Potential symmetry properties of a family of equations occuring in ice sheet dynamics

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Abstract

In this paper we derive some similarity solutions of a nonlinear equation associated with a free boundary problem arising in the shallow-water approximation in glaciology. In addition we present a classical potential symmetry analysis of this second order non-linear degenerate parabolic equation related to non-Newtonian ice sheet dynamics in the isothermal case. After obtaining a general result connecting the thickness function of the ice sheet and the solution of the nonlinear equation (without any unilateral formulation), a particular example of a similarity solution to a problem formulated with Cauchy boundary conditions is described. This allows us to obtain several qualitative properties on the free moving boundary in presence of an accumulation-ablation function with realistic physical properties.

Keywords: Non-linear degenerate equations, Ice flow dynamics; Potential symmetries

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1 A model for ice sheet dynamics

In recent years there has been much interest on modelling ice sheet dynamics especially because of its importance in the understanding of global climate change, global energy balance and circulation models. Although various physical theories for large ice sheet motion have been presented there exists still many open questions related to its mathematical treatment. In this paper we consider an obstacle formulation of slow, isothermal, one dimensional ice slow on a rigid bed due to FOWLER 1992.

The model describing the ice sheet dynamics is formulated in terms of an obstacle problem associated with a one dimensional non-linear degenerate diffusion equation (see CALVO et al. 2002). The original *strong* formulation can be stated in the following terms: let $T > 0$, $L > 0$ be positive fixed real numbers and let $\Omega = (-L, L)$ be an open bounded interval of $I\!R$ (a sufficiently large, fixed spatial domain). Given an accumulation/ablation rate function $a = a(x,t)$ and a function $f(x,t)$ (a sliding velocity, eventually zero) defined on $Q = (0,T) \times (-L, L)$ (a large, fixed, parabolic domain) and an initial thickness $h_0 = h_0(x) \geq 0$ (bounded and with $h_0(x) > 0$ on its support $I(0) \subset \Omega$), find two curves $S_+, S_- \in C^0([0,T])$, with $S_-(t) \leq S_+(t)$, $I(t) := (S_-(t), S_+(t)) \subset \Omega$ for any $t \in [0,T]$, and a sufficiently smooth function $h(x,t)$ defined on the set $Q_T := \bigcup_{t\in(0,T)} I(t)$ such that

$$(SF) := \begin{cases} h_t = \left[\frac{h^{n+2}}{n+2}|h_x|^{n-1}h_x - fh\right]_x + a & \text{in} \quad Q_T, \\ h = (\frac{h^{n+2}}{n+2}|h_x|^{n-1}h_x - fh) = 0, \quad \text{on} & \{S_-(t)\} \cup \{S_+(t)\}, t \in (0,T), \\ h = h_0 & \text{on} \quad I(0), \end{cases}$$

and $h(x,t) > 0$ on Q_T. We recall that n denotes the, so called, Glen exponent, and that several constitutive assumptions are admitted, the most relevant case corresponds to $n = 3$ (see, for example FOWLER 1992).

Notice that, for each fixed $t \in [0,T]$, $I(t) = (S_-(t), S_+(t)) = \{x \in \Omega : h(x,t) > 0\}$ denotes the ice covered region. The curves $S_\pm(t)$ are called the interface curves or free boundaries associated to the problem and are defined by:

$$S_-(t) = \mathrm{Inf}\{x \in \Omega \,:\, h(x,t) > 0\}, \qquad S_+(t) = \mathrm{Sup}\{x \in \Omega \,:\, h(x,t) > 0\}$$

These curves defines the interface separating the regions in which $h(x,t) > 0$ (i.e, ice regions) from those where $h(x,t) = 0$ (i.e. ice-free regions). In the physical context they represent the propagation fronts of the ice sheet.

The qualitative description of solutions of this problem is quite difficult due to the doubly nonlinear terms appearing at the differential operator and, specially, to its formulation involving the unknown fronts $S_\pm(t)$ (the free boundaries). Nevertheless, some mathematical and numerical results are already available in the literature. So, for instance, the physical problem may be characterized by the following properties as have recently been discussed by CALVO et al. 2002 :

- Given an initial ice sheet initial $h(x,0)$, and known $a(x,t)$, $f(x,t)$ the nonlinear partial differential equation determine $h(x,t)$ over its parabolic positivity set.
- The ice free region (melt zone) $h(x,t) = 0$ always exists (from the assumptions on $h(x,0)$) and define the two free boundaries $S_-(t)$ and $S_+(t)$

which are extended to the interval $[0,T]$ if, for $t \in [0,T]$, $a(x,t) > 0$ on some subinterval of Ω.

