t u_\varepsilon^\delta \phi \, dx \, dt + \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \partial_t u_\varepsilon^\delta \phi \, ds \, dt + \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon^\delta \nabla \phi \, dx \, dt$$
$$+ \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \sigma(u_\varepsilon^\delta) \phi \, ds \, dt \tag{4.15}$$
$$= \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} g(x,t) \phi \, ds \, dt + \int_0^T \int_{\Omega_\varepsilon} f \phi \, dx \, dt,$$

where ϕ is an arbitrary test function in $L^2(0,T;H^1(\Omega_\varepsilon, \partial\Omega))$. Let us estimate the limit value of the term $\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \sigma(u_\varepsilon^{\delta,m}) \phi \, ds \, dt$, as $m \to \infty$. By using the Hölder continuity of σ (2.6) we have

$$\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} |\sigma(u_\varepsilon^{\delta,m}) - \sigma(u_\varepsilon^\delta)||\phi| \, ds \, dt$$
$$\leq K\Big(\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} |u_\varepsilon^{\delta,m} - u_\varepsilon^\delta||\phi| \, ds \, dt + \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} |u_\varepsilon^{\delta,m} - u_\varepsilon^\delta|^\alpha |\phi| \, ds \, dt\Big) \tag{4.16}$$
$$\leq K\Big(\varepsilon^{-\frac{\gamma(1+\alpha)}{2}} \|u_\varepsilon^{\delta,m} - u_\varepsilon^\delta\|_{L^2(0,T;L^2(S_\varepsilon))}^\alpha$$
$$+ \varepsilon^{-\gamma} \|u_\varepsilon^{\delta,m} - u_\varepsilon^\delta\|_{L^2(0,T;L^2(S_\varepsilon))}\Big) \|\phi\|_{L^2(0,T;L^2(S_\varepsilon))}.$$

Thus as $m \to \infty$ the right hand side of (4.16) tends to zero. For the solution u_ε^δ we have similar estimates to the ones given in (4.8) and (4.14)

$$\text{ess sup}_{[0,T]} \|(u_\varepsilon^\delta, u_\varepsilon^\delta|_{S_\varepsilon})\|_{H_\varepsilon^\delta}^2 + \int_0^T \|\nabla u_\varepsilon^\delta\|_{L^2(\Omega_\varepsilon)}^2 dt \tag{4.17}$$
$$\leq K(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2),$$

$$\delta \int_0^T \|\partial_t u_\varepsilon^\delta\|_{L^2(\Omega_\varepsilon)}^2 dt + \varepsilon^{-\gamma} \int_0^T \|\partial_t u_\varepsilon^\delta\|_{L^2(S_\varepsilon)}^2 dt + \text{ess sup}_{[0,T]} \|\nabla u_\varepsilon^\delta\|_{L^2(\Omega_\varepsilon)}^2$$
$$\leq K\Big(\|f\|_{H^1(0,T;L^2(\Omega))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^{2\alpha} \tag{4.18}$$
$$+ \|g\|_{L^2(0,T;C(\overline{\Omega}))}^{2\alpha}\Big).$$

Thus, for every $\varepsilon > 0$ there is exist some subsequence $\delta \to 0$ such that

$$u_\varepsilon^\delta \rightharpoonup u_\varepsilon \quad \text{weakly in } L^2(0,T;H^1(\Omega_\varepsilon, \partial\Omega)),$$
$$\partial_t u_\varepsilon^\delta \rightharpoonup \partial_t u_\varepsilon, \quad \text{weakly in } L^2(0,T;L^2(S_\varepsilon)),$$
$$u_\varepsilon^\delta \to u_\varepsilon \quad \text{in } L^2(0,T;L^2(S_\varepsilon)).$$

Taking into account (4.18) we find that

$$\delta \int_0^T \int_{\Omega_\varepsilon} \partial_t u_\varepsilon^\delta \phi \, dx \, dt \to 0, \quad \delta \to 0.$$

Hence, passing to the limit in (4.2) we obtain that u_ε is a strong solution to the problem (2.2) and the estimate (2.8) holds.

5. Auxiliary G_ε^j-capacity problems to adapt the global test function

.

The first step of our adaptation of the master lines of the "oscillating test functions method" consists in to replace the weak formulation of the problem by a kind of "very weak formulation" thanks to the monotonicity in $L^2(S_\varepsilon)$ of the operator given, formally by $u_\varepsilon \to \partial_\nu u_\varepsilon$ (see, e.g. [17, 21] and, more precisely, [3, Lemma 4.1]). By applying [6] (see also [21, Théorème 8.4]) we know that $u_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega))$ is a weak solution of problem (2.2) if and only if for any regular test function, for instance, $\phi \in L^2(0, T; H_0^1(\Omega) \cap C(\overline{\Omega})) \cap H^1(0, T; C(\overline{\Omega}))$, we have

$$\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \partial_t \phi (\phi - u_\varepsilon) \, ds \, dt + \int_{Q_\varepsilon^T} \nabla\phi \nabla(\phi - u_\varepsilon) \, dx \, dt$$

$$+ \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \sigma(\phi)(\phi - u_\varepsilon) \, ds \, dt \tag{5.1}$$

$$\geq \int_{Q_\varepsilon^T} f(\phi - u_\varepsilon) \, dx \, dt + \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} g(\phi - u_\varepsilon) \, ds \, dt - \frac{1}{2}\varepsilon^{-\gamma} \|\phi(x, 0)\|_{L^2(S_\varepsilon)}^2.$$

The second step of our adaptation of this very general set of ideas constituting the "oscillating test functions method" consists in modifying, in a suitable way, any test function (which will be used to check the homogenized equation satisfied by the weak limit function $u_0(x, t)$ obtained in Theorem 3.2) to a different family of test functions better adapted to problem (2.2): the new set of "oscillating test functions" . This step is rather involved and should be carried out in a very sharp way accordingly the problem under consideration. To this purpose, it is useful to simplify the properties satisfied by the departing test function. In our case, it will be enough to consider the given potential test function of the form

$$\phi(x, t) = \varphi(x)\eta(t), \quad \text{where } \varphi \in C^\infty(\overline{\Omega}) \text{ and } \eta \in H^1(0, T). \tag{5.2}$$

As mentioned in the introduction, the identification of the "strange term" operator H$[\phi]$ defined by (3.1) will use the G_0-capacity type exterior nonlinear problem given by (3.2) which we will consider in the next Section. It turns out that it is convenient to approximate the problem (3.2) by a set of auxiliary capacity problems which are sharply adapted to the starting spatial domain $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$: thus, this other set of auxiliary problems will depend on the parameter ε and are built in terms of the cell reference set Y_ε^j. We recall that we assumed that $\overline{G_\varepsilon^j} \subset T_{Ca_\varepsilon}^j \subset T_{\varepsilon/4}^j \subset Y_\varepsilon^j$, where T_r^j is the ball in R^n of radius r with the center at $P_\varepsilon^j = \varepsilon j$ (the center of the cell Y_ε^j), C is a positive constant independent on ε. Then, for every $j \in \Upsilon_\varepsilon$ we

consider the auxiliary problem

$$\Delta w_{\varepsilon,\phi}^j = 0, \quad x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \ t \in (0,T),$$

$$\varepsilon^{-\gamma} \partial_t w_{\varepsilon,\phi}^j + \partial_\nu w_{\varepsilon,\phi}^j - \varepsilon^{-\gamma} \sigma(\phi(P_\varepsilon^j, t) - w_{\varepsilon,\phi}^j)$$
$$= \varepsilon^{-\gamma} (\partial_t \phi(P_\varepsilon^j, t) - g(P_\varepsilon^j, t)), \quad x \in \partial G_\varepsilon^j, \ t \in (0,T), \qquad (5.3)$$

$$w_{\varepsilon,\phi}^j(x,t) = 0, \quad x \in \partial T_{\varepsilon/4}^j, \ t \in (0,T),$$

$$w_{\varepsilon,\phi}^j(x,0) = \phi(P_\varepsilon^j, 0), \quad x \in \partial G_\varepsilon^j.$$

Notice that, again, a more exact notation would be

$$w_{\varepsilon,\phi}^{\mathbf{j}} = w_{\varepsilon,\phi,g,G_0}^{\mathbf{j}},$$

to indicate the important dependence of $w_{\varepsilon,\phi}^{\mathbf{j}}$ with respect also to the datum g and G_0, but we drop such a sophisticated notation for the sake of simplicity. In the next Section we will study the asymptotic resemblance among the solutions $w_{\varepsilon,\phi}^{\mathbf{j}}$ of this new problem and the solutions $\widetilde{w}_\phi(x,y,t)$ of the G_0-capacity type exterior nonlinear problem given by (3.2).

To study problem (5.3) we start by introducing the type of solutions we will consider in this paper: A function $w_{\varepsilon,\phi}^j$ is a strong solution of problem (5.3) if $w_{\varepsilon,\phi}^j \in C([0,T]; L^2(\partial G_\varepsilon^j))$, $w_{\varepsilon,\phi}^j \in L^2(0,T; H^1(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}))$, $\partial_t w_{\varepsilon,\phi}^j \in L^2(0,T; L^2(\partial G_\varepsilon^j))$ and the following integral identity holds for any test function $\psi \in L^2(0,T; H^1(T_{\varepsilon/4}^{\mathbf{j}} \setminus \overline{G_\varepsilon^{\mathbf{j}}}, \partial T_{\varepsilon/4}^{\mathbf{j}}))$,

$$\varepsilon^{-\gamma} \int_0^T \int_{\partial G_\varepsilon^j} \partial_t w_{\varepsilon,\phi}^j \psi \, ds \, dt + \int_0^T \int_{T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}} \nabla w_{\varepsilon,\phi}^j \nabla \psi \, dx \, dt$$

$$- \varepsilon^{-\gamma} \int_0^T \int_{\partial G_\varepsilon^j} \sigma(\phi(P_\varepsilon^j, t) - w_{\varepsilon,\phi}^j) \psi \, ds \, dt \qquad (5.4)$$

$$= \varepsilon^{-\gamma} \int_0^T \int_{\partial G_\varepsilon^j} (\partial_t \phi(P_\varepsilon^j, t) - g(P_\varepsilon^j, t)) \psi \, ds \, dt.$$

Theorem 5.1. *Assume σ Hölder continuous satisfying (2.6). Then, for any given test function $\phi(x,t)$ of the form (5.2), problem (5.3) has a unique strong solution $w_{\varepsilon,\phi}^j$ and the following estimates hold*

$$\|w_{\varepsilon,\phi}^j\|_{L^2(0,T;H^1(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \partial T_{\varepsilon/4}^j))}^2 + \varepsilon^{-\gamma} \|w_{\varepsilon,\phi}^j\|_{L^2(0,T;L^2(\partial G_\varepsilon^j))}^2$$

$$\leq K_0 \varepsilon^n (\|\phi\|_{L^2(0,T;C(\overline{\Omega}))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2),$$

$$\varepsilon^{-\gamma} \|\partial_t w_{\varepsilon,\phi}^j\|_{L^2(0,T;L^2(\partial G_\varepsilon^j))}^2$$

$$\leq K_0 \varepsilon^n \Big(\|\phi\|_{L^2(0,T;C(\overline{\Omega}))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2 + \|\partial_t \phi\|_{L^2(0,T;C(\overline{\Omega}))}^2$$

$$+ \|\phi\|_{L^2(0,T,C(\overline{\Omega}))}^{2\alpha} + \|g\|_{L^2(0,T,C(\overline{\Omega}))}^{2\alpha} + \max_{\overline{\Omega} \times [0,T]} |\phi(x,t)|^2 \Big),$$

$$\|w_{\varepsilon,\phi}^j\|_{L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}))}^2 \leq K_0 \varepsilon^{n+2} \Big(\|\phi\|_{L^2(0,T;C(\overline{\Omega}))}^2 + \|g\|_{L^2(0,T;C(\overline{\Omega}))}^2 \Big),$$

where the positive constant K_0 is independent on ϕ, g and ε.

