On the asymptotic limit of the effectiveness of reaction-diffusion equations in perforated media

J. I. Díaz[a,b,*], D. Gómez-Castro[a,b], A. V. Podolskii[c], T. A. Shaposhnikova[c]

[a]*Dpto. de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid.*
Plaza de las Ciencias 3, 28040 Madrid (Spain)
[b]*Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid*
[c]*Moscow State University,*
Faculty of Mechanics and Mathematics, Moscow, 19992 Russia

Abstract

This paper addresses an investigation of the asymptotic behaviour as $\varepsilon \to 0$ of the solution to the boundary value problem associated with the p-Laplace operator in an ε-periodically perforated domain with a nonlinear Robin-type condition specified on the boundary of the inclusions. Here we consider a non-critical size of the particles. The objective of this paper is two fold. First we study the homogenization of solutions in the case of continuous nonlinearity. Then, we move to studying the homogenization of the effectiveness factor of the reactor, which is of importance in Chemical Engineering.

Keywords: homogenization, p-Laplace diffusion, non-linear boundary reaction, non-critical sizes, effectiveness factor

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Introduction

This paper addresses an investigation of the asymptotic behaviour as $\varepsilon \to 0$ of the solution to the boundary value problem associated with the p-Laplace operator in an ε-periodically perforated domain with a nonlinear Robin-type condition specified on the boundary of the inclusions. Here we consider a non-critical size of the particles. The objective of this paper is two fold.

First, a homogenized problem is constructed and a theorem is proved stating weak convergence as $\varepsilon \to 0$ of the solution of the original problem to the solution of the homogenized. The closest papers in the literature are [31, 32] where the case $p = 2$ was considered, [17, 18, 19, 26] dedicated to the case $2 < p < n$ and [11] where the case $p > n$ was investigated. In contrast to the mentioned papers we consider here that reaction function σ need not be smooth. In order to achieve this result we introduce uniform approximation arguments, that allow us to deal with such reaction functions.

The case when the size of particles are non critical is characterized by the fact that the homogenized problem contains the same nonlinearity as the nonhomogeneous problem. However, there are critical

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cases in which the nature of the nonlinearity changes (see [9, 10, 12, 17, 18, 19, 20, 25, 29, 32]). For further reference see [18, 17, 19]. In the critical case due to the technique used were considered that inclusions are balls, whereas in this noncritical case a general shape is considered.

The second main result of the paper is the analysis of the asymptotic limit of the *effectiveness* functional (as introduce by Aris, see [2, 3]), which extends results in [13, 14] to the cases $p \neq 2$ and σ Hölder continuous.

1. Statement of results

1.1. Problem setting

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = (-\frac{1}{2}, \frac{1}{2})^n$. Let G_0 be a smooth open set such that $\overline{G}_0 \subset Y$. For $\delta > 0$ and $B \subset \mathbb{R}^n$ let $\delta B = \{x \in \mathbb{R}^n : \delta^{-1}x \in B\}$. For $\varepsilon > 0$ we define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$, where ρ is the distance function. Let $a_\varepsilon > 0$, define the set of inclusions

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$, $\mathbb{Z}^n$ is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$, and note that $\overline{G}_\varepsilon^j \subset \overline{Y}_\varepsilon^j$. Finally, we define

$$\Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Notice that the number of inclusions is of the order of ε^{-n}, in the sense that

$$\lim_{\varepsilon \to 0} \frac{|\Upsilon_\varepsilon|}{\varepsilon^{-n}} = |\Omega| \tag{1}$$

Throughout this paper we will write

$$a_\varepsilon \ll b_\varepsilon \iff \lim_{\varepsilon \to 0} a_\varepsilon b_\varepsilon^{-1} = 0$$

$$a_\varepsilon \sim b_\varepsilon \iff \lim_{\varepsilon \to 0} a_\varepsilon b_\varepsilon^{-1} \in (0, +\infty).$$

We will consider that the sizes of the particles is smaller than their repetition, in the sense that

$$a_\varepsilon \ll \varepsilon. \tag{2}$$

Sometimes, this case is known as *tiny holes* (in our case they can be though of as tiny particles). We consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon)g, & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where $\Delta_p u \equiv \text{div}(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε and σ is a continuous nondecreasing function such that $\sigma(0) = 0$, $f \in L^{p'}(\Omega)$ and $g \in W^{1,\infty}(\Omega)$.

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This problem can be obtained as a change in variable $u = 1 - w$, $\sigma(u) = \tilde{\sigma}(1) - \tilde{\sigma}(w)$ of the following problem, which appears in Chemical Engineering in the design of fixed-bed reactors (see, for example, [30])

$$\begin{cases} -\Delta_p w_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\tilde{\sigma}(w_\varepsilon) = 0, & x \in S_\varepsilon, \\ w_\varepsilon = 1, & x \in \partial\Omega. \end{cases} \tag{4}$$

