The Uniform Hopf Inequality for discontinuous coefficients and optimal regularity in bmo for singular problems

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Abstract

We consider some singular second order semilinear problems which includes, among many other special cases, the boundary layer equations such as they were treated by O.A. Oleinik in her pioneering works. We consider diffusion linear operator with possible discontinuous coefficients and prove an optimal criterion to get a quantitative strong maximum principle what we call as Uniform Hopf Inequality UHI. Since the solutions of the singular semilinear problems under consideration are not Lipschtiz continuous we carry out a careful study of the regularity of solutions when the coefficients of the diffusion matrix are merely in the vmo space and bounded. We prove that the gradient of the solution is still p-integrable, in absence of any continuity assumption on the spatial potential coefficient in the singular term. To this end, the UHI property is used several times. We also apply and improve previous a priori estimates due to S. Campanato in 1965.

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1 Introduction

The paper deals with the study of the singular semilinear problem

$$(\mathcal{SP}) = \begin{cases} Lu = \dfrac{a(x)}{u^m} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a regular open domain set of $\mathbb{R}^n$, $a(x)$ satisfies, $m > 0$

$$a \in L^\infty(\Omega),\ a \geqslant 0,\ a \neq 0, \tag{1.1}$$

and L denotes an elliptic second order operator with non-necessarily $C^1(\overline{\Omega})$ coefficients. We shall pay attention meanly to the "case of pure diffusion" in which

$$Lu = -\operatorname{div}(A(x)\nabla u) \tag{1.2}$$

with the coefficients of the "coercive" matrix $A(x) = (a_{ij}(x))_{ij=1}^n$ such that

$$(H_1) \qquad \begin{cases} a_{ij} \in L^\infty(\Omega) \quad \forall\, i,j = 1,\ldots,n \\ \exists\,\alpha > 0, \forall\,\xi = (\xi_1,\ldots\xi_n) \in \mathbb{R}^n,\ \displaystyle\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \quad \text{a.e. in } \Omega. \end{cases}$$

Problem ($\mathcal{S}P$) arises in many applied contexts, as for instance, in the study of the boundary layer in Fluid Mechanics. One of the pioneering works of this type of equations was the 1977 paper by M.G. Crandall, PH. Rabinowitz and L.Tartar [12] in which it was mentioned as applied motivations the case of suitable chemical kinetics (Fulks and Maybee [17]) and the consideration of some similar problems in the context of signal transmission ([34] and [29]). But there are many other contexts in which problem ($\mathcal{SP}$) turn to be relevant. To indicate some other different motivation, we mention here the study of the “boundary layer” flow past a flat plate such as it was considered by H. Blasius [2], in 1908, after the seminal study by L. Prandlt [31] in 1905. It is well-known that the so called “Crocco transformations” [13] lead the problem to the consideration of a suitable class of singular parabolic equations which by different arguments are reduced to the study of equations of the type

$$(P_w)\begin{cases} -w_{\eta\eta}+B(\eta)w_\eta+C(\eta)w=\dfrac{a(\eta)}{w} & in\ (0,1)\\ w(1)=0\\ w'(0)=c_0, \end{cases}$$

for suitable coefficients B and C, for some a satisfying (1.1) and $C_0 \in \mathbb{R}$ (see e.g. the expositions made in the monograph Oleinik and Samoklin [36] or the paper Vajravelu et al. [43]).

In spite of the great relevance of the study of the boundary layer in many problems of engineering, meteorology, oceanography etc., the intensive mathematical treatment was only successive after a series of papers by O.A. Oleinik starting in 1952 (see again the exposition made in Oleinik and Samoklin [36]). In her 1968 paper ([35], see also [27]) she derived the a priori estimate

$$\underline{C}(1-\eta)\sqrt{-Ln\mu(1-\eta)} \leqslant w(\eta) \leqslant \overline{C}(1-\eta)\sqrt{-Ln\mu(1-\eta)} \quad \text{for any } \eta\in(0,1)$$

for suitable positive constants $\underline{C},\ \overline{C},\ \mu$ which shows that

$$w \notin C^{0,1}([0,1]).$$

Since the solution of the problem $(\mathcal{SP})$ is not Lipschitz continuous, it is important to show that it's gradient is in some functional space smaller than $L^2(\Omega)$. A related boundary estimate was proved in Gui and Hua Lin [23] and in [27] for solutions of $(\mathcal{SP})$ when $L=-\Delta$. We will show that the situation is completely different when we consider an operator L with variable coefficients. We shall show (see Theorem 3) that $u\in W^{1,p}(\Omega)$, for any $p\in[1,+\infty)$ assuming furthermore that the coefficients satisfy

$$a_{ij}\in vmo(\Omega),\ \forall i,j=1,\cdot\cdot,n.$$

(see below a full definition).

The regularity $u \in W^{1,p}(\Omega)$ for any $p\in[1,+\infty)$ already improves some previous results in the literature (see e.g. Díaz, Hernández and Rakotoson [15], Bougherara-Giacomoni and Hernández [4], Rakotoson [16], and their references). In fact, under the additional regularity $a_{ij}\in C^{0,1}(\overline{\Omega}),\ \forall i,j=1,\cdot\cdot,n$ we shall show (Theorem 4.3) that $\nabla u\in bmo_r(\Omega)^n$, a functional space already used by Campanato [8] under the name $\mathcal{L}^{2,n}(\Omega)$ and which can be obtained through some variations from the BMO space of John and Nirenberg [20] (see also Chang [10], Chang Dafni and Stein [11]).

One of our key arguments, in this paper, is a suitable application of the so-called "Uniform Hopf Inequality" which says that if v is a very weak solution of the linear problem

$$\begin{cases} Lv=f(x) & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

then there exists a positive constant $C_{\Omega,L}$, depending only on Ω and the coefficients of L, such that

$$v(x) \geqslant C_{\Omega,L}\delta(x)\int_{\Omega} f(y)\delta(y)\,dy \quad \text{a.e. } x\in\Omega. \tag{1.4}$$

Here, $\delta(x)$ stands for the distance of $x \in \Omega$ to the boundary $\partial\Omega$.
This inequality was used by first time in the paper Díaz, Morel and Oswald [33] for the study of a singular semilinear problem (with $L = -\Delta$) in which the singular term arises in the right hand side of the equality. In fact, the detailed proof of the inequality was announced as a separated independent work by Morel and Oswald [33] but it was unpublished. A proof of it (always with $L = -\Delta$) was offered in the paper Brezis and Cabré [5]. The proof is still valid when $L = -\operatorname{div}(A(x)\nabla\cdot)$ and the coefficients a_{ij} of A are in $C^{0,1}(\overline{\Omega})$. Here, we shall show that the Uniform Hopf Inequality holds even for the case in which L has discontinuous coefficients satisfying $a_{ij} \in C^{0,1}(\overline{\Omega}_b)$, $a_{ij} \in L^\infty(\Omega_l)$, where we assume that Ω admits a partition i.e., $\Omega = \overline{\Omega_l} \cup \Omega_b$, with $\Omega_l \subset\subset \Omega$ and Ω_b contained in a neighborhood of $\partial\Omega$. We shall also prove that the condition $a_{ij} \in C^{0,1}(\overline{\Omega}_b)$, $a_{ij} \in L^\infty(\Omega_l)$ is sharp by giving a counter example of it for the case in which a_{ij} are not continuous in some neighborhood of the boundary.

Section 4 and 5 will deal with the main regularity results of this paper (first concerning with u^{-1} as nonlinear term and then, in Section 5, with u^{-m} as a general case).

2 Notations and Preliminaries

We shall consider Ω an open bounded smooth (say $C^{0,1}$ at least) of $\mathbb{R}^n$. We recall some spaces (namely the bounded mean oscillation functions (bmo)) that we shall use later (see e.g. [20], [10] [11], [24], [21], [37], [42], ...)

Definition 1 *(**of** $bmo(\mathbb{R}^n)$) A locally integrable function f on $\mathbb{R}$ is said to be in $bmo(\mathbb{R}^n)$ if*

$$\sup_{0<diam(Q)<1}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \sup_{diam(Q)\geqslant 1}\frac{1}{|Q|}\int_Q |f(x)|\,dx$$
$$\equiv \|f\|_{bmo(\mathbb{R}^n)} < +\infty,$$

where the supremum is taken over all cube $Q \subset \mathbb{R}^n$ whose sides are parallel to the coordinate axis. diam (Q) *stands for the diameter of Q, $|Q|$ the measure of the cube and f_Q the average of f over the cube Q. The cube can be replaced by a ball.*

Definition 2 *(**of** $bmo_r(\Omega)$ **and main property**)*
A locally integrable function f on a Lipschitz bounded domain Ω is said to be in $bmo_r(\Omega)$ (r stands for restriction) if

$$\sup_{0<diam(Q)<1}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \int_\Omega |f(x)|\,dx \equiv \|f\|_{bmo_r(\Omega)} < +\infty,$$

where the supremum is taken over all cubes $Q \subset \Omega$ whose sides are parallel to the coordinate axis.
In this case, there exists a function $\widetilde{f} \in bmo(\mathrm{I\!R}^n)$ such that

$$\widetilde{f}\Big|_{\Omega} = f \text{ and } \|\widetilde{f}\|_{bmo(\mathrm{I\!R}^n)} \leqslant c_\Omega \cdot \|f\|_{bmo_r(\Omega)}.$$

Definition 3 *(of the Campanato space $\mathcal{L}^{2,n}(\Omega)$)*
A function $u \in \mathcal{L}^{2,n}(\Omega)$ if

$$\|u\|_{L^2(\Omega)} + \sup_{x_0\in\Omega, r>0} \left[r^{-n} \int_{Q(x_0,r)\cap\Omega} |u-u_r|^2\,dx \right]^{\frac{1}{2}} = \|u\|_{\mathcal{L}^{2,n}(\Omega)} < +\infty.$$

Here

$$u_r = \frac{1}{|Q(x_0,r)\cap\Omega|} \int_{Q(x_0,r)\cap\Omega} u(x)\,dx.$$

$Q(x_0;r)$ (resp $B(x_0;r)$) is the cube (resp the ball) of center at x_0 of side (resp radius) r_0.

Lemma 2.1 *(Equivalence of the two definitions)*
For a Lipschitz bounded domain Ω, one has:

$$\mathcal{L}^{2,n}(\Omega) = bmo_r(\Omega),$$

with equivalent norms.

This theorem is not essential for our purpose. (we refer to [38] for its proof).

