



# Generalized Gagliardo–Nirenberg inequalities using Lorentz spaces, BMO, Hölder spaces and fractional Sobolev spaces

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## ABSTRACT

The main purpose of this paper is to prove some generalized Gagliardo–Nirenberg interpolation inequalities involving the Lorentz spaces  $L^{p,\alpha}$ , BMO and the fractional Sobolev spaces  $W^{s,p}$ , including also  $\dot{C}^\eta$  Hölder spaces. Although some of the results can be alternatively obtained by using interpolation spaces (specifically, the reiteration theorem), the precise form of the inequalities stated here appears to be novel and, moreover, the proofs given in the present paper are self-contained (save for the use of the John–Nirenberg inequality for the BMO result) in contrast to the other mentioned approach. The use of  $\dot{C}^\eta$  Hölder spaces in such Gagliardo–Nirenberg inequalities seems to be new in the literature.

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## 1. Introduction

The main purpose of this paper is to prove some generalized Gagliardo–Nirenberg interpolation inequalities involving the Lorentz spaces  $L^{p,\alpha}$ , BMO, and the fractional Sobolev spaces  $W^{s,p}$ , including also  $\dot{C}^\eta$  Hölder spaces.

It is well known that the Gagliardo–Nirenberg inequality plays an important role in the analysis of PDEs, see e.g. [10,8,9,1,11,7,3] and the references therein. Thus, any possible improvement of this one could be relevant for many purposes. First of all, let us recall some previous results involving the Gagliardo–Nirenberg inequalities that we shall improve later:

For any  $1 \leq q < p < \infty$ , the following interpolation inequality holds (see Nirenberg [10])

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L^q(\mathbb{R}^n)}^\theta \|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}, \quad \frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right). \quad (1.1)$$

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More recently, in [9], McCormick et al. proved a stronger version of (1.1) involving the weak  $L^q$  space (denoted as  $L^{q,\infty}$ ) as follows:

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^\theta \|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}. \tag{1.2}$$

Concerning the critical case  $s = n/2$ , McCormick et al. [9] obtained

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{q/p} \|f\|_{BMO(\mathbb{R}^n)}^{1-q/p}. \tag{1.3}$$

Note that (1.3) is better than (1.2) since  $\|f\|_{BMO(\mathbb{R}^n)} \leq c \|f\|_{\dot{H}^{n/2}(\mathbb{R}^n)}$ . Furthermore, they also showed a stronger version of inequality (1.3) for the norm  $\|f\|_{L^{p,1}(\mathbb{R}^n)}$  instead of  $\|f\|_{L^p(\mathbb{R}^n)}$  when  $q > 1$ .

Another version of (1.1), in the critical case, was proved by Kozono and Wadade [8] (see also [1]):

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L^q(\mathbb{R}^n)}^{q/p} \|f\|_{\dot{H}^{n/r,r}(\mathbb{R}^n)}^{1-\frac{q}{p}}, \tag{1.4}$$

for any  $1 \leq q < p < \infty$ , and for  $1 < r < \infty$ .

In this paper, we enhance the results of McCormick et al., [9]. We shall prove a stronger version of (1.2)

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq c \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^\theta \|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}. \tag{1.5}$$

In fact, we shall prove an interpolation inequality which implies (1.5) (see Theorem 2.1 below). After that, we shall prove that (1.3) holds for  $\|f\|_{L^{p,\alpha}}$  instead of  $\|f\|_{L^p}$ , for any  $\alpha > 0$ . Finally, we shall study the Gagliardo–Nirenberg type inequality for the case  $sp > n$  of the fractional Sobolev space  $W^{s,p}$ , and also the Lipschitz and Hölder continuous space. Let us point out that although some of the results can be alternatively obtained by using interpolation spaces (specifically, the reiteration theorem), the precise forms of the inequalities stated here appear to be novel and, moreover, the proofs given in the present paper are self-contained (save for the use of the John–Nirenberg inequality for the BMO result, which will be recalled later) in contrast to the other mentioned approach.