A quantity of great interest in the applications is the *effectiveness*, which can be expressed as

$$\mathcal{E}_\varepsilon(\Omega, G_0) = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \tilde{\sigma}(w_\varepsilon) \, \mathrm{d}S, \tag{5}$$

in the nonhomogeneous case and as

$$\mathcal{E}(\Omega, G_0) = \frac{1}{|\Omega|} \int_\Omega \tilde{\sigma}(w) \, \mathrm{d}x, \tag{6}$$

in the homogenized case. It represents the ratio of the actual amount of reactant consumed per unit time in Ω to the amount that would be consumed if the interior concentration were everywhere equal to the ambient concentration. A high effectiveness is desirable in most applications. For isothermal and endothermic reactions, we see that $0 \leq \mathcal{E}_\varepsilon, \mathcal{E} < 1$. This definition was introduced by Aris in the linear case ($p = 2$ and $\sigma = \lambda u$, see [1, 21, 3]). The study of this functional is equivalent to the study of the *ineffectivenes*

$$\eta_\varepsilon = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u_\varepsilon) \, \mathrm{d}S, \qquad \eta = \frac{1}{|\Omega|} \int_\Omega \sigma(u) \, \mathrm{d}x. \tag{7}$$

The mathematical properties have long been studied, see [4, 5, 6, 7, 8]. The aim of this papers is to prove that $\eta_\varepsilon \to \eta$ as $\varepsilon \to 0$.

1.2. Weak formulations

Let us define the energy functional

$$J_\varepsilon(v) = \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla v|^p \, \mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \Phi(v) \, \mathrm{d}S - \int_{\Omega_\varepsilon} fv \, \mathrm{d}x - \beta(\varepsilon) \int_{S_\varepsilon} gv \, \mathrm{d}S \tag{8}$$

where $\Phi(s) = \int_0^s \sigma(\tau)d\tau$. Its subdifferential $A_\varepsilon = \partial J_\varepsilon$ is given by

$$\langle A_\varepsilon v, w \rangle = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla w \, \mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(v)w \, \mathrm{d}S - \int_{\Omega_\varepsilon} fw \, \mathrm{d}x - \beta(\varepsilon) \int_{S_\varepsilon} gw \, \mathrm{d}S. \tag{9}$$

We say that u_ε is a weak solution of (3) if $A_\varepsilon u_\varepsilon = 0$. However, $\sigma(u_\varepsilon)$ is usually not a well behaved sequence. We would rather work with an equivalent formulation that does not include it. In this direction, we have the following characterization of minimizers

Lemma 1. *Let X be a reflexive Banach space, $J : X \to (-\infty, +\infty]$ be a convex functional $A = \partial J : X \to \mathcal{P}(X')$ be its subdifferential. Then the following are equivalent:*

i) *u is a minimizer of J,*

ii) $u \in D(A)$ *and* $0 \in Au$.

If either holds, then

iii) *For every* $v \in D(A)$ *and* $\xi \in Av$

$$\langle \xi, v - u \rangle \geq 0. \tag{10}$$

Furthermore, assume that J is Gateaux-differentiable on X and A is continuous on X then iii) *is also equivalent to* i).

Remark 1. Naturally, if there is uniqueness of iii) then the i)-iii) are also equivalent.

Remark 2. One should not confuse condition iii) with the Stampacchia formulation (see e.g. [6]). For a bilinear form a and a linear function F this function is

$$a(u, v - u) \geq G(v - u) \tag{11}$$

for all v in the correspondent space, whereas with this formulation we have $a(v, v - u)$. The advantage of the representation we consider is that one of the elements can be taken constant as $\varepsilon \to 0$.

We will say that u_ε is a weak solution of (3) if it is a minimizer of J_ε in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

Proposition 1 ([26])**.** *Let $p > 1$. Then there exists an extension operator*

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W_0^{1,p}(\Omega) \tag{12}$$

Furthermore, there exists a constant C independent of ε such that

$$\|\nabla P_\varepsilon u_\varepsilon\|_{L^p(\Omega)} \leq C \|\nabla u_\varepsilon\|_{L^p(\Omega)}. \tag{13}$$

Hence, there exists a subsequence of the original sequence $P_\varepsilon u_\varepsilon$ that admits a weak $W_0^{1,p}(\Omega)$ limit, which we will define as u. The aim of this paper is to characterize u.

1.3. Homogenization of solutions for $1 < p < n$

We state two approximation lemmas, which are key to our arguments.

Lemma 2. *Let $\sigma \in \mathcal{C}(\mathbb{R})$ be nondecreasing such that $\sigma(0) = 0$. Then there exists $\sigma_\varepsilon \in C^1(\mathbb{R})$ non decreasing such that $\sigma_\varepsilon(0) = 0$ and $\|\sigma - \sigma_\varepsilon\| \leq \varepsilon$.*

Let us define the critical values of a_ε and β, for $1 < p < n$

$$a_\varepsilon^* = \varepsilon^{\frac{n}{n-p}}, \qquad \beta^*(\varepsilon) = a_\varepsilon^{-(n-1)} \varepsilon^n, \tag{14}$$

which separates different asymptotic behaviours of the solution. We focus on the cases $a_\varepsilon \gg a_\varepsilon^*$, since the critical case is $a_\varepsilon \sim a_\varepsilon^*$. The value β^* separates the behaviours as shown by the following theorem. In fact, let us define

$$\beta_0 = |\partial G_0| \lim_{\varepsilon \to 0} \beta(\varepsilon)\beta^*(\varepsilon)^{-1}. \tag{15}$$

Theorem 1. *Let $1 < p < n$, $g \in W^{1,\infty}(\Omega)$, $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$, $\sigma \in \mathcal{C}(\mathbb{R})$ nondecreasing such that $\sigma(0) = 0$ and*

$$|\sigma(v)| \leq C(1 + |u|^{p-1}). \tag{16}$$

Then the following results hold:

i) *Let $\beta_0 < +\infty$. Then, up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, where u is the unique solution of*

$$\begin{cases} -\Delta_p u + \beta_0 \sigma(u) = f + \beta_0 g & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{17}$$

ii) *Let $\beta_0 = +\infty$, $g = 0$ and $\sigma \in \mathcal{C}^1$. Then, up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and u satisfies*

$$u(x) \in \sigma^{-1}(0) \tag{18}$$

a.e. in Ω.