Proof. As usual, the uniqueness of the strong solution of the problem (5.3) can be derived from the integral identity by using the monotonicity of the function σ. In order to prove the existence of solutions satisfying the above mentioned estimates, let us introduce the functions $\widetilde{\psi^j_\varepsilon} = \phi(P^j_\varepsilon, t)\psi^j_\varepsilon(x)$, where $\psi^j_\varepsilon \in C^\infty(\overline{T^j_{\varepsilon/4}})$, $\psi^j_\varepsilon(x) = 1$ if $x \in T^j_{Ca_\varepsilon}$ and $\psi^j_\varepsilon(x) = 0$ if $x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{2Ca_\varepsilon}}$, $|\nabla \psi^j_\varepsilon| \leq C_1 a_\varepsilon^{-1}$, C_1 is a positive constant. Then, we make the change of unknowns $w^j_{\varepsilon,\phi} = v^j_{\varepsilon,\phi} + \widetilde{\psi^j_\varepsilon}$, and thus, it is easy to see that $w^j_{\varepsilon,\phi}$ is a strong solution of the problem (5.3) if and only if $v^j_{\varepsilon,\phi}$ is a strong solution to the new problem

$$
\begin{aligned}
\Delta_x(v^j_{\varepsilon,\phi} + \widetilde{\psi^j_\varepsilon}) &= 0, \quad x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \ t \in (0,T), \\
\varepsilon^{-\gamma}\partial_t v^j_{\varepsilon,\phi} &+ \partial_\nu(v^j_{\varepsilon,\phi} + \widetilde{\psi^j_\varepsilon}) - \varepsilon^{-\gamma}\sigma(-v^j_{\varepsilon,\phi}) \\
&= -\varepsilon^{-\gamma}g(P^j_\varepsilon, t), \quad x \in \partial G^j_\varepsilon, \ t \in (0,T), \\
v^j_{\varepsilon,\phi}(x,t) &= 0, \quad x \in \partial T^j_{\varepsilon/4}, \ t \in (0,T), \\
v^j_{\varepsilon,\phi}(x,0) &= 0, \quad x \in \partial G^j_\varepsilon.
\end{aligned}
\tag{5.5}
$$

To prove that problem (5.5) has a strong solution, we consider the approximate auxiliary problem

$$
\begin{aligned}
\delta\partial_t v^{\delta,j}_{\varepsilon,\phi} - \Delta v^{\delta,j}_{\varepsilon,\phi} &= \Delta\widetilde{\psi^j}_{\varepsilon,\phi}, \quad x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \ t \in (0,T), \\
\varepsilon^{-\gamma}\partial_t v^{\delta,j}_{\varepsilon,\phi} &+ \partial_\nu(v^{\delta,j}_{\varepsilon,\phi} + \widetilde{\psi^j}_{\varepsilon,\phi}) - \varepsilon^{-\gamma}\sigma(-v^{\delta,j}_{\varepsilon,\phi}) \\
&= -\varepsilon^{-\gamma}g(P^j_\varepsilon, t), \quad x \in \partial G^j_\varepsilon, \ t \in (0,T), \\
v^{\delta,j}_{\varepsilon,\phi}(x,t) &= 0, \quad x \in \partial T^j_{\varepsilon/4}, \ t \in (0,T), \\
v^{\delta,j}_{\varepsilon,\phi}(x,0) &= 0, \quad x \in T^j_{\varepsilon/4} \setminus G^j_\varepsilon, \\
v^{\delta,j}_{\varepsilon,\phi}(x,0) &= 0, \quad x \in \partial G^j_\varepsilon.
\end{aligned}
\tag{5.6}
$$

The proof of the existence of solution of problem (5.6) can be obtained by a Galerkin method as in Theorem 3.2. If we denote by $\{(w^m_\varepsilon, w^m_\varepsilon|_{\partial G^j_\varepsilon})\}$, $m = 1, 2, \ldots$, the orthonormal basis for the space $H^\delta_\varepsilon = L^2(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}) \times L^2(\partial G^j_\varepsilon)$, then we find the Galerkin approximations $(v^{\delta,m}_{\varepsilon,\phi}, v^{\delta,m}_{\varepsilon,\phi}|_{\partial G^j_\varepsilon}) \in C([0,T]; V^\delta_\varepsilon)$, where $V^\delta_\varepsilon = \{(u, u|_{\partial G^j_\varepsilon}) \in H^\delta_\varepsilon | u \in H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\varepsilon/4})\}$, such that $\partial_t(v^{\delta,m}_{\varepsilon,\phi}, v^{\delta,m}_{\varepsilon,\phi}|_{\partial G^j_\varepsilon}) \in L^2(0,T; H^\delta_\varepsilon)$. Using similar considerations as in the proof of Theorem 3.2 we obtain the following estimates

$$
\begin{aligned}
&\delta\max_{[0,T]}\|v^{\delta,m}_{\varepsilon,\phi}\|^2_{L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})} + \varepsilon^{-\gamma}\max_{[0,T]}\|v^{\delta,m}_{\varepsilon,\phi}\|^2_{L^2(\partial G^j_\varepsilon)} + \|\nabla v^{\delta,m}_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))} \\
&\leq K\varepsilon^n\Big(\|\phi\|^2_{L^2(0,T,C(\overline{\Omega}))} + \|g\|^2_{L^2(0,T;C(\overline{\Omega}))}\Big),
\end{aligned}
\tag{5.7}
$$

$$\delta \|\partial_t v_{\varepsilon,\phi}^{\delta,m}\|^2_{L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}))} + \varepsilon^{-\gamma} \|\partial_t v_{\varepsilon,\phi}^{\delta,m}\|^2_{L^2(0,T;L^2(\partial G_\varepsilon^j))}$$

$$+ \max_{[0,T]} \|\nabla v_{\varepsilon,\phi}^{\delta,m}\|^2_{L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j})}$$

$$\le K\varepsilon^n \Big(\max_{\overline{\Omega} \times [0,T]} |\phi|^2 + \|\partial_t \phi\|^2_{L^2(0,T;C(\overline{\Omega}))} + \|\phi\|^{2\alpha}_{L^2(0,T;C(\overline{\Omega}))}$$

$$+ \|g\|^{2\alpha}_{L^2(0,T;C(\overline{\Omega}))} + \|g\|^2_{L^2(0,T;C(\overline{\Omega}))} \Big),$$

(5.8)

where the constant K does not depend on ε, ϕ, g. From the estimates (5.7), (5.8) it follows that for some subsequence $\{m_s\}$,

$$v_{\varepsilon,\phi}^{\delta,m_s} \rightharpoonup v_{\varepsilon,\phi}^{\delta,j}, \quad \text{weakly in } L^2(0,T;H^1(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j})),$$

$$\partial_t v_{\varepsilon,\phi}^{\delta,m_s} \rightharpoonup \partial_t v_{\varepsilon,\phi}^{\delta,j}, \quad \text{weakly in } L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j})),$$

$$v_{\varepsilon,\phi}^{\delta,m_s} \to v_{\varepsilon,\phi}^{\delta,j} \text{ in } L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j})),$$

$$\partial_t v_{\varepsilon,\phi}^{\delta,m_s} \rightharpoonup \partial_t v_{\varepsilon,\phi}^{\delta,j} \quad \text{weakly in } L^2(0,T;L^2(\partial G_\varepsilon^j)),$$

as $m_s \to \infty$ and $v_{\varepsilon,\phi}^{\delta,j}$ is a strong solution of the problem (5.6) and the similar estimates hold for the limit function $v_{\varepsilon,\phi}^{\delta,j}$. So for fixed $\varepsilon > 0$ by some subsequence $\{v_{\varepsilon,\phi}^{\delta',j}\}$ as $\delta' \to 0$,

$$v_{\varepsilon,\phi}^{\delta',j} \rightharpoonup v_{\varepsilon,\phi}^j, \text{ weakly in } L^2(0,T;H^1(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \partial T_{\varepsilon/4}^j)),$$

$$\partial_t v_{\varepsilon,\phi}^{\delta',j} \rightharpoonup \partial_t v_{\varepsilon,\phi}^j, \quad \text{weakly in } L^2(0,T;L^2(\partial G_\varepsilon^j)),$$

$$v_{\varepsilon,\phi}^{\delta',j}\big|_{\partial G_\varepsilon^j} \to v_{\varepsilon,\phi}^j\big|_{\partial G_\varepsilon^j} \quad \text{in } L^2(0,T;L^2(\partial G_\varepsilon^j)).$$

Taking into account that $\delta' \int_0^T \int_{T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}} \partial_t v_{\varepsilon,\phi}^{\delta',j} \psi \, dx \, dt \to 0$, as $\delta' \to 0$, we obtain that $v_{\varepsilon,\phi}^j$ is a strong solution of the problem (5.5). From estimates (5.7), (5.8), and applying the Friederich inequality, we obtain the estimates in the Theorem 5.1. $\square$

In the following Section we will need some stronger regularity estimates on $w_{\varepsilon,\phi}^j$ which we prove now.