For the reader convenience, we recall here the definition of the functional spaces that we use throughout this paper. We define

$$\|g\|_{L^{q,\alpha}(\mathbb{R}^n)} := \begin{cases} \left( q \int_0^\infty (\lambda^q |\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|)^{\frac{\alpha}{q}} \frac{d\lambda}{\lambda} \right)^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup_{\lambda > 0} \lambda (|\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|)^{1/q} & \text{if } \alpha = \infty. \end{cases}$$

The Lorentz spaces  $L^{q,\alpha}(\mathbb{R}^n)$  includes all measurable functions  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\|g\|_{L^{q,\alpha}(\mathbb{R}^n)} < \infty$ . For a definition of Lorentz spaces using rearrangement techniques see [12].

On the other hand, we recall that the space  $\dot{H}^{s,r}(\mathbb{R}^n)$ , the homogeneous Sobolev space, is defined by

$$\dot{H}^{s,r}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|(-\Delta)^{\frac{s}{2}} f\|_{L^r} < \infty\}.$$

In particular, we shall denote  $\dot{H}^s(\mathbb{R}^n) = \dot{H}^{s,2}(\mathbb{R}^n)$  (see, for instance, [5]).

We denote the space of Lipschitz (or Hölder) continuous functions of order  $\eta \in (0, 1]$  on  $\mathbb{R}^n$  by  $\dot{C}^\eta(\mathbb{R}^n)$ : i.e. functions  $f$  such that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} < \infty.$$

It is useful to introduce the notation

$$\|f\|_{\dot{C}^\eta(\mathbb{R}^n)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta}$$

(see, e.g. section 6.3 of [5]).

On the other hand, if  $s \in (0, 1)$ , then we recall  $W^{s,p}(\mathbb{R}^n)$  the fractional Sobolev space, endowed with the norm:

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \left( \|f\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}.$$

When  $s > 1$ , and  $s$  is not an integer, we write  $s = m + \sigma$ , with  $m$  is an integer and  $\sigma \in (0, 1)$ . Then,  $W^{s,p}(\mathbb{R}^n)$  is endowed with the norm (see, e.g. [2]):

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \left( \|f\|_{W^{m,p}(\mathbb{R}^n)}^p + \sum_{|\alpha|=m} \|D^\alpha f\|_{W^{\sigma,p}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$$

Note that if  $s \geq 0$  is an integer, then  $W^{s,p}(\mathbb{R}^n)$  is the usual Sobolev space.

Finally, concerning the space of functions with bounded mean oscillation (denoted as *BMO*) we refer, for instance, to the presentation made in [4].

### 2. Main results

In this section, we state our main results. First of all, we show an interpolation inequality which is regarded as a generalized version of (1.2).

**Theorem 2.1.** *Let  $0 < q < p < r \leq \infty$  and  $\alpha > 0$ . If  $f \in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n)$ , then  $f \in L^{p,\alpha}(\mathbb{R}^n)$  and*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^\theta \|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{1-\theta}, \tag{2.1}$$

where  $C = C(q, r, p, \alpha) > 0$ , and

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}.$$

As a consequence of Theorem 2.1, we have

**Corollary 2.2.** *Let  $1 \leq q < p$ , and  $s \geq 0$  with  $s > n(1/2 - 1/p)$ . For any  $\alpha > 0$ , there is a constant  $C = C(n, p, q, s, \alpha)$  such that if  $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$  then  $f \in L^{p,\alpha}(\mathbb{R}^n)$  and*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^\theta \|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}, \tag{2.2}$$

with

$$\frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right).$$

Concerning the critical case  $s = n/2$ , we have the following result.

**Theorem 2.3.** *Let  $1 < q < p < \infty$  and  $\alpha > 0$ . If  $f \in L^{q,\infty}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ , then  $f \in L^{p,\alpha}(\mathbb{R}^n)$ , and there is a constant  $C = C(q, p, n, \alpha) > 0$  such that*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}} \|f\|_{BMO(\mathbb{R}^n)}^{1-\frac{q}{p}}. \tag{2.3}$$

Finally, we obtain the Gagliardo–Nirenberg inequality for the  $\eta$ -Hölder space, and the fractional Sobolev space  $W^{s,p}(\mathbb{R}^n)$  with  $s \in (0, 1)$  such that  $sp > n$ .