- The more realistic solutions (from a physical point of view) are non-negative solutions $h\,(x,t) \geq 0$ corresponding to ablation data functions such that $a > 0$ except in a region near the two free boundaries where $a < 0$.

In section 4 we prove that it is possible to obtain estimates on the ice covered region $I(t)$ and the solution $h(x,t)$ (the thickness of the ice sheet) by means of the comparison with the solution $u(x,t)$ of the nonlinear equation

$$\Psi(x,t,u,u_t,u_x,u_{xx}) \equiv u_t - a - \left[\frac{u^{n+2}}{n+2}|u_x|^{n-1}u_x - fu\right]_x = 0. \qquad (1)$$

So, any description of special solutions of the equation (1) (which do not involve obstacle formulation) leads to useful estimates for the more complex formulation for $h(x,t)$. As a matter of fact, the study of the nonlinear equation (1) is of importance in its own right since the equation arises in many other different contexts (with different values of the exponent n) as, for instance, filtration in porous media with turbulent regimes, suitable non-Newtonian flow problems, and so on (see, e.g. the monograph ANTONTSEV et al. 2002 and its references).

We emphasize that very few explicit solutions of the ice sheet free boundary formulation are known in the literature. One of them corresponds to a stationary solution due to PATERSON 1981 and was used as numerical test in the paper . It corresponds to the special case of a no sliding case (i.e. $f = 0$) with $n = 3$ and the following piecewise constant accumulation-ablation function:

$$a(x) = \begin{cases} a_1 & \text{if } 0 \leq |x| < R \\ -a_2 & \text{if } R \leq |x| \leq L, \end{cases}$$

where $L > 1$, $a_1 > 0$, $a_2 > 0$ and $R \in (0,1)$. Moreover, it is assumed that $a_1 R = a_2(1-R)$. Thus, for the particular values $a_1 = 0.01$ and $a_2 = 0.03$, we have the steady state solution

$$h(x) = \begin{cases} H\left[1 - \left(1 + \dfrac{a_1}{a_2}\right)^{1/3} \left(\dfrac{|x|}{L}\right)^{4/3}\right]^{3/8} & \text{if } |x| \leq R \\ H\left(1 + \dfrac{a_2}{a_1}\right)^{1/8} \left(1 - \dfrac{|x|}{L}\right)^{1/2} & \text{if } R \leq |x| \leq 1 \\ 0 & \text{if } 1 \leq |x| \leq L, \end{cases} \qquad (2)$$

where $H = (40\, a_1\, R\,)^{1/8}$ represents the thickness at $x = 0$.

In sections 2-4 of this paper we shall carry out the study of some special transient solutions of the equation (1) of a similarity type which are compatible with the above statements. It should be noted that similarity solutions for problems related to ice-sheet dynamics do exist in the literature. However neither HALIFAR 1981, 1083 or NYE 2000 consider surface accumulation whilst neither HINDMARSCH 1990, 1993 or BUELER et al. 2005 conduct a comprehensive similarity analysis. In this paper similarity solutions will be obtained by conducting a thorough Lie or classical symmetry analysis of (1). The method is described in the next section. In addition BLUMAN et al. 1988, 1989, described how the range of symmetries may be extended whenever a Lie symmetry analysis is conducted on a partial differential equation that may be written in a conserved or a potential form. This is the case with (1) where the corresponding equivalent ice model may be described in terms of the first order potential system, $\mathbf{\Psi} \equiv (\Psi_1, \Psi_2) = \mathbf{0}$ where

$$\begin{aligned} \Psi_1 &= v_x - u + \lambda = 0 \\ \Psi_2 &= v_t - \frac{u^{n+2} \left|u_x\right|^{n-1} u_x}{n+2} + fu = 0 \end{aligned} \tag{3}$$

for a potential function $v = v(x,t)$ and with $\lambda = \lambda(x,t)$ chosen such that

$$a \equiv \lambda_t \tag{4}$$

We recall that, as demonstrated in BLUMAN et al. 1988, 1989, the Lie point symmetries of the potential system induce non-Lie contact symmetries for the original partial differential equation. The treatment presented in Sections 3 and 4 is made independently of the positiveness subset of the solution and so it is carried out directly in terms of equation (1), without any other requisite on the solution (no study on any free boundaries is made in these sections). An application to the strong formulation of the free boundary problem, for some concrete data, is given in the section 5.