Remark 3. In particular, if $\beta_0 = 0$ then the limit problem does not contain any reaction term. If $a_\varepsilon = C_0 \varepsilon^\alpha$ and $\beta(\varepsilon) = \varepsilon^{-\gamma}$ we have

$$\alpha \in \left(1, \frac{n}{n-p}\right)$$

$$\beta_0 = \begin{cases} 0 & \gamma < \alpha(n-p) - n, \\ C_0^{n-1}|\partial G_0| & \gamma = \alpha(n-p) - n, \\ +\infty & \gamma > \alpha(n-p) - n. \end{cases}$$

Remark 4. The same result holds for $p = n$, where the condition on the size a_ε is

$$\varepsilon^{\frac{n}{n-1}} \ln(a_\varepsilon^{-1}\varepsilon) \to 0, \qquad \text{as } \varepsilon \to 0, \tag{19}$$

(see [27]) and for $p > n$, where critical size of inclusions doesn't exist so there is no condition on a_ε^* (see [11]). We can write the critical size for any $p > 1$ as:

$$a_\varepsilon^* = \begin{cases} \varepsilon^{\frac{n}{n-p}} & \text{if } 1 < p < n, \\ \varepsilon e^{-\left(\frac{1}{\varepsilon}\right)^{1-\frac{1}{n}}} & \text{if } p = n, \\ 0 & \text{if } p > n. \end{cases} \tag{20}$$

The value of β^* is still

$$\beta^*(\varepsilon) = a_\varepsilon^{-(n-1)}\varepsilon^n. \tag{21}$$

.

1.4. Homogenization of the effectiveness factor when $p > 1$

We conclude by stating a theorem on homogenization of the *effectiveness* functional that improves previous results by the authors (see [13, 14]). We give conditions so that

$$\frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u_\varepsilon) \, \mathrm{d}S \to \frac{1}{|\Omega|} \int_\Omega \sigma(u) \, \mathrm{d}x \qquad \text{as } \varepsilon \to 0. \tag{22}$$

To achieve this we need a stronger approximation result for the family of Hölder-continuous functions.

Remark 5. If I is a bounded interval then $\mathcal{C}(I) \subset \mathcal{C}^{0,\alpha}(I)$. This is not true if I is unbounded. For example, all functions in $\mathcal{C}^{0,\alpha}(\mathbb{R})$ are sublinear. We introduce the following condition

$$|\sigma(t) - \sigma(s)| \le C(|t - s|^\alpha + |t - s|^p) \qquad \forall t, s \in \mathbb{R}, \tag{23}$$

that represents "local Hölder" continuity, in the sense that there is no need for the function to be differentiable. On the other hand, as $|s - t| \to +\infty$, the function σ behaves like a power, and then σ can be a non sublinear.

Lemma 3. Let $\sigma \in \mathcal{C}(\mathbb{R})$, nondecreasing and there exists $0 < \alpha \le 1$, $p > 1$ such that (23) holds. Then, for every $0 < \varepsilon < \frac{1}{4C}$ there exists $\sigma_\varepsilon \in \mathcal{C}(\mathbb{R})$ (piecewise linear) such that

$$\|\sigma_\varepsilon - \sigma\|_{\mathcal{C}(\mathbb{R})} \le \varepsilon, \tag{24}$$

$$0 \le \sigma_\varepsilon' \le D\varepsilon^{1 - \frac{1}{\alpha}}, \tag{25}$$

where D depends only on the C, α, p.

Theorem 2. Let $p > 1$, $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$, $\beta \sim \beta^*$ and σ be continuous such that $\sigma(0) = 0$. Let u_ε and u be the solutions of (3) and (17). Lastly, assume either:

i) σ is uniformly Lipschitz continuous ($\sigma' \in L^\infty$), or

ii) $\sigma \in \mathcal{C}(\mathbb{R})$ and there exists $0 < \alpha \le 1$ and $q > 1$ such that we have (23) and

$$(\sigma(t) - \sigma(s))(t - s) \ge C|t - s|^q, \qquad \forall t, s \in \mathbb{R}. \tag{26}$$

Then (22) *holds.*

Remark 6. Even though roots $\sigma(s) = |s|^{q-1}s$ do not satisfy (26), but a continuous linear cutoff

$$\sigma(s) = \begin{cases} |s|^{q-1}s & |s| \le s_0, \\ \sigma_0 + \lambda s & |s| > s_0, \end{cases} \tag{27}$$

does satisfy this kind of behaviour. Hence, the result for $\sigma(s) = |s|^{q-1}s$ where $q < 1$ holds, at least for uniformly bounded solutions. This must hold, for example, in Chemical Engineering since u typically represents a concentration, so $0 \le u \le 1$.