We shall also use the associated Sobolev space

$$W_0^1 bmo_r(\Omega) := \Big\{ u : \Omega \to \mathrm{I\!R};\; u \in W_0^{1,1}(\Omega) \text{ and } \nabla u \in bmo_r(\Omega)^n \Big\}.$$

As in [42], we also introduce the space.

$$vmo(\Omega) := \Big\{ f \in bmo_r(\Omega) \text{ and } \lim_{R\to 0} \sup_{\substack{r\leqslant R\\ x_0\in\Omega}} \frac{1}{r^n} \int_{B(x_0,r)\cap\Omega} |f-f_r|\,dx \to 0 \Big\}.$$

We recall that the Sobolev-Poincaré inequality implies that $W^{1,n}(\Omega) \hookrightarrow vmo_{loc}(\Omega)$. This gives how we can construct elements *vmo.*

For a measurable set E in $\mathrm{I\!R}^n$ we denote by $|E|$ its Lebesgue measure, and for a measurable function u from the open bounded set Ω into $\mathrm{I\!R}^n$ we define the following auxiliary functions :

1. The distribution function of u. It is a $m : \mathbb{R} \to]0, |\Omega|[$, such that
$$m(t) = \text{ meas } \{x \in \Omega : u(x) > t\} = |u > t|,$$

2. The monotone rearrangement of u (denoted by u_*), is the generalized inverse of m, i.e.
$$u_*(s) = \inf\{t \in \mathbb{R} : |u > t| \leqslant s, \},\ s \in]0, |\Omega|[,$$
$$u_*(0) = \text{ess}\ \sup_\Omega u.$$
We also define $|u|_{**} = \dfrac{1}{t}\displaystyle\int_0^t |u|_*(s)\, ds$ for $t > 0$.
The Lorentz spaces $L^{p,q}(\Omega)$ are defined, for $1 \leqslant p < +\infty,\ 1 \leqslant q < +\infty$, as
$$\begin{aligned} L^{p,q}(\Omega) &= \left\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_0^{|\Omega|} \left[t^{\frac{1}{p}} |u|_{**}(t)\right]^q \frac{dt}{t} < +\infty\right\}, \\ L^{p,\infty}(\Omega) &= \left\{u : \Omega \to \mathbb{R} \text{ measurable } : \sup_{t \leqslant |\Omega|} t^{\frac{1}{p}} |u|_{**}(t) < +\infty\right\}, \\ W_0^1 L^{p,q}(\Omega) &= \left\{u \in W_0^{1,1}(\Omega) : |\nabla u| \in L^{p,q}(\Omega)\right\}. \end{aligned}$$

We shall also use the usual notation
$$\begin{aligned} C^{0,1}(\overline{\Omega}) &:= \left\{u : \Omega \to \mathbb{R} \text{ measurable }, \exists K > 0; |u(x) - u(y)| \leqslant K|x - y|,\ \text{for any } x, y \in \overline{\Omega}\right\} \\ &:= \left\{u : \Omega \to \mathbb{R} : \exists \widetilde{u} \in C^{0,1}(\mathbb{R}^n),\ \widetilde{u} \text{ restricted to } \Omega \text{ is } u : \widetilde{u}|_\Omega = u\right\}. \end{aligned}$$

We define the following operator
$$L^* = -\operatorname{div}(A^* \nabla \cdot);\ A^* \text{ is the adjoint matrix of } A.$$

It is well-known that we can define the Green function associated to those operators and Dirichlet boundary conditions :

Theorem 2.1 *(Green function for L^* and L)([41], [26],[44])*

There exists a unique function $G_{L^*} : \Omega \times \Omega \longrightarrow \overline{\mathbb{R}}$ such that

1) $\forall y \in \Omega,\ G_{L^*}(\cdot, y) \in W_0^1 L^{n',\infty}(\Omega)$ and $\sup_y \|G_{L^*}(\cdot, y)\|_{W_0^1 L^{n',\infty}(\Omega)} \leqslant C(\Omega)$, satisfying
$$\int_\Omega A(x) \nabla G_{L^*}(x, y) \nabla \varphi\, dx = \varphi(y), \forall \varphi \in W_0^1 L^{n,1}(\Omega).$$

2) $G_{L^*}(\cdot, y) \in C(\overline{\Omega}\backslash\{y\}) \cap H^1(\Omega\backslash B(y,r)),\ \forall r > 0.$

3) $\forall\varphi \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ *such that* $L^*\varphi \in C(\overline{\Omega})$, *we have*

$$\int_\Omega G_{L^*}(x,y)L^*\varphi(x)\,dx = \varphi(y).$$

4) $G_L(x,y) = G_{L^*}(y,x),\ \forall(x,y) \in \Omega^2.$

5) Given $f \in L^2(\Omega)$, *if* $u \in H_0^1(\Omega)$ *verifies* $Lu = f$ *then*

$$u(x) = \int_\Omega G_L(x,y)f(y)\,dy.$$

G_L*(resp* G_{L^*}*) is called the Green kernel associated to* L *(resp* L^**) for Dirichlet conditions.*

Remark 1 *Statement 3) in Theorem 2.1 is due to G. Stampacchia [41]. Here the definition of the Green kernel is given according to Stampacchia. But, such as it is pointed by this author, this definition is stable by approximation (his proof relies on the approximation of A). Nevertheless, it is already known that it holds for measure data problems, (here the measure is e.g. the Dirac measure). In this case the problem stated in 1) has a solution whenever* $A(x) \in L^\infty(\Omega)^{n^2}$ *(see [39], [40], [41], [44]).*

Theorem 2.2 is then a combination of all those properties.

Theorem 2.2 *(Comparison of Green kernel)[41]*
Suppose we have the following operator defined by $L^1\varphi = -\operatorname{div}(A^1(x)\nabla\varphi)$ *such that*

$$A^1(x) = \left(a^1_{ij}(x)\right)_{i,j},\ a^1_{ij} \in C^{0,1}(\overline{\Omega}),$$

with the coercivity condition in all the domain Ω,

$$\sum_{i,j=1}^n a^1_{ij}(x)\zeta_i\zeta_j \geqslant \alpha|\zeta|^2, \forall\zeta \in \mathbb{R}^n, \alpha > 0$$

and let G_{L^1} *be the Green function associated to* L^1. *Then for any relatively open compact set* $\Omega'_{\ell,0}$ *of* Ω *there exists a constant* $K_1 = K_1(\Omega'_{\ell,0}) > 0$ *such that*

$$K_1^{-1}G_{-\Delta}(x,y) \leqslant G_{L^{1*}}(x,y) \leqslant K_1 G_{L^*}(x,y), \qquad \forall(x,y) \in \overline{\Omega'}_{\ell,0} \times \overline{\Omega'}_{\ell,0}.$$

In all this paper, we shall use the notation $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. We shall also use the following,

Lemma 2.2 ***Hardy Inequality*** *([16, 30],)*
Let Ω be of class $C^{0,1}$. Then, $\exists c > 0$ such that $\forall u \in C_c^1(\Omega)$

$$\int_\Omega \left(\frac{|u(x)|}{\delta(x)}\right)^2 dx \leqslant c \int_\Omega |\nabla u|^2\, dx.$$

Moreover, for $a > 1$, $\exists C_a(\Omega) > 0$, such that $\forall\, u \in C_c^1(\Omega)$.

$$\int_\Omega \frac{|u(x)|}{\delta^a}\, dx \leqslant C_a(\Omega) \int_\Omega |\nabla u| \delta^{1-a}\, dx.$$

Lemma 2.3 *(***Iteration*** [22, 8, 14]) Let $\Phi(\rho)$ be a non negative and non decreasing function. Assume that for some non negative constants A, α, β, r_0, B with $\beta < \alpha$, we have $\forall\, r \in]\,0, r_0[$, $\forall \rho \in]\,0, r\,[$*

$$\Phi(\rho) \leqslant A \left(\frac{\rho}{r}\right)^\alpha \Phi(r) + Br^\beta.$$

Then, there exists $c > 0$ such that

$$\Phi(\rho) \leqslant c \left(\frac{\rho}{r}\right)^\beta \Phi(r) + B\rho^\beta, \qquad \forall\ 0 < \rho \leqslant r \leqslant r_0.$$

3 On the Uniform Hopf Inequality

To solve $(\mathcal{SP})$, our approach will not be based on the notion of sub-solution and super-solution such as it is done in [25], [27], when $A(x) = (a_{ij}(x))$, $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$ (smooth). We shall apply the following inequality :

Definition 4 *(Uniform Hopf Inequality)*
We say that the operator L satisfies the Uniform Hopf Inequality if there exists a constant $C_{\Omega,L} > 0$ such that for all $f \in L^\infty_+(\Omega)$, the unique solution $v \in H_0^1(\Omega)$ of $-\operatorname{div}(A\nabla v) = f$ in $\mathcal{D}'(\Omega)$ satisfies

$$v(x) \geqslant C_{\Omega,L}\delta(x) \int_\Omega f(y)\delta(y)\, dy, \quad a.e\ x \in \Omega. \tag{3.1}$$

The inequality (3.1) still holds for (v, f) which can be approximate pointwise almost everywhere by a sequence of $(v_n, f_n) \in H_0^1(\Omega) \times L^\infty(\Omega)$ with the same matrix A.

Inequality (3.1) holds true if the coefficients of the matrix A are Lipschitz as it is shown in

Theorem 3.1 *([33], [7]).*
Suppose that $f \in L^\infty_+(\Omega)$, and consider L^1 the operator given in Theorem 2.2. Let v be a solution of

$$\begin{cases} L^1 v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$v(x) \geqslant C\delta(x)\int_\Omega f(y)\delta(y)\,dy, \quad a.e.\ x \in \Omega,$$

where $C > 0$ is a constant depending on Ω and $\delta(x) = dist(x, \partial\Omega)$.

Remark 2 *(on the proof of Theorem 3.1) In the mentioned references, the proofs are given for the Laplacian operator they can be modified to hold for the case where $A = A^1$. An alternative proof can be given using the equivalence of Green functions (see [28]).*

Our first result wants to point out that if A is only bounded near the boundary but not Lipschitz continuous then (3.1) may fail to be true.

Theorem 3.2
There exist a smooth open set $\Omega \subset \mathbb{R}^2$, a matrix A with bounded coefficients, $f \in L^{\frac{3}{2}}(\Omega)$ and $u \in H_0^1(\Omega)$ solution of $\displaystyle\int_\Omega A(x)\nabla u\nabla\varphi\,dx = \int_\Omega f\varphi\,dx$ $(\forall\varphi \in H_0^1(\Omega))$, such that the Uniform Hopf Inequality fails to be true.

Proof. Consider $\Omega := \{(x, y) \in \mathbb{R}^2$ such that $x > 0$ and $x^2 + y^2 < 1\}$.
Define the function

$$v(x, y) = \frac{x}{(x^2 + y^2)^{\frac{1}{4}}} - x := v_1(x, y) + v_2(x, y),$$

and the following matrix (already used in Meyers [32]).

$$A(x, y) := \frac{1}{4(x^2 + y^2)}\begin{pmatrix} 4x^2 + y^2 & 3xy \\ 3xy & x^2 + 4y^2 \end{pmatrix}.$$

We have $v \in H_0^1(\Omega)$. Now, we claim that $-\operatorname{div}(A\nabla v) = f \geqslant 0$ on Ω. Indeed, since

$$\nabla v_1 = \begin{pmatrix} \dfrac{x^2 + 2y^2}{2(x^2 + y^2)^{\frac{5}{4}}} \\[2ex] \dfrac{-xy}{2(x^2 + y^2)^{\frac{5}{4}}} \end{pmatrix},$$

then

$$A\nabla v_1 = \begin{pmatrix} \dfrac{2x^2 + y^2}{2(x^2 + y^2)^{\frac{5}{4}}} \\[2ex] \dfrac{xy}{4(x^2 + y^2)^{\frac{5}{4}}} \end{pmatrix}.$$

Moreover

$$\frac{\partial}{\partial x}\left(\frac{2x^2+y^2}{2(x^2+y^2)^{\frac{5}{4}}}\right)=\frac{-2x^3+3xy^2}{4(x^2+y^2)^{\frac{9}{4}}},$$
$$\frac{\partial}{\partial y}\left(\frac{xy}{4(x^2+y^2)^{\frac{5}{4}}}\right)=\frac{2x^3-3xy^2}{4(x^2+y^2)^{\frac{9}{4}}}.$$

Consequently, we have

$$-\operatorname{div}(A\nabla v_1)=0 \text{ in } \Omega. \tag{3.2}$$

On the other hand, for $v_2(x,y)=-x$, we have

$$\nabla v_2=\begin{pmatrix}-1\\0\end{pmatrix},$$

and then

$$A\nabla v_2=\begin{pmatrix}-\dfrac{4x^2+y^2}{4(x^2+y^2)}\\[2ex] \dfrac{-3xy}{4(x^2+y^2)}\end{pmatrix}$$

In a similar way, we have

$$-\operatorname{div}(A\nabla v_2)=\frac{3x}{4(x^2+y^2)} \text{ in } \Omega. \tag{3.3}$$

Thus, by (3.2)-(3.3) we conclude that

$$-\operatorname{div}(A\nabla v)=\frac{3x}{4(x^2+y^2)}:=f\geqslant 0,\ \text{ in } \Omega.$$

We have $f\in L^{\frac{3}{2}}(\Omega)$ since, by using polar coordinates, we have that

$$\begin{aligned}\int_\Omega |f(x,y)|^{\frac{3}{2}}\,dx\,dy &\leqslant c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\left|\frac{3r\cos\theta}{4r^2}\right|^{\frac{3}{2}} r\,dr\,d\theta\\ &\leqslant c\int_0^1 r^{\frac{-1}{2}}\,dr<+\infty.\end{aligned}$$