**Theorem 2.4.** *Let  $0 < q < p < \infty$ ,  $\alpha > 0$ , and  $\eta \in (0, 1)$ . If  $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{C}^\eta(\mathbb{R}^n)$ , then  $f \in L^{p,\alpha}(\mathbb{R}^n)$  and*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{\eta+\frac{n}{p}}{\eta+\frac{n}{q}}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{\frac{\frac{n}{q}-\frac{n}{p}}{\frac{n}{q}+\eta}}. \tag{2.4}$$

We point out that the use of Hölder spaces in such Gagliardo–Nirenberg’s inequalities seems to be new in the literature. As a consequence of [Theorem 2.4](#) and the embedding  $W^{s,r}(\mathbb{R}^n) \subset \dot{C}^\eta(\mathbb{R}^n)$ , with  $\eta = \frac{sr-n}{r}$  (see [\[2\]](#)), we get the following result.

**Corollary 2.5.** *Let  $0 < q < p < \infty$ , and  $\alpha > 0$ . Let  $s, r > 0$  be such that  $sr > n$ , and  $\eta = \frac{sr-n}{r}$ . If  $f \in L^{q,\infty}(\mathbb{R}^n) \cap W^{s,r}(\mathbb{R}^n)$ , then  $f \in L^{p,\alpha}(\mathbb{R}^n)$  and*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{\eta+\frac{n}{p}}{\eta+\frac{n}{q}}} \|f\|_{W^{s,r}(\mathbb{R}^n)}^{\frac{\frac{n}{q}-\frac{n}{p}}{\frac{n}{q}+\eta}}. \tag{2.5}$$

### 3. Proof of theorems

Before giving the proof of [Theorem 2.1](#), we also note that there is an alternative approach using interpolation spaces (specifically, the reiteration theorem). We emphasize that in contrast to that approach, the proofs given in this paper are self-contained.

**Proof of Theorem 2.1.** Let us write

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)}^\alpha = p \int_0^{\lambda_0} \lambda^\alpha |\{ |f| > \lambda \}|^{\alpha/p} \frac{d\lambda}{\lambda} + p \int_{\lambda_0}^\infty \lambda^\alpha |\{ |f| > \lambda \}|^{\alpha/p} \frac{d\lambda}{\lambda}. \tag{3.1}$$

Since  $f \in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n)$ , we have

$$\int_0^{\lambda_0} \lambda^\alpha |\{ |f| > \lambda \}|^{\alpha/p} \frac{d\lambda}{\lambda} \leq \int_0^{\lambda_0} \lambda^\alpha \left( \frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^q}{\lambda^q} \right)^{\alpha/p} \frac{d\lambda}{\lambda} = \frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\alpha q/p}}{\alpha(1-q/p)} \lambda_0^{\alpha(1-q/p)}, \tag{3.2}$$

and

$$\int_{\lambda_0}^\infty \lambda^\alpha |\{ |f| > \lambda \}|^{\alpha/p} \frac{d\lambda}{\lambda} \leq \int_{\lambda_0}^\infty \lambda^\alpha \left( \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^r}{\lambda^r} \right)^{\alpha/p} \frac{d\lambda}{\lambda} = \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\alpha r/p}}{\alpha(r/p-1)} \lambda_0^{\alpha(1-r/p)}. \tag{3.3}$$

A combination of [\(3.1\)](#), [\(3.2\)](#) and [\(3.3\)](#) yields

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)}^\alpha \leq p \left( \frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\alpha q/p}}{\alpha(1-q/p)} \lambda_0^{\alpha(1-q/p)} + \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^{\alpha r/p}}{\alpha(r/p-1)} \lambda_0^{\alpha(1-r/p)} \right).$$

Now, we equalize the right hand side of the above inequality by choosing

$$\lambda_0^{r-q} = \frac{\|f\|_{L^{r,\infty}(\mathbb{R}^n)}^r}{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^q}.$$

This puts an end to the proof of [Theorem 2.1](#). ■

**Proof of Corollary 2.2.** For any  $r > 2$ , it follows from the Sobolev embedding theorem

$$\|f\|_{L^r(\mathbb{R}^n)} \leq C(n, r)\|f\|_{\dot{H}^s(\mathbb{R}^n)}, \quad \text{with } s = n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Take  $r > p$ , we have  $s > n\left(\frac{1}{2} - \frac{1}{p}\right)$ . By noting that  $\|f\|_{L^r, \infty(\mathbb{R}^n)} \leq \|f\|_{L^r(\mathbb{R}^n)}$ , and inserting the last inequality into the right hand side of (2.1) yield the result. ■

**Proof of Theorem 2.3.** It suffices to prove that Theorem 2.3 holds for  $\|f\|_{BMO(\mathbb{R}^n)} = 1$ , then we obtain (2.3) by applying the resulting inequality to  $f/\|f\|_{BMO(\mathbb{R}^n)}$ .