2 Ice sheet equation, conservation and potential symmetry analysis

In the classical Lie group method, one-parameter infinitesimal point transformations,with group parameter ε are applied to the dependent and independent variables (x,t,u,v). In this case the transformation, including that of the potential variable are

$$\begin{aligned} \bar{x} &= x + \varepsilon\eta_1(x,t,u,v) + O\left(\varepsilon^2\right) & \bar{t} &= t + \varepsilon\eta_2(x,t,u,v) + O\left(\varepsilon^2\right) \\ \bar{u} &= u + \varepsilon\phi_1(x,t,u,v) + O\left(\varepsilon^2\right) & \bar{v} &= v + \varepsilon\phi_2(x,t,u,v) + O\left(\varepsilon^2\right) \end{aligned} \tag{5}$$

and the Lie method requires form invariance of the solution set:

$$\Sigma \equiv \{u(x,t), v(x,t),\ \mathbf{\Psi} = 0\} \tag{6}$$

This results in a system of over-determined, linear equations for the infinitessimal η_1, η_2, ϕ_1 and ϕ_2. The corresponding Lie algebra of symmetries is the set of vector fields

$$\mathcal{X} = \eta_1 (x,t,u,v) \frac{\partial}{\partial x} + \eta_2 (x,t,u,v) \frac{\partial}{\partial t} + \phi_1 (x,t,u,v) \frac{\partial}{\partial u} + \phi_2 (x,t,u,v) \frac{\partial}{\partial v} \quad (7)$$

The condition for invariance of (1) is the equation

$$\mathcal{X}_E^{(1)} (\mathbf{\Psi}) \,|_{\Psi_1=0,\Psi_2=0} = 0 \quad (8)$$

where the first prolongation operator $\mathcal{X}_E^{(1)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi_1^{[t]} \frac{\partial}{\partial u_t} + \phi_1^{[x]} \frac{\partial}{\partial u_x} + \phi_2^{[t]} \frac{\partial}{\partial v_t} + \phi_2^{[x]} \frac{\partial}{\partial v_x} \quad (9)$$

where $\phi_1^{[t]}$, $\phi_1^{[x]}$ and $\phi_2^{[t]}$, $\phi_2^{[x]}$ are defined through the transformations of the partial derivatives of u and v. In particular to the first order in ε:

$$\begin{aligned} \bar{u}_{\bar{x}} &= u_x + \varepsilon \phi_1^{[x]} (x,t,u,v) & \qquad \bar{u}_{\bar{t}} &= u_t + \varepsilon \phi_1^{[t]} (x,t,u,v) \\ \bar{v}_{\bar{x}} &= v_x + \varepsilon \phi_2^{[x]} (x,t,u,v) & \qquad \bar{v}_{\bar{t}} &= v_t + \varepsilon \phi_2^{[t]} (x,t,u,v) \end{aligned} \quad (10)$$

Once the infinitesimals are determined the symmetry variables may be found from condition for invariance of surfaces $u{=}u(x,t)$ and $v{=}v(x,t)$:

$$\begin{aligned} \Omega_1 &= \phi_1 - \eta_1 u_x - \eta_2 u_t = 0 \\ \Omega_2 &= \phi_2 - \eta_1 v_x - \eta_2 v_t = 0 \end{aligned} \quad (11)$$

In the following both Macsyma and Maple software have been used to calculate the determining equations. In the case of the ice equation (3) there are nine over-determined linear determining equations. From these equations it may be shown that:

$$\eta_1 = \eta_1 (x,t) = (c_0 - z(t))\, x + s \quad (12)$$

$$\eta_2 = \eta_2 (t) \quad (13)$$

$$\phi_1 = \phi_1 (t,u) = z(t)\, u \quad (14)$$

$$\phi_2 = \phi_2 (x,t,v) = g(x,t) + c_0 v \quad (15)$$

where c_0 is an arbitrary constant such that:

$$(3n+2)\, z(t) + \eta_{2_t} - (n+1)\, c_0 = 0 \quad (16)$$

$$(z(t)\, x - s(t) - c_0 x)\, \lambda_x - \eta_2 (t)\, \lambda_t + z(t)\, \lambda - g(t)\,(x,t)_x = 0 \quad (17)$$

$$x \lambda z(t)_t - \lambda s_t - g(x,t)_t = 0 \quad (18)$$

$$\begin{aligned} &(z(t)\, x - s(t) - c_0 x)\, f_x - \eta_2 (t)\, f_t \\ &= -f\,((3n+1)\, z(t) - n c_0) + x z(t)_t - s(t)_t \end{aligned} \quad (19)$$

When it is assumed that $s(t)$ and $z(t)$ are known (see the next section) then equation (16) may be used to determine $\eta_2(t)$ whilst (17) to (19) may be used to determine $g(x,t)$, $\lambda(x,t)$ and the sliding velocity $f(x,t)$.