Let us calculate $\inf\limits_{(x,y)\in\Omega}\dfrac{v(x,y)}{\delta(x,y)}$. We observe that since $v(x,y)\geqslant 0$, then

$$\frac{v(x,y)}{\delta(x,y)}\geqslant 0 \text{ a.e. on } \Omega.$$

By using polar coordinates again, we get

$$v(x,y)=r\cos\theta\left(\frac{1}{\sqrt{r}}-1\right).$$

Then,

$$\frac{v(r,\theta)}{\delta(r,\theta)} \leqslant \sqrt{r}\cos\theta \longrightarrow 0 \text{ as } r \searrow 0.$$

Therefore,

$$\inf_{(x,y)\in\Omega} \frac{v(x,y)}{\delta(x,y)} = 0. \tag{3.4}$$

Arguing by contradiction, (3.4) infers that the Uniform Hopf Inequality cannot hold in this case. □

We shall assume that

$$(H_2) \quad \begin{cases} \text{there exists a matrix } A^1(x) = \Big(a^1_{ij}(x)\Big)_{i,j},\ x \in \overline{\Omega},\ \text{with} \\ \begin{cases} a^1_{ij} \in C^{0,1}(\overline{\Omega}),\ \alpha\text{-coercive i.e } \forall\, \xi \in \mathbb{R}^n\ (A^1(x)\xi,\xi) \geqslant \alpha|\xi|^2,\ \forall\, x \in \overline{\Omega}, \\ a^1_{ij} \text{ restricted to } \Omega_b \text{ coincides with } a_{ij},\ \forall\, i,j : a_{ij}\Big|_{\Omega_b} = a^1_{ij}. \end{cases} \end{cases}$$

Here $\Omega = \overline{\Omega}_\ell \cup \Omega_b$, $\Omega_\ell \subset\subset \Omega$.
In Ω, we shall associate to A^1 the operator $L^1 = -div(A^1(x)\nabla\,\cdot\,)$. The main result of this section is the following :

Theorem 3.3 *Under the above assumptions* (H_1), *and* (H_2) *there exists* $C_{\Omega,L} > 0$, *such for any* $f \in L^\infty_+(\Omega)$, *the solution* $u \in H^1_0(\Omega)$ *of (1.3) satisfies for a.e.* $\forall y \in \Omega$

$$u(y) \geqslant C_{\Omega,L}\delta(y)\int_\Omega f(x)\delta(x)\,dx. \tag{3.5}$$

For its proof, for $\Omega_\ell \subset\subset \Omega$, we shall consider the open set $\Omega'_{\ell,0} \subset\subset \Omega$ such that $\overline{\Omega}_\ell \subset \Omega'_{\ell,0}$. In addition, for $\Omega_b = \Omega\backslash\overline{\Omega}_\ell$ we consider its subset $\Omega'_{b,0} = \Omega\backslash\overline{\Omega'}_{\ell,0}$. We shall need the following lemmas to prove the inequality (3.5).

Lemma 3.1 *Under the same assumptions as in Theorem 3.3, and if* $\Omega'_{\ell,0}$ *is given as above, the constant* K_1 *given in Theorem 2.2, is such that* $K_1 = K_1(\Omega'_{\ell,0}) > 0$ *and*

$$K_1 G_{L^*}(x,y) \geqslant G_{L^{1*}}(x,y),\ \forall x \in \Omega'_{b,0},\ \forall y \in \overline{\Omega'}_{\ell,0}.$$

Proof. Let $\varphi \in W^1_0L^{n,1}(\Omega'_{b,0})$ and let $\widetilde{\varphi}$ its extension to Ω by zero.
Then, $\widetilde{\varphi} \in W^1_0L^{n,1}(\Omega)$. For $y \in \overline{\Omega}_l$ fixed, let $w(x) = K_1G_{L^*}(x,y) - G_{L^{1*}}(x,y),\ x \in \Omega'_{b,0}$.
Then $w \in H^1(\Omega'_{b,0})$ and

$$\int_\Omega A(x)\nabla w \cdot \nabla\widetilde{\varphi}\,dx = K_1\int_\Omega A(x)\nabla G_{L^*}(x,y)\nabla\widetilde{\varphi}\,dx - \int_\Omega A(x)\nabla G_{L^{1*}}(x,y)\nabla\widetilde{\varphi}\,dx.$$

Since $A(x) = A^1(x)$ on $\Omega'_{b,0}$, thanks to Theorem 2.1, we obtain

$$\int_\Omega A(x)\nabla w \cdot \nabla\widetilde{\varphi}\, dx = K_1\widetilde{\varphi}(y) - \int_{\Omega'_{b,0}} A^1(x)\nabla G_{L^{1*}}(x,y)\nabla\varphi\, dx, \qquad \text{for any } y \in \overline{\Omega'}_{\ell,0}$$

Using again $\widetilde{\varphi}$ in the last term and Theorem 2.1

$$\begin{aligned}\int_\Omega A(x)\nabla w \cdot \nabla\widetilde{\varphi}\, dx &= K_1\widetilde{\varphi}(y) - \int_\Omega A^1(x)\nabla_x G_{L^{1*}}(x,y)\nabla\widetilde{\varphi}\, dx \\ &= K_1\widetilde{\varphi}(y) - \widetilde{\varphi}(y) = 0.\end{aligned}$$

Thus,

$$\int_{\Omega'_{b,0}} A(x)\nabla w \cdot \nabla\varphi\, dx = 0. \tag{3.6}$$

Moreover, its trace verifies that, $\gamma_0 w(x) = 0$ on $\partial\Omega$, and from Theorem 2.2

$$w(x) = K_1 G_{L^*}(x,y) - G_{L^{1*}}(x,y) \geqslant 0 \text{ for } x \in \partial\Omega'_{\ell,0}.$$

Consequently, by the maximum principle, $w \geqslant 0$ on $\Omega'_{b,0}$. □

Corollary 3.1 *(of Theorem 2.2 and Lemma 3.1)*
Under the same assumptions as in Theorem 3.3, the constant $K_1 > 0$ found in Lemma 3.1 satisfies

$$K_1 G_{L^*}(x,y) \geqslant G_{L^{1*}}(x,y),\ \forall x \in \Omega,\ \forall y \in \overline{\Omega'}_{\ell,0}.$$

Proof. Since $\Omega = \Omega'_{b,0} \cup \overline{\Omega'}_{\ell,0}$ and we have shown that $\forall y \in \overline{\Omega'}_{\ell,0}$,

$$w(x) = K_1 G_{L^*}(x,y) - G_{L^{1*}}(x,y) \geqslant 0,\ \forall x \in \Omega'_{b,0},$$

and $w(x) \geqslant 0,\ \forall x \in \overline{\Omega'}_{\ell,0}$, (from Theorem 2.2), thus, $w(x) \geqslant 0$ on $\Omega'_{b,0} \cup \overline{\Omega'}_{\ell,0} = \Omega$. □

Corollary 3.2 *(lower estimates on the subset Ω_l)*
Let $f \in L^\infty_+(\Omega)$, and let u and v in $H^1_0(\Omega)$ satisfying $Lu = f$ and $L^1 v = f$ respectively. Then, a.e. $y \in \overline{\Omega}_\ell$

1) $u(y) \geqslant \dfrac{1}{K_1}\displaystyle\int_\Omega G_{L^1}(y,x) f(x)\, dx,$

2) *there exists $C_{\Omega,L^1} > 0$ such that*

$$v(y) = \int_\Omega G_{L^1}(y,x) f(x)\, dx \geqslant C_{\Omega,L^1}\delta(y)\int_\Omega f(x)\delta(x)\, dx.$$

In particular,

$$u(y) \geqslant C_{\Omega'_{\ell,0}}\int_\Omega f(x)\delta(x)\, dx > 0. \tag{3.7}$$

Proof. Let $f \in L^\infty_+(\Omega)$. Then from Corollary 3.1, for a.e. $y \in \overline{\Omega}_\ell$, after integrating over Ω, we have

$$u(y) = \int_\Omega G_{L^*}(x,y)f(x)\,dx \geqslant \frac{1}{K_1}\int_\Omega G_{L^{1*}}(x,y)f(x)\,dx = v(y). \tag{3.8}$$

Recalling that the coefficients of L^1 are Lipschitz continuous, we can apply Theorem 3.1 to obtain that $\exists C_{\Omega,L^1} > 0$ such that, for a.e. $y \in \overline{\Omega}_\ell$,

$$u(y) \geqslant v(y) \geqslant C_{\Omega,L^1}\delta(y)\int_\Omega f(x)\delta(x)\,dx. \tag{3.9}$$

□

Lemma 3.2 *(Lower estimates on Ω_b near the boundary)*
Under the same assumptions as in Theorem 3.3, there exists $C_{\Omega,L} > 0$ such that

$$u(y) \geqslant C_{\Omega,L}\delta(y)\int_\Omega f(x)\delta(x)\,dx, \ \text{a.e. } y \in \Omega_b = \Omega\backslash\overline{\Omega}_\ell.$$

Proof. This procedure is inspired by the method of proof used by Brezis-Cabré[4]. Let $\Gamma_b := \partial\Omega \cup \partial\Omega_\ell = \partial\Omega_b$ and introduce the function $w \in H^1(\Omega)$ solution of:

$$\begin{cases} L^1 w = -\operatorname{div}(A^1(x)\nabla w) = 0 & \text{in } \Omega_b,\\ w = 0 & \text{on } \partial\Omega,\\ w = 1 & \text{on } \partial\Omega_\ell. \end{cases}$$

Since the coefficients of $A = A^1$ are Lipschitz continuous on $\overline{\Omega}_b$, then by the Hopf strong maximum principle, there exists $C'_{\Omega_b} > 0$ such that

$$w(y) \geqslant C'_{\Omega_b}\delta(y), \ \forall y \in \Omega_b. \tag{3.10}$$

Now, let us set $\underline{w}(y) = \left[C'_{\Omega_b}\int_\Omega f(x)\delta(x)\,dx\right]^{-1} u(y)$. By the linearity of operator L^1 and since $f \geqslant 0$:

$$\begin{cases} L^1\underline{w} = \left[C'_{\Omega_b}\int_\Omega f(x)\delta(x)\,dx\right]^{-1} L^1 u \geqslant 0 = L^1 w \text{ in } \Omega_b,\\[2ex] \underline{w}|_{\partial\Omega_b} \geqslant w|_{\partial\Omega_b}. \end{cases}$$

Thus, thanks to the maximum principle, we obtain $\underline{w} \geqslant w$ on Ω_b. This means that for all $y \in \Omega_b$

$$u(y) \geqslant C'_{\Omega_\ell}C'_{\Omega_b}\delta(y)\int_\Omega f(x)\delta(x)\,dx. \tag{3.11}$$

Finally, combining (3.11) with relation (3.9), we get

$$u(y) \geqslant C\delta(y)\int_\Omega f(x)\delta(x)\,dx, \quad \text{a.e. } y \in \Omega.$$

□

4 On the singular problem : the case of u^{-1} as nonlinear term.

The Uniform Hopf Inequality (UHI) is very useful to derive regularity results for the singular semilinear problem

$$(\mathcal{P}) = \begin{cases} -\operatorname{div}(A(x)\nabla u) = \dfrac{a(x)}{u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

An existence and uniqueness results are also proved in [3, 9] for linear operators and in [18, 19] for non linear operators. The main difference with our results is double. Indeed, firstly our method to prove the positivity of the solution is different than the above mentioned papers. For instance, in [3] it is obtained by a monotonicity result, nevertheless our method of proof can be extended to a general operator as $Lu = -div(A\nabla u) + B\nabla u + div(Cu) + a_0 u$, (as it will be presented in the Nada El Berdan's thesis). Secondly the additional regularity that we shall obtain in the following theorem (the term $\dfrac{a}{u} \in H^{-1}(\Omega)$ or the results given in Theorem 5.2 below) is not mentioned in the above papers.