Now, we mimic the proof of Theorem 9.1, [9]. Let us split  $f = f_{1-} + f_{1+}$ , with  $f_{1-} = f\chi_{\{|f| \leq 1\}}$ , and  $f_{1+} = f\chi_{\{|f| > 1\}}$ , with  $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$

Then,

$$\|f_{1-}\|_{L^{p, \alpha}(\mathbb{R}^n)}^\alpha = p \int_0^\infty \lambda^\alpha |\{|f_{1-}| > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} = p \int_0^1 \lambda^\alpha |\{|f_{1-}| > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda}. \tag{3.4}$$

On the other hand, we have from  $f \in L^{q, \infty}(\mathbb{R}^n)$

$$|\{|f_{1-}| > \lambda\}| \leq |\{|f| > \lambda\}| \leq \frac{\|f\|_{L^{q, \infty}(\mathbb{R}^n)}^q}{\lambda^q}. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\|f_{1-}\|_{L^{p, \alpha}(\mathbb{R}^n)}^\alpha \leq p \int_0^1 \lambda^\alpha \left( \frac{\|f\|_{L^{q, \infty}(\mathbb{R}^n)}^q}{\lambda^q} \right)^{\alpha/p} \frac{d\lambda}{\lambda} = \frac{p}{\alpha(1 - q/p)} \|f\|_{L^{q, \infty}(\mathbb{R}^n)}^{\alpha q/p}.$$

Or

$$\|f_{1-}\|_{L^{p, \alpha}(\mathbb{R}^n)} \leq C_1(p, q, \alpha) \|f\|_{L^{q, \infty}(\mathbb{R}^n)}^{q/p}. \tag{3.6}$$

Next, we estimate

$$\|f_{1+}\|_{L^{p, \alpha}(\mathbb{R}^n)}^\alpha = p \int_0^\infty \lambda^\alpha |\{|f_{1+}| > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} = p \int_1^\infty \lambda^\alpha |\{|f_{1+}| > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda}. \tag{3.7}$$

To estimate the level set  $\{|f_{1+}| > \lambda\}$  involving  $\|f\|_{BMO}$ , we recall the following result, which is a consequence of John–Nirenberg inequality, [6] (see also Corollary 2.2, [8]):

**Lemma 3.1.** *If  $g \in BMO(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ , then there exists a constant  $C = C(n) > 0$  such that*

$$|\{ |g| > \lambda \}| \leq C e^{-C\lambda/\|g\|_{BMO(\mathbb{R}^n)}} \|g\|_{L^1(\mathbb{R}^n)},$$

for all  $\lambda > \|g\|_{BMO(\mathbb{R}^n)}$ .

We refer the reader to the proof of Lemma 3.1 to [8] (see also in [9]).

Now, we apply Lemma 3.1 to  $f_{1+}$  to get

$$|\{ |f_{1+}| > \lambda \}| \leq C e^{-C\lambda/\|f_{1+}\|_{BMO}} \|f_{1+}\|_{L^1(\mathbb{R}^n)}.$$

Note that  $\|f_{1-}\|_{BMO} \leq 2\|f_{1-}\|_{L^\infty(\mathbb{R}^n)} \leq 2$ , so

$$\|f_{1+}\|_{BMO(\mathbb{R}^n)} \leq \|f_{1+}\|_{BMO(\mathbb{R}^n)} + \|f_{1-}\|_{BMO(\mathbb{R}^n)} \leq 3.$$

This leads to

$$|\{ |f_{1+}| > \lambda \}| \leq C e^{-C\lambda/3} \|f_{1+}\|_{L^1(\mathbb{R}^n)}. \tag{3.8}$$

Moreover, we have from Chapter 1 in Grafakos, [4] (see also Lemma 3.1, [9])

$$\|f_{1+}\|_{L^1(\mathbb{R}^n)} \leq \frac{q}{q-1} 1^{1-q} \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^q. \tag{3.9}$$

