On the mathematical analysis of an elastic-gravitational layered Earth model for magmatic intrusion: The stationary case

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Short title: **Analysis of an elastic-gravitational model**

Abstract. In early eighties Rundle (1980, 1981a, b, and 1982) developed the techniques needed for calculation of displacements and gravity changes due to internal sources of strain in layered linear elastic-gravitational media. The approximation of the solution for the half-space was obtained by using the propagator matrix technique. The Earth model considered is elastic-gravitational, composed by several homogeneous layers overlying a bottom half-space. Two dislocation sources types can be considered, representing magma intrusions and faults. In the last decades theoretical and computational extensions of that methodology have been developed by Rundle and co-workers (e.g., Fernández and Rundle, 1994a, b; Fernández et al., 1997; Tiampo et al., 2004; Fernández et al., 2005; Charco et al., 2006, 2007a, b). The source can be located at any depth in the media. In this work we prove that the perturbed equations representing the elastic-gravitational deformation problem, with the natural boundary and transmission conditions, leads to a well-posed problem even for very domains and general data. We give a constructive proof of the existence and we show the uniqueness and the continuous dependence with respect to the data of weak solutions of the coupled elastic-gravitational field equations.

Keywords: Gravity changes, elastic-gravitational earth model, displacement, weak solution.

1. Introduction

Geological hazards have a great destructive power (e.g., Sigurdsson et al., 2000; national research council of the national academies, 2003), being able to cause an instantaneous and total destruction of the life in the proximities of a volcano, fault or landslide. Geodetic techniques for the measurement of surface displacements, starting from classical terrestrial ones to the most modern space techniques like GPS campaigns, continuous GPS observation (e.g., Segall et al., 1997; Dixon et al., 1997; Sagiya et al., 2000; Larson et al., 2001; Fernández et al., 2004), satellite radar interferometry (e.g., Puglisi et al., 2001; Pritchard and Simons, 2002; Wright, 2002; Dzurisin, 2003; Fernández et al., 2005; Manzo et al., 2006; Tamisiea et al., 2007), or their combination (e.g., Gudmunsson et al., 2002; Lundgren and Stramondo, 2002; Bustin et al., 2004; Lanari et al., 2004; Samsonov and Tiampo, 2006) they have broadly demonstrated his capacity of detection of surface displacements useful to study seismic and volcanic events. These techniques allow determining displacements with a precision of a centimeter or better.

Now a day, there is a clear tendency to make a joint interpretation of displacement and variations of gravity, considering the clear improvements obtained in the results (see e.g., Rundle, 1982; Fernández et al., 2001; Yoshiyuki et al., 2001; Tiampo et al., 2004; Charco et al., 2006). The subject is of continuous interest (Tamisiea et al., 2007).

If we concentrate on hazards geological associated with volcanism, surface ground deformation and gravity changes can be indicators of volcanic activity as well as precursors of an eruptions. Usually they appear together with other volcanic activity indicators, such as seismicity, gas emission, fumarolic activity, etc. Considering that, ideal monitoring should consider all the possible parameters allowing to detect their changes on the active area and to obtain information about the magmatic source below surface from them (see e.g., Sigurdsson et al., 2000; Díaz and Talenti, 2004; Tiampo et al., 2004); deformations, gravity or temperature changes, emitted gases, etc. within the dangerous zone.

Thus volcanic eruptions are the outcome of significant physical and geological processes. Among others there is magma formation in the mantle or crust, as well as its ascent to more superficial zones. These phenomenons become apparent through changes in the volcanic building and the surroundings alike. One of the main challenges is to determine if an intrusion process will entail or not an eruption. On the other hand, in order to interpret geodetic anomalies (displacements, gravity changes, etc.) which may be tied to volcanic activity, mathematical models allowing the resolution of the inverse problem which consists of obtaining a volcanic intrusion's properties from surface observation, are necessary. Therefore we need analytical models. Numerical models more realistic in some aspects allow a better approximation to the real problem in cases in which more time is available than in a critic situation. Each model is characterized by a series of mathematical equations describing the problem's physics. Specifically, the model studied in this work is a deformation model in which surface deformation and gravity change, understood as possible symptoms of a future eruption, are coupled. This model responds to a system of partial differential equations.

As first elastic models we can consider Love's work (1911). He showed that displacement field produced by a center of expansion within an elastic medium may be obtained from a suitable Green's function. With base on these works, Rundle (1980) obtains and solves the equations that represent the elastic-gravitational problem for point sources in an elastic-gravitational half space, stratified in flat, isotropic and homogeneous layers. In order to introduce the layered medium in the problem a matrix method is used to propagate the solutions from one layer to the next. That is to say, he obtains the solution on each layer and with the aforesaid matrices he joins these solutions together to obtain a global solution on the whole domain. Rundle (1981a) achieves the numerical evaluation of this problem. He also studies the problem of obtaining vertical displacements for a rectangular fault (Rundle, 1981b). This model (Rundle, 1982, 1983) allows to study variation on the displacements, on the perturbed potential and gravity changes, as well as the sea level variation caused by volcanic loading. Rundle (1982) proves the uniqueness of solutions for the elastic-gravitational case, but considering only a infinite medium, the basic solutions for the used methodology, but not for the case of a layered model.

The goal of this paper is to complete the work developed by Rundle (1982) and, applying techniques of the weak solutions of partial differential equations theory, to prove the existence and uniqueness of solutions of the coupled elastic-gravitational model for the layered configuration on a general spatial section domain Ω. To consider a layered medium made necessary the consideration of weak solution instead of the classical solutions in the whole of the spatial domain. Let's point out that it is a necessary study to be done considering the broad applications of this elastic-gravitational deformation model.

Therefore, the deformation model we are going to work with consist of an Earth model composed by several elastic-gravitational layers overlying an elastic-gravitational half space. We consider the contribution of source term will be magmatic intrusion, corresponding to body forces acting on the medium. This will be due to both volumetric change of wall of the chamber and sudden place of a mass into the medium result of injection of material into the chamber. This way, a force will be added to both equations due to increase of pressure into the chamber which called $\mathbf{f}_u$ and f_ϕ, giving way to final coupled system (Aki et al., 2002):

$$\begin{cases} -\Delta\mathbf{u} - \dfrac{1}{1-2\nu}\nabla\left(div\mathbf{u}\right) - \dfrac{\rho g}{\mu}\nabla\left(\mathbf{u}\cdot\mathbf{e}_z\right) + \dfrac{\rho g}{\mu}\mathbf{e}_z div\mathbf{u} = \dfrac{\rho}{\mu}\nabla\phi + \mathbf{f}_u \\ -\Delta\phi = 4\pi\rho G div\mathbf{u} + f_\phi \end{cases} \tag{1}$$

The associate dynamic system will be the object of a different paper (Arjona et al., 2007).

2. Weak formulation

Let us define spatial domain in the following way: we will assume p layers "overstrike", that we will denote as Ω_i $\forall i = 1, \ldots, p$, and which union determines global domain Ω, $\Omega = \bigcup_{i=1}^{p} \Omega_i$.

Each layer is given through common horizontal set: a open $\omega \subset \mathbb{R}^2$ and so

$$\Omega_1 := \omega \times \left(d_1, d_1 + d_2\right), \ \ \Omega_2 := \omega \times \left(d_1 + d_2, d_1 + d_2 + d_3\right), \ \text{etc.}, \tag{2}$$

that is

$$\Omega_i := \omega \times \left(\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j\right) \subset \mathbb{R}^3, \text{when} \quad i = 1, \ldots, p-1, \tag{3}$$

and

$$\Omega_p := \omega \times \left(H, H + d_r\right), \tag{4}$$

when $H := \sum_{j=1}^{i-1} d_j$ and d_r can be equal to $+\infty$.

Let $\mathbf{u}^i{:}\Omega_i \longrightarrow \mathbb{R}^3$ be the displacement vector in each layer, $\mathbf{u}^i = \left(u_x^i, u_y^i, u_z^i\right)$ and $\mathbf{f}_u^i = \left(f_x^i, f_y^i, f_z^i\right)$. Both functions depend on $\mathbf{x} = (x, y, z)$.

Let us describe the boundary of our domain to establish the boundary conditions of the problem. We distinguish, for each layer comprised between the first and the $(p-1)$-th, side, upper and bottom boundary by means of the following notation (see figure1):

$$\left\{\begin{array}{ll} \partial_+\Omega_i = \omega \times \left\{ \sum_{j=1}^{i-1} d_j \right\}, & \text{top boundary,} \\ \partial_-\Omega_i = \omega \times \left\{ \sum_{j=1}^{i} d_j \right\}, & \text{bottom boundary,} \\ \partial_l\Omega_i = \partial\omega \times \left[\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i} d_j \right], & \text{side lateral boundary.} \end{array}\right. \tag{5}$$

Then:

$$\partial\Omega_i = \partial_+\Omega_i \cup \partial_-\Omega_i \cup \partial_l\Omega_i \qquad \forall i = 1, \ldots, p-1 \tag{6}$$

For the last layer, that is, the p-th one we have:

$$\left\{\begin{array}{l} \partial_+\Omega_p = \omega \times \{H\}, \\ \partial_-\Omega_p = \omega \times \{H + d_p\}. \end{array}\right. \tag{7}$$

INSERT FIGURE 1

Let us denote the displacement vector and the gravitational perturbed potential in the following manner: $\mathbf{u}^i(\mathbf{x})$ represents the displacement vector field on each point of the layer i, for $i = 1, \ldots, p$. So actually the unknown we look for is $\mathbf{u} \equiv (\mathbf{u}^i)_{i=1,\ldots,p}$. To simplify the notation, we will use $\mathbf{u}$ when it is not ambiguous. Again to simplify equation the same way we denote $\phi^i(\mathbf{x})$ vector as gravitational perturbed potential on the point $\mathbf{x}$ of the layer i, that is the unknown we look for is: $\phi \equiv (\phi^i)_{i=1,\ldots,p}$. Again, to simplify the notation, we use ϕ when there is not ambiguity. Constitutive constants of the different layers take the following notation: $\rho \equiv (\rho^i)_{i=1,\ldots,p}$, $\mu \equiv (\mu^i)_{i=1,\ldots,p}$ and $\nu \equiv (\nu^i)_{i=1,\ldots,p}$. In relation to the functions due to magmatic intrusion we use: $\mathbf{f}_u \equiv (\mathbf{f}_u^i)_{i=1,\ldots,p}$ and $f_\phi \equiv \left(f_\phi^i\right)_{i=1,\ldots,p}$.