Theorem 5.2. *As a matter of facts, $w_{\varepsilon,\phi}^j \in L^\infty((T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}) \times (0,T))$ and we have*

$$\operatorname{ess\,sup}_{(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}) \times (0,T)} |w_{\varepsilon,\phi}^j| \le 2 \max_{\overline{\Omega} \times [0,T]} |\phi(x,t)| + 2 \int_0^T \max_{\overline{\Omega}} |g(x,t)| dt. \quad (5.9)$$

Proof. We introduce the function $h_{\varepsilon,\phi}^j = \phi(P_\varepsilon^j, t) - w_{\varepsilon,\phi}^j - \int_0^t g(P_\varepsilon^j, s) ds$. Then $h_{\varepsilon,\phi}^j$ is a strong solution of the problem

$$\Delta h_{\varepsilon,\phi}^j = 0, \quad x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \ t \in (0,T),$$

$$\varepsilon^{-\gamma} \partial_t h_{\varepsilon,\phi}^j + \partial_\nu h_{\varepsilon,\phi}^j + \varepsilon^{-\gamma} \sigma(h_{\varepsilon,\phi}^j + \int_0^t g(P_\varepsilon^j, s) ds) = 0, \quad x \in \partial G_\varepsilon^j, \ t \in (0,T),$$

$$h_{\varepsilon,\phi}^j = \phi(P_\varepsilon^j, t) - \int_0^t g(P_\varepsilon^j, s) ds, \quad x \in \partial T_{\varepsilon/4}^j, \ t \in (0,T),$$

$$h_{\varepsilon,\phi}^j(x,0) = 0, \quad x \in \partial G_\varepsilon^j.$$

(5.10)

We set $K = \max_{\overline{\Omega} \times [0,T]} |\phi(x,t)| + \int_0^T \max_{\overline{\Omega}} |g(x,t)| dt$. Then

$$(h_{\varepsilon,\phi}^j - K)^+ \in L^2(0,T; H^1(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \partial T_{\varepsilon/4}^j))$$

and taking it as test function in the corresponding integral identity associated with problem (5.10) we obtain

$$
\frac{\varepsilon^{-\gamma}}{2} \|(h_{\varepsilon,\phi}^j - K)^+(x,\tau)\|_{L^2(\partial G_\varepsilon^j)}^2 + \int_0^\tau \|\nabla(h_{\varepsilon,\phi}^j - K)^+\|_{L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j})}^2 dt
$$
$$
+ \varepsilon^{-\gamma} \int_0^\tau \int_{\partial G_\varepsilon^j} \sigma(h_{\varepsilon,\phi}^j + \int_0^t g(x,s)ds)(h_{\varepsilon,\phi}^j - K)^+ ds dt = 0.
$$
(5.11)

Using that $h_{\varepsilon,\phi}^j + \int_0^\tau g(x,s)ds \geq 0$ on the set where $h_{\varepsilon,\phi}^j \geq K$ and since σ is non-decreasing we obtain

$$\sigma(h_{\varepsilon,\phi}^j + \int_0^t g(x,s)ds) \geq 0.$$

Thus on the left hand side of (5.11) we have a sum of nonnegative terms which is equal to zero. So we conclude that $(h_{\varepsilon,\phi}^j - K)^+ = 0$ a.e. $x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}$ and $t \in (0,T)$. Arguing in a similar way we obtain also that $(h_{\varepsilon,\phi}^j + K)^- = 0$. Hence we obtain the statement of Theorem 5.2. $\square$

6. G_0-CAPACITY FOR THE EXTERIOR PROBLEM

Now, in this section, we will prove the existence and uniqueness of $\widetilde{w}_\phi(x,y,t)$ solution of exterior problem (3.2), mentioned in the Introduction, by assuming $g \in L^2(0,T; C(\overline{\Omega}))$, and for a general test function $\phi \in L^2(0,T; L^2(\Omega))$.

To define the notion of strong solution of problem (3.2), we denote by $\mathfrak{M}$ the set of functions $w \in C^\infty(R^n \setminus \overline{G_0})$ such that $w(y) = 0$ for $y \in R^n \setminus \overline{T_R^0}$, for some ball T_R^0, such that $\overline{G_0} \subset T_R^0$. We denote by $\mathcal{M}$ the closure of $\mathfrak{M}$ with the norm $\|w\|_{\mathcal{M}} = \|\nabla w\|_{L^2(\mathbb{R}^n \setminus \overline{G_0})}$.

By applying Hardy inequality on the rings $T_R^0 \setminus \overline{T_{R_0}^0}$ and making $R \to \infty$, we arrive to the following well-known result.

Lemma 6.1. *Let $G_0 \subset R^n$, $n \geq 3$, be a smooth bounded domain, star-shaped with respect to a ball T_R^0, $T_R^0 \subset G_0$. Then there exist a constant K_0, only dependent of n, such that*

$$\int_{\mathbb{R}^n \setminus \overline{G_0}} |y|^{-2} w^2 dy \leq K_0 \|w\|_{\mathcal{M}}^2, \quad \forall w \in \mathcal{M}.$$
(6.1)

By a strong solution to (3.2) we mean a function $\widetilde{w}_\phi(x,y,t)$ such that for a.e. $x \in \Omega$, $\widetilde{w}_\phi \in C([0,T]; L^2(\partial G_0))$, $\widetilde{w}_\phi \in L^2(0,T; \mathcal{M})$ and $\partial_t \widetilde{w}_\phi \in L^2(0,T; L^2(\partial G_0))$, $\widetilde{w}_\phi(x,y,0) = 0$ for $y \in \partial G_0$, and the following integral identity holds for any test function $v \in L^2(0,T; \mathcal{M})$:

$$
\int_0^T \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla \widetilde{w}_\phi \nabla v \, dy dt + C_0 \int_0^T \int_{\partial G_0} \partial_t \widetilde{w}_\phi v \, ds \, dt + C_0 \int_0^T \int_{\partial G_0} \sigma(\widetilde{w}_\phi) v \, ds \, dt
$$
$$
= \int_0^T \phi(x,t) \int_{\partial G_0} \partial_\nu \kappa v ds dt.
$$
(6.2)

Theorem 6.2. *Assume σ is Hölder continuous satisfying* (2.6) *and let g belong to $L^2(0,T;C(\overline{\Omega}))$ and ϕ belong to $L^2(0,T;L^2(\Omega))$. Then problem* (3.2) *has a unique strong solution and for a.e. $x \in \Omega$ the following estimates hold*

$$\|\widetilde{w}_\phi\|_{C([0,T];L^2(\partial G_0))} + \|\widetilde{w}_\phi\|_{L^2(0,T;\mathcal{M})}$$
$$\leq K(\|\phi(\cdot,x)\|_{L^2(0,T)} + \|g(\cdot,x)\|_{L^2(0,T)}),$$
$$\|\partial_t \widetilde{w}_\phi\|_{L^2(0,T;L^2(\partial G_0))} \leq K\Big(\|\phi(\cdot,x)\|_{L^2(0,T)} + \|g(\cdot,x)\|_{L^2(0,T)} \tag{6.3}$$
$$+ \|\phi(\cdot,x)\|_{L^2(0,T)}^\alpha + \|g(\cdot,x)\|_{L^2(0,T)}^\alpha \Big).$$

In addition, if we define

$$w_\phi(x,y,t) = \kappa(y)\phi(x,t) - \widetilde{w}_\phi(x,y,t), \tag{6.4}$$

with $\kappa(y)$ solution of standard G_0-capacity exterior problem (3.3) *then, for a.e. $x \in \Omega$, we have*

$$\int_0^T \Big(\int_{\partial G_0} \partial_\nu w_\phi(x,y,t)ds_y\Big)^2 dt \leq K\Big(\|\phi(\cdot,x)\|_{L^2(0,T)}^2 + \|g(\cdot,x)\|_{L^2(0,T)}^2\Big), \tag{6.5}$$

where the positive constant K does not depend on ϕ and g.

Proof. It is a simple variation of the proof of Theorem 3.2 (with $f \equiv 0$ and $\Omega = R^n \setminus \overline{G_0}$). Notice that the application of the Hardy inequality (6.1) allows to extend the conclusion to this spatial domain even if it is unbounded. Then, for instance, the uniqueness results, once again, from the monotonicity of the operator mentioned in the proof of Theorem 3.2. Estimate (6.3) is consequence of the corresponding estimate (2.8). In addition, since $\widetilde{w}_\phi$ is a strong solution, we have

$$\int_{\partial G_0} \partial_\nu w_\phi(x,y,t)ds = \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla w_\phi \nabla \kappa dy$$
$$= \phi(x,t)\int_{\mathbb{R}^n \setminus \overline{G_0}} |\nabla\kappa(y)|^2 dy - \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla\widetilde{w}_\phi \nabla\kappa dy$$
$$= \phi(x,t)\lambda_{G_0} - \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla\widetilde{w}_\phi \nabla\kappa dy.$$

From here we obtain

$$\Big(\int_{\partial G_0} \partial_\nu w_\phi ds\Big)^2 \leq 2\lambda_{G_0}^2 \phi^2(x,t) + 2\|\widetilde{w}_\phi\|_{\mathcal{M}}^2 \lambda_{G_0}^2.$$

Therefore,

$$\int_0^T \Big(\int_{\partial G_0} \partial_\nu w_\phi(x,y,t)ds\Big)^2 dt \leq 2\lambda_{G_0}^2(\|\phi(\cdot,x)\|_{L^2(0,T)}^2 + \|\widetilde{w}_\phi\|_{L^2(0,T;\mathcal{M})}^2).$$

Applying estimate of (6.3) we obtain (6.5). □

Later, we will need some continuous dependence of w_ϕ with respect to ϕ.

Theorem 6.3. *Assume σ is Hölder continuous satisfying* (2.6), *and let ϕ_1, ϕ_2 blong to $L^2(0,T;L^2(\Omega))$. Then, for a.e. $x \in \Omega$, we have the following estimates*

$$\|\widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}\|_{C([0,T];L^2(\partial G_0))} + \|\widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}\|_{L^2(0,T;\mathcal{M})}$$
$$\leq K\|[\phi_1 - \phi_2](\cdot,x)\|_{L^2(0,T)}. \tag{6.6}$$

$$\|\partial_t \widetilde{w}_{\phi_1} - \partial_t \widetilde{w}_{\phi_2}\|_{L^2(0,T;L^2(\partial G_0))}$$
$$\leq K\Big(\|[\phi_1 - \phi_2](\cdot, x)\|_{L^2(0,T)} + \|[\phi_1 - \phi_2](\cdot, x)\|_{L^2(0,T)}^\alpha \Big),$$
$$\int_0^T \Big(\int_{\partial G_0} (\partial_\nu w_{\phi_1} - \partial_\nu w_{\phi_2}) ds \Big)^2 dt \leq K\|[\phi_1 - \phi_2](\cdot, x)\|_{L^2(0,T)}^2, \qquad (6.7)$$

where the constant K is independent on function g.

Proof. We set $v = \widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}$. Then v is a solution of the problem

$$\Delta_y v = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0}, \ t \in (0,T),$$
$$C_0 \partial_t v + \partial_\nu v + C_0(\sigma(\widetilde{w}_{\phi_1}) - \sigma(\widetilde{w}_{\phi_2}))$$
$$= (\phi_1(x,t) - \phi_2(x,t))\partial_\nu \kappa(y), \quad y \in \partial G_0, \ t \in (0,T), \qquad (6.8)$$
$$v(y,x,0) = 0, \quad y \in \partial G_0,$$
$$v \to 0, \quad |y| \to \infty,$$

for a.e. $x \in \Omega$. Now we can argue as in the proof of Theorem 3.2 and we obtain, for a.e. $x \in \Omega$,

$$\|v\|_{L^2(0,T;\mathcal{M})}^2 + \max_{t \in [0,T]} \|v(t,\cdot)\|_{L^2(\partial G_0)}^2 \leq K\|[\phi_1 - \phi_2](\cdot, x)\|_{L^2(0,T)}^2, \qquad (6.9)$$

$$\operatorname{ess\,sup}_{t \in [0,T]} \|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n \setminus \overline{G_0})}^2 + \|\partial_t v\|_{L^2(0,T;L^2(\partial G_0))}^2$$
$$\leq K\Big(\|[\phi_1 - \phi_2](\cdot, x)\|_{L^2(0,T)}^2 + \|[\phi_1 - \phi_2](\cdot, x)\|_{L^2(0,T)}^{2\alpha} \Big). \qquad (6.10)$$

Taking into account that

$$\int_{\partial G_0} \partial_\nu(w_{\phi_1} - w_{\phi_2}) ds$$
$$= \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla(w_{\phi_1} - w_{\phi_2}) \nabla \kappa \, dy$$
$$= \lambda_{G_0}(\phi_1(x,t) - \phi_2(x,t)) - \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla(\widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}) \nabla \kappa \, dy,$$

and using estimate (6.9) we obtain (6.7). $\square$

As in [11, Lemma 4.6] we have a priori estimate for the solutions of problem (3.2) on the interior of the set $\mathbb{R}^n \setminus \overline{G_0}$.