We observe that we have shown that the potential symmetries of the conserved form of the ice dynamics equations (3) are entirely equivalent to those of single equation.(1). This is so because according to BLUMAN et al. 1989 additional symmetries can only be induced by the potential system when:

$$\eta_{1_v}^2 + \eta_{2_v}^2 + \phi_{1_v}^2 \neq 0 \tag{20}$$

Clearly substitution of equations (12), (13) and (14) demonstrate that this is not the case.

In addition that a differential consequence of equations (17) and (18) incorporating the relation (4) is the differential equation for a, similar in form to (**??**), namely:

$$\left(z(t)\,x - s(t) - c_0 x\right) a_x - \eta_2(t)\, a_t = -a\,(n+1)\left(3z(t) - c_0\right) \tag{21}$$

Moreover, we note that equation (17) may be obtained directly by differentiating the second surface invariant condition (11) with respect to x and then applying (3), (12) -(15) together with the first of (11).

In summary, the results (16) to (21) together with the first invariant condition at (11) may be simplified by eliminating $z(t)$ using (16) and combined to give three first order partial differential equations which $u(x,t)$, $a(x,t)$ and $f(x,t)$ must satisfy, namely:

$$\left(s(t) + \frac{\left((2n+1)\,c_0 + r_t(t)\right)}{3n+2}x\right) u_x + r(t)u_t = \frac{(n+1)\,c_0 - r_t(t)}{3n+2}u \tag{22}$$

$$\left(s(t) + \frac{\left((2n+1)\,c_0 + r_t(t)\right)}{3n+2}x\right) a_x + r(t)a_t = \frac{(n+1)}{3n+2}\left(c_0 - 3r_t(t)\right) a \tag{23}$$

$$\begin{aligned}&\left(s(t) + \tfrac{\left((2n+1)c_0 + r(t)_t\right)}{3n+2}x\right) f_x + r(t)f_t \\ &= \tfrac{\left((2n+1)c_0 - (3n+1)r_t(t)\right)}{3n+2}f + \tfrac{x r_{tt}(t)}{3n+2} + s_t(t)\,,\end{aligned} \tag{24}$$

where $r(t) \equiv \eta_2(t)$ has been used to simplify the notation.

3 Symmetry analysis results for the case $n = 3$

As stated in Section 1 the exponent n which occurs in (1) is Glen's exponent and FOWLER 1992 suggests that $n \approx 3$ in physically realistic situations Thus in the following we will assume that $n = 3$ although the analysis is unchanged for any non-Newtonian values $n > 1$. The results presented assumed that each of the functions u, a and f explicitly depend on x and t.

3.1 The case $f(x,t) = 0$

Firstly, substitution of $f(x,t) = 0$ into equation (24) gives $r(t) = c_1 t + c_2$ and $s(t) = c_3$.

3.1.1 The subcase $7c_0 + c_1 \neq 0$, $c_1 \neq 0$

The solution of (22) and (23) may be expressed in terms of the similarity variable $\omega = \omega\left(x,t\right)$ for which:

$$\omega\left(x,t\right) = \left(x + c_3\right)\left(c_1 t + c_2\right)^{-\frac{7c_0+c_1}{11c_1}} \quad \text{when} \quad 7c_0 + c_1 \neq 0 \tag{25}$$

with:

$$u\left(x,t\right) = \psi\left(\omega\left(x,t\right)\right)\left(c_1 t + c_2\right)^{\frac{4c_0-c_1}{11c_1}} \tag{26}$$

$$a\left(x,t\right) = A\left(\omega\left(x,t\right)\right)\left(c_1 t + c_2\right)^{\frac{4c_0-12c_1}{11c_1}} \tag{27}$$

Substituting the relationships into equation (1) with $n = 3$ gives rise to the ordinary differential equation :

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5} + \frac{\left(c_1 + 7c_0\right)\omega\psi}{11}\right\} - c_0\psi - A = 0 \tag{28}$$

3.1.2 The subcase $7c_0 + c_1 = 0$, $c_1 \neq 0$

For this subcase it may be shown that:

$$\omega\left(x,t\right) = x + c_3 \ln\left(c_1 t + c_2\right) \qquad \text{when} \quad 7c_0 + c_1 = 0 \tag{29}$$

with

$$u\left(x,t\right) = \psi\left(\omega\left(x,t\right)\right)\left(c_1 t + c_2\right)^{-\frac{1}{7}} \tag{30}$$

$$a\left(x,t\right) = A\left(\omega\left(x,t\right)\right)\left(c_1 t + c_2\right)^{-\frac{8}{7}} \tag{31}$$