On each layer $\Omega_i, i = 1, \ldots, p$, the following system of equations holds:

$$\begin{cases} -\Delta \mathbf{u}^i(\mathbf{x}) - \dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(\mathbf{x})\right) - \dfrac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right) + \dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x}) = \dfrac{\rho^i}{\mu^i}\nabla\phi^i(\mathbf{x}) + \mathbf{f}_u^i(\mathbf{x}), \\ -\Delta\phi^i(\mathbf{x}) = 4\pi\rho^i G div\mathbf{u}^i(\mathbf{x}) + f_\phi^i(\mathbf{x}), \text{in } \Omega_i. \end{cases} \tag{8}$$

To the set of partial differential equations we will add the boundary conditions that we will specify. With regard to displacement field we assume that:
on the side boundary, $\partial_l\Omega_i$, of $i = 1, \ldots, p$, so let:

$$\mathbf{u}^i(\mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_l\Omega_i, \tag{9}$$

on upper boundary of the first layer $\partial_+\Omega_1$:

$$\frac{\partial\mathbf{u}^1(\mathbf{x})}{\partial z} = \mathbf{0}, \mathbf{x} \in \partial_+\Omega_1, \tag{10}$$

and on bottom boundary, $\partial_-\Omega_p$, gives:

$$\mathbf{u}^p(\mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_-\Omega_p. \tag{11}$$

In general, we can assure only that the first derivatives of $\mathbf{u}$ are continuous on the boundaries of the layers, that is, on the boundary between layers. We will require "transmission conditions" between both upper and bottom boundaries of the layers excepting on the first layer and the last layer. So, we must have, on $\partial_-\Omega_i = \partial_+\Omega_{i+1}$ with $i = 1, \ldots, p-1$, the next conditions:

$$\begin{cases} \mathbf{u}^i(\mathbf{x}) = \mathbf{u}^{i+1}(\mathbf{x}), \mathbf{x} \in \partial_-\Omega_i, \\ \dfrac{\partial\mathbf{u}^i(\mathbf{x})}{\partial z} = \dfrac{\partial\mathbf{u}^{i+1}(\mathbf{x})}{\partial z}, \mathbf{x} \in \partial_-\Omega_i. \end{cases} \tag{12}$$

In relation to gravitational perturbed potential we will assume that: on side boundary $\partial_l\Omega_i$ for $i = 1, \ldots, p$:

$$\phi(\mathbf{x}) = 0, \mathbf{x} \in \partial_l\Omega_i, \tag{13}$$

on the upper boundary of the first layer $\partial_+\Omega_1$:

$$\phi^1(\mathbf{x}) = \phi_0(\mathbf{x}), \mathbf{x} \in \partial_+\Omega_1, \tag{14}$$

and on the bottom boundary, $\partial_{-}\Omega_p$:

$$\phi^p(\mathbf{x}) = 0, \mathbf{x} \in \partial_{-}\Omega_p. \tag{15}$$

Like before, we will require a transmission conditions between upper boundary and bottom boundary of the next layers excepting on the first layer and the last layers. So, we must have, on $\partial_{-}\Omega_i = \partial_{+}\Omega_{i+1}$ with $i = 1, \ldots, p-1$, the following conditions:

$$\begin{cases} \phi^i(\mathbf{x}) = \phi^{i+1}(\mathbf{x}), \mathbf{x} \in \partial_{-}\Omega_i, \\ \dfrac{\partial \phi^i(\mathbf{x})}{\partial z} = \dfrac{\partial \phi^{i+1}(\mathbf{x})}{\partial z}, \mathbf{x} \in \partial_{-}\Omega_i. \end{cases} \tag{16}$$

Remark 1. *In what follows, we shall work with the boundary data ϕ_0 by extending it to the interior of the domain Ω_1: i.e. we assume that there exists a function $\widehat{\phi}_0(\mathbf{x})$ defined on the upper layer Ω_1 such that*

$$\widehat{\phi}_0 \in H^1(\Omega_1),\ \widehat{\phi}_0(\mathbf{x}) = \phi_0(\mathbf{x}) \ \text{on}\ \partial_{+}\Omega_1 \ \text{and}\ \widehat{\phi}_0(\mathbf{x}) = 0 \ \text{on}\ \partial_{-}\Omega_1 \cup \partial_l\Omega_1. \tag{17}$$

Here, and in what follows, $H^1(\Omega)$ denotes the Sobolev space given by

$$H^1(\Omega) = \{\phi \in L^2(\Omega_1), \frac{\partial \phi}{\partial x_i} \in L^2(\Omega_1) \forall i = 1, 2, 3\}. \tag{18}$$

(see, e.g. Brézis, 1984, for more details).

It is clear that under the above conditions any classical solution does not need to exist. So we have to introduce the notion of weak solution which allows a greater generality.

Firstly, we define the space formed by the test functions (which we shall denote as *space of energy*), for both displacement vector and gravitational perturbed potential, denoting by V_u and V_ϕ. In order to simplify the presentation of the results we shall assume that the horizontal projection ω is bounded, connected and "regular":

$$\begin{aligned} &V_u := \{(\mathbf{u}^1, \phi^1), \ldots, (\mathbf{u}^p, \phi^p) \in \prod_{i=1}^{p} H^1(\Omega_i)^3 \times H^1(\Omega_i) \ \text{such that}\ \mathbf{u}^i \ \text{verifies (9) to (12)}\}, \\ &V_\phi := \{((\mathbf{u}^1, \phi^1), \ldots, (\mathbf{u}^p, \phi^p)) \in \prod_{i=1}^{p} H^1(\Omega_i)^3 \times H^1(\Omega_i) \ \text{such that}\ \phi^i \\ &\text{verifies (13), (15), (16) and}\ \phi^i \equiv 0 \ \text{on}\ \partial_{+}\Omega_1\}. \end{aligned} \tag{19}$$

Remark 2. *V_u and V_ϕ are Hilbert's spaces with the natural inner product (for instance, for V_u with the inherited one of the $\prod_{i=1}^{p} H^1(\Omega_i)^3 \times H^1(\Omega_i)$ space), that is:*

$$\begin{array}{l} (\mathbf{u},\mathbf{w})_{V_u} = \sum\limits_{i=1}^{p} \left\{ \int_{\Omega_i} \nabla \mathbf{u}^i : \nabla \mathbf{w}^i \, d\mathbf{x} + \int_{\Omega_i} \mathbf{u}^i \cdot \mathbf{w}^i \, d\mathbf{x} \right\}, \\ (\phi,\theta)_{V_\phi} = \sum\limits_{i=1}^{p} \left\{ \int_{\Omega_i} \nabla \phi^i \cdot \nabla \theta^i \, d\mathbf{x} + \int_{\Omega_i} \phi^i \theta^i \, d\mathbf{x} \right\}. \end{array} \tag{20}$$

Moreover, if we introduce the dual space we will have the following embedding chain (Brézis, 1984):

$$\begin{array}{l} V_u \subset \prod\limits_{i=1}^{p} H^1(\Omega_i)^3 \subset L^2(\Omega)^3 = (L^2(\Omega)^3)' \subset \prod\limits_{i=1}^{p} H^{-1}(\Omega_i)^3 \subset V_u' \\ V_\phi \subset \prod\limits_{i=1}^{p} H^1(\Omega_i) \subset L^2(\Omega) = (L^2(\Omega))' \subset \prod\limits_{i=1}^{p} H^{-1}(\Omega_i) \subset V_\phi', \end{array} \tag{21}$$

where we have used the fact of that $\prod_{i=1}^{p} L^2(\Omega_i)$ can be identified with $L^2(\Omega)$ since:

$$\int_\Omega \phi^2(\mathbf{x}) \, d\mathbf{x} = \sum_{i=1}^{p} \int_{\Omega_i} \phi_i^2(\mathbf{x}) \, d\mathbf{x} \tag{22}$$

Here $H^{-1}(\Omega_i)$ denotes the dual space of $H_0^{-1}(\Omega_i)$.