Theorem 6.4. *Assume that* $\phi(x,t) = \psi(x)\eta(t)$, $\psi(x) \in C^\infty(\overline{\Omega})$, $\eta \in C^1([0,T])$, $Q^T = \Omega \times (0,T)$. *Then, a.e.* $t \in (0,T)$, $x \in \Omega$

$$|\widetilde{w}_\phi(x,y,t)| \leq \frac{R_0^{n-2} K(\max_{\overline{Q^T}} |\phi(x,t)| + \int_0^T \max_{\overline{\Omega}} |g(x,t)| dt)}{|y|^{n-2}}, \qquad (6.11)$$

for all $y \in \mathbb{R}^n \setminus \overline{G_0}$, *and for* $y > 2R_0$, $R_0 = \max_{y \in \partial G_0} |y|$,

$$|\nabla_y w_\phi(x,y,t)| \leq \frac{K(\max_{\overline{Q^T}} |\phi(x,t)| + \int_0^T \max_{\overline{\Omega}} |g(x,t)| dt)}{|y|^{n-1}}, \qquad (6.12)$$

where w_ϕ *is defined by* (6.4) *and* K *is a positive constant only dependent on* n.

Proof. Let w_ϕ be defined by (6.4), i.e. $\widetilde{w}_\phi = \phi(x,t)\kappa(y) - w_\phi$. Then w_ϕ is a strong solution of the problem

$$\Delta_y w_\phi = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0}, \ t \in (0,T),$$

$$C_0 \partial_t w_\phi + \partial_\nu w_\phi - C_0 \sigma(\phi - w_\phi)$$
$$= -C_0 g(x,t) + C_0 \partial_t \phi(x,t), \quad y \in \partial G_0, \ t \in (0,T), \qquad (6.13)$$
$$w_\phi(x,y,0) = \phi(x,0), \quad y \in \partial G_0,$$
$$w_\phi \to 0, \quad |y| \to \infty.$$

We consider the sequence of solutions $\{w_{\phi,R}\}$ to the problems

$$\Delta_y w_{\phi,R} = 0, \quad y \in T_R^0 \setminus \overline{G_0}, \ t \in (0,T),$$

$$C_0 \partial_t w_{\phi,R} + \partial_\nu w_{\phi,R} - C_0 \sigma(\phi - w_{\phi,R})$$
$$= -C_0 g(x,t) + C_0 \partial_t \phi(x,t), \quad y \in \partial G_0, \ t \in (0,T), \qquad (6.14)$$
$$w_{\phi,R}(x,y,0) = \phi(x,0), \quad y \in \partial G_0,$$
$$w_{\phi,R}(x,y,t) = 0, \quad y \in \partial T_R^0, \ t \in (0,T).$$

Arguing as in the proof of the estimate (5.9) in Theorem 5.2, we find that

$$|w_{\phi,R}| \le K_0 \equiv 2 \max_{\overline{\Omega} \times [0,T]} |\phi(x,t)| + 2 \int_0^T \max_{\overline{\Omega}} |g(x,t)| dt.$$

Using the maximum principle, we derive the estimate

$$|w_{\phi,R}| \le \frac{K_0 R_0}{|y|^{n-2}}, \quad y \in T_R^0 \setminus \overline{G_0}, \ t \in (0,T). \qquad (6.15)$$

We define

$$W_{\phi,R} = \begin{cases} w_{\phi,R}, & y \in T_R^0 \setminus \overline{G_0}, t \in (0,T), \\ 0, & |y| \ge R, \ t \in (0,T). \end{cases}$$

Noting that $W_{\phi,R} \in L^2(0,T;\mathcal{M})$ and taking into account that $W_{\phi,R} \rightharpoonup w_\phi$ in $L^2(0,T;\mathcal{M})$ as $R \to \infty$, we deduce the estimate

$$|w_\phi| \le \frac{K_0 R_0}{|y|^{n-2}}.$$

From here we obtain (6.11). Using the inequality (6.11) and the mean-value theorem for the harmonic function $\partial_{y_i} w_\phi(x,y,t)$, $i = 1,\ldots,n$, on a ball we obtain estimate (6.12). □

As the previous regularity result we have the following.

Theorem 6.5. *Assume* (3.9). *Let* $\widetilde{w}_\phi(x_1,y,t)$ *be the weak solution to the problem* (3.2). *Then for any* $x_1, x_2 \in \Omega$,

$$\|\widetilde{w}_\phi(x_1,\cdot,\cdot) - \widetilde{w}_\phi(x_2,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})} \le K\{|x_1 - x_2| + \|\phi(x_1,\cdot) - \phi(x_2,\cdot)\|_{L^2(0,T)}\}.$$

Proof. The function $\widetilde{w}_\phi(x_1,y,t) - \widetilde{w}_\phi(x_2,y,t)$ satisfies the following problem

$$\Delta_y(\widetilde{w}_\phi(x_1,y,t) - \widetilde{w}_\phi(x_2,y,t)) = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0},$$

$$C_0 \partial_t(\widetilde{w}_\phi(x_1,y,t) - \widetilde{w}_\phi(x_2,y,t)) + \partial_\nu(\widetilde{w}_\phi(x_1,y,t) - \widetilde{w}_\phi(x_2,y,t))$$
$$+ C_0(\sigma(\widetilde{w}_\phi(x_1,y,t)) - \sigma(\widetilde{w}_\phi(x_2,y,t)))$$
$$= C_0(g(x_1,t) - g(x_2,t)) + (\phi(x_1,t) - \phi(x_2,t))\partial_\nu \kappa(y), \quad y \in \partial G_0,$$

$$\widetilde{w}_\phi(x_1, y, 0) - \widetilde{w}_\phi(x_2, y, 0) = 0, \quad x \in \partial G_0,$$
$$\widetilde{w}_\phi(x_1, y, t) - \widetilde{w}_\phi(x_2, y, t) \to 0, \quad |y| \to \infty.$$

From the definition of weak solution for this problem we deduce that

$$\int_0^t \int_{\mathbb{R}^n \setminus \overline{G_0}} |\nabla_y(\widetilde{w}_\phi(x_1, y, t) - \widetilde{w}_\phi(x_2, y, t))|^2 dy d\tau$$

$$+ \frac{C_0}{2} \int_{\partial G_0} |\widetilde{w}_\phi(x_1, y, t) - \widetilde{w}_\phi(x_2, y, t)|^2 ds$$

$$+ C_0 \int_0^t \int_{\partial G_0} (\sigma(\widetilde{w}_\phi(x_1, y, t)) - \sigma(\widetilde{w}_\phi(x_2, y, t)))(\widetilde{w}_\phi(x_1, y, t) - \widetilde{w}_\phi(x_2, y, t)) \, ds \, d\tau$$

$$= C_0 \int_0^t \int_{\partial G_0} (g(x_1, t) - g(x_2, t))(\widetilde{w}_\phi(x_1, y, t) - \widetilde{w}_\phi(x_2, y, t)) \, ds \, d\tau$$

$$+ \int_0^t \int_{\partial G_0} (\phi(x_1, t) - \phi(x_2, t))\partial_\nu \kappa(y)(\widetilde{w}_\phi(x_1, t) - \widetilde{w}_\phi(x_2, y, t)) \, ds \, d\tau.$$

Using the monotonicity of σ, (3.9) and the Gronwall's lemma we obtain the conclusion of the Theorem. $\square$

The following result shows that the solution $\widetilde{w}_\phi(x, y, t)$ has a Lipschitz continuous dependence on function g without requiring the assumption (3.9).

Theorem 6.6. *Let* $\widetilde{w}_\phi^g(x, y, t)$, $\widetilde{w}_\phi^{\widehat{g}}(x, y, t)$ *be the weak solutions to the problem* (3.2) *corresponding to the boundary data* $g, \widehat{g} \in L^2(0, T; C(\overline{\Omega}))$. *Then for a.e.* $x \in \Omega$,

$$\|\widetilde{w}_\phi^g(x, \cdot, \cdot) - \widetilde{w}_\phi^{\widehat{g}}(x, \cdot, \cdot)\|_{L^2(0,T;\mathcal{M})} \le K\|g(x, \cdot) - \widehat{g}(x, \cdot)\|_{L^2(0,T)}.$$

Proof. The function $\widetilde{w}_\phi^g(x, y, t) - \widetilde{w}_\phi^{\widehat{g}}(x, y, t)$ satisfies the problem

$$\Delta_y(\widetilde{w}_\phi^g(x, y, t) - \widetilde{w}_\phi^{\widehat{g}}(x, y, t)) = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0},$$
$$C_0 \partial_t(\widetilde{w}_\phi^g(x, y, t) - \widetilde{w}_\phi^{\widehat{g}}(x, y, t)) + \partial_\nu(\widetilde{w}_\phi^g(x, y, t) - \widetilde{w}_\phi^{\widehat{g}}(x, y, t))$$
$$+ C_0(\sigma(\widetilde{w}_\phi^g(x, y, t)) - \sigma(\widetilde{w}_\phi^{\widehat{g}}(x, y, t)))$$
$$= C_0(g(x, t) - \widehat{g}(x, t)) \quad y \in \partial G_0,$$
$$\widetilde{w}_\phi(x_1, y, 0) - \widetilde{w}_\phi(x_2, y, 0) = 0, \quad x \in \partial G_0,$$
$$\widetilde{w}_\phi(x_1, y, t) - \widetilde{w}_\phi(x_2, y, t) \to 0, \quad |y| \to \infty.$$

From the definition of weak solution for this problem (taking $\widetilde{w}_\phi^g(x, y, t) - \widetilde{w}_\phi^{\widehat{g}}(x, y, t)$ as test function) we obtain

$$\int_0^t \int_{\mathbb{R}^n \setminus \overline{G_0}} |\nabla_y(\widetilde{w}_\phi^g(x, y, \tau) - \widetilde{w}_\phi^{\widehat{g}}(x, y, \tau))|^2 dy d\tau$$

$$+ \frac{C_0}{2} \int_{\partial G_0} |\widetilde{w}_\phi^g(x, y, t) - \widetilde{w}_\phi^{\widehat{g}}(x, y, t)|^2 ds$$

$$+ C_0 \int_0^t \int_{\partial G_0} (\sigma(\widetilde{w}_\phi^g(x, y, \tau)) - \sigma(\widetilde{w}_\phi^{\widehat{g}}(x, y, \tau)))(\widetilde{w}_\phi^g(x, y, \tau) - \widetilde{w}_\phi^{\widehat{g}}(x, y, \tau)) \, ds \, d\tau$$

$$= C_0 \int_0^t \int_{\partial G_0} (g(x, \tau) - \widehat{g}(x, \tau))(\widetilde{w}_\phi^g(x, y, \tau) - \widetilde{w}_\phi^{\widehat{g}}(x, y, \tau)) \, ds \, d\tau.$$

Then, using the monotonicity of σ, Friedrich's inequality and the Gronwall's lemma we obtain the conclusion of the Theorem. $\qquad\square$

7. Properties of the operator $\mathrm{H}[\phi]$

By applying the results of previous sections we obtain different properties on the operator $\mathrm{H}[\phi]$ defined by (3.1).