2. Auxiliary results and estimates

2.1. Estimates on the boundary integrals

First let us introduce a uniform trace information in S_ε

Proposition 2. *Let $p > 1$ and assume (2). Then*

i) *There exists C, independent of ε, such that, for $u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, it holds that*

$$\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^p \, \mathrm{d}S \leq C \int_{\Omega_\varepsilon} |\nabla u|^p \, \mathrm{d}x. \tag{28}$$

ii) *If $v_\varepsilon \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ and $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$. Then*

$$\beta^*(\varepsilon) \int_{S_\varepsilon} v_\varepsilon \, \mathrm{d}S \to |\partial G_0| \int_\Omega v \, \mathrm{d}x. \tag{29}$$

Remark 7. Notice that the natural trace in S_ε is not well behaved with respect to $\int_{S_\varepsilon} \cdot \, \mathrm{d}S$, but rather with $\frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \cdot \, \mathrm{d}S$

Lemma 4. *Let $0 < r < s$. Then, there exists C, independent of ε, such that*

$$\left(\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^r \, \mathrm{d}S \right)^{\frac{1}{r}} \leq C \left(\beta^* \int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{1}{s}}. \tag{30}$$

Proof. Let $q = \frac{s}{r} > 1$. Then $q' = \frac{s}{s-r}$. Applying Hölder's inequality we find that

$$\int_{S_\varepsilon} |u|^r \, \mathrm{d}S \leq C \left(\int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{r}{s}} \left(\int_{S_\varepsilon} 1^{\frac{s}{s-r}} \, \mathrm{d}S \right)^{\frac{s-r}{s}}$$

$$\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^r \, \mathrm{d}S \leq C\beta^*(\varepsilon)^{\frac{1}{q}+\frac{1}{q'}} \left(\int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{r}{s}} |S_\varepsilon|^{\frac{s-r}{s}}$$

$$\leq C\beta^*(\varepsilon)^{\frac{1}{q}+\frac{1}{q'}} \left(\int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{r}{s}} |S_\varepsilon|^{\frac{s-r}{s}}$$

$$\leq C \left(\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{r}{s}} (\beta^*(\varepsilon)|S_\varepsilon|)^{\frac{s-r}{s}}$$

$$\leq C \left(\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{r}{s}},$$

which concludes the result. $\qquad\square$

With this results we can proof that

Proposition 3. *Let $p > 1$. Then, for every $\varepsilon > 0$ there exists a unique weak solution of (3) $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Furthermore, there exists a constant C independent of ε such that*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{p-1} \leq C(\|f\|_{L^{p'}(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|g\|_{L^\infty(\mathbb{R})}). \tag{31}$$

2.2. Characterization of solutions

The proof of the *furthermore* statement can be found in [16]. In fact we state the the following characterization, which could improve the regularity required, but that we do not apply due to the homogenization techniques applied.

Lemma 5 (Proposition 2.2 in [16])**.** *Let us assume that $J = J_1 + J_2$ and J_1 and J_2 being l.s.c. convex functions on a convex set $\mathcal{C}$ into $\mathbb{R}$, J_1 being Gateaux-differentiable with differential J_1'. Then $u \in \mathcal{C}$, the following three conditions are equivalent to each other:*

i) *u is a minimizer of J,*

ii) *For every $v \in \mathcal{C}$*

$$\langle J_1'(u), v - u \rangle + J_2(v) - J_2(u) \geq 0, \tag{32}$$

iii) *For every $v \in \mathcal{C}$*

$$\langle J_1'(v), v - u \rangle + J_2(v) - J_2(u) \geq 0. \tag{33}$$

We have the following lemma

Lemma 6. *Let $1 < p < +\infty$ and σ be a nondecreasing function. Then if*

$$X = W^{1,p}(\Omega_\varepsilon, \partial\Omega) \qquad \mathcal{C} = \{v \in X : \Phi(v) \in L^1(S_\varepsilon)\}$$

$$J(v) = E_\varepsilon(v) \qquad Av = A_\varepsilon v$$

$$J_1(v) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla v|^p \, \mathrm{d}x - \int_{\Omega_\varepsilon} fv \, \mathrm{d}x - \beta(\varepsilon)\int_{S_\varepsilon} gv \, \mathrm{d}S$$

$$J_1'(v)(w) = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla w \, \mathrm{d}x - \int_{\Omega_\varepsilon} fw \, \mathrm{d}x - \beta(\varepsilon)\int_{S_\varepsilon} gw \, \mathrm{d}S$$

$$J_2(v) = \beta(\varepsilon)\int_{S_\varepsilon} \Phi(v) \, \mathrm{d}S$$

$$\langle J_2'(v), w \rangle = \beta(\varepsilon)\int_{S_\varepsilon} \sigma(v)w \, \mathrm{d}S,$$

or

$$X = W^{1,p}(\Omega), \qquad \mathcal{C} = \{v \in X : \Phi(v) \in L^1(\Omega)\}$$

$$J = J_1 + J_2$$

$$J_1(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x - \beta_0\int_\Omega gv \, \mathrm{d}x$$

$$\langle J_1'(v), w \rangle = \frac{1}{p}\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla w \, \mathrm{d}x - \int_\Omega fw \, \mathrm{d}x - \beta(\varepsilon)\int_{S_\varepsilon} gw \, \mathrm{d}x$$

$$J_2(v) = \beta_0\int_\Omega \Phi(v) \, \mathrm{d}x$$

$$J_2'(v)(w) = \beta(\varepsilon)\beta_0\int_\Omega \sigma(v)w \, \mathrm{d}x,$$

we have that $J_1, J_2 : \mathcal{C} \to \mathbb{R}$, J_1 are convex, J_1 is Gateaux differentiable. Furthermore, if (16) holds then $\mathcal{C} = X$, J_2 is Gateaux-differentiable in X and J' is continuous on X.