We also remark that the $H^{-1}(\Omega_i)$ spaces are Hilbert spaces and their norm is alternatively given in the following way: if $f^i \in H^{-1}(\Omega_i)$ then

$$\begin{array}{l} f^i(\mathbf{x}) = f_0^i(\mathbf{x}) + \sum\limits_{k=1}^{3} \dfrac{\partial f_k^i(\mathbf{x})}{\partial x_k} \text{ with } f_j^i \in L^2(\Omega_i), j = 0,1,2,3 \quad \text{and} \\ \|f^i\|_{H^{-1}(\Omega_i)} = \|f_0^i\|_{L^2(\Omega_i)} + \sum\limits_{k=1}^{3} \|f_k^i\|_{L^2(\Omega_i)}. \end{array} \tag{23}$$

We will assume the following regularity on the data:

$$\mathbf{f}_u \in \prod_{i=1}^{p} H^{-1}(\Omega_i)^3, \tag{24}$$

$$f_\phi \in \prod_{i=1}^{p} H^{-1}(\Omega_i), \tag{25}$$

$$\phi_0 \in \prod_{i=1}^{p} H^1(\Omega_i) \text{ and satisfy (17).} \tag{26}$$

In order to justify the definition of following weak solution, we shall consider, for a while, that $(\mathbf{u},\phi)$ is a classical solution of the system. Then we take a test functions $(\mathbf{w},\theta) \in V$. We multiply first equation of the system (8) by $\mathbf{w}^i$ and apply Green's theorem:

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{w} = \int_{\boldsymbol{\Omega}} \Delta\mathbf{u}\cdot\mathbf{w} + \int_{\partial\boldsymbol{\Omega}} \nabla\mathbf{u}\,\mathbf{w}\cdot\,\mathbf{n} = \int_{\boldsymbol{\Omega}} \Delta\mathbf{u}\cdot\mathbf{w} + \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial n}\cdot\mathbf{w}\,. \tag{27}$$

If we assume transmission conditions (12) and (16), and lateral side boundary $\partial_l\Omega_i$, the sum of normal derivatives vanishes since the normal vector to upper layers are opposites.

If moreover we assume that the test function vanishes on ∂_Ω, we conclude that:

$$\begin{aligned}
&\textstyle\int_{\Omega_i}\left(-\Delta\mathbf{u}^i(\mathbf{x}) - \dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(\mathbf{x})\right) - \dfrac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right) + \dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\right)\cdot\mathbf{w}^i(\mathbf{x})\;d\mathbf{x}\\
&= \textstyle\int_{\Omega_i}(\nabla\mathbf{u}^i(\mathbf{x}) : \nabla\mathbf{w}^i(\mathbf{x}) + \dfrac{1}{1-2\nu^i}div\mathbf{u}^i(\mathbf{x})\;div\mathbf{w}^i(\mathbf{x}) - \dfrac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right)\cdot\mathbf{w}^i(\mathbf{x}) +\\
&+\dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\;\mathbf{w}^i(\mathbf{x})\;d\mathbf{x})\,.
\end{aligned} \tag{28}$$

We operate analogously with the second equation of (8) but now multiplying by a test function θ^i:

$$-\int_{\Omega_i} \Delta\phi^i(\mathbf{x})\;\theta^i(\mathbf{x})\;\;d\mathbf{x} = \int_{\Omega_i} \nabla\phi^i(\mathbf{x})\cdot\nabla\theta^i(\mathbf{x})\;d\mathbf{x}\,. \tag{29}$$

Same way, we multiply by $\mathbf{w}^i$ and by θ^i to the right hand side of equations of the system (8). By adding them we originate the terms:

$$\begin{array}{l}
\sum\limits_{i=1}^{p}\left(-\dfrac{\rho^i}{\mu^i}\right)\int_{\Omega_i}\nabla\phi^i(\mathbf{x})\cdot\mathbf{w}^i(\mathbf{x})\;d\mathbf{x} + \langle\mathbf{f}_u,\mathbf{w}\rangle_{V_u'\times V_u}\,,\\
\sum\limits_{i=1}^{p}\left(-4\pi\rho^i G\right)\int_{\Omega_i} div\mathbf{u}^i(\mathbf{x})\;\theta^i(\mathbf{x})\;d\mathbf{x} + \langle f_\phi,\theta\rangle_{V_\phi'\times V_\phi}\,.
\end{array} \tag{30}$$

We reach to some integral equalities which any classical solution must verify. So, we are going to use it in order to define the notion of weak solution (without requiring the existence of any classical second derivative).

Definition 3. *We assume the regularity (24), (25) and (26), on the functions* $\mathbf{f}_u$, f_ϕ *and* ϕ_0. *We say that* $(\mathbf{u},\phi)$ *is a weak solution of the problem (8) with the boundary conditions (9)-(16) if* $(\mathbf{u},\phi-\phi_0) \in V$ *and for any test function* $(\mathbf{w},\theta) \in V$ *the following identities hold:*

$$\sum_{i=1}^{p}\left[\int_{\Omega_i}(\nabla\mathbf{u}^i(\mathbf{x}):\nabla\mathbf{w}^i(\mathbf{x})+\frac{1}{1-2\nu^i}div\mathbf{u}^i(\mathbf{x})\ div\mathbf{w}^i(\mathbf{x})-\frac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right)\cdot\mathbf{w}^i(\mathbf{x})\right.$$
$$+\frac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\ \mathbf{w}^i(\mathbf{x})\ d\mathbf{x})=\sum_{i=1}^{p}\left(-\frac{\rho^i}{\mu^i}\right)\int_{\Omega_i}\nabla\phi^i(\mathbf{x})\cdot\mathbf{w}^i(\mathbf{x})\ d\mathbf{x}+\langle\mathbf{f}_u,\mathbf{w}\rangle_{V_u'\times V_u}, \tag{31}$$

and

$$\sum_{i=1}^{p}\int_{\Omega_i}\nabla\phi^i(\mathbf{x})\cdot\nabla\theta^i(\mathbf{x})\ d\mathbf{x}\ =-\sum_{i=1}^{p}4\pi\rho^i G\int_{\Omega_i}div\mathbf{u}^i(\mathbf{x})\ \theta^i(\mathbf{x})\ d\mathbf{x}+\langle f_\phi,\theta\rangle_{V_\phi'\times V_\phi}. \tag{32}$$

The mathematical treatment of the problem will require to apply some technical results which may not be valid in some special cases. Due to that, it will be useful to introduce a change of scale $\mathbf{y}=\lambda\mathbf{x}$ allowing to define a re-scaling function $\mathbf{v}(\mathbf{y})=\mathbf{u}(\lambda\mathbf{x})$ which makes emerge some coefficients λ associated to first derivatives of $\mathbf{u}$ and λ^2 associated to second derivatives of $\mathbf{u}$. The new system of equations satisfies by $\mathbf{v}(\mathbf{y})$ are:

$$\left\{\begin{array}{l}-\Delta\mathbf{v}^i(\mathbf{y})-\dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{v}^i(\mathbf{y})\right)-\dfrac{\rho^i g\lambda}{\mu^i}\nabla\left(\mathbf{v}^i(\mathbf{y})\cdot\mathbf{e}_z\right)\\ +\dfrac{\rho^i g\lambda}{\mu^i}\mathbf{e}_z div\mathbf{v}^i(\mathbf{y})=-\dfrac{\rho_i\lambda}{\mu_i}\nabla\phi^i\left(\lambda^{-1}\mathbf{y}\right)+\lambda^2\mathbf{f}_u^i\left(\lambda^{-1}\mathbf{y}\right)\end{array}\right.\quad\text{in }\Omega_i,\qquad\text{(Dilatated equation)}$$

The main goal of this paper is to prove that under the above assumptions the system (8) is well possed (in the sense of Hadamard) on the space V.

Theorem 4. *Assume the regularity (24), (25) and (26) on the data* $\mathbf{f}_u$, f_ϕ *and* ϕ_0. *Then there exists a unique weak solution* $\{\mathbf{u},\phi\}$ *of the problem (8). Moreover, we have the following estimate on the continuous dependence with respect to the data:*

$$\begin{aligned}&\sum_{i=1}^{p}2\pi\rho^i G\left\|\nabla\mathbf{u}^i\right\|^2_{L^2(\Omega_i)}+\sum_{i=1}^{p}\frac{\rho^i}{2\mu^i}\left\|\nabla\phi^i\right\|^2_{L^2(\Omega_i)}\\ \leq\ & K(2\left\|\mathbf{f}_u\right\|^2_{V_u'}+\frac{1}{2}\left\|f_\phi\right\|^2_{V_\phi'}+4C\rho^1\left\|\phi^0\right\|_{H^{1/2}(\partial_+\Omega_1)}),\end{aligned} \tag{33}$$

where K *is a constant which depends on the scale* λ *and where* C *is the constant of the traze embedding* $H^1(\Omega_1)\to H^{1/2}(\partial_+\Omega_1)$.

Firstly, we shall prove the uniqueness of the solution of weak solution. Then we shall get the estimate on the continuous dependence with respect to the data. Finally we shall prove the existence of weak solutions by means of an iterative method which can be useful for numerical purposes.