Theorem 7.1. *Assume* $\phi \in L^2(0, T; L^2(\Omega))$. *Then, for a.e.* $x \in \Omega$,

$$\|\mathrm{H}[\phi](\cdot, x)\|_{L^2(0,T)} \le K(\|\phi(\cdot, x)\|_{L^2(0,T)} + \|g(\cdot, x)\|_{L^2(0,T)}). \tag{7.1}$$

Moreover, for $\phi_1, \phi_2 \in L^2(0, T; L^2(\Omega))$ *and for a.e.* $x \in \Omega$ *we have*

$$\|(\mathrm{H}[\phi_1] - \mathrm{H}[\phi_2])(\cdot, x)\|_{L^2(0,T)} \le K\|(\phi_1 - \phi_2)(\cdot, x)\|_{L^2(0,T)}. \tag{7.2}$$

Proof. By taking in the integral identity for $\widetilde{w}_\phi$ as a test function the solution of problem (3.3) we obtain

$$\int_{\partial G_0} \partial_\nu \widetilde{w}_\phi \, ds = \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla_y \widetilde{w}_\phi \nabla \kappa \, dy.$$

So, we have

$$\mathrm{H}[\phi](x, t) = \lambda_{G_0} \phi(x, t) - \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla_y \widetilde{w}_\phi \nabla \kappa \, dy.$$

Applying estimate (6.3) we obtain

$$\|\mathrm{H}[\phi](\cdot, x)\|_{L^2(0,T)} \le K(\|\phi(\cdot, x)\|_{L^2(0,T)} + \|g(\cdot, x)\|_{L^2(0,T)}),$$

where K is independent from $\phi(\cdot, x)$ and $g(\cdot, x)$. To obtain the inequality (7.2) we note that

$$\int_{\partial G_0} \partial_\nu (\widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}) ds = \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla(\widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}) \nabla \kappa \, dy.$$

From here we conclude that

$$\mathrm{H}[\phi_1] - \mathrm{H}[\phi_2] = (\phi_1 - \phi_2)\lambda_{G_0} - \int_{\mathbb{R}^n \setminus \overline{G_0}} \nabla(\widetilde{w}_{\phi_1} - \widetilde{w}_{\phi_2}) \nabla \kappa \, dy.$$

Applying (6.6) we obtain (7.2). $\qquad\square$

Remark 7.2. Note that if G_0 is a ball, as in Remark 3.1, and $\mathcal{H}$ is a solution of the Cauchy problem (3.4), then we can prove, in a different way, that independently of the regularity of σ, the function $\mathcal{H}$ is Lipcshitz continuous. Indeed, for any $\phi_1, \phi_2 \in L^2(0, T)$ we have

$$\partial_t(\mathcal{H}_{\phi_1} - \mathcal{H}_{\phi_2}) + \frac{n-2}{C_0}(\mathcal{H}_{\phi_1} - \mathcal{H}_{\phi_2}) + (\sigma(\mathcal{H}_{\phi_1}) - \sigma(\mathcal{H}_{\phi_2})) = \frac{n-2}{C_0}(\phi_1 - \phi_2).$$

Multiplying this equality by $(\mathcal{H}_{\phi_1} - \mathcal{H}_{\phi_2})$, integrating on $(0, t)$ and using the monotonicity property of σ, we obtain

$$(\mathcal{H}_{\phi_1}(x, t) - \mathcal{H}_{\phi_2}(x, t))^2 \le \frac{2(n-2)}{C_0} \int_0^t (\phi_1 - \phi_2)(\mathcal{H}_{\phi_1} - \mathcal{H}_{\phi_2}) d\tau.$$

Applying the Gronwall's Lemma, we conclude that

$$\max_{[0,T]} |\mathcal{H}_{\phi_1} - \mathcal{H}_{\phi_2}| \le K\|\phi_1 - \phi_2\|_{L^2(0,T)}.$$

Remark 7.3. If we consider the similar problem for Robin type boundary conditions (see [15]), then, if G_0 is a ball, the new nonlinear term in the homogenized problem must solve the functional equation

$$\mathrm{H} = \lambda\sigma(u - \mathrm{H}), \ \lambda = \frac{C_0}{n-2} > 0.$$

Setting $V = u - \mathrm{H}$, we derive the equation

$$\lambda\sigma(V) + V = u.$$

Let us consider the difference of two equalities $\lambda\sigma(V_1) + V_1 = u_1$ and $\lambda\sigma(V_2) + V_2 = u_2$, i.e.

$$\lambda(\sigma(V_1) - \sigma(V_2)) + V_1 - V_2 = u_1 - u_2.$$

Multiplying this equality by $V_1 - V_2$, and using the monotonicity of σ, we obtain the Lipschitz continuity of V.

Theorem 7.4. *For any $\phi_1(x,t), \phi_2(x,t)$, defined a.e. in $\Omega \times (0,T)$ such that for a.e. $x \in \Omega$, $\phi_i(x,\cdot) \in L^2(0,T)$, for a.e. $x \in \Omega$, $i = 1,2$, we have*

$$\int_0^T (\mathrm{H}[\phi_1](x,t) - \mathrm{H}[\phi_2](x,t))(\phi_1(x,t) - \phi_2(x,t))dt \geq 0. \tag{7.3}$$

Proof. By denseness, without loss of generality we can assume that $\phi_i(x,\cdot) \in C^1([0,T])$, $i = 1,2$. We consider w_{ϕ_1}, w_{ϕ_2}. Then $w_{\phi_1} - w_{\phi_2}$ is a weak solution to the problem

$$\begin{aligned}
\Delta_y(w_{\phi_1} - w_{\phi_2}) &= 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0}, \\
C_0\partial_t(w_{\phi_1} - w_{\phi_2}) &+ \partial_\nu(w_{\phi_1} - w_{\phi_2}) \\
- C_0(\sigma(\phi_1 - w_{\phi_1}) &- \sigma(\phi_2 - w_{\phi_2})) \\
&= C_0\partial_t(\phi_1 - \phi_2), \quad y \in \partial G_0, \ t \in (0,T), \\
(w_{\phi_1} - w_{\phi_2})(x,y,0) &= \phi_1(x,0) - \phi_2(x,0), \quad y \in \partial G_0, \\
w_{\phi_1} - w_{\phi_2} &\to 0, \quad |y| \to \infty.
\end{aligned} \tag{7.4}$$

Taking $w_{\phi_1} - w_{\phi_2}$ as a test function in the integral identity corresponding to notion of weak solution of problem (7.4), we obtain

$$\begin{aligned}
\|w_{\phi_1} - w_{\phi_2}\|^2_{L^2(0,T;\mathcal{M})} &+ C_0\int_0^T \int_{\partial G_0} \partial_t(w_{\phi_1} - w_{\phi_2})(w_{\phi_1} - w_{\phi_2})\, ds\, dt \\
- C_0\int_0^T \int_{\partial G_0} &(\sigma(\phi_1 - w_{\phi_1}) - \sigma(\phi_2 - w_{\phi_2}))(w_{\phi_1} - w_{\phi_2})\, ds\, dt \\
&= C_0\int_0^T \int_{\partial G_0} \partial_t(\phi_1 - \phi_2)(w_{\phi_1} - w_{\phi_2})\, ds\, dt.
\end{aligned} \tag{7.5}$$

From this equality we deduce that

$$\begin{aligned}
\|w_{\phi_1} &- w_{\phi_2}\|^2_{L^2(0,T;\mathcal{M})} \\
&+ C_0\int_0^T \int_{\partial G_0} \partial_t((w_{\phi_1} - \phi_1) - (w_{\phi_2} - \phi_2))((w_{\phi_1} - \phi_1) - (w_{\phi_2} - \phi_2))\, ds\, dt \\
&+ C_0\int_0^T \int_{\partial G_0} (\sigma(\phi_1 - w_{\phi_1}) - \sigma(\phi_2 - w_{\phi_2}))((\phi_1 - w_{\phi_1}) - (\phi_2 - w_{\phi_2}))\, ds\, dt
\end{aligned}$$

$$= C_0 \int_0^T \int_{\partial G_0} \partial_t((\phi_1 - w_{\phi_1}) - (\phi_2 - w_{\phi_2}))(\phi_1 - \phi_2)\, ds\, dt$$

$$+ C_0 \int_0^T \int_{\partial G_0} (\sigma(\phi_1 - w_{\phi_1}) - \sigma(\phi_2 - w_{\phi_2}))(\phi_1 - \phi_2)\, ds\, dt$$

$$= \int_0^T (\mathrm{H}[\phi_1] - \mathrm{H}[\phi_2])(\phi_1 - \phi_2)\, dt.$$

Using that σ is monotone we obtain that the left-hand side of last equality is nonnegative and completes the proof. $\qquad\square$

We obtain a stronger regularity on the operator $\mathrm{H}[\phi]$ when the boundary datum g is more regular.

Theorem 7.5. *Assume* (3.9) *and let* $x_1, x_2 \in \Omega$. *Then there exists* $K > 0$ *such that*

$$\|\mathrm{H}[\phi](x_1,t) - \mathrm{H}[\phi](x_2,t)\|_{L^2(0,T)} \leq K\{|x_1 - x_2| + \|\phi(x_1,t) - \phi(x_2,t)\|_{L^2(0,T)}\}.$$

Proof. Note that

$$\mathrm{H}[\phi](x_1,t) - \mathrm{H}[\phi](x_2,t)$$

$$= (\phi(x_1,t) - \phi(x_2,t))\lambda_{G_0} + \int_{\mathbb{R}^n \backslash G_0} \nabla(\widetilde{w}_\phi(x_1,y,t) - \widetilde{w}_\phi(x_2,y,t))\nabla\kappa(y)\, dy.$$

Applying Theorem 6.5 to the right-hand side of this equality we obtain the statement of the theorem. $\qquad\square$

As in the previous section, we can prove that the operator $\mathrm{H}[\phi]$ has a Lipschitz continuous dependence on function g without requiring assumption (3.9).

Theorem 7.6. *Let* $\mathrm{H}_\phi^g(x,t), \mathrm{H}_\phi^{\widehat{g}}(x,t)$ *be the associate operators corresponding to the boundary data* $g, \widehat{g} \in L^2(0,T;C(\overline{\Omega}))$. *Then there exists* $K > 0$ *such that for a.e.* $x \in \Omega$,

$$\|\mathrm{H}_\phi^g(x,\cdot) - \mathrm{H}_\phi^{\widehat{g}}(x,\cdot)\|_{L^2(0,T)} \leq K\|g(x,\cdot) - \widehat{g}(x,\cdot)\|_{L^2(0,T)}.$$

Proof. Note that

$$\mathrm{H}_\phi^g(x,t) - \mathrm{H}_\phi^{\widehat{g}}(x,t) = \int_{\mathbb{R}^n \backslash G_0} \nabla(\widetilde{w}_\phi^g(x,y,t) - \widetilde{w}_\phi^{\widehat{g}}(x,y,t))\nabla\kappa(y)\, dy.$$

Then it suffices to apply Theorem 6.6 to the right-hand side of this equality to obtain the statement of the theorem. $\qquad\square$

8. Asymptotic similarity between the capacity functions $w_{\varepsilon,\phi}^j$ and

$$w_\phi(P_\varepsilon^j, \tfrac{x-P_\varepsilon^j}{a_\varepsilon}, t)$$

The following results give some estimates on the differences of the two types of capacities functions introduced in previous sections.