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Remark 8. The *furthermore* part was first proved in [15]. Condition (16) is given by the fact that, $v \mapsto G(v)$ is $L^r(\Omega) \to L^t(\Omega)$ is continuous if $|G| \leq C(1 + |v|^{\frac{r}{t}})$. Notice that, for $r = p$ and $t = p'$ we have $\frac{r}{t} = p - 1$. In this case, J satisfies the continuity condition for $L^p \to L^1$, which is enough to make J continuous. It is likely that (16) is purely a technical requirement so that iii) implies i).

2.3. On the coercivity of the p-Laplacian, when $1 < p < 2$

We will need the following auxiliary lemma, that deals with the coercivity of the p-Laplace operator:

Lemma 7. *Let $1 < p < 2$ and $u, v \in W^{1,p}(\Omega)$. Then*

$$
\int_\Omega |\nabla(u - v)|^p \, \mathrm{d}x \leq C \left(\int_\Omega \frac{|\nabla(u - v)|^2}{|\nabla u|^{2-p} + |\nabla v|^{2-p}} \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{p}}
$$

$$
\leq C \left(\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u - v) \, \mathrm{d}x \right)^{\frac{p}{2}}
$$

$$
\times \left(\int_\Omega (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{p}}. \tag{34}
$$

Proof. The first inequality is a direct consequence of the Hölder inequality

$$
\left(\int_\Omega \frac{|\nabla(u - v)|^p}{(|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2}}} \left(|\nabla u|^{2-p} + |\nabla v|^{2-p}\right)^{\frac{p}{2}} \, \mathrm{d}x \right)
$$

$$
\leq \left(\int_\Omega \frac{|\nabla(u - v)|^2}{|\nabla u|^{2-p} + |\nabla v|^{2-p}} \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{p}},
$$

and the second one is due to the estimate for vectors, $\xi, \eta \in \mathbb{R}^n$, not both zero:

$$
\frac{|\xi - \eta|^2}{|\xi|^{2-p} + |\eta|^{2-p}} \leq C \left(|\eta|^{p-2}\eta - |\xi|^{p-2}\xi \right) \cdot (\eta - \xi),
$$

this concludes the proof. □

2.4. Comparison of solutions with different kinetics

We have the following comparison lemma for the solutions:

Lemma 8. *Let $\sigma, \hat\sigma$ be continuous functions, σ satisfies (26) for some $q > 1$ and let u_ε and $\hat u_\varepsilon$ be the corresponding solutions of (3) with $\beta \sim \beta^*$. Then*

$$
\beta(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - \hat u_\varepsilon|^q \, \mathrm{d}s \leq C \|\sigma - \hat\sigma\|_{C(\mathbb{R})}^{\frac{q}{q-1}}. \tag{35}
$$

Proof. We use $u - \hat{u}$ as a test function, and via the monotonicity of σ we have

$$\beta(\varepsilon) \int_{S_\varepsilon} (\sigma(u) - \sigma(\hat{u}))(u - \hat{u}) \, \mathrm{d}S \leq \int_{\Omega_\varepsilon} |\nabla(u - \hat{u})|^p \, \mathrm{d}S + \beta(\varepsilon) \int_{S_\varepsilon} (\sigma(u) - \sigma(\hat{u}))(u - \hat{u}) \, \mathrm{d}S$$

$$\leq \beta(\varepsilon) \int_{S_\varepsilon} (\hat{\sigma}(\hat{u}) - \sigma(\hat{u}))(u - \hat{u}) \, \mathrm{d}S$$

$$\leq \|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})} \beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}| \, \mathrm{d}S$$

$$\leq C\|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})} \left(\beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}|^q \, \mathrm{d}S \right)^{\frac{1}{q}}.$$

Due to (26) we have that

$$\beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}|^q \, \mathrm{d}S \leq C\|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})} \left(\beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}|^q \, \mathrm{d}S \right)^{\frac{1}{q}}$$

$$\left(\beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}|^q \, \mathrm{d}S \right)^{1 - \frac{1}{q}} \leq C\|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})},$$

which concludes the result. $\square$

Lemma 9. *Let $\sigma, \hat{\sigma}$ be continuous nondecreasing functions such that $\sigma(0) = 0$ and $u, \hat{u}$ be their respective solutions of (3). Then, there exists constants C depending on p, but independent of ε, such that*

i) *If $1 < p < 2$*

$$\|\nabla(u_\varepsilon - \hat{u}_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})} \left(\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{2-p} + \|\nabla \hat{u}_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{2-p} \right)^{\frac{2}{p}}. \tag{36}$$

ii) *If $p \geq 2$ then*

$$\|\nabla(u_\varepsilon - \hat{u}_\varepsilon)\|_{L^p(\Omega_\varepsilon)}^{p-1} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})}. \tag{37}$$