3. Uniqueness of solution

Let us first prove the uniqueness of solutions of the coupled system. We assume two weak solutions for system (8), $\mathbf{u}_1^i, \mathbf{u}_2^i, \phi_1^i$ and ϕ_2^i with $i = 1, \ldots, p$, and let:

$$\begin{cases} \mathbf{u}^i(\mathbf{x}) = \mathbf{u}_1^i(\mathbf{x}) - \mathbf{u}_2^i(\mathbf{x}), \\ \phi^i(\mathbf{x}) = \phi_1^i(\mathbf{x}) - \phi_2^i(\mathbf{x}). \end{cases} \tag{34}$$

Since $\mathbf{u}_1^i, \mathbf{u}_2^i, \phi_1^i$ y ϕ_2^i are weak solutions, the verify (14), so by subtracting we obtain:

$$\begin{aligned} &\sum_{i=1}^{p}(-\left(\Delta\mathbf{u}_1^i(\mathbf{x}) - \Delta\mathbf{u}_2^i(\mathbf{x})\right) - \left(\frac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}_1^i(\mathbf{x})\right) - \nabla\left(div\mathbf{u}_2^i(\mathbf{x})\right)\right) - \\ &\frac{\rho^i g}{\mu^i}\left(\nabla\left(\mathbf{u}_1^i(\mathbf{x})\cdot\mathbf{e}_z\right) - \nabla\left(\mathbf{u}_2^i(\mathbf{x})\cdot\mathbf{e}_z\right)\right) + \frac{\rho^i g}{\mu^i}\left(\mathbf{e}_z div\mathbf{u}_1^i(\mathbf{x}) - \mathbf{e}_z div\mathbf{u}_2^i(\mathbf{x})\right)) \\ = &\sum_{i=1}^{p}(\frac{\rho^i}{\mu^i}\left(\nabla\phi_1^i(\mathbf{x}) - \nabla\phi_2^i(\mathbf{x})\right). \end{aligned} \tag{35}$$

Then, by the linearity of the differential operators we have:

$$\begin{aligned} &\sum_{i=1}^{p}\left(-\Delta\mathbf{u}^i(\mathbf{x}) - \frac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(\mathbf{x})\right) - \frac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right) + \frac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\right) \\ = &\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\nabla\phi^i(\mathbf{x}). \end{aligned} \tag{36}$$

Analogously:

$$\sum_{i=1}^{p}\left(-\Delta\phi^i(\mathbf{x})\right) = \sum_{i=1}^{p}4\pi\rho^i G div\mathbf{u}^i(\mathbf{x}). \tag{37}$$

Concerning the boundary conditions, since $\mathbf{u}_1^i, \mathbf{u}_2^i, \phi_1^i$ and ϕ_2^i verify the same boundary conditions of the system. So, on the lateral side boundary $\partial_l\Omega_i$, for $i = 1, \ldots, p$, we have

$$\begin{aligned} \mathbf{u}_1^i(\mathbf{x}) &= \mathbf{0}, \mathbf{x}\in\partial_l\Omega_i, \\ \phi^i(\mathbf{x}) &= 0, \mathbf{x}\in\partial_l\Omega_i. \end{aligned} \tag{38}$$

The transmission conditions on the top and bottom boundary, except first layer and last layer, i.e. on $\partial_{-}\Omega_1 \cup (\partial_{+}\Omega_2 \cup \partial_{-}\Omega_2) \cup \ldots \cup \partial_{+}\Omega_p$ for $i = 1, \ldots, p-1$ lead to:

$$\left\{\begin{array}{rcl} \mathbf{u}^i(\mathbf{x}) & = & \mathbf{u}^{i+1}(\mathbf{x}), \\ \dfrac{\partial \mathbf{u}^i(\mathbf{x})}{\partial z} & = & \dfrac{\partial \mathbf{u}^{i+1}(\mathbf{x})}{\partial z}, \\ \phi^i(\mathbf{x}) & = & \phi^{i+1}(\mathbf{x}), \\ \dfrac{\partial \phi^i(\mathbf{x})}{\partial z} & = & \dfrac{\partial \phi^{i+1}(\mathbf{x})}{\partial z}, \end{array}\right. \tag{39}$$

on top boundary, $\partial_{+}\Omega_1$:

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{u}^i(\mathbf{x})}{\partial z} = \mathbf{0}, \mathbf{x} \in \partial_{+}\Omega_1, \\ \phi^i(\mathbf{x}) = 0, \mathbf{x} \in \partial_{+}\Omega_1 \end{array}\right. \tag{40}$$

and on bottom boundary, $\partial_{-}\Omega_p$:

$$\left\{\begin{array}{l} \mathbf{u}^p(\mathbf{x}) = \mathbf{0}, \mathbf{x} \in \partial_{-}\Omega_p, \\ \phi^p(\mathbf{x}) = 0, \mathbf{x} \in \partial_{-}\Omega_p. \end{array}\right. \tag{41}$$

Multiplying the equation (36) by the term $4\pi\rho^i G\mathbf{u}^i$ and the equation (37) by $\dfrac{\rho^i}{\mu^i}\phi^i$, and we conclude that

$$\begin{array}{l} \sum\limits_{i=1}^{p}\left(-\Delta \mathbf{u}^i(\mathbf{x}) - \dfrac{1}{1-2\nu^i}\nabla(div\mathbf{u}^i(\mathbf{x})) - \dfrac{\rho^i g}{\mu^i}\nabla(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z) + \dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\right)\cdot 4\pi\rho G\mathbf{u}^i(\mathbf{x}) \\ = \sum\limits_{i=1}^{p}\dfrac{\rho^i}{\mu^i}\nabla\phi^i(\mathbf{x})\cdot 4\pi\rho G\mathbf{u}^i(\mathbf{x}). \end{array} \tag{42}$$

Let us omit for a while the symbol $\sum\limits_{i=1}^{p}$. Multiplying term by term and integrating on the domain we obtain:

$$\begin{array}{l} -\int_{\Omega_i} 4\pi\rho^i G\Delta\mathbf{u}^i(\mathbf{x})\cdot\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} - \int_{\Omega_i}\dfrac{4\pi\rho^i G}{1-2\nu^i}\nabla(div\mathbf{u}^i(\mathbf{x}))\cdot\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} \\ -\int_{\Omega_i}\dfrac{4\pi(\rho^i)^2 Gg}{\mu^i}\nabla(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z)\cdot\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} + \int_{\Omega_i}\dfrac{4\pi(\rho^i)^2 Gg}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\,\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} \\ = \int_{\Omega_i}\dfrac{\rho^i}{\mu^i}\nabla\phi^i(\mathbf{x})\cdot(4\pi\rho^i G)\,\mathbf{u}^i(\mathbf{x})\,d\mathbf{x}, \\ 4\pi\rho^i G(-\int_{\Omega_i}\Delta\mathbf{u}^i(\mathbf{x})\cdot\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} + \int_{\Omega_i}\dfrac{1}{1-2\nu^i}div\mathbf{u}^i(\mathbf{x})\,div\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} \\ -\int_{\Omega_i}\dfrac{\rho^i g}{\mu^i}\nabla(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z)\cdot\mathbf{u}^i(\mathbf{x})\,d\mathbf{x} + \int_{\Omega_i}\dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\,\mathbf{u}^i(\mathbf{x})\,d\mathbf{x}) \\ = \int_{\Omega_i}\dfrac{4\pi(\rho^i)^2 G}{\mu^i}\nabla\phi^i(\mathbf{x})\cdot\mathbf{u}^i(\mathbf{x})\,d\mathbf{x}. \end{array} \tag{43}$$

Then:

$$4\pi\rho^i G a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)=\frac{4\pi\left(\rho^i\right)^2 G}{\mu^i}\int_{\Omega_i}\nabla\phi^i\left(\mathbf{x}\right)\cdot\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x}\,, \tag{44}$$

where we used $a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)$ to denote to the corresponding bilinear form on $\mathbf{u}^i$. Integrating by parts we can see that terms of the second member of the equation can be simplified. Bearing in mind the boundary conditions we get:

$$\sum_{i=1}^{p}4\pi\rho^i G a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)=\sum_{i=1}^{p}\left(\frac{4\pi\left(\rho^i\right)^2 G}{\mu^i}[(\int_{\partial\Omega_i}\phi^i\left(\mathbf{s}\right)\mathbf{u}^i\left(\mathbf{s}\right)\cdot\mathbf{n}ds)-\int_{\Omega_i}\phi^i\left(\mathbf{x}\right)\,div\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x}]\right), \tag{45}$$

and then:

$$\sum_{i=1}^{p}4\pi\rho^i G a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)=-\sum_{i=1}^{p}\frac{4\pi\left(\rho^i\right)^2 G}{\mu^i}\int_{\Omega_i}\phi^i\left(\mathbf{x}\right)\,div\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x}\,. \tag{46}$$

We proceed with second equation:

$$\sum_{i=1}^{p}\left(-\Delta\phi^i\left(\mathbf{x}\right)\frac{\rho^i}{\mu^i}\phi^i\left(\mathbf{x}\right)\right)\;=\;\sum_{i=1}^{p}4\pi\rho^i G div\mathbf{u}^i\left(\mathbf{x}\right)\frac{\rho^i}{\mu^i}\phi^i\left(\mathbf{x}\right), \tag{47}$$

integrating on the domain Ω_i:

$$\sum_{i=1}^{p}\left(-\frac{\rho^i}{\mu^i}\int_{\Omega_i}\Delta\phi^i\left(\mathbf{x}\right)\,\phi^i\,d\mathbf{x}\right)\;=\;\sum_{i=1}^{p}\frac{4\pi\rho^i G}{\mu^i}\int_{\Omega_i}\phi^i\left(\mathbf{x}\right)\,div\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x}\,. \tag{48}$$

Using Green's theorem and the boundary conditions we get:

$$\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\nabla\phi^i\left(\mathbf{x}\right)\cdot\nabla\phi^i\left(\mathbf{x}\right)\,d\mathbf{x}=\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\left|\nabla\phi^i\left(\mathbf{x}\right)\right|^2\,d\mathbf{x}\;=\;\sum_{i=1}^{p}\frac{4\pi\rho^i G}{\mu^i}\int_{\Omega_i}\phi^i\left(\mathbf{x}\right)\,div\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x}\,. \tag{49}$$