Theorem 8.1. *Let* $v_{\varepsilon,\phi}^j = w_{\varepsilon,\phi}^j - w_\phi(P_\varepsilon^j, \tfrac{x-P_\varepsilon^j}{a_\varepsilon}, t)$, *where* $w_\phi(x,y,t)$ *is the solution of* (6.13). *Then*

$$\sup_{(T_{\varepsilon/4}^j \backslash \overline{G_\varepsilon^j}) \times (0,T)} |v_{\varepsilon,\phi}^j| \leq \sup_{\partial T_{\varepsilon/4}^j \times (0,T)} \left| w_\phi\left(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon}, t\right) \right|. \qquad (8.1)$$

Here, $\phi(x,t) = \psi(x)\eta(t)$, *with* $\psi \in C^\infty(\overline{\Omega})$, $\eta \in C^1([0,T])$.

The proof of the above theorem can be find in [34].

Theorem 8.2.

$$\varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \max_{t \in [0,T]} \|v_{\varepsilon,\phi}^j\|_{L^2(\partial G_\varepsilon^j)}^2 + \sum_{j \in \Upsilon_\varepsilon} \|\nabla v_{\varepsilon,\phi}^j\|_{L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}))}^2$$

$$\leq K\varepsilon^2 (\max_{Q^T} \phi^2(x,t) + \int_0^T \max_{\overline{\Omega}} g^2(x,t)dt), \tag{8.2}$$

$$\sum_{j \in \Upsilon_\varepsilon} \|v_{\varepsilon,\phi}^j\|_{L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}))}^2 \leq K\varepsilon^4 (\max_{Q^T} |\phi(x,t)|^2 + \int_0^T \max_{\overline{\Omega}} |g(x,t)|^2 dt), \tag{8.3}$$

where the positive constant K is independent of ε, ϕ and g.

Proof. From the Green formulas we have

$$\int_0^t \int_{T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}} |\nabla v_{\varepsilon,\phi}^j|^2 dx d\tau + \varepsilon^{-\gamma} \int_0^t \int_{\partial G_\varepsilon^j} \partial_t v_{\varepsilon,\phi}^j v_{\varepsilon,\phi}^j \, ds \, d\tau$$

$$- \varepsilon^{-\gamma} \int_0^t \int_{\partial G_\varepsilon^j} \Big(\sigma(\phi(P_\varepsilon^j, \tau) - w_{\varepsilon,\phi}^j) - \sigma(\phi(P_\varepsilon^j, \tau)$$

$$- w_\phi(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon}, \tau)) \Big) v_{\varepsilon,\phi}^j \, ds \, d\tau \tag{8.4}$$

$$= - \int_0^t \int_{\partial T_{\varepsilon/4}^j} w_\phi(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon}, \tau) \partial_\nu v_{\varepsilon,\phi}^j \, ds \, d\tau.$$

Taking into account that σ is monotone and applying the Green formula, we obtain

$$\int_0^t \int_{T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}} |\nabla v_{\varepsilon,\phi}^j|^2 dx d\tau + \frac{\varepsilon^{-\gamma}}{2} \|v_{\varepsilon,\phi}^j\|_{L^2(\partial G_\varepsilon^j)}^2$$

$$\leq - \int_0^t \int_{\partial T_{\varepsilon/4}^j} w_\phi(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon}, \tau) \partial_\nu v_{\varepsilon,\phi}^j d d\tau \tag{8.5}$$

$$= - \int_0^t \int_{T_{\varepsilon/4}^j \setminus \overline{T_{\varepsilon/8}^j}} \nabla v_{\varepsilon,\phi}^j \nabla w_\phi dx d\tau + \int_0^t \int_{\partial T_{\varepsilon/8}^j} \partial_\nu v_{\varepsilon,\phi}^j w_\phi \, ds \, d\tau.$$

Using the estimate (6.11) we obtain

$$|v_{\varepsilon,\phi}^j(x,t)| \leq \sup_{\partial T_{\varepsilon/4}^j \times (0,T)} \Big| w_\phi(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon}, t) \Big|$$

$$\leq K\varepsilon^2 (\max_{Q^T} |\phi(x,t)| + \int_0^T \max_{\overline{\Omega}} |g(x,t)|dt), \tag{8.6}$$

where K is a constant independent on ε, ϕ and g. From (8.6) we deduce for $x_0 \in \partial T_{\varepsilon/8}^j$

$$|\partial_{x_i} v_{\varepsilon,\phi}^j(x_0, t)| = |T_{\varepsilon/16}^{x_0}|^{-1} \Big| \int_{T_{\varepsilon/16}^{x_0}} \partial_{x_i} v_{\varepsilon,\phi}^j dx \Big|$$

$$\leq K\varepsilon (\max_{Q^T} |\phi(x,t)| + \int_0^T \max_{\overline{\Omega}} |g(x,t)|dt). \tag{8.7}$$

Consequently,

$$\left| \int_0^T \int_{\partial T_{\varepsilon/8}^j} \partial_\nu v_{\varepsilon,\phi}^j w_\phi \, ds \, dt \right| \leq K\varepsilon^{n+2} (\max_{\overline{Q^T}} \phi^2(x,t) + \int_0^T \max_{\overline{\Omega}} g^2(x,t) dt). \quad (8.8)$$

Applying (6.12) we obtain, that for $x \in T_{\varepsilon/4}^j \setminus \overline{T_{\varepsilon/8}^j}$,

$$\left| \nabla_x w_\phi(P_\varepsilon^j, \frac{x - P_\varepsilon^j}{a_\varepsilon}, t) \right| \leq K\varepsilon (\max_{\overline{Q^T}} |\phi(x,t)| + \int_0^T \max_{\overline{\Omega}} |g(x,t)| dt).$$

Therefore,

$$\left| \int_0^T \int_{T_{\varepsilon/4}^j \setminus \overline{T_{\varepsilon/8}^j}} \nabla v_{\varepsilon,\phi}^j \nabla w_\phi \, dx \, dt \right|$$

$$\leq \frac{1}{2} \int_0^T \int_{T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}} |\nabla v_{\varepsilon,\phi}^j|^2 \, dx \, dt + K\varepsilon^{n+2} (\max_{\overline{Q^T}} |\phi(x,t)|^2 + \int_0^T \max_{\overline{\Omega}} |g(x,t)|^2 dt).$$

From here and (8.4) we obtain the estimate

$$\varepsilon^{-\gamma} \max_{[0,T]} \|v_{\varepsilon,\phi}^j\|_{L^2(\partial G_\varepsilon^j)}^2 + \|\nabla v_{\varepsilon,\phi}^j\|_{L^2(0,T;L^2(T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}))}^2$$

$$\leq K\varepsilon^{n+2} (\max_{\overline{Q^T}} |\phi(x,t)|^2 + \int_0^T \max_{\overline{\Omega}} |g(x,t)|^2 dt).$$

Hence, we obtain (8.2). The estimate (8.3) is then obtained from the Friedrichs inequality. □

9. Proof of Theorem 3.2

First step. We start by assuming the additional condition (3.9) on g. As mentioned in Section 5, the very weak formulation of problem (2.2) leads to the inequality

$$\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \partial_t \phi(\phi - u_\varepsilon) \, ds \, dt + \int_{Q_\varepsilon^T} \nabla \phi \nabla(\phi - u_\varepsilon) \, dx \, dt$$

$$+ \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \sigma(\phi)(\phi - u_\varepsilon) \, ds \, dt$$

$$\geq \int_{Q_\varepsilon^T} f(\phi - u_\varepsilon) \, dx \, dt + \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} g(x,t)(\phi(x,t) - u_\varepsilon(x,t)) \, ds \, dt$$

$$- \frac{1}{2}\varepsilon^{-\gamma} \|\phi(x,0)\|_{L^2(S_\varepsilon)}^2, \quad (9.1)$$

for any smooth test function $\phi(x,t)$. By density, we can assume, for instance, that $\phi(x,t) = \psi(x)\eta(t)$, for some $\psi \in C_0^\infty(\Omega)$, $\eta \in C^1([0,T])$.

Second step. Again, under condition (3.9), we introduce the auxiliary function

$$W_{\varepsilon,\phi}(x,t) = \begin{cases} w_{\varepsilon,\phi}^j(x,t), & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \ t \in (0,T), \ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \cup_{j \in \Upsilon_\varepsilon} \overline{T_{\varepsilon/4}^j}, \ t \in (0,T), \end{cases}$$

where $w_{\varepsilon,\phi}^j$ is a solution of (5.3). Using Theorem 5.1 and the estimates from it we have that $P_\varepsilon W_{\varepsilon,\phi} \in L^2(0,T;H_0^1(\Omega))$ and $P_\varepsilon W_{\varepsilon,\phi} \rightharpoonup 0$ weakly in $L^2(0,T;H_0^1(\Omega))$ as $\varepsilon \to 0$.

By taking $(\phi - W_{\varepsilon,\phi})$ as a test function in (9.1), we obtain

$$
\varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \partial_t(\phi - W_{\varepsilon,\phi})(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, ds\, dt
$$
$$
+ \int_{Q_\varepsilon^T} \nabla(\phi - W_{\varepsilon,\phi})\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, dx\, dt
$$
$$
+ \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} \sigma(\phi - W_{\varepsilon,\phi})(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, ds\, dt \tag{9.2}
$$
$$
\geq \int_{Q_\varepsilon^T} f(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, dx\, dt + \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon} g(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, ds\, dt
$$
$$
- \frac{1}{2}\varepsilon^{-\gamma}\|\phi(x,0) - W_{\varepsilon,\phi}(x,0)\|^2_{L^2(S_\varepsilon)}.
$$

Taking into account properties of $W_{\varepsilon,\phi}(x,t)$ we obtain

$$
\lim_{\varepsilon \to 0} \int_{Q_\varepsilon^T} f(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, dx\, dt = \int_{Q^T} f(\phi - u_0)\, dx\, dt, \tag{9.3}
$$
$$
\lim_{\varepsilon \to 0} \int_{Q_\varepsilon^T} \nabla\phi\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, dx\, dt = \int_{Q^T} \nabla\phi\nabla(\phi - u_0)\, dx\, dt. \tag{9.4}
$$

Since $w_{\varepsilon,\phi}^j$ is associated with the strong solution of problem (5.3), we have

$$
- \int_{Q_\varepsilon^T} \nabla W_{\varepsilon,\phi}\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)\, dx\, dt
$$
$$
= - \sum_{j \in \Upsilon_\varepsilon} \int_0^T \int_{T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}} \nabla w_{\varepsilon,\phi}^j \nabla(\phi - w_{\varepsilon,\phi}^j - u_\varepsilon)\, dx\, dt
$$
$$
= -\varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int_0^T \int_{\partial G_\varepsilon^j} \big(\partial_t(\phi - w_{\varepsilon,\phi}^j) - g + \sigma(\phi(P_\varepsilon^j, t) - w_{\varepsilon,\phi}^j)\big) \tag{9.5}
$$
$$
\times (\phi - w_{\varepsilon,\phi}^j - u_\varepsilon)\, ds\, dt - \sum_{j \in \Upsilon_\varepsilon} \int_0^T \int_{\partial T_{\varepsilon/4}^j} \partial_\nu w_{\varepsilon,\phi}^j(\phi - u_\varepsilon)\, ds\, dt.
$$

Taking into account that as $\varepsilon \to 0$, we have

$$
I_\varepsilon \equiv \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int_0^T \int_{\partial G_\varepsilon^j} \Big(\partial_t(\phi(x,t) - \phi(P_\varepsilon^j, t)) + (\sigma(\phi(x,t) - w_{\varepsilon,\phi}^j)
$$
$$
- \sigma(\phi(P_\varepsilon^j, t) - w_{\varepsilon,\phi}^j) - (g(x,t) - g(P_\varepsilon^j, t))\Big)(\phi(x,t) - w_{\varepsilon,\phi}^j - u_\varepsilon)\, ds\, dt
$$
$$
\to 0,
$$

we conclude, that the sum of all integrals over $S_\varepsilon \times (0,T)$ tends to zero, as $\varepsilon \to 0$. Thus from here and from (9.2)-(9.5) we have that u_0 satisfies the inequality

$$
\int_{Q^T} \nabla\phi\nabla(\phi - u_0)\, dx\, dt - \lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int_0^T \int_{\partial T_{\varepsilon/4}^j} \partial_\nu w_{\varepsilon,\phi}^j(\phi - u_\varepsilon)\, ds\, dt
$$
$$
\geq \int_{Q^T} f(\phi - u_0)\, dx\, dt. \tag{9.6}
$$

Third step. We still assume condition (3.9).To estimate the second term we will apply the following Lemma.