By (3.7), (3.8) and (3.9), there is a constant  $C_2 = C_2(p, q, r, n) > 0$  such that

$$\|f_{1+}\|_{L^{p,\alpha}(\mathbb{R}^n)}^\alpha \leq C_2 \int_1^\infty \lambda^\alpha e^{-C\lambda\alpha/3p} \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\alpha q/p} \frac{d\lambda}{\lambda} \leq C_3 \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\alpha q/p}.$$

Thus,

$$\|f_{1+}\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C_4 \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{q/p}. \tag{3.10}$$

It follows from (3.6) and (3.10) that

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C(p, q, \alpha, n) \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{q/p},$$

which completes the proof of Theorem 2.3. ■

**Proof of Theorem 2.4.** For any  $\varepsilon > 0$ , let us put

$$\rho_\varepsilon(x) := \frac{\chi_{\{|x| \leq \varepsilon\}}(x)}{|B_\varepsilon(0)|}.$$

**Step 1:** If  $q > 1$ , by [9, Proposition 4.2], we have

$$\|f * \rho_\varepsilon\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)} \|\rho_\varepsilon\|_{L^{p_0}(\mathbb{R}^n)},$$

where

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{p_0}.$$

Note that  $\|\rho_\varepsilon\|_{L^{p_0}(\mathbb{R}^n)} = C(n, p_0) \varepsilon^{-n(\frac{1}{q} - \frac{1}{p})}$ . Thus,

$$\|f * \rho_\varepsilon\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C \varepsilon^{-n(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^{q,\infty}(\mathbb{R}^n)}. \tag{3.11}$$

Similarly, we also have

$$\|f * \rho_\varepsilon\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}. \tag{3.12}$$

Applying (2.1) to the function  $(f - f * \rho_\varepsilon)$  in Theorem 2.1 yields

$$\begin{aligned} \|f - f * \rho_\varepsilon\|_{L^p(\mathbb{R}^n)} &\leq C \|f - f * \rho_\varepsilon\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}} \|f - f * \rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{q}{p}} \\ &\leq C_1 \left( \|f\|_{L^{q,\infty}(\mathbb{R}^n)} + \|f * \rho_\varepsilon\|_{L^{q,\infty}(\mathbb{R}^n)} \right)^{\frac{q}{p}} \|f - f * \rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{q}{p}}. \end{aligned}$$

It follows from the last inequality and (3.12) that there exists a constant, still denoted by  $C$  such that

$$\|f - f * \rho_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}} \|f - f * \rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{q}{p}}. \tag{3.13}$$

Since for all  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} |f(x) - f * \rho_\varepsilon(x)| &\leq \int_{B_\varepsilon(0)} |f(x-y) - f(x)| dy \\ &\leq \int_{B_\varepsilon(0)} |y|^\eta \|f\|_{\dot{C}^\eta(\mathbb{R}^n)} dy \\ &\leq C\varepsilon^\eta \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}. \end{aligned}$$

Or

$$\|f - f * \rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq C\varepsilon^\eta \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}.$$

By inserting the last inequality into (3.13), we obtain

$$\|f - f * \rho_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon^{\eta(1-\frac{q}{p})} \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{1-\frac{q}{p}}. \tag{3.14}$$

Combining (3.14) with (3.11) yields

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C \left( \varepsilon^{-n(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^{q,\infty}(\mathbb{R}^n)} + \varepsilon^{\eta(1-\frac{q}{p})} \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{1-\frac{q}{p}} \right). \tag{3.15}$$

By choosing  $\varepsilon = \left( \frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}}{\|f\|_{\dot{C}^\eta(\mathbb{R}^n)}} \right)^{\frac{1}{\frac{n}{q}+\eta}}$  in (3.15), we obtain Theorem 2.4 with  $\alpha = \infty$  and  $q > 1$ .