Summarizing, we have following equations:

$$\begin{aligned}&\sum_{i=1}^{p}4\pi\rho^i G a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)=-\sum_{i=1}^{p}\frac{4\pi\left(\rho^i\right)^2 G}{\mu^i}\int_{\Omega_i}\phi^i\left(\mathbf{x}\right)\,div\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x},\\&\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega}\left|\nabla\phi^i\left(\mathbf{x}\right)\right|^2=\sum_{i=1}^{p}\frac{4\pi\rho^i G}{\mu^i}\int_{\Omega_i}\phi^i\,div\mathbf{u}^i\left(\mathbf{x}\right)\,d\mathbf{x}.\end{aligned} \tag{50}$$

Adding both relations we necessary reach that:

$$\sum_{i=1}^{p}4\pi\rho^i G a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)+\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\left|\nabla\phi^i\left(\mathbf{x}\right)\right|^2 d\mathbf{x}=0, \tag{51}$$

and using the coercive inequality satisfied by $a_u^i\left(\mathbf{u}^i,\mathbf{u}^i\right)$ (which we shall show later) we get:

$$\sum_{i=1}^{p}4\pi\rho^i G K\left\|\nabla\mathbf{u}^i\right\|_{L^2(\Omega_i)}^2+\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\left|\nabla\phi^i\left(\mathbf{x}\right)\right|^2 d\mathbf{x}=0, \tag{52}$$

for some constant K which can depend on the spatial scale. So we deduce that

$$\mathbf{u}^{i}\left(\mathbf{x}\right)=\mathbf{0},\text{ and }\phi^{i}\left(\mathbf{x}\right)=0. \tag{53}$$

since $\nabla\mathbf{u}^{i}=0$ and from the boundary conditions we conclude that $\mathbf{u}^{i}\left(\mathbf{x}\right)=\mathbf{0}$.. Similarly $\left|\nabla\phi^{i}\left(\mathbf{x}\right)\right|^{2}=0$ implies that $\phi^{i}\left(\mathbf{x}\right)=cte$. But as $\phi^{i}\left(\mathbf{x}\right)\equiv 0$ on the upper surface, necessarily $\phi^{i}\left(\mathbf{x}\right)=0$ holds in all Ω_{i}. We conclude that $\mathbf{u}_{1}^{i}=\mathbf{u}_{2}^{i}$ y $\phi_{1}^{i}=\phi_{2}^{i}$ $\forall i=1,\ldots,p$, this prove the uniqueness of weak solution.

4. Continuous dependence estimate

The argument of cancellation to prove the uniqueness of solutions can be applied in the same way to every possible weak solutions $\mathbf{u}^{i},\phi^{i}$. Now, it appears f_{ϕ}^{i} and $\mathbf{f}_{u}^{i}$, the contributions of the body force terms and the term of the integration by parts $\partial_{+}\Omega_{1}$. In particular, on top of the first layer appears the next inequality

$$\begin{aligned}&\sum_{i=1}^{p}4\pi\rho^{i}GA(\lambda)\left\|\nabla\mathbf{u}^{i}\right\|_{L^{2}(\Omega_{i})}^{2}+\sum_{i=1}^{p}\frac{\rho^{i}}{\mu^{i}}\int_{\Omega_{i}}\left|\nabla\phi^{i}\left(\mathbf{x}\right)\right|^{2}d\mathbf{x}\\ \leq\ &\sum_{i=1}^{p}4\pi\rho^{i}G\left\langle\mathbf{f}_{u}^{i},\mathbf{u}^{i}\right\rangle+\sum_{i=1}^{p}\frac{\rho^{i}}{\mu^{i}}\left\langle f_{\phi}^{i},\phi^{i}\right\rangle+\frac{4\pi\left(\rho^{1}\right)^{2}G}{\mu^{1}}\int_{\partial_{+}\Omega_{1}}\phi^{0}\left(\mathbf{s}\right)\phi^{1}\left(\mathbf{s}\right)\cdot\mathbf{n}ds,\end{aligned} \tag{54}$$

where $A(\lambda)$ is a positive constant depending on the scale. Applying Young inequality (with $\varepsilon=\frac{1}{4}$ in the first and third term and $\varepsilon=1$ in the second one) and using the theorem of traces $H^{1}(\Omega_{1})\rightarrow H^{1/2}(\partial_{+}\Omega_{1})$, the estimate follows without difficulty.

5. Existence of weak solution

To prove the existence of a weak solution we are going to divide the proof in two different uncoupled problems: the first one when gravitational perturbed is known and the second one in which the displacements are known. In both cases we shall use the Lax-Milgram's theorem (see, e.g. Brézis, 1984) which for the sake of the reader we recall here:

Let H be a Hilbert space and $a\left(u,v\right):H\times H\longrightarrow\mathbb{R}$ being a continuous and coercive bilinear form on H. Let $L:H\longrightarrow\mathbb{R}$ be a linear and continuous form on H. Then there exists a solutionu $\in H$ such that $a\left(u,v\right)=L\left(v\right)\ \forall v\in H$. We shall also use an extension of this result stated in terms of the Fredholm alternative is presented in Gilbarg et al., 1977.

5.1 Uncoupled problem for the potential (u assumed to be known).

Firstly, we are going to consider the following problem namely $(P1)\left[\phi_0^1, \mathbf{u}_0^i, f_\phi^i\right]$, on the space of energy V_ϕ, where we assume that $\mathbf{u}$ is a priori known.

$$(P1)\left[\phi_0^1, \mathbf{u}_0^i, f_\phi^i\right]\begin{cases} -\Delta\phi^i(\mathbf{x}) = 4\pi\rho^i G div\mathbf{u}^i(\mathbf{x}) + f_\phi^i(\mathbf{x}) & \text{in } \Omega_i, \\ \phi^i(\mathbf{x}) = 0 & \text{on } \partial_l\Omega_i \forall i = 1,\ldots,p, \\ \begin{array}{l} \phi^i(\mathbf{x}) = \phi^{i+1}(\mathbf{x}) \\ \dfrac{\partial\phi^i(\mathbf{x})}{\partial z} = \dfrac{\partial\phi^{i+1}(\mathbf{x})}{\partial z} \end{array} & \text{on } \partial_+\Omega_i = \partial_-\Omega_{i+1}\, \forall i = 1,\ldots,p-1, \\ \phi^1(\mathbf{x}) = \phi_0^1(\mathbf{x}) & \text{on } \partial_+\Omega_1, \\ \phi^p(\mathbf{x}) = 0 & \text{on } \partial_-\Omega_p. \end{cases} \tag{55}$$

Definition 5. *We assume the above regularity (25)and (26) on the data f_ϕ and ϕ_0 . We say that function ϕ is a weak solution of the problem (55) if $\phi^* := \phi - \phi_0 \in V_\phi$ and for every test function $\theta \in V_\phi$ the following integral identity holds:*

$$\sum_{i=1}^{p}\int_{\Omega_i}\nabla\phi^{*i}(\mathbf{x})\cdot\nabla\theta^i(\mathbf{x})\,d\mathbf{x} = \sum_{i=1}^{p}\left(-4\pi\rho^i G\right)\int_{\Omega_i} div\mathbf{u}^i(\mathbf{x})\,\theta^i(\mathbf{x})\,d\mathbf{x} + \langle f_\phi, \theta\rangle_{V_\phi'\times V_\phi}. \tag{56}$$

Theorem 6. *Assumed the (25) and (26) on the data f_ϕ and ϕ_0 , there exists a unique weak solution, ϕ, of problem $(P1)\left[\phi_0^1, \mathbf{u}_0^i, f_\phi^i\right]$.*

Proof. In order to apply the Lax-Milgram theorem we define the bilinear form $a_\phi \;:\; V_\phi \times V_\phi \longrightarrow \mathbb{R}$ and the linear form $L_\phi \;:\; V_\phi \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned} a_\phi(\phi^*,\theta) &:= \sum_{i=1}^{p} a_\phi^i(\phi^{*i},\theta^i) = \sum_{i=1}^{p}\int_{\Omega_i}\nabla\phi^{*i}(\mathbf{x})\cdot\nabla\theta^i(\mathbf{x})\,d\mathbf{x} \\ &= \int_\Omega \nabla\phi^*(\mathbf{x})\cdot\nabla\theta(\mathbf{x})\,d\mathbf{x} \\ L_\phi(\theta) &:= -\sum_{i=1}^{p} 4\pi\rho^i G\int_{\Omega_i} div\mathbf{u}^i(\mathbf{x})\,\theta^i(\mathbf{x})\,d\mathbf{x} + \langle f_\phi, \theta\rangle_{V_\phi'\times V_\phi} \end{aligned} \tag{57}$$

To apply this theorem we have to prove that the bilinear form $a_\phi(\cdot,\cdot)$ is continuous and coercive, and that the linear form $L_\phi(\cdot)$ is continuous. Let us see it:

i) $a_\phi(\cdot,\cdot)$ is bilinear form. It is easy to see that

$$\begin{cases} a_\phi(\lambda\phi_1+\mu\phi_2,\theta)=\lambda a_\phi(\phi_1,\theta)+\mu a_\phi(\phi_2,\theta), \\ a_\phi(\phi,\lambda\theta_1+\mu\theta_2)=\lambda a_\phi(\phi,\theta_1)+\mu a_\phi(\phi,\theta_2). \end{cases} \tag{58}$$

ii) To prove that $a_\phi(\cdot,\cdot)$ is continuous we should prove that there exists a constant C such that:

$$|a_\phi(\phi^*,\theta)|\leq C\|\phi^*\|_{V_\phi}\|\theta\|_{V_\phi}\quad \forall\phi,\theta\in V_\phi. \tag{59}$$