Lemma 9.1. *Assume σ is Hölder continuous in (2.6). Let $\mathrm{H}[\phi](x,t)$ be the value of $\mathrm{H}[\phi]$ given by (3.1) for regular functions ϕ. Then, $\mathrm{H}[\phi] \in L^2(0,T;L^2(\Omega))$ and for any sequence $\{h_\varepsilon\}$, $h_\varepsilon \in L^2(0,T;H_0^1(\Omega))$, such that $h_\varepsilon \rightharpoonup h$ weakly in $L^2(0,T;H_0^1(\Omega))$ as $\varepsilon \to 0$, we have*

$$\Big| \sum_{j\in\Upsilon_\varepsilon} \int_0^T \int_{\partial G_\varepsilon^j} \partial_\nu w_\phi(P_\varepsilon^j, \frac{x-P_\varepsilon^j}{a_\varepsilon}, t)h_\varepsilon \, ds\, dt$$
$$+ C_0^{n-2} \int_0^T \int_\Omega \Big[\int_{\partial G_0} \partial_\nu w_\phi(x,y,t)ds_y\Big] h(x,t)\, dx\, dt \Big| \to 0, \tag{9.7}$$

as $\varepsilon \to 0$, where ν is the unit outward normal vector to $\partial T_{\varepsilon/4}^j$ (respectively to ∂G_0) directed along the radius of the ball $T_{\varepsilon/4}^j$ (respectively to the exterior of G_0).

The conclusion of this lemma can be obtained by different methods: here we point out that its proof is an easy variant of [11, Lemma 5.7] (which follows the main lines of a result of [25]). A detailed proof when σ is Lipschitz continuous can be obtained also thought [33, Lemma 9]. Nevertheless, the adaptation to the more general case of σ Hölder continuous is automatic thanks to the estimates given in Theorem 6.5 of the present paper.

Using Theorem 7.5 and the previous Lemma we obtain

$$\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_0^T \int_{\partial T_{\varepsilon/4}^j} \partial_\nu w_{\varepsilon,\phi}^j(\phi - u_\varepsilon)\, ds\, dt$$
$$= \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_0^T \int_{\partial T_{\varepsilon/4}^j} \partial_\nu w_\phi(x, \frac{x-P_\varepsilon^j}{a_\varepsilon}, t)(\phi - u_\varepsilon)\, ds\, dt \tag{9.8}$$
$$= -C_0^{n-2} \int_{Q^T} \mathrm{H}[\phi](x,t)(\phi - u_0)\, dx\, dt.$$

From (9.6) and (9.8) we conclude that u_0 satisfies

$$\int_{Q^T} \nabla\phi\nabla(\phi - u_0)\, dx\, dt + C_0^{n-2} \int_{Q^T} \mathrm{H}[\phi](\phi - u_0)\, dx\, dt \geq \int_{Q^T} f(\phi - u_0)\, dx\, dt, \tag{9.9}$$

for any smooth test function $\phi(x,t) = \psi(x)\eta(t)$, $\psi \in C_0^\infty(\Omega)$, $\eta \in C^1[0,T]$. By denseness this inequality holds for any $\phi \in L^2(0,T;H_0^1(\Omega))$. Then, applying again the characterization of solutions given by monotone operators (see [6], or [21]) we deduce that u_0 is a weak solution of problem (3.11).

Fourth step. Assuming (3.9), to characterize the homogenized limit u_0 it is important to prove that under the assumptions of Theorem 5.1 problem (3.11) has an unique solution. This is consequence of the following continuous dependence result: if we suppose that $u_{0,1}$ and $u_{0,2}$ are two weak solutions of the problem (3.11) corresponding to f_1, f_2 satisfying (2.4) then, by multiplying by $(u_{0,1} - u_{0,2})$ the corresponding equations, by the monotonicity of the operator $\mathrm{H}[\phi](x,t)$, we obtain

$$\|\nabla(u_{0,1} - u_{0,2})\|_{L^2(0,T;L^2(\Omega))} \leq \|f_1 - f_2\|_{L^2(Q^T)}^2.$$

This proves the uniqueness of solutions and the proof of Theorem 5.1 under the additional condition (3.9) ends.

Fifth step. Given $g \in L^2(0,T; C(\overline{\Omega}))$, let $g_m \in L^2(0,T; W^{1,\infty}(\Omega))$ (i.e., satisfying (3.9)) such that $g_m \to g$ in $L^2(0,T; C(\overline{\Omega}))$ and such that

$$\|g_m\|_{L^2(0,T;C(\overline{\Omega}))} \leq \|g\|_{L^2(0,T;C(\overline{\Omega}))} + 1. \tag{9.10}$$

By the monotonicity in of the abstract operator associated with problem (2.2) (see, e.g. [3]) we know that if $u_{\varepsilon,m}$ is the solution of problem (2.2) corresponding to the boundary data g_m, then

$$\begin{aligned}
&\|u_\varepsilon - u_{\varepsilon,m}\|^2_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))} + \varepsilon^{-\gamma}\|u_\varepsilon - u_{\varepsilon,m}\|^2_{L^2(0,T;L^2(S_\varepsilon))} \\
&\leq K(\|g - g_m\|^2_{L^2(0,T;C(\overline{\Omega}))}).
\end{aligned} \tag{9.11}$$

By applying the four previous steps to the boundary data g_m we know the existence of a unique solution $u_{0,m} \in L^2(0,T; H_0^1(\Omega))$ of the family of problems (depending of the parameter $t \in (0,T)$)

$$\begin{aligned}
-\Delta_x u_{0,m}(x,t) + C_0^{n-2} H_m[u_{0,m}](x,t) &= f(x,t), \quad x \in \Omega, t \in (0,T), \\
u_{0,m}(x,t) &= 0, \quad x \in \partial\Omega, t \in (0,T),
\end{aligned} \tag{9.12}$$

with $H_m[u_0](x,t)$ defined by the nonlocal operator (3.1), corresponding to the boundary data g_m, for a.e. $x \in \Omega$. We define $u_0 \in L^2(0,T; H_0^1(\Omega))$ be the unique weak solution of the family of problems (depending of the parameter $t \in (0,T)$) associate to the boundary data $g \in L^2(0,T; C(\overline{\Omega}))$,

$$\begin{aligned}
-\Delta_x u_0(x,t) + C_0^{n-2} H[u_0](x,t) &= f(x,t), \quad x \in \Omega, t \in (0,T), \\
u_0(x,t) &= 0, \quad x \in \partial\Omega, t \in (0,T),
\end{aligned} \tag{9.13}$$

with $H[u_0](x,t)$ defined by the nonlocal operator (3.1) for a.e. $x \in \Omega$. Note that u_0 is well defined for g merely in $L^2(0,T; C(\overline{\Omega}))$ (i.e., without requiring condition (3.9)). Then we have

$$\begin{aligned}
\Delta(u_0 - u_{0,m}) &= C_0^{n-2}[H[u_0] - H_m[u_{0,m}]] \\
&= C_0^{n-2}[H_m[u_0] - H_m[u_{0,m}]] + C_0^{n-2}[H[u_0] - H_m[u_0]].
\end{aligned}$$

As usual, by multiplying by $u_0 - u_{0,m}$ and applying Poincaré inequality we obtain that $u_{0,m} \to u_0$ in $L^2(0,T; H_0^1(\Omega))$ (recall the estimates given in Theorems 7.1 and 7.6 and the fact that the corresponding constants are uniform in m). Then, given $g \in L^2(0,T; C(\overline{\Omega}))$ and a test function $v \in L^2(0,T; H_0^1(\Omega))$ we have

$$\begin{aligned}
&\int_0^T \int_\Omega (P_\varepsilon u_\varepsilon^g - u_0^g) v \, dx \, dt \\
&= \int_0^T \int_\Omega (P_\varepsilon u_\varepsilon^g - P_\varepsilon u_\varepsilon^{g_m}) v \, dx \, dt + \int_0^T \int_\Omega (P_\varepsilon u_\varepsilon^{g_m} - u_0^{g_m}) v \, dx \, dt \\
&\quad + \int_0^T \int_\Omega (u_0^{g_m} - u_0^g) v \, dx \, dt \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Thus, we obtain that for any $\delta > 0$, and for any $\varepsilon > 0$ there exists a $m_0 \in \mathbb{N}$ such that, for any $m \geq m_0$, we have

$$|I_1| \leq \delta$$

(note that m_0 is independent on ε since $g_m \to g$ in $L^2(0,T; C(\overline{\Omega}))$, estimate (9.10) and the continuous dependence estimate (9.11)). On the other hand, thanks to the

fourth step, we know that for any $\delta > 0$, there exists $\varepsilon_0 > 0$ (independent on m) such that for any $\varepsilon \geq \varepsilon_0$, we have

$$|I_2| \leq \delta$$

(the independence on m of ε_0 comes from the fact that the a priori estimates for passing to the weak limit in Theorem 2.1 are only dependent on $\|g\|_{L^2(0,T;C(\overline{\Omega}))}$ and we know the uniform estimate (9.10)). Finally, we know that $u_0^{g_m} \to u_0^g$ in $L^2(0,T;H_0^1(\Omega))$ and thus, for any $\delta > 0$, there exists a $\widehat{m}_0 \in \mathbb{N}$ such that, for any $m \geq \widehat{m}_0$, we have

$$|I_3| \leq \delta.$$

In conclusion, for any $\delta > 0$, and for any $\varepsilon \geq \varepsilon_0$ we know that by taking as intermediate step the approximation of g by g_m, with $m \geq \max\{m_0, \widehat{m}_0\}$, we obtain that

$$\Big| \int_0^T \int_\Omega (P_\varepsilon u_\varepsilon^g - u_0^g) v \, dx \, dt \Big| \leq 3\delta,$$

which implies the weak convergence for a boundary datum $g \in L^2(0,T;C(\overline{\Omega}))$ and the proof of Theorem 5.1 is complete.