Substituting the relationships into equation (1) with $n = 3$ gives rise to the ordinary differential equation :

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5} + 7c_0c_3\psi\right\} - c_0\psi - A = 0 \tag{32}$$

3.1.3 The subcase $c_1 = 0$

Without loss of generality consider the case $c_2 = 1$. The solution of (22) and (23) may be expressed in terms of the similarity variable $\omega = \omega\left(x,t\right)$ for which:

$$\omega\left(x,t\right) = \left(x + c_3\right)e^{-\frac{7c_0 t}{11}} \tag{33}$$

with:

$$u\left(x,t\right) = \psi\left(\omega\left(x,t\right)\right)e^{\frac{4c_0 t}{11}} \tag{34}$$

$$a\left(x,t\right)=A\left(\omega\left(x,t\right)\right)e^{\frac{4c_0 t}{11}} \tag{35}$$

Substituting the relationships into equation (1) with $n=3$ gives rise to the ordinary differential equation :

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5}+\frac{7c_0\omega\psi}{11}\right\}-c_0\psi-A=0 \tag{36}$$

3.2 The case $s\left(t\right)=0$, $r\left(t\right)\neq 0$, $f(x,t)\neq 0$

In this case equations (22) to (24) may be integrated immediately to give solutions in terms of the similarity variable $\omega=\omega\left(x,t\right)$ for which:

$$\omega\left(x,t\right)=xr(t)^{-\frac{1}{11}}\exp\left(-\frac{7c_0}{11}\int\frac{dt}{r\left(t\right)}\right) \tag{37}$$

with:

$$u\left(x,t\right)=\psi\left(\omega\left(x,t\right)\right)r\left(t\right)^{-\frac{1}{11}}\exp\left(\frac{4c_0}{11}\int\frac{dt}{r\left(t\right)}\right) \tag{38}$$

$$a\left(x,t\right)=A\left(\omega\left(x,t\right)\right)r\left(t\right)^{-\frac{12}{11}}\exp\left(\frac{4c_0}{11}\int\frac{dt}{r\left(t\right)}\right) \tag{39}$$

$$f\left(x,t\right)=\left[\frac{\omega(x,t)r_t(t)}{11}+F\left(\omega\left(x,t\right)\right)\right]r\left(t\right)^{-\frac{10}{11}}\exp\left(\frac{7c_0}{11}\int\frac{dt}{r\left(t\right)}\right) \tag{40}$$

Substituting the relationships into equation (1) with $n=3$ gives rise to the ordinary differential equation :

$$\frac{3\psi^5\psi_\omega^2\psi_{\omega\omega}}{5}+\psi^4\psi_\omega^4+\frac{7c_0\omega\psi_\omega}{11}-\frac{4c_0\psi}{11}-\psi F_\omega-\psi_\omega F-A=0 \tag{41}$$

That is:

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5}+\frac{7c_0\omega\psi}{11}-\psi F\right\}-c_0\psi-A=0 \tag{42}$$

3.3 The case $s\left(t\right)\neq 0$, $r\left(t\right)\neq 0$, $f(x,t)\neq 0$

In this case the similarity variable has the form:

$$\omega\left(x,t\right)=xr(t)^{-\frac{1}{11}}\exp\left(-\frac{7c_0}{11}\int\frac{dt}{r\left(t\right)}\right)-b\left(t\right) \tag{43}$$

where

$$b\left(t\right)=\int\left\{\frac{s\left(t\right)}{r\left(t\right)^{\frac{12}{11}}}\exp\left(-\frac{7c_0}{11}\int\frac{dt}{r\left(t\right)}\right)\right\}dt \tag{44}$$

and the solutions (380 and (39) for $u\left(x,t\right)$ and $a\left(x,t\right)$ still apply. However the solution for $f\left(x,t\right)$ now becomes:

$$f\left(x,t\right)=\left[\frac{\omega(x,t)r_t(t)}{11}+F\left(\omega\left(x,t\right)\right)+h\left(t\right)\right]r\left(t\right)^{-\frac{10}{11}}\exp\left(\frac{7c_0}{11}\int\frac{dt}{r\left(t\right)}\right) \tag{45}$$

where

$$h\left(t\right)=\frac{\left(r\left(t\right)_{t}+7c_{0}\right)}{11}b+r(t)b_{t} \tag{46}$$

The resulting ordinary differential equation is once again (42).