Theorem 4.1 *Let $a \in L^\infty_+(\Omega), a \neq 0$. Then, there exists a unique solution $u \in H^1_0(\Omega)$ of $(\mathcal{P})$, such that*

i) $\dfrac{a}{u} \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega),\ u > 0$ *in* Ω,

ii) $\displaystyle\int_\Omega A(x)\nabla u \cdot \nabla\varphi\, dx = \int_\Omega \frac{a(x)\varphi}{u}\, dx, \quad \forall \varphi \in H^1_0(\Omega).$

Proof. Let us start with the uniqueness of u. If $u, \bar{u}$ satisfy ii) then by the coercivity condition on A and choosing $\varphi = u - \bar{u}$;

$$\alpha \int_\Omega |\nabla(u-\bar{u})|^2\, dx \leqslant \int_\Omega a(x) \left[\frac{1}{u} - \frac{1}{\bar{u}}\right](u-\bar{u})\, dx \leqslant 0.$$

This implies that, necesarely, $u = \bar{u}$. For the existence part, we introduce the following regularized problem:

$$(\mathcal{P}_\varepsilon) = \begin{cases} Lu_\varepsilon = -\operatorname{div}(A(x)\nabla u_\varepsilon) = \dfrac{a}{|u_\varepsilon| + \varepsilon} & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with, $\varepsilon > 0$. The weak (variational) formulation for $(\mathcal{P}_\varepsilon)$ reads

$$a_L(u_\varepsilon, \varphi) := \int_\Omega A(x)\nabla u_\varepsilon \cdot \nabla\varphi\, dx = \int_\Omega \frac{a(x)\varphi}{|u_\varepsilon| + \varepsilon}\, dx, \quad \forall \varphi \in H^1_0(\Omega). \tag{4.1}$$

Using the Schauder fixed point theorem (see e.g. [23]), we get the existence of u_ε. In addition, if we apply the weak maximum principle [23], we obtain that $u_\varepsilon \geqslant 0$. The same argument as the used for u ensures that u_ε is unique.

Estimate on u_ε: Taking $\varphi = u_\varepsilon$ as test function in (4.1), we can write

$$a_L(u_\varepsilon, u_\varepsilon) = \int_\Omega \frac{a u_\varepsilon}{u_\varepsilon + \varepsilon}\, dx,$$

and then

$$\alpha \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leqslant \|a\|_{L^\infty} \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + \varepsilon}\, dx \leqslant C_\Omega \|a\|_{L^\infty}. \tag{4.2}$$

Therefore, u_ε is uniformly bounded in $H^1_0(\Omega)$, and then there exists $u \in H^1_0(\Omega)$ such that (for a subsequence) u_ε converges to u a.e.

Now, we shall prove that $\dfrac{a}{u_\varepsilon}$ remains in a bounded set of $L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$. By Theorem 2.1 (see also Theorem 9.3 of [41]), we have

$$u_\varepsilon(x) = \int_\Omega G_L(x,y) \frac{a(y)}{(u_\varepsilon(y) + \varepsilon)}\, dy \text{ a.e. } x \in \Omega, \tag{4.3}$$

where G_L is the Green function associated to L as it is defined in section 1.
Let Ω' be a relatively compact open set in Ω. Let $f_0(y) = \dfrac{a}{u_\varepsilon + \varepsilon} \chi_{\Omega'}(y)$ and consider the following problem

$$\begin{cases} -\Delta w(x) = f_0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique solution w in $H^1_0(\Omega) \cap H^2(\Omega)$. According to the Uniform Hopf Inequality (see Theorem 3.1), there exist $C_{\Omega,\Delta} > 0$ such that

$$w(x) \geqslant C_{\Omega,\Delta} \delta(x) \int_{\Omega'} \frac{a}{u_\varepsilon + \varepsilon} \delta(y)\, dy, \qquad \text{a.e. } x \in \Omega.$$

Returning to Theorem 2.1 and to the inequality in Theorem 2.2, then for $K = K(\Omega') > 0$

$$\begin{aligned} K^{-1} w(x) = K^{-1} \int_\Omega G_{-\Delta}(x,y) f_0(y)\, dy \quad &\leqslant \int_\Omega G_L(x,y) f_0(y)\, dy \\ &\leqslant \int_\Omega G_L(x,y) \frac{a(y)}{u_\varepsilon(y) + \varepsilon}\, dy = u_\varepsilon(x), \text{ a.e. } x \in \Omega. \end{aligned}$$

Consequently,

$$+\infty > u_\varepsilon(x) \geqslant K^{-1} w(x) \geqslant C_{\Omega,\Delta} \delta(x) \int_{\Omega'} \frac{a}{u_\varepsilon + \varepsilon} \delta(y)\, dy, \text{ a.e. } x \in \Omega', \tag{4.4}$$

which yields that $\dfrac{a}{u_\varepsilon} \in L^1_{loc}(\Omega)$ and $u_\varepsilon > 0$ a.e. in Ω. To prove that $\dfrac{a}{u} \in L^1_{loc}(\Omega)$, it is enough to pass to the limit at (4.4), since from Fatou's Lemma, we get

$$+\infty > u(x) \geqslant C_{\Omega,\Delta} \delta(x) \int_{\Omega'} \frac{a(y)}{u(y)}\, dy, \text{ a.e. } x \in \Omega'.$$

In particular, this implies that $\inf\limits_{0<\varepsilon<\varepsilon_1} \int_{\Omega'} \frac{a}{u_\varepsilon + \varepsilon} \delta(y)\, dy > 0$. We now have

$$\frac{a}{u_\varepsilon + \varepsilon} \xrightarrow[\varepsilon \to 0]{} \frac{a}{u} \quad \text{a.e., on } \Omega,$$

and for all Ω' relatively compact in Ω, since $\inf\limits_{y \in \Omega'} \delta(y) > 0$, then (4.4) yields for a.e. $x \in \Omega'$

$$0 \leqslant \frac{a}{u_\varepsilon + \varepsilon}(x) \leqslant \frac{a}{u_\varepsilon}(x) \leqslant C_{\Omega'} \frac{a}{\displaystyle\int_{\Omega'} \frac{a(x)}{u_\varepsilon(x) + \varepsilon}(y)\, dy} \leqslant C_{\Omega'}\; a(x) < +\infty.$$

Then, by using the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \frac{a}{u_\varepsilon + \varepsilon} \psi(x)\, dx \xrightarrow[\varepsilon \to 0]{} \int_\Omega \frac{a}{u} \psi(x)\, dx, \quad \forall \psi \in \mathcal{D}(\Omega).$$

Now, we want to show that

$$\int_\Omega A(x) \nabla u \cdot \nabla \psi = \int_\Omega \frac{a}{u} \psi, \quad \forall \psi \in H_0^1(\Omega).$$

For this we observe that $\frac{a}{u_\varepsilon + \varepsilon}$ belongs to a bounded subset of $H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ since, $\forall \psi \geqslant 0,\ \psi \in H_0^1(\Omega)$ we have

$$\int_\Omega \frac{a\psi}{u_\varepsilon + \varepsilon}\, dx = \left| \int_\Omega A(x) \nabla u_\varepsilon \cdot \nabla \psi\, dx \right| \leqslant C \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \leqslant C \|\nabla \psi\|_{L^2(\Omega)}.$$

But, $\frac{a}{u_\varepsilon + \varepsilon}$ converges to $\frac{a}{u}$ a.e. Thus by Fatou's lemma, we deduce (knowing that $|\psi| \in H_0^1(\Omega)$ once that $\psi \in H_0^1(\Omega)$)

$$\sup_{\substack{\psi \in H_0^1 \\ \|\nabla\psi\|_{L^2(\Omega)} = 1}} \left| \int_\Omega \frac{a\psi}{u}\, dx \right| \leqslant C_\Omega < +\infty.$$

This shows that $\frac{a}{u} \in H^{-1}(\Omega)$ and following the property defined by Brezis-Browder ([6]), $\frac{a\psi}{u} \in L^1(\Omega)$ for $\psi \in H_0^1(\Omega)$
and also

$$< \frac{a}{u}; \psi >_{H^{-1},H^1} = \int_\Omega \frac{a\psi}{u}\, dx,\ \forall \psi \in H_0^1(\Omega).$$

But $\forall \psi \in \mathcal{D}(\Omega)$,

$$\int_\Omega A(x) \nabla u \cdot \nabla \psi\, dx = \int_\Omega \frac{a}{u} \psi\, dx.$$

So, by density, we have

$$\lim_{\substack{\psi_n \in \mathcal{D} \\ \psi_n \to \psi}} < \frac{a}{u}; \psi_n > = < \frac{a}{u}; \psi >,\ \forall \psi \in H_0^1(\Omega).$$

This ends the proof. □

Our next results show that the gradient of the solution is more regular. We start with the study in the $L^p(\Omega)$ spaces.

Theorem 4.2 *Assume furthermore that operator L satisfies the Uniform Hopf Inequality, $\partial\Omega$ is C^1 and that $a_{ij} \in vmo(\Omega)$. Then, $\forall p \in [1, \infty[$, $u \in W^{1,p}(\Omega)$, and*

$$\|u\|_{W_0^{1,p}(\Omega)} \leqslant C_\Omega^1(A,p) \frac{\|a\|_\infty}{\displaystyle\int_\Omega \frac{a(x)\delta(x)}{u(x)}\,dx}, \tag{4.5}$$

for some positive constant $C_\Omega^1(A,p)$.

Proof. Since L satisfies the Uniform Hopf Inequality, there exists a constant $C_L > 0$ such that we have

$$0 \leqslant \frac{a(x)}{u_\varepsilon(x)} \leqslant C_L \frac{\|a\|_\infty}{\delta(x) \displaystyle\int_\Omega \frac{a(y)\delta(y)}{(u_\varepsilon + \varepsilon)(y)}\,dy}, \text{ a.e. } x \in \Omega.$$

Then, if $\psi \in W_0^{1,p'}(\Omega)$ with $\dfrac{1}{p'} = 1 - \dfrac{1}{p}$, $\psi \geqslant 0$, we then have,

$$\begin{aligned} 0 \leqslant \int_\Omega \frac{a(x)}{u_\varepsilon(x)} \psi(x)\,dx &\leqslant C_L \int_\Omega \frac{a(x)\psi(x)}{\delta(x) \displaystyle\int_\Omega \frac{a(y)}{(u_\varepsilon + \varepsilon)(y)}\,dy}\,dx \\ &\leqslant C_L \frac{\|a\|_\infty}{\displaystyle\int_\Omega \frac{a(y)}{(u_\varepsilon + \varepsilon)(y)}\,dy} \int_\Omega \frac{\psi(x)}{\delta(x)}\,dx \\ &\leqslant C_{\Omega,L}(a) \int_\Omega \frac{\psi(x)}{\delta(x)}\,dx. \end{aligned}$$

By using the Hardy inequality, we obtain :

$$0 \leqslant \int_\Omega \frac{a(x)}{u_\varepsilon(x)} \psi(x)\,dx \leqslant C_{\Omega,L}(a) \|\nabla\psi\|_{L^{p'}(\Omega)}, \tag{4.6}$$

for some positive constant $C_{\Omega,L}(a)$. Therefore, we deduce

$$\frac{a}{u_\varepsilon + \varepsilon} \text{ belongs to a bounded set of } W^{-1,p}(\Omega).$$

From the regularity result applied to the associed linear equation, the unique solution u_ε of $(\mathcal{P}_\varepsilon)$ is in $W^{1,p}(\Omega)$ (see Simader [40], Auscher-Quafsaoui [1] and Byun [5]) and

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqslant C_\Omega \|\frac{a}{u_\varepsilon + \varepsilon}\|_{W^{-1,p}(\Omega)} \leqslant C_\Omega \frac{\|a\|_\infty}{\displaystyle\int_\Omega \frac{a(y)\delta(y)}{u_\varepsilon + \varepsilon}\,dy} \leqslant C(\Omega, a). \tag{4.7}$$