Proof. By considering the difference of weak formulations we can write, for the test function $u_2 - u_1$,

$$\int_\Omega (|\nabla u_2|^{p-2}\nabla u_2 - |\nabla u_1|^{p-2}\nabla u_1) \cdot \nabla(u_2 - u_1) \, \mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} (\sigma_2(u_2) - \sigma_2(u_1))(u_2 - u_1) \, \mathrm{d}S$$

$$= \beta(\varepsilon) \int_{S_\varepsilon} (\sigma_1(u_1) - \sigma_2(u_1))(u_2 - u_1) \, \mathrm{d}S.$$

Applying monotonicity, Proposition 2 and Lemma 4

$$\int_\Omega (|\nabla u_2|^{p-2}\nabla u_2 - |\nabla u_1|^{p-2}\nabla u_1) \cdot \nabla(u_2 - u_1) \, \mathrm{d}x$$

$$\leq \beta(\varepsilon)\|\sigma_2 - \sigma_1\|_\infty \beta^*(\varepsilon)^{-1} \left(\beta^*(\varepsilon) \int_{S_\varepsilon} |u_1 - u_2|^p \, \mathrm{d}S \right)^{\frac{1}{p}}$$

$$\leq C\beta(\varepsilon)\|\sigma_2 - \sigma_1\|_\infty \beta^*(\varepsilon)^{-1}\|\nabla(u_1 - u_2)\|_{L^p(\Omega_\varepsilon)},$$

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Part ii) follows directly. Let us prove part i). Applying Lemma 7 we have that

$$\|\nabla(u_1 - u_2)\|_{L^p}^p \leq C \left(\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_2 - \sigma_1\|_\infty\|\nabla(u_1 - u_2)\|_{L^p}\right)^{\frac{p}{2}}$$
$$\times \left(\int_\Omega (|\nabla u_1|^{2-p} + |\nabla u_2|^{2-p})^{\frac{p}{2-p}} \, dx\right)^{\frac{2-p}{p}}$$

$$\|\nabla(u_1 - u_2)\|_{L^p} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_2 - \sigma_1\|_\infty \left(\int_\Omega (|\nabla u_1|^{2-p} + |\nabla u_2|^{2-p})^{\frac{p}{2-p}} \, dx\right)^{\frac{2}{p}\frac{2-p}{p}}$$

$$\leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_2 - \sigma_1\|_\infty \left(\|\nabla u_1\|_{L^p}^{2-p} + \|\nabla u_2\|_{L^p}^{2-p}\right)^{\frac{2}{p}},$$

which proves the result. □

2.5. Proof of the approximation lemmas

There is extensive literature on the approximation of functions in bounded intervals, in particular approximation that preserve the monotonicity. For example, it is known that Bernstein polynomials of a monotone function are also monotone, and the convolution with a positive kernel also preserves global monotonicity. Finer results are known as to the approximation of function which are piecewise monotone by functions that share their monotonicity (i.e. *comonotone* functions. In this direction see, e.g. [22, 23, 24, 28]).

One of the canonical options in this directions is the Yosida approximation, but, in general this only converges pointwise. This is natural, since one can approximate a discontinues function, which is Lipschitz continuous, and therefore the limit cannot be uniform. We choose, locally, a convolution with mollifiers.

Proof of Lemma 2. Let $\sigma_{\varepsilon,0} \in \mathcal{C}^1([-1,1])$ be an approximation of σ such that

$$\begin{cases} \sigma_{\varepsilon,0} = \sigma \text{ in } \{-1,0,1\}, \\ \|\sigma_{\varepsilon,0} - \sigma\|_{\mathcal{C}([-1,1])} \leq \varepsilon \\ \sigma_{\varepsilon,0} \text{ is nondecreasing.} \end{cases}$$

This can be done, since, for example, the convolution of σ with nonnegative mollifiers are nondecreasing. Let $\sigma_{\varepsilon,1} \in \mathcal{C}^1([1,2])$ be an approximation of σ in $[1,2]$ such that

$$\begin{cases} \sigma_{\varepsilon,1} = \sigma \text{ in } \{1,2\}, \\ \sigma'_{\varepsilon,1}(1) = \sigma'_{\varepsilon,0}(1), \\ \|\sigma_{\varepsilon,1} - \sigma\|_{\mathcal{C}([1,2])} \leq \varepsilon, \\ \sigma_{\varepsilon,1} \text{ is nondecreasing.} \end{cases}$$

We proceed analogously in $[n, n+1], [-(n+1), -n]$ for $n \in \mathbb{N}$. We finally construct $\sigma_\varepsilon \in \mathcal{C}^1(\mathbb{R})$ by matching the pieces. □