Remark 9.2. Our treatment remains valid if G_0 is the union of a finite family of sets G_0^m, $m = 1,2,\ldots,M$, satisfying the same geometric properties (especially the critical size assumption 2.3). Indeed, the adaptation of the test function made in Step 1 of the proof of Theorem 3.2 is local and can be done, separately over disjoint neighborhoods of the associate G_ε^m, $m = 1,2,\ldots,M$. In this way

$$\mathrm{H}[\phi](x,t) = \sum_{m=1}^M \mathrm{H}_m[\phi](x,t), \ \ with \ \mathrm{H}_m[\phi](x,t) = \int_{\partial G_0^m} \partial_\nu w_\phi^m(x,y,t) ds_y,$$

where $w_\phi^m(y,t)$ is the corresponding solution of the associate problem (3.2).

Remark 9.3. If $n = 2$, $a_\varepsilon = \varepsilon \exp(-\alpha^2/\varepsilon^2)$ and u_ε is a weak solution of the problem

$$
\begin{aligned}
-\Delta u_\varepsilon &= f(x,t), \quad (x,t) \in Q_\varepsilon^T, \\
\beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(x,u_\varepsilon) &= \beta(\varepsilon)g(x,t), \quad (x,t) \in S_\varepsilon \times (0,T), \\
u_\varepsilon &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
u_\varepsilon(x,0) &= 0, \quad x \in S_\varepsilon,
\end{aligned}
\tag{9.14}
$$

where $\beta(\varepsilon) = \varepsilon \exp(\alpha^2/\varepsilon^2)$, then, arguing as in [16] the pair $(u_0, \mathrm{H}(u_0))$, defined in (2.9) and (3.1) respectively, is a weak solution of the problem

$$
\begin{aligned}
-\Delta u_0 + \frac{2\pi}{\alpha^2}(u_0(x,t) - \mathrm{H}[u_0](x,t)) &= f(x,t), \quad (x,t) \in Q^T, \\
\partial_t \mathrm{H}[u_0] + \frac{2\pi}{\alpha^2|\partial G_0|}\mathrm{H}[u_0] + \sigma(x,\mathrm{H}[u_0]) &= g(x,t) + \frac{2\pi}{\alpha^2|\partial G_0|}u_0, \\
x \in \Omega, \ t &\in (0,T), \\
u_0(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
\mathrm{H}[u_0](x,0) &= 0, \quad x \in \Omega.
\end{aligned}
\tag{9.15}
$$

Remark 9.4. We point out that although we do not need to justify the existence of solutions of problem (3.11), since u_0 was built by passing to the limit in u_ε, nevertheless it is useful to analyze some properties of the associated operators since

they are useful , for instance, to the extension to the case in which f is merely in $L^1(0,T;L^1(\Omega))$ following, for instance, the abstract results of [7]. The existence of a weak solution to the problem (3.11) follows from the monotonicity of H$[\phi]$ understood in the sense of Theorem 7.1 and the method of monotone operators. If we introduce the operator $A : L^2(0,T;H_0^1(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$ given by

$$(A(u),v) = \int_{Q^T} \nabla u \nabla v \, dx \, dt + C_0^{n-2} \int_{Q^T} \text{H}[u]v \, dx \, dt,$$

then A is a monotone operator, since by Theorem 7.1,

$$(A(u)-A(v),u-v) = \int_{Q^T} |\nabla(u-v)|^2 \, dx \, dt + C_0^{n-2} \int_{Q^T} (\text{H}[u]-\text{H}[v])(u-v) \, dx \, dt \geq 0.$$

Moreover, A is a coercive operator since

$$(A(u),u) = \int_{Q^T} |\nabla u|^2 \, dx \, dt + C_0^{n-2} \int_{Q^T} \text{H}[u]u \, dx \, dt \geq \|u\|^2_{L^2(0,T;H_0^1(\Omega))},$$

and hence

$$\frac{(A(u),u)}{\|u\|_{L^2(0,T;H_0^1(\Omega))}} \geq \|u\|_{L^2(0,T;H_0^1(\Omega))} \to \infty,$$

as $\|u\|_{L^2(0,T;H_0^1(\Omega))} \to \infty$.

From (7.1) we conclude that

$$\int_0^T \int_\Omega H^2[u](x,t) \, dx \, dt \leq K(\|u\|^2_{L^2(0,T;L^2(\Omega))} + \|g\|^2_{L^2(0,T;L^2(\Omega))}). \qquad (9.16)$$

Using this inequality we obtain

$$|(A(u),v)| \leq \|u\|_{L(0,T;H_0^1(\Omega))}\|v\|_{L^2(0,T;H_0^1(\Omega))} + (\|u\|_{L^2(0,T;L^2(\Omega))}$$
$$+ \|g\|_{L^2(0,T;L^2(\Omega))})\|v\|_{L^2(0,T;L^2(\Omega))},$$

which proves that A is a nonlinear bounded operator. Notice that this also implies the existence of a weak solution of (3.11) even if $f \in L^2(0,T;H^{-1}(\Omega))$ (see [21]).

Remark 9.5. We conjecture that the main result of this paper remains valid when $\sigma : D(\sigma) \subset \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a maximal monotone graph of $\mathbb{R}^2$ such that $\sigma(0) \ni 0$ for $x \in \Omega$, as, for instance, σ corresponds to the zero-order chemical reactions ($\alpha = 0$) or when σ represents the case of Signorini type boundary conditions. By adapting to this framework some abstract results on the Cauchy problem associated with subdifferential operators (see, e.g., [17, 18, 3] and their references) it is possible to prove that, for any $\varepsilon > 0$, problem (2.2) has a unique strong solution and the following estimate holds

$$\varepsilon^{-\gamma}\|\partial_t u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))} + \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))} + \varepsilon^{-\gamma}\|u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))} \leq K, \quad (9.17)$$

where K is a constant independent on ε. Indeed, such as shown in [17] and [3], we know that the operator $u_\varepsilon \to \partial_\nu u_\varepsilon + \sigma(u_\varepsilon)$ is not only a maximal monotone operator on $L^2(S_\varepsilon)$ (such as it can be deduced from the results of [21]) but, in fact, it is the subdifferential of the convex function is the subdifferential of the lower semicontinuous convex and proper function $\Phi_t : L^2(S_\varepsilon) \to \overline{\mathbb{R}}$,

$$\Phi_t(v_\varepsilon) = \begin{cases} \frac{\varepsilon^\gamma}{2}\int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2 dx - \varepsilon^\gamma \int_{\Omega_\varepsilon} f(x,t)u_\varepsilon(x)\,dx + \int_{S_\varepsilon} j(v_\varepsilon(\sigma))\,d\sigma \\ \quad \text{if } u_\varepsilon \in H^1(\Omega_\varepsilon,\partial\Omega), \ u_\varepsilon = v_\varepsilon, \ j(v_\varepsilon) \in L^1(S_\varepsilon), \\ +\infty \quad \text{otherwise,} \end{cases}$$

for each $t \in [0, T]$, where $\partial j(s) = \sigma(s)$ for any s in the domain of σ. Note that the t-dependence in $\Phi_t(v_\varepsilon)$ because of the presence of the term $f(x, t)$ satisfying (2.4), which obviously is independent on v_ε. Then, problem (2.2) can be rewritten as a particular case of an abstract Cauchy problem of the form

$$\frac{dv_\varepsilon}{dt} + \partial\Phi_t(v_\varepsilon) \ni g(., t) \quad \text{in } L^2(S_\varepsilon), \text{ for } t \in (0, T),$$
$$v_\varepsilon(0) = 0. \tag{9.18}$$

Then, since $v_\varepsilon(0) = 0 \in D(\Phi_t)$, a.e. $t \in (0, T)$, and $g \in L^2(0, T; L^2(S_\varepsilon))$, by [5, Théorème 3.6] (if Φ_t is time independent) or [30, Theorem 1] we know that the mild solution $v_\varepsilon \in C([0, T]; L^2(S_\varepsilon))$ of the Cauchy problem (9.18) is, in fact, a strong solution such that $\frac{dv_\varepsilon}{dt} \in L^2(0, T; L^2(S_\varepsilon))$ and that the application $t \to \Phi_t(v_\varepsilon)$ is absolutely continuous on $[0, T]$. It seems possible to generalize all the arguments used in this paper to get the conclusion of Theorem 5.1. That was carried out in [13] for the stationary case associated with (2.2) by means of an argument of regularization of σ (as to be Lipschitz continuous) and then by passing to the limit. Notice that all the arguments of the present paper were obtained directly for the case of σ non-Lipschitz continuous but satisfying merely the Hölder regularity condition (2.6). The extension to the case σ a maximal monotone graph of $\mathbb{R}^2$, including in particular the case of Signorini type boundary conditions, will be the object of a separated study.

Remark 9.6. Another possible generalization of the results of this paper concerns the consideration of nonzero initial conditions in the formulation of problem (2.2):

$$-\Delta_x u_\varepsilon = f(x, t), \quad (x, t) \in Q_\varepsilon^T,$$
$$\varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = \varepsilon^{-\gamma}g(x, t), \quad (x, t) \in S_\varepsilon^T,$$
$$u_\varepsilon(x, t) = 0, \quad (x, t) \in \Gamma^T,$$
$$u_\varepsilon(x, 0) = U_0(x), \quad x \in S_\varepsilon, \tag{9.19}$$

for some $U_0 \in H^{1/2}(S_\varepsilon)$, $U_0 \neq 0$. Such as it was pointed out in [14, Remark B.2] a new curious "strange phenomenon" arise then: $P_\varepsilon u_\varepsilon \rightharpoonup U$, weakly in $L^2(0, T; H_0^1(\Omega))$, in fact $U \in C([0, T]; H_0^1(\Omega))$, the homogenized equation is exactly of the same type than problem (3.11) if $t > 0$ but, in general, $U(x, 0) \neq U_0(x)$ since $U(x, 0)$ solves the modified problem

$$-\Delta_x U_0(x, 0) + A_0(U_0(x, 0) - U_0(x)) = f(x, 0), \quad x \in \Omega,$$
$$U_0(., 0) = 0, \quad x \in \partial\Omega, \tag{9.20}$$

for a suitable constant $A_0 > 0$. The main steps of the proof of this result (getting a "linear strange term") were indicated in [14] (see, for instance, page 12 and notice that in our case $p = 2$). To avoid additional technical details we are not developing this property here. Notice that, in fact, if $f(x, 0) \neq 0$, and $U_0 \equiv 0$ this "strange initial datum" arises since $u_0(x, 0) \neq 0$ on Ω, even if $u_\varepsilon(x, 0) = 0$ on S_ε.

Remark 9.7. We point out that the extension of the results of this paper (with particles of general shape) to the case in which the diffusion operator is replaced by a degenerate quasilinear operator, as for instance the p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, remains as an open problem. As a matter of fact, it is already an open problem for the easier case in which the boundary conditions on the boundary

of the particles is not dynamic (see Remark 3.17 and Section 4.7.4 of [14]). For the case of dynamic boundary conditions and particles given by balls see [26].

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