$\Big($Notice that

$$\lim_{\varepsilon_1\to 0}\left[\inf_{0<\varepsilon<\varepsilon_1}\int_\Omega \frac{a(y)\delta(y)}{u_\varepsilon+\varepsilon}dy\right]\geqslant \int_\Omega \frac{a(y)\delta(y)}{u(y)}dy>0$$

which implies the uniform estimates in (4.7)$\Big)$.
Therefore, ∇u_ε is bounded in $L^p(\Omega)^n$, and since u_ε converges to u weakly in $H_0^1(\Omega)$, thus ∇u_ε converges weakly to ∇u in $L^p(\Omega)^n$. Then

$$\begin{aligned}\|\nabla u\|_{L^p(\Omega)} &\leqslant \liminf_{\varepsilon\to 0}\|\nabla u_\varepsilon\|_{L^p(\Omega)}\\ &\leqslant \liminf_{\varepsilon\to 0}\frac{C_\Omega\|a\|_\infty}{\displaystyle\int_\Omega \frac{a(y)\delta(y)}{u_\varepsilon+\varepsilon}\,dy}=C_\Omega(a).\end{aligned}$$

This gives relation (4.5). □

Theorem 4.3 *Assume that $a_{ij}\in C^{0,1}(\overline{\Omega}),\forall i,j$ and $\partial\Omega$ is $C^{1,1}$. Then the solution satisfying ii) of Theorem 4.1, belongs to $W_0^1 bmo_r(\Omega)$.*

Proof. First let us notice that according to Theorem 3.1, the operator L satisfies the Uniform Hopf Inequality. Therefore the uniform estimates given by (4.6) and Theorem 4.2 hold true. We will prove that $\nabla u\in bmo_r(\Omega)^n$. For this statement, we will partly use some arguments from Campanato [8]. We shall establish two new *a priori* estimates:

i) **Interior local estimate:**

Lemma 4.1 *For any open smooth sets $\Omega_0,\widetilde{\Omega_0}$ with $\overline{\Omega_0}\subset\widetilde{\Omega_0}$ and $\overline{\widetilde{\Omega_0}}\subset\Omega$, for all $1\leqslant p<\infty$, there exists a constant $C(p;\widetilde{\Omega_0})$ such that:*

$$\|D^2u_\varepsilon\|_{L^p(\Omega_0)}\leqslant C(p;\widetilde{\Omega_0}).$$

Proof. Consider $\widetilde{\delta}=\mathrm{dist}(\widetilde{\Omega}_0;\partial\Omega)$ and introduce function θ_0 be such that:

$$\begin{cases}\theta_0\in C_c^\infty(\widetilde{\Omega}_0),\quad 0\leqslant\theta_0\leqslant 1 \text{ and } \mathit{supp}\ \theta_0\subset\widetilde{\Omega}_0\subset\Omega,\\[2ex] \theta_0=1 \text{ on } \Omega_0,\quad |D^\alpha\theta_0|\leqslant\dfrac{M}{\widetilde{\delta}^{|\alpha|}}\ ,\quad \alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n,\ |\alpha|=\alpha_1+\ldots+\alpha_n.\end{cases}$$

Let $v_\varepsilon=u_\varepsilon\theta_0\in H_0^1(\Omega)$. Then v_ε verifies the local problem:

$$(\mathcal{P}_l):\begin{cases}-div(A(x)\nabla v_\varepsilon)=\dfrac{a\theta_0}{u_\varepsilon+\varepsilon}-div(A(x)u_\varepsilon\nabla\theta_0) & \text{in } \widetilde{\Omega_0},\\ v_\varepsilon=0 & \text{on } \partial\widetilde{\Omega_0}.\end{cases}$$

Let $F_\varepsilon := \dfrac{a\theta_0}{u_\varepsilon+\varepsilon} - div(A(x)u_\varepsilon \nabla\theta_0)$, and let $U_0 = A(x)\nabla\theta_0$. Then $F_\varepsilon = \dfrac{a\theta_0}{u_\varepsilon+\varepsilon} + U_0\nabla u_\varepsilon + u_\varepsilon div(U_0)$, support $U_0 \subset \widetilde{\Omega_0}\setminus\Omega_0$ and

$$\|div(U_0)\|_{L^\infty} + \|U_0\|_{L^\infty} \leqslant \frac{M}{\widetilde{\delta^2}} = M(\widetilde{\Omega_0}).$$

Therefore,using estimates (4.7)

$$\|F_\varepsilon\|_{L^p(\widetilde{\Omega}_0)} \leqslant C_a(\widetilde{\Omega_0})\left(\|\frac{\theta_0}{\widetilde{\delta}}\|_{L^p(\Omega)} + C(\Omega,a)\right). \tag{4.8}$$

By the well known Agmon-Douglis-Nirenberg regularity results we have,

$$\|v_\varepsilon\|_{W^{2,p}(\widetilde{\Omega_0})} \leqslant C_\Omega(p)\|F_\varepsilon\|_{L^p(\widetilde{\Omega}_0)}. \tag{4.9}$$

Since $D^\alpha v_\varepsilon = D^\alpha u_\varepsilon$ on Ω_0, then relation (4.9) leads finally to

$$\|u_\varepsilon\|_{W^{2,p}(\Omega_0)} \leqslant C\|v_\varepsilon\|_{W^{2,p}(\widetilde{\Omega_0})} \leqslant C_p(\widetilde{\Omega_0}).$$

□

As a consequence of Lemma 4.1 on has,

Lemma 4.2 *For all $p \geqslant 1$, for all open smooth set Ω_0 relatively compact in Ω, the sequence u_ε remains in a bounded set of $W^{2,p}(\Omega_0)$. Moreover,the sequence remains in a bounded set of $C^1(\overline{\Omega_0})$.*

Proof. It is a consequence of Lemma 4.1 and the Sobolev embedding,

$$W^{2,p}(\Omega_0) \hookrightarrow C^1(\overline{\Omega_0}), \quad p > n.$$

□

ii) **Estimates in a neighborhood of the boundary:** Since we assume that Ω is of class $C^{1,1}$, for every $x \in \partial\Omega$, we can find, an open neighborhood of x denoted by $\Omega_{0,1}(x)$ and a bijection

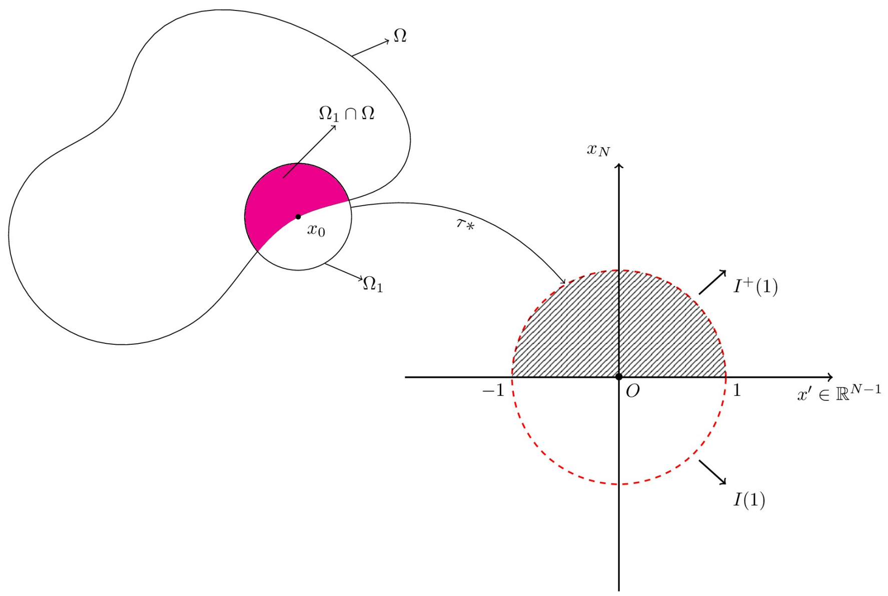
$$\tau : \Omega_{0,1} = \Omega_{0,1} \longrightarrow I^+(1),$$

such that

$$\tau \in C^{1,1}\left(\Omega_{0,1}(x)\right)^n, \quad \tau^{-1} \in C^{1,1}\left(\overline{I^+(1)}\right)^n \text{ and } \tau(\partial\Omega\cap\Omega_{0,1}) = \Gamma_1,$$

where

$$I^+(1) := \left\{x = (x', x_n) \in \mathbb{R}^n \text{ such that } |x| < 1 \text{ and } x_n > 0\right\}$$



and

$$\Gamma_1 := \Big\{x = (x', 0) : |x'| \leqslant 1\Big\}.$$

This means that we can continuously deform the boundary to an hyperplane and that this transformation is regular. After this transformation the problem $(\mathcal{P}_\varepsilon)$ reads

$$\tau(\mathcal{P}_\varepsilon) : \begin{cases} -\operatorname{div}(B(y)\nabla w_\varepsilon) = \dfrac{\widetilde{a}(y)}{w_\varepsilon + \varepsilon} & \text{in } I^+(1), \\ w_\varepsilon = 0 & \text{on } \Gamma_1, \end{cases}$$

with $B(y) \in C^{0,1}\left(\overline{I^+(1)}\right), \widetilde{a}(y) \in L^\infty(I^+(1))$ and $\forall \zeta = (\zeta_1, \cdot \cdot \cdot, \zeta_n) \in \mathbb{R}^n, \exists \nu > 0$ such that $\sum\limits_{i,j} b_{ij}\zeta_i\zeta_j \geqslant \nu|\zeta|^2$. On each ball $I^+(0, r) = I^+(r)$, more generally, for $x_0 \in \overline{I^+(1)}$, we set

$$I^+(x_0, r) := \Big\{x \in I^+(1) : |x - x_0| < r\Big\}, \quad \Gamma_r = \Big\{x \in \overline{I^+(r)} : x_n = 0\Big\},$$

$I(x_0, r)$ is the ball of radius r centered at x_0.
We will construct two Dirichlet problems with constant coefficients such that the sum of the two solutions of these two problems coincides with w_ε.
Let us fix $0 < R < 1$, $x_0 \in \overline{I^+(R)}$, R closed to 1. The first problem will be defined without the right-hand side of $\tau(\mathcal{P}_\varepsilon)$, and having the same trace of w_ε, i.e.,

$$\tau(\mathcal{P}_\varepsilon)_1 : \begin{cases} -\operatorname{div}(B(x_0)\nabla w_\varepsilon^1) = 0 & \text{in } I^+(x_0, r), \\ w_\varepsilon^1 = w_\varepsilon & \text{on } \partial I^+(x_0, r), \end{cases}$$

Here, $0 < r \leqslant \dfrac{1-R}{2}$.

Theorem 4.4 *Campanato [pages 338, 352]*
$\tau(\mathcal{P}_\varepsilon)_1$ admits a positive solution w_ε^1 with

$$\int_{I^+(x_0;r)} |\nabla w_\varepsilon^1| dx \leqslant c_B \int_{I^+(x_0;r)} |\nabla w_\varepsilon|^2 dx \leqslant c(\Omega, a) \text{ (independent of } x_0, \text{ and } r).$$

There exists $c(\nu) > 0, \forall\ 0 < \rho < r$ such that

1. *If $\Gamma_R \cap I(x_0, r) = \emptyset$ then*

$$\|\nabla w_\varepsilon^1 - \{\nabla w_\varepsilon^1\}_\rho\|^2_{L^2(I^+(x_0,\rho))} \leqslant c(\nu) \left(\frac{\rho}{r}\right)^{n+2} \|\nabla w_\varepsilon^1 - \{\nabla w_\varepsilon^1\}_r\|^2_{L^2(I^+(x_0,r))}. \quad (4.10)$$

2. *If $\Gamma_R \cap I(x_0, r) \neq \emptyset$ then*

$$\int_{I^+(x_0;\rho)} \left|D_j w_\varepsilon^1\right|^2 dx \leqslant c(\nu) \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} \left|D_j w_\varepsilon^1\right|^2 dx, \quad j = 1, \ldots, n-1, \quad (4.11)$$

$$\int_{I^+(x_0;\rho)} \left|D_n w_\varepsilon^1 - \{D_n w_\varepsilon^1\}_\rho\right|^2 \leqslant c(\nu) \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;\rho)} \left|D_n w_\varepsilon^1 - \{D_n w_\varepsilon^1\}_r\right|^2 dx. \quad (4.12)$$

Here D_j denotes the partial derivative in the x_j-direction, $j = 1, \ldots n$ and $\{\cdot\}_\rho$ is the average over $I^+(x_0, \rho)$.