3.4 The case $r\left(t\right)=0$, $f(x,t)\neq 0$

In the following only the non-trivial case $c_0 \neq 0$ is considered. Equations (22) to (24) may be integrated immediately to give the following solutions:

$$u\left(x,t\right)=m\left(11s+7c_{0}x\right)^{\frac{4}{7}} \tag{47}$$

$$a\left(x,t\right)=n\left(11s+7c_{0}x\right)^{\frac{4}{7}} \tag{48}$$

$$f\left(x,t\right)=p\left(11s+7c_{0}x\right)-\frac{xs_{t}}{7c_{0}} \tag{49}$$

where the relationship between the functions $m=m\left(t\right)$, $n=n\left(t\right)$ and $p=p\left(t\right)$ may be found upon substitution of equations (47) to (49) into (1). The following equation holds:

$$m_{t}=-11c_{0}mp-n+\frac{704c_{0}^{4}m^{8}}{5} \tag{50}$$

4 A comparison result and some particular examples.

We start by showing a useful result connecting the solution of the obstacle problem and the solutions of the nonlinear equation (1).

Theorem 1 *Let $a\in L^{\infty}(Q)$, $f\in L^{\infty}(Q)$ and a compactly supported initial data $h_0\in L^{\infty}(\Omega)$. Let $h(x,t)$ be the unique solution of the obstacle problem (SF). Also let $u(x,t)$ be any continuous solution of the equation (1) corresponding to an ablation function $\widetilde{a}\in L^{\infty}(Q)$ and for which there exists two Lipschitz curves $x_{\pm}(t)$ such that*

$$u(x_{\pm}(t),t)=0 \text{ and } u(x,t)>0 \text{ for a.e. } x\in(x_{-}(t),x_{+}(t)) \text{ and any } t\in[0,T].$$

Assume that

$$\widetilde{a}(x,t)\leq a(x,t) \text{ for a.e. } (x,t)\in Q,$$

$$u(x,0)\leq h_{0}(x) \text{ for a.e. } x\in(x_{-}(0),x_{+}(0)).$$

Then, if $S_{\pm}(t)$ denotes the free boundaries generated by function $h(x,t)$ we have that

$$S_{-}(t)\leq x_{-}(t)\leq x_{+}(t)\leq S_{+}(t) \text{ and any } t\in[0,T],$$

and

$$h(x,t)\geq u(x,t) \text{ for a.e. } x\in(x_{-}(t),x_{+}(t)) \text{ and any } t\in[0,T].$$

Moreover, if

$$\widetilde{a}(x,t)=a(x,t) \;\; a.e. \;\; (x,t)\in Q, \;\; u(x,0)=h_0(x) \;\; a.e. \;\; x\in(x_-(0),x_+(0)), \quad (51)$$

and

$$\frac{u^{n+2}}{n+2}|u_x|^{n-1}u_x - fu = 0 \quad on \quad \{(x\_(t),t)\}\cup\{(x_+(t),t)\}, \;\; for\ t\in(0,T), \quad (52)$$

then $S_-(t)=x_-(t)$, $x_+(t)=S_+(t)$ *and* $h(x,t)=u(x,t)$ *for a.e.* $x\in(x_-(t),x_+(t))$ *any for any* $t\in[0,T]$.

Proof. We shall assume, additionally that $h_t, u_t \in L^1(Q)$ and that $f\equiv 0$. The general case, without this information, follows some technical arguments which can be found, for instance, in CARRILLO et al. 1999. We take as a test function the following approximation of the $sign_0^+(u^m-h^m)$ function (with $m=2(n+1)/n$) given by $\Psi_\delta(\eta):=\min(1,\max(0,\frac{\eta}{\delta}))$, for $\delta>0$ small. Then we define $v=\Psi_\delta(u^m-h^m)$. Notice that $v\in L^\infty\left(\cup_{t\in[0,T]}(x_-(t),x_+(t))\times\{t\}\right)$ and that $v(.,t)\in W_0^{1,p}\left((x_-(t),x_+(t))\right))$ for $p=n+1$ with

$$v_x=\begin{cases}\frac{1}{\delta}(u^m-h^m)_x & \text{if } 0<u-h<\delta\\ 0 & \text{otherwise.}\end{cases}$$