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Proof of Lemma 3. Let $\varepsilon < \frac{1}{4C}$ and $\delta = \left(\frac{\varepsilon}{4C}\right)^{\frac{1}{\alpha}} < 1$. If $|x - y| \le \delta$ then

$$|\sigma(x) - \sigma(y)| \le D(|x-y|^\alpha + |x-y|^p) \le D(\delta^\alpha + \delta^p)$$

$$\le 2D\delta^\alpha = \varepsilon.$$

We define

$$\sigma_\varepsilon(n\delta) = \sigma(n\delta), \qquad n \in \mathbb{Z} \tag{38}$$

and linear in $(n, n+1)$. Since σ is nondecreasing so is σ_ε. For $x \in [\delta(n-1), \delta n]$ we have

$$|\sigma(x) - \sigma_\varepsilon(x)| \le |\sigma(x) - \sigma(\delta n)| + |\sigma(\delta n) - \sigma_\varepsilon(x)|$$

$$\le \frac{\varepsilon}{2} + |\sigma_\varepsilon(\delta n) - \sigma_\varepsilon(x)|$$

$$\le \frac{\varepsilon}{2} + \big(\sigma_\varepsilon(\delta n) - \sigma_\varepsilon(\delta(n-1))\big)$$

$$= \frac{\varepsilon}{2} + \big(\sigma(\delta n) - \sigma(\delta(n-1))\big)$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

On the other hand, for $x \in (\delta(n-1), \delta n)$ we have

$$0 \le \sigma_\varepsilon'(x) = \frac{\sigma(\delta n) - \sigma(\delta(n-1))}{\delta} \le C(\delta^{\alpha-1} + \delta^{p-1}) \le D\varepsilon^{1-\frac{1}{\alpha}},$$

which concludes the result. $\qquad\square$

3. Proof of Theorem 1

Proof of Theorem 1. We rewrite the problem, due to Lemma 1 as

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\, \mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(v)(v - u_\varepsilon)\, \mathrm{d}S$$

$$\ge \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\, \mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g(v - u_\varepsilon)\, \mathrm{d}S \qquad \forall v \in W_0^{1,p}(\Omega). \tag{39}$$

Let us start by considering $\sigma \in \mathcal{C}^1(\mathbb{R})$. If either $\beta_0 = +\infty$ and $g = 0$ or $\beta_0 < +\infty$, we can apply Proposition 3 to show that $P_\varepsilon u_\varepsilon$ are uniformly bounded in $W_0^{1,p}(\Omega)$, and therefore there exists $u \in W_0^{1,p}(\Omega)$ and a subsequence of $P_\varepsilon u_\varepsilon$ (denoted as the original sequence) such that

$$P_\varepsilon u_\varepsilon \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0. \tag{40}$$

Then it is known that (see [26, 29, 32]), for $v \in W_0^{1,\infty}(\Omega)$ we have

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\, \mathrm{d}x \to \int_\Omega |\nabla v|^{p-2}v \cdot \nabla(v - u)\, \mathrm{d}x$$

$$\int_{\Omega_\varepsilon} f(v - u_\varepsilon)\, \mathrm{d}x \to \int_\Omega f(v - u)\, \mathrm{d}x$$

$$\beta^*(\varepsilon)\int_{S_\varepsilon} \sigma(v)(v - u_\varepsilon)\, \mathrm{d}S \to |\partial G_0|\int_\Omega \sigma(v)(v - u)\, \mathrm{d}x$$

$$\beta^*(\varepsilon)\int_{S_\varepsilon} g(v - u_\varepsilon)\, \mathrm{d}S \to |\partial G_0|\int_\Omega g(v - u_\varepsilon)\, \mathrm{d}x,$$

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as $\varepsilon \to 0$. If $\beta_0 < +\infty$ we can pass to the limit in (39) as $\varepsilon \to 0$ and obtain

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)\,\mathrm{d}x + \beta_0 \int_\Omega \sigma(v)(v-u)\,\mathrm{d}x$$

$$\geq \int_\Omega f(v-u)\,\mathrm{d}x + \beta_0 \int_\Omega g(v-u)\,\mathrm{d}x \qquad \forall v \in W_0^{1,\infty}(\Omega). \tag{41}$$

Applying density, Lemma 6 and Lemma 1 this is equivalent to u being a solution of (17).

If $\beta_0 = +\infty$ and $g = 0$ then we write (39) as

$$\beta^*(\varepsilon)\beta(\varepsilon)^{-1}\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u_\varepsilon)\,\mathrm{d}x + \beta^*(\varepsilon)\int_{S_\varepsilon} \sigma(v)(v-u_\varepsilon)\,\mathrm{d}S$$

$$\geq \beta^*(\varepsilon)\beta(\varepsilon)^{-1}\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x.$$

By passing to the limit we obtain

$$\int_\Omega \sigma(v)(v-u_\varepsilon)\,\mathrm{d}x \geq 0, \quad v \in W_0^{1,\infty}(\Omega).$$

Again, applying Lemma 1 we deduce the result.

Let $\sigma \in \mathcal{C}(\mathbb{R})$ and $\beta_0 < +\infty$. By Lemma 2 there exists nondecreasing functions $(\sigma_m) \subset \mathcal{C}^1(\mathbb{R})$ such that $\sigma_m(0) = 0$, $\sigma_m \to \sigma$ in $\mathcal{C}(\mathbb{R})$. Let $u_{\varepsilon,m}$ and u_m be the solutions of (3) and (17) with kinetic σ_m, which by the previous proof satisfy

$$P_\varepsilon u_{\varepsilon,m} \rightharpoonup u_m \qquad \text{in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0. \tag{42}$$

Applying Lemma 9 we have that

$$\|\nabla(u_\varepsilon - u_{m,\varepsilon})\|_{L^p(\Omega)} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} \qquad \text{if } 1 < p < 2,$$

$$\|\nabla(u_\varepsilon - u_{m,\varepsilon})\|_{L^p(\Omega)}^{p-1} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} \qquad \text{if } 2 \leq p < n.$$