**Step 2:** If  $0 < q \leq 1$ , we set for any  $\delta \in (0, q)$ :

$$g = |f|^{q-1-\delta} f, \quad p_1 = \frac{p}{q-\delta}, \quad q_1 = \frac{q}{q-\delta}, \quad \eta_1 = \eta(q-\delta).$$

Thanks to Step 1, we obtain

$$\|g\|_{L^{p_1,\infty}(\mathbb{R}^n)} \leq C \|g\|_{L^{q_1,\infty}(\mathbb{R}^n)}^{\frac{\eta_1 + \frac{n}{p_1}}{\eta_1 + \frac{n}{q_1}}} \|g\|_{\dot{C}^{\eta_1}(\mathbb{R}^n)}^{\frac{\frac{n}{q_1} - \frac{n}{p_1}}{\frac{n}{q_1} + \eta_1}}. \tag{3.16}$$

Next, we have the following inequality

$$\left| |a|^{s-1} a - |b|^{s-1} b \right| \leq C_s |a - b|^s, \quad \forall a, b \in \mathbb{R}^n, \quad s \in (0, 1).$$

By applying this inequality with  $a = f(x)$ ,  $b = f(y)$ , and  $s = q - \delta$ , we obtain

$$\left| |f(x)|^{q-1-\delta} f(x) - |f(y)|^{q-1-\delta} f(y) \right| \leq |f(x) - f(y)|^{q-\delta},$$

which implies

$$\|g\|_{\dot{C}^{\eta_1}(\mathbb{R}^n)} = \| |f|^{q-1-\delta} f \|_{\dot{C}^{\eta_1}(\mathbb{R}^n)} \leq \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{q-\delta}. \tag{3.17}$$

Note that

$$\|f\|_{L^{m,\infty}(\mathbb{R}^n)} = \| |f|^{s-1} f \|_{L^{\frac{m}{s},\infty}(\mathbb{R}^n)}^{1/s}, \quad \forall m, s > 0. \tag{3.18}$$

Then, a combination of (3.16), (3.17), and (3.18) yields

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)}^{q-\delta} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{(q-\delta)\frac{\eta+\frac{n}{p}}{\eta+\frac{n}{q}}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{(q-\delta)\frac{\frac{n}{q}-\frac{n}{p}}{\frac{n}{q}+\eta}}.$$

Or, Theorem 2.4 holds for  $\alpha = \infty$ , and  $0 < q \leq 1$ .

**Step 3:** Finally, we prove [Theorem 2.4](#) holds for any  $\alpha > 0$  and  $q > 0$ .

In fact, let  $q < p_2 < p < p_3 < \infty$  be such that

$$\frac{1}{p} = \frac{\theta}{p_2} + \frac{1-\theta}{p_3}.$$

By [Theorem 2.1](#), we have

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{p_2,\infty}(\mathbb{R}^n)}^\theta \|f\|_{L^{p_3,\infty}(\mathbb{R}^n)}^{1-\theta}. \tag{3.19}$$

By Step 1 and Step 2, we proved that [\(2.4\)](#) is true with  $\alpha = \infty$ , so

$$\|f\|_{L^{p_i,\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{\eta+\frac{n}{p_i}}{\eta+\frac{n}{q}}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{\frac{\frac{n}{q}-\frac{n}{p_i}}{\frac{n}{q}+\eta}}, \quad i = 2, 3.$$

From the last inequality and [\(3.19\)](#), we deduce

$$\begin{aligned} \|f\|_{L^{p,\alpha}(\mathbb{R}^n)} &\leq C \left( \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{\eta+\frac{n}{p_2}}{\eta+\frac{n}{q}}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{\frac{\frac{n}{q}-\frac{n}{p_2}}{\frac{n}{q}+\eta}} \right)^\theta \left( \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{\eta+\frac{n}{p_3}}{\eta+\frac{n}{q}}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{\frac{\frac{n}{q}-\frac{n}{p_3}}{\frac{n}{q}+\eta}} \right)^{1-\theta} \\ &= C \|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{\eta+\frac{n}{p}}{\eta+\frac{n}{q}}} \|f\|_{\dot{C}^\eta(\mathbb{R}^n)}^{\frac{\frac{n}{q}-\frac{n}{p}}{\frac{n}{q}+\eta}}, \end{aligned}$$

which implies [\(2.4\)](#). ■

Finally, [Corollary 2.5](#) follows immediately from [Theorem 2.4](#) using the embedding  $W^{s,r} \subset \dot{C}^\eta$ , with  $\eta = \frac{sr-n}{r}$ . Then we leave it to the reader.

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