Then, defining the set

$$A_\delta:=\{(x,t), \text{ such that } t\in[0,T], \; x\in(x_-(t),x_+(t)) \text{ and } 0<u(t,x)-h(t,x)<\delta\},$$

and multiplying the difference of both partial differential equations and integrating by parts (that is, by taking v as a test function) we find

$$\int_0^T\int_{(x_-(t),x_+(t))}(u_t-h_t)\,\Psi_\delta(u^m-h^m)dxdt+I(\delta)\le 0$$

where

$$I(\delta)=\frac{1}{\delta}\int_0^T\int_{A_\delta}\{\phi((u^m)_x)-\phi((h^m)_x\}\,((u^m)_x-(h^m)_x)dxdt,$$

with $\phi(r)=\mu\,|r|^{n-1}\,r$, $\mu=n^n/[2^n(n+1)^n(n+2)]$ and where we used the fact that $u(x_\pm(t),t)=0\le h(x_\pm(t),t)$ for any $t\in[0,T]$. Then, from the monotonicity of $\phi(r)$ we can pass to the limit when $\delta\searrow 0$ and conclude that

$$\int_{(x_-(t),x_+(t))}\max\{u(t,x)-h(t,x),0\}dxdt\le 0.$$

which implies that $u\le h$ on the set $(x_-(t),x_+(t))$.
In the special case of u satisfying (51) and (52) we find that the function $u^\#(x,t)$ defined as

$$u^\#(x,t)=\begin{cases}u(x,t) & \text{if } x\in(x_-(t),x_+(t)),\, t\in[0,T],\\ 0 & \text{otherwise,}\end{cases}$$

satisfies all the conditions required to be weak solution of the obstacle problem and by the uniqueness of such solutions we also find that $h(x,t)=u^\#(x,t)$.

Remark 2 *We note that no information on the global boundary conditions satisfied by u on $\partial\Omega \times [0,T]$ is required in the above result.*

Remark 3 *Notice also that the conditions satisfied by $h(x,t)$ on the free boundary $S_{\pm}(t)$ indicate that the Cauchy problem on the curves $\cup_{t\in[0,T]}(S_{\pm}(t),t)$ do not satisfy the unique continuation property since h is identically zero to the left or the right sides of those curves. Some sharper information on the growth with t and the study of the differential equation satisfied by the free boundaries can be found by means of some arguments involving Lagrangian coordinates. This is the main object of the work DIAZ ET AL. 2008 concerning a different simplified obstacle problem.*

We consider now the particular example of a non-sliding ice sheet at the base so that $f(x,t) = 0$ and consider the values, $c_0 = -0.1$, $c_1 = 1$, $c_2 = 1$ and $c_3 = 0$ with the initial condition for the ice sheet profile:

$$u\left(x,0\right) = \psi\left(\omega\left(x,0\right)\right) = \frac{1}{2}\cos\left(\frac{\omega\left(x,0\right)}{4}\right) \tag{53}$$

Note that the cosine has been chosen because it is a simple mathematical example of an initial profile with with compact support. Clearly in a more detailed enquiry it would interesting to discuss broader classes of families of initial conditions with this property. However for this particular example and according to the subcase $7c_0 + c_1 \neq 0$, $c_1 \neq 0$ and equations (25), (26) the similarity solution is

$$\omega\left(x,t\right) = \frac{x}{\left(1+t\right)^{0.0273}} \tag{54}$$

$$u\left(x,t\right) = \frac{\psi\left(\omega\left(x,t\right)\right)}{\left(1+t\right)^{0.1272}} \tag{55}$$

with accumulation-ablation function (which now is denoted by $\widetilde{a}\left(x,t\right)$) given by (27) and (28) using $c_0 = -0.1$, $c_1 = 1$, $c_2 = 1$ and $c_3 = 0$ so:

$$\widetilde{a}\left(x,t\right) = A\left(\omega\left(x,t\right)\right)\left(1+t\right)^{-1.1273} \tag{56}$$

with

$$\begin{aligned} A(\omega) &= 0.153\times 10^{-4}\cos^4\left(\frac{\omega}{4}\right)\sin^4\left(\frac{\omega}{4}\right) - 0.916\times 10^{-5}\cos^6\left(\frac{\omega}{4}\right)\sin^2\left(\frac{\omega}{4}\right) \\ &\quad + 0.636\times 10^{-1}\cos\left(\frac{\omega}{4}\right) - 0.341\times 10^{-2}\omega\sin\left(\frac{\omega}{4}\right). \end{aligned} \tag{57}$$

In this case the propagation fronts of the ice sheet region are are found from:

$$\psi(x,t) = 0 \tag{58}$$

so

$$x_{\pm}(t) = \pm 2\pi\left(1+t\right)^{0.0273} \tag{59}$$

and the finite velocity is:

$$\frac{d}{dt}x_{\pm}(t) = \pm 0.0546\pi\,(1+t)^{-0.973}\,. \tag{60}$$

Figures 1-3 illustrate the time evolution of the ice sheet $u\,(x,t)$ and also the accumulation-oblation function $\widetilde{a}\,(x,t)$.