Proof. The problem $\tau(\mathcal{P}_\varepsilon)_1$ is identical to the one considered by Campanato [8]. Therefore, his proof can be reproduced line by line to get (4.10), $\Big($this estimate is proven in p.338 by Campanato [8], see relation (8.12) for the local estimate, observing in that case $I(x_0, r) = B(x_0; r) \subset I^+(1)\Big)$.
The second set of relations (4.11) and (4.12), are given in page 352 (Corollary I.11 and Lemma II.11) of Campanato [8].

□

Now, we construct the second problem as follows:

$$\tau(\mathcal{P}_\varepsilon)_2 := \begin{cases} -\operatorname{div}(B(x_0)\nabla w_\varepsilon^2) = \dfrac{\widetilde{a}(y)}{w_\varepsilon + \varepsilon} + \operatorname{div}((B(y) - B(x_0))\nabla w_\varepsilon) & \text{in } I^+(x_0, r), \\ w_\varepsilon^2 = 0 & \text{on } \partial I^+(x_0, r). \end{cases}$$

Theorem 4.5
$\tau(\mathcal{P}_\varepsilon)_2$ admits a unique solution $w^2_\varepsilon \in H^1_0(I^+(x_0;r))$ and for all $\lambda \in [0,n[$,

$$\|\nabla w^2_\varepsilon\|^2_{L^2(I^+(x_0;r))} \leqslant c\int_{I^+(x_0;r)} \widetilde{a}(y)\,dy + r^{\lambda+2}C_\Omega(\lambda,a),$$

for some $c>0$, which depends only on Ω.

Proof. This problem is well-posed since the right hand side is in $H^{-1}(I^+(x_0,r))$ and admits a unique solution $w^2_\varepsilon \in H^1_0(I^+(x_0,r))$ (by using Lax-Milgram Theorem). Note that we have $w_\varepsilon = w^1_\varepsilon + w^2_\varepsilon$.

Estimate on ∇w^2_ε: By multiplying $\tau(\mathcal{P}_\varepsilon)_2$ by w^2_ε, we get

$$\nu\int_{I^+(x_0,r)} |\nabla w^2_\varepsilon|^2\,dy \leqslant \int_{I^+(x_0,r)} \frac{\widetilde{a}(y)w^2_\varepsilon(y)}{w_\varepsilon+\varepsilon}\,dy + \int_{I^+(x_0,r)} (B(y)-B(x_0))\nabla w_\varepsilon\nabla w^2_\varepsilon\,dy.$$

Since $w^2_\varepsilon = w_\varepsilon - w^1_\varepsilon$ and $w^1_\varepsilon \geqslant 0$ then $w^2_\varepsilon \leqslant w_\varepsilon$ the Lipschitz continuity condition on B, Cauchy-Schwartz inequality and Young's inequality, yield

$$\begin{aligned}\int_{I^+(x_0,r)} |\nabla w^2_\varepsilon|^2\,dy &\leqslant c\int_{I^+(x_0,r)} \frac{\widetilde{a}(y)w^2_\varepsilon}{w_\varepsilon+\varepsilon}\,dy + \int_{I^+(x_0,r)} cr|\nabla w_\varepsilon.\nabla w^2_\varepsilon|\,dy,\\ &\leqslant c\int_{I^+(x_0,r)} \widetilde{a}(y)\,dy + \frac{c^2r^2}{2}\int_{I^+(x_0,r)} |\nabla w_\varepsilon|^2\,dy + \frac{1}{2}\int_{I^+(x_0,r)} |\nabla w^2_\varepsilon|^2\,dy.\end{aligned}$$

Thus

$$\int_{I^+(x_0;r)} |\nabla w^2_\varepsilon|^2\,dy \leqslant c\int_{I^+(x_0;r)} \widetilde{a}(y)\,dy + cr^2\int_{I^+(x_0;r)} |\nabla w_\varepsilon|^2\,dy.$$

Next, we want to show that

$$\int_{I^+(x_0;r)} |\nabla w_\varepsilon|^2 dx \leqslant r^\lambda c_\Omega(\lambda,a). \tag{4.13}$$

From relation (4.6) of the proof of Theorem 4.2, we have proved that

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqslant C_\Omega(p,a), \qquad \forall p\in[1,+\infty[.$$

Lemma 4.3 *Let $\lambda\in[0,n[$. Then, $L^p(\Omega)\hookrightarrow L^{2,\lambda}(\Omega)$ provided for $p\geqslant \dfrac{2n}{n-\lambda}$. Moreover, there exists $C(\Omega)>0$ such that*

$$\sup_{\substack{x\in\Omega\\ r>0}}\left[r^{-\lambda}|v|_{L^2(B(x,r)\cap\Omega)}\right] \leqslant C(\Omega)|v|_{L^p(\Omega)}.$$

Proof. Setting $\Omega_{0,r} = B(x,r) \cap \Omega$, we have

$$r^{-\lambda} \int_{\Omega \cap B(x,r)} |v|^2 \, dx = r^{-\lambda} \int_{\Omega} |v|^2 \chi_{\Omega_{0,r}} \, dx \leqslant r^{-\lambda} \|v\|^2_{L^p(\Omega)} |\Omega_{0,r}|^{\frac{p-2}{p}} \leqslant c(\Omega) \|v\|^2_{L^p(\Omega)}.$$

Thus,

$$\sup_{\substack{r>0\\ x_0 \in \Omega}} (r^{-\lambda} \int_{\Omega \cap B(x,r)} |v|^2 \, dx) \leqslant c(\Omega) \|v\|^2_{L^p(\Omega)}.$$

□

Using Lemma 4.3 and relation (4.6), we obtain that

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega \cap B(x,r))} \leqslant r^\lambda C_\Omega(n, \lambda, a). \tag{4.14}$$

By applying the homeomorphism function τ, we obtain

$$\|\nabla w_\varepsilon\|^2_{L^2(I^+(x_0;r))} \leqslant r^\lambda C_\Omega(\lambda, a). \tag{4.15}$$

Therefore,

$$\|\nabla w^2_\varepsilon\|^2_{L^2(I^+(x_0;r))} \leqslant c \int_{I^+(x_0;r)} \widetilde{a}(y) \, dy + r^{\lambda+2} C_\Omega(\lambda, a). \tag{4.16}$$

□

Next, we want prove that for all $R < 1$

$$\sup_{\substack{\forall x_0 \in \overline{I^+(R)}\\ \forall \rho > 0}} \rho^{-n} \|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\|^2_{L^2(I^+(x_0,\rho))} \leqslant C(R) < \infty. \tag{4.17}$$

Let $x_0 \in \overline{I^+(R)}$, se set $\delta_0 = \dfrac{1-R}{2}$ with $0 < R < 1$. We have two cases to be analyzed :

1^{st} case : $\rho \geqslant \dfrac{1-R}{2}$

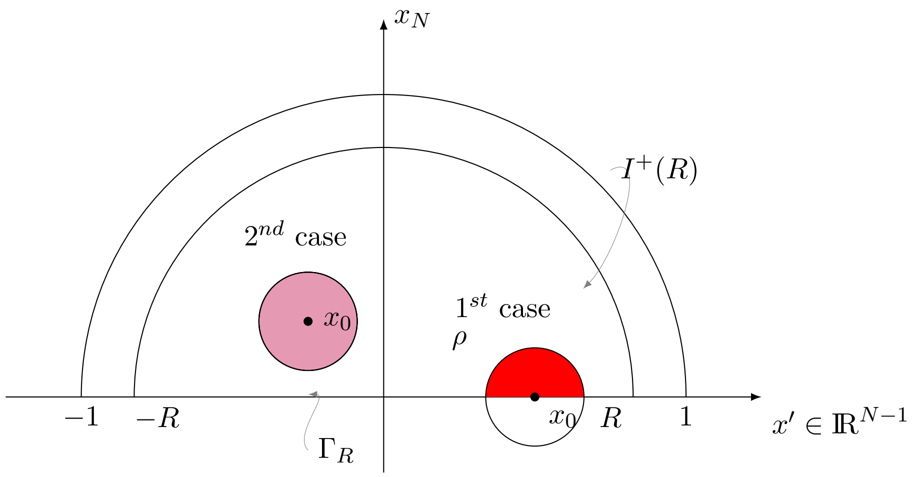
In this case, we have for any x_0

$$\rho^{-n} \Big\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\Big\|^2_{L^2(I^+(x_0;\rho))} \leqslant \left(\frac{2}{1-R}\right)^n \|\nabla w_\varepsilon\|_{L^2(I^+(1))} \leqslant c_\Omega \frac{\|a\|_\infty}{(1-R)^n}. \tag{4.18}$$

2^{nd} case : $0 < \rho < \dfrac{1-R}{2}$. Let r be such $0 < \rho < r \leqslant \dfrac{1-R}{2}$. We have two subcases : either $\Gamma_R \cap I(x_0;r) = \emptyset$ or $\Gamma_R \cap I(x_0;r) \neq \emptyset$.

(a) For the first case, $\Gamma_R \cap I(x_0;r) = \emptyset$, we first write $w_\varepsilon = w^1_\varepsilon + w^2_\varepsilon$ and apply estimate (4.10) of Theorem 4.4 to derive

$$\begin{aligned}\Big\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\Big\|^2_{L^2(I_+(x_0;\rho))} &\leqslant c(\nu) \left(\frac{\rho}{r}\right)^{n+2} \Big\|\nabla w^1_\varepsilon - \{\nabla w^1_\varepsilon\}_r\Big\|^2_{L^2(I(x_0;r))} \\ &\quad + \Big\|\nabla w^2_\varepsilon - \{\nabla w^2_\varepsilon\}_\rho\Big\|^2_{L^2(I(x_0;\rho))}.\end{aligned}$$



Applying Theorem 4.5 and the decomposition $w_\varepsilon^1 = w_\varepsilon - w_\varepsilon^2$,

$$\begin{aligned} \|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\|^2_{L^2(I^+(x_0,\rho))} &\leqslant c(\nu)\left(\frac{\rho}{r}\right)^{n+2}\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_r\|^2_{L^2(I^+(x_0,r))} \\ &\quad + r^{\lambda+2}C_\Omega(\lambda,a) + 2\int_{I^+(x_0,r)} \widetilde{a}(y)\,dy. \end{aligned}$$

Choosing $\lambda = n-1$, one has for all $0<\rho<r$

$$\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\|^2_{L^2(I^+(x_0,\rho))} \leqslant c(\nu)\left(\frac{\rho}{r}\right)^{n+2}\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_r\|^2_{L^2(I^+(x_0,r))} + c(\Omega)r^n. \tag{4.19}$$

Applying the iteration Lemma 2.3 (see [8] [14]) on (4.13) with

$$\Phi(\rho) = \|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\|^2_{L^2(I^+(x_0,\rho))},$$

we get

$$\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\|^2_{L^2(I^+(x_0,\rho))} \leqslant c\left(\frac{\rho}{r}\right)^{n}\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_r\|^2_{L^2(I^+(x_0,r))} + c(\Omega)\rho^n.$$

Dividing by ρ^n, we obtain

$$\frac{1}{\rho^n}\Big\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\Big\|^2_{L^2(I^+(x_0,\rho))} \leqslant \frac{c}{r^n}\|\nabla w_\varepsilon\|^2_{L^2(I^+(1))} + c(\Omega). \tag{4.20}$$

(b) For the second case, $\Gamma_R \cap I(x_0; r) \neq \emptyset$, we need to use relation (4.11) of Theorem 4.4 by distinguishing the x_j-direction, $j \leqslant n-1$ and x_n-direction. For $j = 1, \dots, n-2$, since $w_\varepsilon = w_\varepsilon^1 + w_\varepsilon^2$, we have

$$\int_{I^+(x_0;\rho)} |D_j w_\varepsilon|^2 dx \leqslant 2 \int_{I^+(x_0;\rho)} |D_j w_\varepsilon^1|^2 dx + 2 \int_{I^+(x_0;\rho)} |D_j w_\varepsilon^2|^2 dx \qquad (4.21)$$