Passing to the limit as $\varepsilon \to 0$ in this estimations we get

$$\|\nabla(u - u_m)\|_{L^p(\Omega)} \leq C\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} \qquad \text{if } 1 < p < 2,$$

$$\|\nabla(u - u_m)\|_{L^p(\Omega)}^{p-1} \leq C\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} \qquad \text{if } 2 \leq p < n.$$

By uniform boundedness there exists $\hat{u} \in W_0^{1,p}(\Omega)$ such that $u_m \rightharpoonup \hat{u}$ in $W^{1,p}(\Omega)$ as $m \to +\infty$. By continuity of the equation with respect to the kinetic we know that $\hat{u}$ is the solution of (17). From the previous estimate we have that $u = \hat{u}$, which concludes the proof. $\square$

Remark 9. Notice that condition (16) is only used to show that (41) implies that u is a solution of (17). However, if we show that (41) has a unique solution then condition (16) can be removed. Also, if u is bounded then this condition can also be removed.

4. Proof of Theorem 2

Proof of Theorem 2. Applying the results of this paper for the case $1 < p < n$, which extend naturally to $p = n$ (see Remark 4) and [11] for the case $p > n$ we have that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$.

First, let us suppose that $\sigma' \in L^\infty$. Then $\sigma(u_\varepsilon)$ is bounded in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Hence, it is easy to show that $P_\varepsilon \sigma(u_\varepsilon) \rightharpoonup \sigma(u)$ in $W_0^{1,p}(\Omega)$. We have

$$|S_\varepsilon| = |\Upsilon_\varepsilon||a_\varepsilon \partial G_0| = a_\varepsilon^{n-1}|\Upsilon_\varepsilon||\partial G_0| \tag{43}$$

Taking into account (1) we get

$$\frac{|S_\varepsilon|}{\beta^*(\varepsilon)|\Omega||\partial G_0|} \to 1. \tag{44}$$

Hence, applying Proposition 2 we have the result for σ uniformly Lipschitz.

Let $\sigma \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ such that (23), (26) are satisfied. According to Lemma 3 there exist a sequence of nondecreasing functions $(\sigma_m) \subset \mathcal{C}(\mathbb{R})$ such that $\sigma_m' \in L^\infty$ and $\sigma_m \to \sigma$ in $\mathcal{C}(\mathbb{R})$.

Let $u_{\varepsilon,m}$ be the corresponding solution of (3) with kinetic σ_m. Then we have

$$\left| \beta(\varepsilon) \int_{S_\varepsilon} \sigma(u) \, \mathrm{d}S - \beta(\varepsilon) \int_{S_\varepsilon} \sigma_m(u_{\varepsilon,m}) \, \mathrm{d}S \right| \le \beta(\varepsilon) \int_{S_\varepsilon} |\sigma(u_\varepsilon) - \sigma_m(u_\varepsilon)| \, \mathrm{d}S$$

$$\le \beta(\varepsilon) \int_{S_\varepsilon} |\sigma(u_\varepsilon) - \sigma(u_{\varepsilon,m})| \, \mathrm{d}S + \beta(\varepsilon) \int_{S_\varepsilon} |\sigma(u_{\varepsilon,m}) - \sigma_m(u_{\varepsilon,m})| \, \mathrm{d}S$$

$$\le C\beta(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - u_{\varepsilon,m}|^\alpha \, \mathrm{d}S + \beta(\varepsilon)|S_\varepsilon| \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})}$$

$$\le C \left(\beta(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - u_{\varepsilon,m}|^q \, \mathrm{d}S \right)^{\frac{\alpha}{q}} + \beta(\varepsilon)|S_\varepsilon| \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})}$$

$$\le C \left(\|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})}^{\frac{\alpha}{q-1}} + \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})} \right).$$

In particular, taking any $m \in \mathbb{Z}$ we show that up to a subsequence following convergence holds

$$\eta_\varepsilon = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u) \, \mathrm{d}S \to \eta_0 \qquad \text{as } \varepsilon \to 0.$$

Applying the first part of the proof, we have that

$$\left| \eta_0 - \frac{1}{|\Omega|} \int_\Omega \sigma_m(u_m) \, \mathrm{d}x \right| \le C \left(\|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})}^{\frac{\alpha}{q-1}} + \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})} \right).$$

Due to Lemma 9 we have that, as $m \to +\infty$, $u_m \to u$ in $L^p(\Omega)$. Also, due (23) we have that $\sigma(u_m) \to \sigma(u)$ in $L^1(\Omega)$. Hence

$$\|\sigma_m(u_m) - \sigma(u)\|_{L^1(\Omega)} \le \|\sigma_m(u_m) - \sigma(u_m)\|_{L^1(\Omega)} + \|\sigma(u_m) - \sigma(u)\|_{L^1(\Omega)}$$

$$\le \|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} + \|\sigma(u_m) - \sigma(u)\|_{L^1(\Omega)}.$$

Therefore, $\sigma_m(u_m) \to \sigma(u)$ in $L^1(\Omega)$. Hence,

$$\eta_0 = \frac{1}{|\Omega|} \int_\Omega \sigma(u) \, \mathrm{d}x.$$

Since every convergent subsequence of (η_ε) has the same limit η_0 we conclude the proof. $\qquad \square$

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