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Fig 1. *Plots of the initial ice sheet profile,* $u\,(x,0)$ *[upper curve] and also the initial accumulation-ablation function,* $a\,(x,0)$ *[lower curve] versus* x

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Fig 2. *Plots of the ice sheet profile,* $u\,(x,100)$ *[upper curve] and also the accumulation-ablation function,* $a\,(x,100)$ *[lower curve] at time* $t=100$ *versus* x

As a consequence of Theorem 1 we have

Corollary 1. *Let* $\Omega=(-L,L)$ *with* $L>2\pi$ *and* $f(x,t)\equiv 0$*. Let* $a\in L^{\infty}(Q)$ *with* $\widetilde{a}(x,t)\leq a(x,t)$ for a.e. $(x,t)\in Q$, *where* $\widetilde{a}(x,t)$ *is given by (56) and assume that*

$$h_0(x)\geq\begin{cases}\frac{1}{2}\cos\left(\frac{x}{4}\right) & \text{if } x\in(-2\pi,2\pi),\\ 0 & \text{if } x\in(-L-2\pi)\cup(L,2\pi).\end{cases}$$

Let $h(x,t)$ *be the (unique) solution of the obstacle formulation (with* $f(x,t)\equiv 0$*) associated to the data* a *and* h_0*. Then*

$$S_-(t)\leq -2\pi\,(1+t)^{0.0273}<2\pi\,(1+t)^{0.0273}\leq S_+(t) \text{ for any } t\in[0,T],$$

and

$$h(x,t)\geq\frac{\psi\left(\frac{x}{(1+t)^{0.0273}}\right)}{(1+t)^{0.1272}}$$
$$\text{for a.e. } x\in\left(-2\pi\,(1+t)^{0.0273},\ 2\pi\,(1+t)^{0.0273}\right) \text{ and any } t\in[0,T],$$

where $\psi(\omega)$ *satisfies (28).*

This example clearly demonstrates the useful properties of the closed form solutions of (1) for an accumulation-ablation function which changes sign and is negative near the propagation fronts.

Remak. The research will be continued elsewhere and in the next phase we are seeking similarity solutions corresponding to the strong formulation of the problem, when it is written (in other equivalent terms, as indicated in DIAZ-SCHIAVI 1995) by using the multivalued maximal monotone graph $\beta(u)$ of $\mathbb{R}^2$

given by $\beta(r) = \phi$ (the empty set) if $r < 0$, $\beta(0) = (-\infty, 0]$ and $\beta(r) = \{0\}$ if $r > 0$. Then, the formulation is

$$\begin{cases} \Psi(x,t,u,u_t,u_x,u_{xx}) \equiv u_t - a - \left[\frac{u^{n+2}}{n+2}|u_x|^{n-1}u_x - fu\right]_x + \varphi = 0 \\ \text{with } \varphi(x,t) \in \beta(u(x,t)) \text{ a.e. } (x,t) \in \Omega \times (0,T). \end{cases}$$

In this case the focus is on both a classical and a non-classical symmetry reduction of the equation. If, for instance, we assume that $f = 0$ then it is possible to find sharper assumptions on function $a(x,t)$ guarantying the formation of the free boundary. This is the main goal of the paper DIAZ-WILTSHIRE 2008 which deals with the multivalued formulation of general obstacle problems (also arising in many other contexts: see, e.g. DUVAUT-LIONS, 1972).

5 Summary and conclusions

In this paper we have concentrated on the problem of determining closed form similarity solutions of equation (1) (using potential symmetries) and its connections with the thickness function $h(x,t)$ of ice sheets as solution of the associate obstacle problem. The main aim has been to demonstrate that classes of such solutions exist and that they contain physically realistic properties. We observe that equation (1) contains certain modelling deficiencies (with respect the obstacle problem formulation) because inadmissible solutions for which $u\,(x,t) < 0$ (in some subset) are possible. Certainly the similarity solution approach presented here demonstrates the possibility of such unrealistic solutions for equation (1) and so we obtain only some estimates for the physical relevant function $h(x,t)$. We use some comparison results in order to extend several conclusions to the case of the free boundary formulation. Our paper illustrate the powerful of the exact solution technics based in classical Lie groups when they coupled with an ad hoc additional analysis for free boundary formulations.

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