But

$$|a_\phi(\phi^*,\theta)| \ \leq\ \int_\Omega|\nabla\phi^*(\mathbf{x})\cdot\nabla\theta(\mathbf{x})|\,d\mathbf{x}\leq\|\nabla\phi^*\|_{L^2}\|\nabla\theta\|_{L^2} \tag{60}$$

and by some well-known results (see, e.g. Lions, 1981) we know that norm on $H^1(\Omega)$ is equivalent to the space V_ϕ.

iii) To prove that $a_\phi(\cdot,\cdot)$ is coercive we have to prove that there exists a constant $\alpha>0$ such that:

$$a_\phi(\phi^*,\phi^*)\geq\alpha\|\phi^*\|^2\quad\forall\phi^*\in V_\phi. \tag{61}$$

However by Poincare's inequality we have that

$$a_\phi(\phi^*,\phi^*)\ =\ \int_\Omega|\nabla\phi^*(\mathbf{x})|^2\,d\mathbf{x}=\|\phi^*\|^2_{V_\phi} \tag{62}$$

and so, by taking alpha equal to $\alpha=1.$, we obtain it.

iv) It is easy to see that $L_\phi(\cdot)$ is a linear.

v) To prove that $L_\phi(\theta)$ is continuous we have to prove that there exists a constant $D>0$ such that

$$L_\phi(\theta)\leq D\|\theta\|_{V_\phi}\quad\forall\theta\in V_\phi. \tag{63}$$

But

$$\begin{aligned} L_\phi(\theta(\mathbf{x})) &\leq \int_\Omega|(4\pi\rho G div\mathbf{u}(\mathbf{x})\,\theta(\mathbf{x})+f_\phi(\mathbf{x})\,\theta(\mathbf{x}))|\,d\mathbf{x} \\ &\leq 4\pi(\max_{i=1,\ldots,p}\rho^i)G\|\mathbf{u}\|_{L^2(\Omega)^3}\|\theta\|_{V_\phi}+\|f_\phi\|_{V_\phi'}\|\theta\|_{V_\phi}, \end{aligned} \tag{64}$$

so, by taking D as

$$D=4\pi(\max_{i=1,\ldots,p}\rho^i)G\|\mathbf{u}\|_{L^2(\Omega)^3}+\|f_\phi\|_{V_\phi'} \tag{65}$$

we get (63) ■

5.2 Uncoupled problem for the potential (ϕ assumed to be known).

Now, we consider the next problem on V_u:

$$
(P2)\left[\phi_0^i,\mathbf{f}_u^i\right]\begin{cases}
-\Delta\mathbf{u}^i(\mathbf{x})-\dfrac{1}{1-2\nu^i}\nabla\left(div\mathbf{u}^i(\mathbf{x})\right)-\dfrac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right) & \\
+\dfrac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})=-\dfrac{\rho_i}{\mu_i}\nabla\phi^i(\mathbf{x})+\mathbf{f}_u^i(\mathbf{x}) & \text{in } \Omega_i,\\
\mathbf{u}^i(\mathbf{x})=\mathbf{0} & \text{on } \partial_l\Omega_i\forall i=1,\dots,p,\\
\mathbf{u}^i(\mathbf{x})=\mathbf{u}^{i+1}(\mathbf{x}) & \text{on } \partial_+\Omega_i=\partial_-\Omega_{i+1}\\
\dfrac{\partial\mathbf{u}^i(\mathbf{x})}{\partial z}=\dfrac{\partial\mathbf{u}^{i+1}(\mathbf{x})}{\partial z} & \forall i=1,\dots,p-1,\\
\mathbf{u}^1(\mathbf{x})=\mathbf{0} & \text{on } \partial_+\Omega_1,\\
\mathbf{u}^p(\mathbf{x})=\mathbf{0} & \text{on } \partial_-\Omega_p.
\end{cases}
\tag{66}
$$

Definition 7. *We assume the regularity (24) and (26) on the data f_ϕ and ϕ_0. Given ϕ, with $\phi-\phi_0\in V_\phi$, we say that function $\mathbf{u}$ is a weak solution of the problem (66) if $\mathbf{u}\in V_u$ and for every test function $\mathbf{w}\in V_u$ the following integral identity holds:*

$$
\begin{aligned}
&\sum_{i=1}^{p}[\int_{\Omega_i}(\nabla\mathbf{u}^i(\mathbf{x}):\nabla\mathbf{w}^i(\mathbf{x})+\frac{1}{1-2\nu^i}div\mathbf{u}^i(\mathbf{x})\ div\mathbf{w}^i(\mathbf{x})-\frac{\rho^i g}{\mu^i}\nabla\left(\mathbf{u}^i(\mathbf{x})\cdot\mathbf{e}_z\right)\cdot\mathbf{w}^i(\mathbf{x})\\
&+\frac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\ \mathbf{w}^i(\mathbf{x})\ d\mathbf{x})]=-\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\nabla\phi^i(\mathbf{x})\cdot\mathbf{w}^i(\mathbf{x})\ d\mathbf{x}+\langle\mathbf{f}_u,\mathbf{w}\rangle_{V_u'\times V_u}.
\end{aligned}
\tag{67}
$$

Theorem 8. *Assume (24) and (26) on the data f_ϕ and ϕ_0. Assume also that*

$$
H(\rho,\mu,\nu)\quad\frac{(\max_{i=1,\dots,p}\dfrac{\rho^i}{\mu^i})g}{2(\min_{i=1,\dots,p}\dfrac{1}{1-2\nu^i})}\ \textit{is enough small.}\tag{68}
$$

Then there exist a unique weak solution, $\mathbf{u}$, of problem $(P2)\left[\phi_0^i,\mathbf{f}_u^i\right]$.

Proof. We define the bilinear form $a_u : V_u \times V_u \longrightarrow \mathbb{R}$ and the linear form $L_u : V_u \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned} a_u(\mathbf{u},\mathbf{w}) &:= \sum_{i=1}^{p} [\int_{\Omega_i} (\nabla \mathbf{u}^i(\mathbf{x}) : \nabla \mathbf{w}^i(\mathbf{x}) + \frac{1}{1-2\nu^i} div\mathbf{u}^i(\mathbf{x})\ div\mathbf{w}^i(\mathbf{x}) - \\ &\quad -\frac{\rho^i g}{\mu^i}\nabla(\mathbf{u}^i(\mathbf{x})\cdot \mathbf{e}_z)\cdot \mathbf{w}^i(\mathbf{x}) + \frac{\rho^i g}{\mu^i}\mathbf{e}_z div\mathbf{u}^i(\mathbf{x})\ \mathbf{w}^i(\mathbf{x})\ d\mathbf{x})], \\ L_u(\mathbf{w}) &:= -\sum_{i=1}^{p}\frac{\rho^i}{\mu^i}\int_{\Omega_i}\nabla\phi^i(\mathbf{x})\cdot\mathbf{w}^i(\mathbf{x})\ d\mathbf{x} + \langle \mathbf{f}_u,\mathbf{w}\rangle_{V_u'\times V_u} \end{aligned} \tag{69}$$

We shall apply the version of the Lax-Milgram theorem given in Gilbarg et al. (1977). We have to prove that the bilinear form $a_u(\cdot,\cdot)$ is continuous and coercive, and the lineal form $L_u(\cdot)$ is continuous. Indeed:

i) It is easy to see that $a_u(\cdot,\cdot)$ is a bilinear form.

ii) To prove that $a_u(\cdot,\cdot)$ is continuous we have to prove the existence of a constant C such that:

$$|a_u(\mathbf{u},\mathbf{w})| \le C\,\|\mathbf{u}\|_{V_u}\,\|\mathbf{w}\|_{V_u} \quad \forall \mathbf{u},\mathbf{w} \in V_u\,. \tag{70}$$

We have

$$\begin{aligned} |a_u(\mathbf{u},\mathbf{w})| &\le \|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{w}\|_{L^2} + (\max_{i=1,\dots,p}\frac{1}{1-2\nu^i})\|div\mathbf{u}\|_{L^2}\|div\mathbf{w}\|_{L^2} \\ &\quad +(\max_{i=1,\dots,p}\frac{\rho^i}{\mu^i})g[\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{w}\|_{L^2} + \|div\mathbf{u}\|_{L^2}\|\mathbf{w}\|_{L^2}]. \end{aligned} \tag{71}$$

So, by taking the constant C as

$$C = 1 + (\max_{i=1,\dots,p}\frac{1}{1-2\nu^i}) + 2(\max_{i=1,\dots,p}\frac{\rho^i}{\mu^i})g. \tag{72}$$

the inequality (70) holds.

iii) To prove that $a_u(\cdot,\cdot)$ is coercive we have to prove the existence of a constant $\alpha > 0$ such that:

$$a_u(\mathbf{u},\mathbf{u}) \ge \alpha\,\|\mathbf{u}\|_{V_u}^2 \quad \forall \mathbf{u} \in V_u\,. \tag{73}$$