Using Theorem 4.4 relation (4.11) and Theorem 4.5, we derive from this last relation that

$$\begin{aligned} \int_{I^+(x_0;\rho)} |D_j w_\varepsilon|^2 &\leqslant c \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} |D_j w_\varepsilon^1|^2 dx + cr^n \\ &\leqslant c \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} |D_j w_\varepsilon|^2 dx + c \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} |D_j w_\varepsilon^2|^2 dx + cr^n. \end{aligned} \qquad (4.22)$$

Using again Theorem 4.5 with relation (4.21), we deduce

$$\int_{I^+(x_0;\rho)} |D_j w_\varepsilon|^2 dx \leqslant c \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} |D_j w_\varepsilon|^2 dx + cr^n. \qquad (4.23)$$

This last relation is valid for all $0 < \rho < r$ we may appeal the iteration lemma (see [14], [38]), to derive

$$\int_{I^+(x_0;\rho)} |D_j w_\varepsilon|^2 dx \leqslant c \left(\frac{\rho}{r}\right)^{n} \int_{I^+(x_0;r)} |D_j w_\varepsilon|^2 dx + c.\rho^n \qquad (4.24)$$

Thus

$$\rho^{-n} \int_{I^+(x_0;\rho)} \left| D_j w_\varepsilon - \left\{ D_j w_\varepsilon \right\}_\rho \right|^2 dx \leqslant \rho^{-n} \int_{I^+(x_0;\rho)} |D_j w_\varepsilon|^2 \leqslant c \frac{1}{r^n} \int_{I^+(1)} \left| D_j w_\varepsilon \right|^2 dx + c. \qquad (4.25)$$

In the x_n-direction, we have from Theorem 4.4, relation (4.12) and Theorem 4.5 with $\lambda = n-1$, for all $0 < \rho < r$

$$\begin{aligned} \int_{I^+(x_0;\rho)} \left| D_n w_\varepsilon - \{D_n w_\varepsilon\}_\rho \right|^2 &\leqslant \int_{I^+(x_0;\rho)} \left| D_n w_\varepsilon^1 - \{D_n w_\varepsilon^1\}_\rho \right|^2 dx + \int_{I^+(x_0;\rho)} \left| D_n w_\varepsilon^2 \right|^2 dx \\ &\leqslant c \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} \left| D_n w_\varepsilon^1 - \{D_n w_\varepsilon^1\}_r \right|^2 dx + cr^n \\ &\leqslant c \left(\frac{\rho}{r}\right)^{n+2} \int_{I^+(x_0;r)} \left| D_n w_\varepsilon - \{D_n w_\varepsilon\}_r \right|^2 dx. \end{aligned} \qquad (4.26)$$

Thus, we may appeal the iteration Lemma 2.3 with $\Phi(\rho) = \int_{I^+(x_0;\rho)} \left| D_n w_\varepsilon - \{D_n w_\varepsilon\}_\rho \right|^2 dx$ to derive

$$\rho^{-n} \int_{I^+(x_0;\rho)} \left| D_n w_\varepsilon - \{D_n w_\varepsilon\} \right|^2 dx \leqslant \frac{c}{r^n} \int_{I^+(1)} |D_n w_\varepsilon|^2 dx + c. \qquad (4.27)$$

In all the cases, from relations (4.20), (4.25), (4.27), there exists a constant c , $\forall\, 0 < \rho < r$, for all $x_0 \in \overline{I^+(R)}$

$$\rho^{-n}\Big\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\Big\|^2_{L^2(I^+(x_0;\rho))} \leqslant c\frac{1}{r^n}\|\nabla w_\varepsilon\|^2_{L^2(I^+(1))} + c, \tag{4.28}$$

which infer that

$$\sup_{\rho>0,\ x_0\in\overline{I^+(R)}} \rho^{-n}\Big\|\nabla w_\varepsilon - \{\nabla w_\varepsilon\}_\rho\Big\|^2_{L^2(I^+(x_0;\rho))} \leqslant \frac{c}{(1-R)^n} < \infty.$$

This ends the proof of (4.17). □

Applying τ^{-1} on relation (4.17), we derive that

$$u_\varepsilon \text{ remains in a bounded set of } W^1bmo_r\Big(\tau^{-1}(I^+(R))\Big) \text{ for all } R<1.$$

In the local estimate, we have proved that $\nabla u_\varepsilon \in bmo_r(\Omega_0)^n$, $\forall \Omega_0 \subset\subset \Omega$, and in the estimate in a neighborhood of the boundary $\tau^{-1}(I^+(R)) = \Omega_{0,R}(x)$ for all $R<1$, we proved that $\nabla u_\varepsilon \in bmo_r(\Omega_{0,R}(x))^n$ with $x \in \partial\Omega$. Collecting both results on local estimates and boundary estimates, we can conclude as in Campanato [8] that ∇u_ε remains in a bounded set of $bmo_r(\Omega)^n$. This implies that u_ε belongs to a bounded set of $W^1_0 bmo_r(\Omega)$. This shows that $u \in W^1_0 bmo_r(\Omega)$. □

5 Case where the right hand side is $a(x)u^{-m}(x),\ m>0$

In this paragraph, we want to discuss the existence and the regularity of solution for the following problem

$$\begin{cases} Lu = -\operatorname{div}(A(x)\nabla u) = a(x)u^{-m}(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As we shall see the regularity of the solution relies not only on the value of m but also on the regularity of the coefficients of $A(x) = (a_{ij}(x))_{i,j}$ the domain is still a Lipschitz one. For an alternative proof see also [3]. More precisely, we want to show :

Theorem 5.1 *(Existence)*
Let $a \in L^\infty_+(\Omega)$, $m>0$, and let $A(x) = (a_{ij}(x))_{i,j}$ be an α-coercive matrix with bounded coefficients. Then there exists a positive function $u \in H^1_{loc}(\Omega)$ such that

1) $u \in L^q(\Omega),\ q = \dfrac{2^*}{2}(m+1),\ \Big[2^* \textit{ is the Sobolev exponent}\Big],\ \dfrac{a}{u^m} \in L^1_{loc}(\Omega)$, *and*

$$\int_\Omega |u(x)|^q\,dx \leqslant C_\Omega\left(\frac{\|a\|_\infty}{\alpha}\right)^{\frac{2^*}{2}}.$$

2) $\forall\psi\in\mathcal{D}(\Omega)$, $\displaystyle\int_\Omega A(x)\nabla u\cdot\nabla\psi\,dx=\int_\Omega\frac{a(x)\psi(x)}{u^m(x)}\,dx$, *and* $u^{\frac{m+1}{2}}\in H_0^1(\Omega)$. *Moreover,* $\displaystyle\int_\Omega|\nabla u|^2u^{m-1}\,dx\leqslant\frac{\|a\|_\infty}{\alpha m}\left(\frac{m+1}{2}\right)^2.$

In the special case of $0<m\leqslant 1$, *then* $\dfrac{a}{u}\in L^1(\Omega,\delta)$ *and* $u\in H_0^1(\Omega)$.

Proof. Let $\varepsilon>0$. Then, there exists a non negative function $u_\varepsilon\in H_0^1(\Omega)\cap L^\infty(\Omega)$ satisfying

$$\int_\Omega A(x)\nabla u_\varepsilon\cdot\nabla\varphi\,dx=\int_\Omega\frac{a(x)\varphi(x)}{u_\varepsilon^m+\varepsilon}\,dx,\qquad\forall\varphi\in H_0^1(\Omega).\tag{5.1}$$

Then, one has the following *a priori* estimates

$$\int_\Omega|\nabla(u_\varepsilon^{\frac{m+1}{2}})|^2\,dx\leqslant\frac{\|a\|_\infty}{\alpha m}\left(\frac{m+1}{2}\right)^2,\tag{5.2}$$

and there exists $C_\Omega>0$, such that

$$\int_\Omega|u_\varepsilon|^q\,dx\leqslant C_\Omega\left(\frac{\|a\|_\infty}{\alpha}\right)^{\frac{2^*}{2}}\tag{5.3}$$

with $q=\dfrac{2^*}{2}(m+1)$, again [2^* is the Sobolev exponent].
Indeed, for the first inequality, we choose $\varphi=u_\varepsilon^m$ as a test function and we use the coercivity condition on A to derive that

$$m\alpha\int_\Omega\nabla u_\varepsilon\cdot\nabla u_\varepsilon^m\,dx\leqslant\int_\Omega A(x)\nabla u_\varepsilon\cdot\nabla u_\varepsilon^m\,dx=\int_\Omega\frac{a(x)u_\varepsilon^m}{u_\varepsilon^m+\varepsilon}\leqslant\|a\|_\infty.$$

Therefore,

$$\int_\Omega|\nabla u_\varepsilon|^2u_\varepsilon^{m-1}\,dx\leqslant\frac{\|a\|_\infty}{\alpha m}.$$

Consequently,

$$\int_\Omega|\nabla(u_\varepsilon^{\frac{m+1}{2}})|^2\,dx=\left(\frac{m+1}{2}\right)^2\int_\Omega|\nabla u_\varepsilon|^2u_\varepsilon^{m-1}\,dx\leqslant\left(\frac{m+1}{2}\right)^2\frac{\|a\|_\infty}{\alpha m}.\tag{5.4}$$

While for the second inequality, we shall set $v_\varepsilon=u_\varepsilon^{\frac{m+1}{2}}$ and we have

$$\int_\Omega|u_\varepsilon|^q\,dx=\int_\Omega|v_\varepsilon|^{2^*}\,dx\leqslant C_\Omega\left(\int_\Omega|\nabla v_\varepsilon|^2\,dx\right)^{\frac{2^*}{2}}\leqslant C_\Omega\left(\frac{\|a\|_\infty}{\alpha m}\right)^{\frac{2^*}{2}}\left(\frac{m+1}{2}\right)^2.$$

Using the Rellich-Kondrachov compactness, we may assume that there exists $v\in H_0^1(\Omega)$ such that $v\geqslant 0$ and $v_\varepsilon(x)\longrightarrow v(x)$ a.e. in Ω. Then,

$$v_\varepsilon^{\frac{2}{m+1}}=u_\varepsilon\longrightarrow v(x)^{\frac{2}{m+1}}=u(x),\ \text{ a.e. and } v_\varepsilon\longrightarrow v\text{ weakly in }H_0^1(\Omega).$$

So,

$$\int_\Omega |\nabla v(x)|^2\,dx \leqslant \frac{\|a\|_\infty}{\alpha m}\left(\frac{m+1}{2}\right)^2.$$

By Fatou's lemma, we have from the estimate (5.5)

$$\int_\Omega |u|^q\,dx \leqslant C_\Omega \left(\frac{\|a\|_\infty}{\alpha m}\right)^{\frac{2^*}{2}}\left(\frac{m+1}{2}\right). \tag{5.5}$$

Then we deduce from the Theorem 2.2 that for all open set Ω_0 relatively compact in Ω, we have for a.e. $y \in \overline{\Omega}_0$

$$u_\varepsilon(y) \geqslant C'_{\Omega_0}\int_{\Omega_0}\frac{a(x)\delta(x)}{u_\varepsilon^m+\varepsilon}\,dx > 0. \tag{5.6}$$

We know from the Egorov theorem that there is a set B in Ω_0 of positive measure on which $\sup\limits_{y\in B}\sup\limits_{\varepsilon>0} u_\varepsilon(y) = M$ is finite. Thus, there is a constant $C(\Omega_0) > 0$ such that $\forall\varepsilon > 0$

$$\int_{\Omega_0}\frac{a(x)\delta(x)}{u_\varepsilon^m+\varepsilon}\,dx \leqslant C(\Omega_0). \tag{5.7}$$

By Fatou's lemma, we deduce

$$0 < \int_{\Omega_0}\frac{a(x)}{u^m(x)}\delta(x)\,dx \leqslant C(\Omega_0) < +\infty. \tag{5.8}$$

• If $0 < m \leqslant 1$, we can choose $\varphi = u_\varepsilon$ as a test function and get

$$\alpha\int_\Omega |\nabla u_\varepsilon|^2\,dx \leqslant \int_\Omega A(x)(\nabla u_\varepsilon)^2\,dx = \int_\Omega \frac{a(x)u_\varepsilon}{u_\varepsilon^m+\varepsilon} \leqslant \|a\|_\infty\int_\Omega u_\varepsilon^{1-m}\,dx.$$

This implies that

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx \quad\leqslant\quad \frac{\|a\|_\infty}{\alpha}\int_\Omega u_\varepsilon^{1-m}\,dx \leqslant C(\Omega,a,\alpha), \quad \Big(\text{since } 0 \leqslant 1-m < \frac{2^*}{2}(m+1)\Big).$$

This and (5.7) yield that $u \in H_0^1(\Omega)$ and that u_ε converges weakly to u in $H_0^1(\Omega)$.