But due to:

$$\sum_{i=1}^{p}\frac{\rho^{i}g}{\mu^{i}}\int_{\Omega_i}[-\nabla\left(\mathbf{u}^{i}\left(\mathbf{x}\right)\cdot\mathbf{e}_{z}\right)\cdot\mathbf{u}^{i}\left(\mathbf{x}\right)+\mathbf{e}_{z}div\mathbf{u}^{i}\left(\mathbf{x}\right)\;\mathbf{u}^{i}\left(\mathbf{x}\right)\;d\mathbf{x})]$$
$$= 2\sum_{i=1}^{p}\frac{\rho^{i}g}{\mu^{i}}\int_{\Omega_i}\mathbf{e}_{z}div\mathbf{u}^{i}\left(\mathbf{x}\right)\;\mathbf{u}^{i}\left(\mathbf{x}\right)\;d\mathbf{x} \tag{74}$$

we have that,

$$\left|\sum_{i=1}^{p}\int_{\Omega_i}\frac{\rho^{i}g}{\mu^{i}}[-\nabla\left(\mathbf{u}^{i}\left(\mathbf{x}\right)\cdot\mathbf{e}_{z}\right)\cdot\mathbf{u}^{i}\left(\mathbf{x}\right)+\mathbf{e}_{z}div\mathbf{u}^{i}\left(\mathbf{x}\right)\;\mathbf{u}^{i}\left(\mathbf{x}\right)\;d\mathbf{x})]\right|\leq 2(\max_{i=1,\dots,p}\frac{\rho^{i}}{\mu^{i}})g\left\|div\mathbf{u}\right\|_{L^2}\left\|\mathbf{u}\right\|_{L^2}. \tag{75}$$

So, by applying the Young's inequality $ab\leq\varepsilon a^{2}+\frac{1}{4\varepsilon}b^{2}$ with

$$\varepsilon=\frac{(\min_{i=1,\dots,p}\frac{1}{1-2\nu^{i}})}{2(\max_{i=1,\dots,p}\frac{\rho^{i}}{\mu^{i}})g} \tag{76}$$

we deduce that

$$a_{u}\left(\mathbf{u},\mathbf{u}\right)\;\;\geq\;\;\left\|\nabla\mathbf{u}\right\|_{L^2}^{2}-C\left\|\mathbf{u}\right\|_{L^2}^{2} \tag{coercive}$$

with

$$C=\frac{(\max_{i=1,\dots,p}\frac{\rho^{i}}{\mu^{i}})g}{2(\min_{i=1,\dots,p}\frac{1}{1-2\nu^{i}})}. \tag{77}$$

If we use the equivalence of norms in V_u

$$\left\|\nabla\mathbf{u}\right\|_{L^2}^{2}\geq\Theta(\left\|\nabla\mathbf{u}\right\|_{L^2}^{2}+\left\|\mathbf{u}\right\|_{L^2}^{2}) \tag{78}$$

we deduce that if

$$\Theta>\frac{(\max_{i=1,\dots,p}\frac{\rho^{i}}{\mu^{i}})g}{2(\min_{i=1,\dots,p}\frac{1}{1-2\nu^{i}})} \tag{ratio}$$

then

$$a_{u}\left(\mathbf{u},\mathbf{w}\right)\;\;\geq\;\;(\Theta-\frac{(\max_{i=1,\dots,p}\frac{\rho^{i}}{\mu^{i}})g}{2(\min_{i=1,\dots,p}\frac{1}{1-2\nu^{i}})})\left\|\nabla\mathbf{u}\right\|_{L^2}^{2} \tag{79}$$

and taking

$$\alpha = \Theta - \frac{(\max_{i=1,,p} \dfrac{\rho^i}{\mu^i})g}{2(\min_{i=1,\ldots,p} \dfrac{1}{1-2\nu^i})}. \tag{80}$$

iv) It is easy to see that $L_u(\cdot)$ is a linear form.

v) To prove that $L_u(\cdot)$ is continuous we have to prove the existence of a constant $D > 0$ such that:

$$L_u(\mathbf{w}) \leq D \|\mathbf{w}\|_{V_u} \quad \forall \mathbf{w} \in V_u\,. \tag{81}$$

But we have

$$\begin{aligned} L_u(\mathbf{w}) &\leq \|\nabla\phi(\mathbf{x})\|_{L^2} \|\mathbf{w}(\mathbf{x})\|_{L^2} + \|\mathbf{f}_u(\mathbf{x})\|_{L^2} \|\mathbf{w}(\mathbf{x})\|_{L^2} \\ &\leq (\max_{i=1,\ldots,p} \frac{\rho^i}{\mu^i}) \|\nabla\phi\|_{L^2} \|\mathbf{w}\|_{L^2} + \|\mathbf{f}_u\|_{V_u'} \|\mathbf{w}\|_{V_u} \end{aligned} \tag{82}$$

and so, is enough to take

$$D = (\max_{i=1,\ldots p} \frac{\rho^i}{\mu^i}) \|\nabla\phi\|_{L^2} + \|\mathbf{f}_u\|_{V_u'}\,. \tag{83}$$

To treat the general case we can take the Θ constant as

$$\Theta = \frac{1}{2}\min\{\frac{1}{C(\Omega)}, 1\} \tag{84}$$

where $C(\Omega)$ is the Poincaré constant on the domain Ω. We introduce the change of scale $\mathbf{y} = \lambda\mathbf{x}$ what allows to define a rescaling function $\mathbf{v}(\mathbf{y}) = \mathbf{u}(\lambda\mathbf{x})$ which make to emerge terms in λ associated to the first derivative of $\mathbf{u}$ and terms in λ^2 associated to second derivatives of $\mathbf{u}$. In this way, equations satisfied by $\mathbf{v}(\mathbf{y})$ are:

$$\left\{ \begin{array}{l} -\Delta\mathbf{v}^i(\mathbf{y}) - \dfrac{1}{1-2\nu^i}\nabla(div\mathbf{v}^i(\mathbf{y})) - \dfrac{\rho^i g\lambda}{\mu^i}\nabla(\mathbf{v}^i(\mathbf{y})\cdot\mathbf{e}_z) \\ +\dfrac{\rho^i g\lambda}{\mu^i}\mathbf{e}_z div\mathbf{v}^i(\mathbf{y}) = -\dfrac{\rho_i\lambda}{\mu_i}\nabla\phi^i(\lambda^{-1}\mathbf{y}) + \lambda^2\mathbf{f}_u^i(\lambda^{-1}\mathbf{y}) \end{array} \right. \quad \text{in } \Omega_i, \qquad \text{(Dilatation equation)}$$

Next, we remark that new constant $C(\lambda\Omega)$ can be taken as $\lambda C(\Omega)$ since it depends only on the diameter of Ω. So, in the new system of equations we have to require that the following inequality is verified:

$$\frac{1}{2}\min\{\frac{1}{\lambda C(\Omega)}, 1\} > \frac{\lambda(\max_{i=1,\ldots,p} \dfrac{\rho^i}{\mu^i})g}{2(\min_{i=1,\ldots,p} \dfrac{1}{1-2\nu^i})}. \tag{85}$$

But this is obtained by taking λ small enough. This allows to avoid consider the hypothesis on $H(\rho,\mu,\nu)$. ■

Remark 9. *If we do not have hypothesis $H(\rho,\mu,\nu)$ Fredholm's alternative (as stated e.g. in Gilbarg et al., 1977) can be also applied. Uniqueness of solutions of the problem with zero data would lead to the existence of solution of the problem for arbitrary data.*

Remark 10. *In fact, by remarking that the last inequality and the change of variable do not modify the contour of level of $\mathbf{u}$, without loss of generality, we can assume that the coercive constant is $\alpha \geq 1$.*

Once proved the above theorems on the uncoupled problems we proceed with the proof of the main theorem of this paper, that is, the existence of weak solutions for the coupled system (8).

5.3 General idea of the proof of existence of solutions of the coupled system

The existence of weak solutions for both cases (ϕ known, problem $(P1)\left[\phi_0^1,\mathbf{u}_0^i,f_\phi^i\right]$, and $\mathbf{u}$ known, problem $(P2)\left[\phi_0^i,\mathbf{f}_u^i\right]$) has been proved. To prove the existence of weak solutions of the coupled system we will use an iterative scheme which, as matter of facts, is also interesting for the numerical analysis of the system. Firstly, we shall construct two sequences $\{\mathbf{u}^n(\mathbf{x})\}$ and $\{\phi^n(\mathbf{x})\}$ in following way. We start with the vector $\phi^0(\mathbf{x})$ which has the boundary date $\phi_0(\mathbf{x})$ as a first component and the rest of the components 0. With this vector and problem $(P2)\left[\phi_0^i,\mathbf{f}_u^i\right]$ we obtain a unique vector $\mathbf{u}^1(\mathbf{x})$. Then, putting it in problem $(P1)\left[\phi_0^1,\mathbf{u}_0^i,f_\phi^i\right]$ we obtain a unique vector ϕ^1. In way we build the sequences:

$$\phi^0(\mathbf{x}) = \begin{pmatrix} \phi_0(\mathbf{x}) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \overset{(P2)[\phi_0^i,\mathbf{f}_u^i]}{\longrightarrow} \mathbf{u}^1(\mathbf{x}) = \begin{pmatrix} \mathbf{u}_1^1(\mathbf{x}) \\ \mathbf{u}_2^1(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}_p^1(\mathbf{x}) \end{pmatrix} \overset{(P1)[\phi_0^1,\mathbf{u}_0^i,f_\phi^i]}{\longrightarrow} \phi^1(\mathbf{x}) = \begin{pmatrix} \phi_1^1(\mathbf{x}) \\ \phi_2^1(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ \phi_p^1(\mathbf{x}) \end{pmatrix} \overset{(P2)[\phi_0^i,\mathbf{f}_u^i]}{\longrightarrow}$$