• If $m > 1$, then from (5.7) and (5.8), we have

$$\lambda_0 = \inf_{\varepsilon<\varepsilon_1}\int_{\Omega_0}\frac{a(x)\delta(x)}{u_\varepsilon^m(x)+\varepsilon}\,dx > 0, \quad \text{for some } \varepsilon_1 > 0.$$

From (5.4) and (5.6), we have

$$\frac{\|a\|_\infty}{\alpha} \geqslant \int_{\Omega_0}|\nabla u_\varepsilon|^2 u_\varepsilon^{m-1}\,dx \geqslant {C'_{\Omega_0}}^{m-1}\lambda_0^{m-1}\int_{\Omega_0}|\nabla u_\varepsilon|^2\,dx. \tag{5.9}$$

We then deduce that

$$\int_{\Omega_0} |\nabla u_\varepsilon|^2 \, dx \leqslant C_{\Omega_0} < +\infty,$$

and jointly with the estimate on the L^q norm of u_ε, we deduce that u_ε remains in a bounded set of $H^1_{loc}(\Omega)$. Therefore, since for all $\psi \in \mathcal{D}(\Omega)$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega \frac{a(x)\psi(x)}{u_\varepsilon^m(x)+\varepsilon} \, dx = \int_\Omega \frac{a(x)\psi(x)}{u^m(x)} \, dx$$

$\forall\ \Omega_0 \subset\subset \Omega$, recalling that $u_\varepsilon(y) \geqslant C'_{\Omega_0}\lambda_0 > 0$, and u_ε converges weakly to u in $H^1(\Omega_0)$ for all $\Omega_0 \subset\subset \Omega$, we get the statement 2) of the theorem.

$\square$

Remark 3 *(the uniqueness of u_ε and u.)*
The function u_ε is unique since the mapping $t \to \dfrac{a(x)}{t^m+\varepsilon}$ is decreasing and the regularity of u_ε allows at to choose $\varphi = u_\varepsilon - \overline{u_\varepsilon}$ as a test function whenever u_ε and $\overline{u_\varepsilon}$ are two differents solutions. The same remarks holds for u when $0 < m \leqslant 1$.

Next, we want to study the regularity of the function $u = \lim u_\varepsilon$ constructed in Theorem 5.1.

Theorem 5.2 *(regularity)*
Assume that the operator L satisfies the Uniform Hopf Inequality. Then, the function $u = \lim u_\varepsilon$ satisfies

1) $\dfrac{a}{u^m} \in L^1(\Omega;\delta)$ *and* $u(y) \geqslant C_\Omega \delta(y) \displaystyle\int_\Omega \frac{a\delta}{u^m}(x)\, dx$ *for a.e.* $y \in \Omega$.

2) *For all* $m \geqslant 1$,

$$\int_\Omega |\nabla u|^2 \delta^{m-1} \, dx \leqslant C'_\Omega \frac{\|a\|_\infty}{m\alpha \left(\displaystyle\int_\Omega \frac{a\delta}{u^m}(x)\, dx \right)^{m-1}}.$$

In particular, $u \in W^{1,2}(\Omega;\delta^{m-1})$.

3) *Assume that* $a_{ij} \in vmo(\Omega) \cap L^\infty(\Omega)$ *and* $\partial\Omega$ *is* C^1, *then*

   a) *if* $0 < m \leqslant 1$, *then* $u \in W_0^{1,p}(\Omega)$ *for all* $1 \leqslant p < +\infty$,

   b) *if* $1 < m < 2$, *then* $u \in W_0^1 L^{\frac{1}{m-1},\infty}(\Omega)$.

4) *Assume that* $a_{ij} \in C^{0,1}(\overline{\Omega})$ *and* $\partial\Omega$ *is* $C^{1,1}$. *Then, if* $0 < m \leqslant 1$, *we have*

$$u \in W_0^1 bmo_r(\Omega).$$

Proof. By Fatou's lemma, we have

$$\liminf_{\varepsilon} \int_\Omega \frac{a\delta}{u_\varepsilon^m+\varepsilon}(x)\,dx \geqslant \int_\Omega \frac{a\delta}{u^m}(x)\,dx > 0, \quad \text{since } a \neq 0, \tag{5.10}$$

and $u(x) < +\infty$ a.e. On the other hand, since L satisfies the Uniform Hopf Inequality, we get

$$u_\varepsilon(x) \geqslant C_\Omega \delta(x) \int_\Omega \frac{a\delta}{u_\varepsilon^m+\varepsilon}(y)\,dy. \tag{5.11}$$

Relations (5.10) and (5.11) with the fact that $u_\varepsilon(x)$ converges to $u(x)$ a.e., infers statement 1).

Now combining this last inequality with (5.4), one has, for $m \geqslant 1$,

$$m\alpha C_\Omega^{m-1}\left(\int_\Omega \frac{a\delta}{u_\varepsilon^m+\varepsilon}(y)\,dy\right)^{m-1}\int_\Omega |\nabla u_\varepsilon|^2\delta^{m-1}\,dx \leqslant \|a\|_\infty. \tag{5.12}$$

Thanks to (5.10) and (5.12), we deduce that u_ε remains in a bounded set of $W^{1,2}(\Omega;\delta^{m-1}) = \{\varphi \in L^2(\Omega;\delta^{m-1}) : \int_\Omega |\nabla\varphi|^2\delta^{m-1}\,dx < +\infty\}$. Therefore, u_ε converges weakly to u in $W^{1,2}(\Omega;\delta^{m-1})$ and from relation (5.12), we then have

$$m\alpha C_\Omega^{m-1}\left(\int_\Omega \frac{a\delta}{u^m}(y)\,dy\right)^{m-1}\int_\Omega |\nabla u|^2\delta^{m-1}\,dx \leqslant \|a\|_\infty. \tag{5.13}$$

This proves the second statement.

To prove 3.a), let us show that $\dfrac{a}{u_\varepsilon^m+\varepsilon} \in W^{-1,p}(\Omega)$ with $1 < p < +\infty$. If $0 < m \leqslant 1$, one has from (5.11), by taking $\varphi \in W_0^{1,p'}(\Omega)$, $\dfrac{1}{p}+\dfrac{1}{p'} = 1$, $\varphi \geqslant 0$,

$$0 \leqslant \int_\Omega \frac{a}{u_\varepsilon^m+\varepsilon}\varphi\,dx \leqslant \frac{C_\Omega^0\|a\|_\infty}{\left(\displaystyle\int_\Omega \frac{a\delta}{u_\varepsilon^m+\varepsilon}\,dy\right)^m}\int_\Omega \frac{\varphi}{\delta^m}\,dx \leqslant C_\Omega^1\left(\int_\Omega \frac{\varphi}{\delta^m}\,dx\right). \tag{5.14}$$

If $0 < m \leqslant 1$, we have $\dfrac{1}{\delta^m} = \dfrac{1}{\delta}\delta^{1-m} \leqslant C_\Omega\dfrac{1}{\delta}$, so the Hardy inequality leads to

$$0 \leqslant \int_\Omega \frac{a}{u_\varepsilon^m+\varepsilon}\varphi\,dx \leqslant C_\Omega^1\int_\Omega \frac{\varphi}{\delta}\,dx \leqslant C_\Omega^2\left\|\frac{\varphi}{\delta}\right\|_{L^{p'}(\Omega)} \leqslant C\|\nabla\varphi\|_{W^{1,p'}(\Omega)},\ p' > 1,$$

which implies that

$$\left\|\frac{a}{u_\varepsilon^m+\varepsilon}\right\|_{W^{-1,p}(\Omega)} \leqslant C(p) \quad \text{uniformly in } \varepsilon.$$

Now, to prove 3.b), let us show that $\dfrac{a}{u_\varepsilon^m+\varepsilon} \in W^{-1}L^{\frac{1}{m-1},+\infty}(\Omega)$, with $W^{-1}L^{\frac{1}{m-1},+\infty}(\Omega)$ being the dual of $W_0^1L^{\frac{1}{m-1},+\infty}(\Omega)$.

If $1 < m < 2$, we deduce from relation (5.11) and Hardy inequality with weights (see statement 2, section 1) $\forall \varphi \in C^1_c(\Omega)$

$$\begin{aligned} 0 \leqslant \int_\Omega \frac{a}{u_\varepsilon^m + \varepsilon}|\varphi|\,dx &\leqslant C'_\Omega \int_\Omega |\varphi|\delta^{-m}\,dx \leqslant C'_\Omega \int_\Omega |\nabla\varphi|\delta^{1-m}\,dx \\ &\leqslant C'_\Omega |\nabla\varphi|_{L^{n_m,1}(\Omega)} \|\delta^{1-m}\|_{L^{\frac{1}{m+1},\infty}(\Omega)},\ n_m = \frac{1}{2-m}, \end{aligned}$$

according to Diáz-Rakotoson ([16], p.53-54), $\delta^{1-m} \in L^{\frac{1}{m-1},+\infty}(\Omega)$ Thus,

$$\frac{a}{u_\varepsilon^m + \varepsilon} \in W^{-1}L^{\frac{1}{m-1},\infty}(\Omega).$$

Now, we apply the regularity result to u_ε satisfying $-\operatorname{div}(A(x)\nabla u_\varepsilon) = \dfrac{a}{u_\varepsilon^m + \varepsilon}$.

If $1 < m < 2$,

$$\|\nabla u_\varepsilon\|_{L^{\frac{1}{m-1},\infty}(\Omega)} \leqslant C_\Omega \|\delta^{1-m}\|_{L^{\frac{1}{m-1},\infty}(\Omega)} < +\infty.$$

If $0 < m \leqslant 1$, for $1 < p < +\infty$,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqslant C_\Omega(p) < +\infty.$$

Finally, for statement 4), if $a_{ij} \in C^{0,1}(\overline{\Omega})$, then u_ε remains in a bounded set of $W^1 bmo_r(\Omega)$. Indeed, we write $\dfrac{a}{u_\varepsilon^m + \varepsilon} = \dfrac{a}{u_\varepsilon + \varepsilon} \cdot \dfrac{u_\varepsilon + \varepsilon}{u_\varepsilon^m + \varepsilon} = \dfrac{a_\varepsilon}{u_\varepsilon + \varepsilon}$. We have

$$0 \leqslant a_\varepsilon = a \cdot \frac{u_\varepsilon + \varepsilon}{u_\varepsilon^m + \varepsilon} \leqslant \|a\|_\infty u_\varepsilon^{1-m} \leqslant C_\Omega.$$

If $0 < m \leqslant 1$, since u_ε remains in a bounded set of $W_0^{1,p}(\Omega)$, $p > n$ according to statement 3.a), thus $u_\varepsilon \in W_0^{1,p}(\Omega)$, $\forall p < +\infty$ and satisfies

$$-\operatorname{div}(A(x)\nabla u_\varepsilon) = \frac{a_\varepsilon}{u_\varepsilon + \varepsilon}, \tag{5.15}$$

with a_ε remaining in a bounded set of $L^\infty(\Omega)$. Thus, we conclude (following the $bmo_r(\Omega)$ result previously proved) that the solution u_ε of (5.15) is in a bounded set of $W_0^1 bmo_r(\Omega)$.

□

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