$$\mathbf{u}^2(\mathbf{x}) = \begin{pmatrix} \mathbf{u}_1^2(\mathbf{x}) \\ \mathbf{u}_2^2(\mathbf{x}) \\ . \\ . \\ . \\ \mathbf{u}_p^2(\mathbf{x}) \end{pmatrix} ...\phi^{n-1}(\mathbf{x}) = \begin{pmatrix} \phi_1^{n-1}(\mathbf{x}) \\ \phi_2^{n-1}(\mathbf{x}) \\ . \\ . \\ . \\ \phi_p^{n-1}(\mathbf{x}) \end{pmatrix} \overset{(P2)[\phi_0^i,\mathbf{f}_u^i]}{\longrightarrow} \mathbf{u}^n(\mathbf{x}) = \begin{pmatrix} \mathbf{u}_1^n(\mathbf{x}) \\ \mathbf{u}_2^1(\mathbf{x}) \\ . \\ . \\ . \\ \mathbf{u}_p^n(\mathbf{x}) \end{pmatrix} \overset{(P1)[\phi_0^1,\mathbf{u}_0^i,f_\phi^i]}{\longrightarrow}$$

$$\phi^n(\mathbf{x}) = \begin{pmatrix} \phi_1^n(\mathbf{x}) \\ \phi_2^n(\mathbf{x}) \\ . \\ . \\ . \\ \phi_p^n(\mathbf{x}) \end{pmatrix}.$$

Then we prove weak convergence on V_u and V_ϕ

$$\{\mathbf{u}^n\} \overset{V_u}{\rightharpoonup} \mathbf{u}, \ \ \{\phi^n\} \overset{V_\phi}{\rightharpoonup} \phi \tag{86}$$

to some functions $(\mathbf{u}, \phi)$ and more later we will see $\{\mathbf{u}, \phi\}$ is weak solution of the coupled system.

5.3.1 A priori estimates on $\{\mathbf{u}^n, \phi^n\}$

Now, we shall construct an iterative scheme by defining a linear operator L on the vector $\mathbf{u}$ as following way:

$$L\mathbf{u}(\mathbf{x}) := -\Delta\mathbf{u}(\mathbf{x}) - \frac{1}{1-2\nu}\nabla(div\mathbf{u}(\mathbf{x})) - \frac{\rho g}{\mu}\nabla(\mathbf{u}(\mathbf{x})\cdot\mathbf{e}_z) + \frac{\rho g}{\mu}\mathbf{e}_z div\mathbf{u}(\mathbf{x}) . \tag{87}$$

We also define

$$\mathbf{F}^{n-1}(\mathbf{x}) := \frac{\rho}{\mu}\nabla\phi^{n-1}(\mathbf{x}) + \mathbf{f}_u(\mathbf{x}) . \tag{88}$$

Assuming $\phi^{n-1}(\mathbf{x})$ given, then (by the above theorem) we can define $\mathbf{u}^n(\mathbf{x})$ as the unique weak solution of:

$$\begin{cases} L\mathbf{u}^n(\mathbf{x}) = \mathbf{F}^{n-1}(\mathbf{x}), \\ + \text{Boundary conditions (9)-(12)}. \end{cases} \tag{89}$$

On the other hand, by defining $J^n(\mathbf{x}) := 4\pi\rho G div\mathbf{u}^n(\mathbf{x}) + f_\phi(\mathbf{x})$, we can define $\phi^n(\mathbf{x})$ as unique weak solution of:

$$\begin{cases} -\Delta\phi^n(\mathbf{x}) = J^n(\mathbf{x}), \\ + \text{(13)-(16)}. \end{cases} \tag{90}$$

The iterative scheme we consider is the following

$$\begin{array}{ccccccccc} \phi^0 & & \phi^1 & & \phi^2 & \ldots & \phi^{n-1} & & \phi^n \\ & \searrow & \uparrow & \searrow & \uparrow & & \uparrow & \searrow & \uparrow \\ & & \mathbf{u}^1 & & \mathbf{u}^2 & \ldots & \mathbf{u}^{n-1} & & \mathbf{u}^n \end{array} \tag{91}$$

where the step $2n$ is determined by iterative scheme (89) and the step $2n+1$ is determined by (90). We are going to obtain some a priori estimates (independent on n) in order to pass to the limit.

Lemma 11. *We assume that*

$$\varepsilon := \frac{4\pi^2 G}{\alpha_u}(\max_{i=1,\ldots,p} \frac{(\rho^i)^2}{\mu^i}) < 1 \tag{92}$$

Then, for any natural n

$$\|(\phi^*)^n\|_{V_\phi} \le \varepsilon \left\|(\phi^*)^{n-1}\right\|_{V_\phi} + \delta_u, \tag{93}$$

$$\|\mathbf{u}^n\|_{V_u} \le \varepsilon \left\|\mathbf{u}^{n-1}\right\|_{V_u} + \delta_\phi, \tag{94}$$

$$\begin{aligned} \delta_u &= \frac{(\max_{i=1,\ldots,p} \frac{\rho^i}{\mu^i})}{\alpha_u} \|f_\phi\|_{V'_\phi} + \frac{1}{\alpha_u} \|\mathbf{f}_u^i\|_{V'_u} + \frac{(\max_{i=1,\ldots,p} \frac{\rho^i}{\mu^i})}{\alpha_u} \|\phi_0\|_{L^2(\Omega)} \\ \textit{and } \delta_\phi &= \frac{4\pi G(\max_{i=1,\ldots,p} \rho^i)}{\alpha_u} \|\mathbf{f}_u^i\|_{V'_u} + \frac{8\pi G(\max_{i=1,\ldots,p} \frac{(\rho^i)^2}{\mu^i})}{\alpha_u} \|\phi_0\|_{L^2(\Omega)} + \|f_\phi\|_{V'_\phi}. \end{aligned} \tag{95}$$

In particular,

$$\|(\phi^*)^n\|_{V_\phi} \le \frac{\delta}{1-\varepsilon}, \tag{96}$$

$$\|\mathbf{u}^n\|_{V_u} \le \frac{\delta}{1-\varepsilon}. \tag{97}$$

The proof of Lemma 11 is given in the Appendix.

Remark 12. *If hypothesis of the last lemma is not verified we can carry out a change of scale in the spatial variable* $\mathbf{y} = \lambda\mathbf{x}$ *so that the final coupled system for rescaled functions* $\varphi(\mathbf{y}) = \phi(\lambda\mathbf{x})$ *and* $\mathbf{v}(\mathbf{y}) = \mathbf{u}(\lambda\mathbf{x})$ *lead to some new constants (now dependent on* λ*) which implies that the new* ε *verifies this hypothesis. Then, we can always reconsider the system in an appropriate scale and, in this way, we can conclude that the subsequences* $\{\mathbf{u}^n, \phi^n\}$ *are uniformly bounded on the space* V*.*

5.4 Passing to the limit.

As V is an Hilbert space, from the a priori estimates we can say there exists a subsequence which converge weakly

$$\begin{cases} \mathbf{u}^m \rightharpoonup \mathbf{u} \text{ in } V_u, \\ \phi^m \rightharpoonup \phi \text{ in } V_\phi. \end{cases} \tag{98}$$

From the compact Sobolev embedding $H^1 \subset L^2$ we can say that this subsequence $\{\mathbf{u}^m, \phi^m\}$ converges strongly in L^2. Now, if we multiply by any test functions we can pass to the limit in all expressions and so the vectorial function $(\mathbf{u}, \phi)$ is a weak solution of the problem. Moreover, from the uniqueness of solutions (already proved), we can affirm that any subsequence of $\{\mathbf{u}^n, \phi^n\}$ has to converge to the same vectorial function $(\mathbf{u}, \phi)$. In this way the proof of the Theorem 4 is now finished.

6. Conclusion

We have proved the existence and uniqueness of solutions of an elastic-gravitational model representing an ideal Earth layered. We have now completed a part of the work started by Rundle in 1982. We have applied some techniques of the weak solutions of partial differential equations theory give a rigorous proof about the well-posedness of the model. Moreover, we have given a constructive proof of the existence which will allow us to construct a computational method, by means of an iterative scheme which we show to be convergent, to compute the coupled effects of gravity and elastic deformations from possible sources embedded in the Earth. We also discover that there are suitable spatial scales in which the model is better determined than in others due to the delicate balance between the second and first differential terms in the displacement equation.

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Appendix: *Proof of Lemma 11*

Let ϕ^{*n} be such that

$$a_\phi\left(\phi^{*n},\theta\right)=\left\langle J^n,\theta\right\rangle_{V_\phi'\times V_\phi}\quad\forall\theta\in V_\phi\,. \tag{99}$$

By taking $\theta=\phi^{*n}$ then

$$a_\phi\left(\left(\phi^*\right)^n,\left(\phi^*\right)^n\right)=\left\langle J^n,\left(\phi^*\right)^n\right\rangle_{V_\phi'\times V_\phi}\leq\left\|J^n\right\|_{V_\phi'}\left\|\left(\phi^*\right)^n\right\|_{V_\phi}\,. \tag{100}$$

But, on the other hand, as the bilinear form is coercive (with $\alpha_\phi=1$) we have

$$\left\|\left(\phi^*\right)^n\right\|_{V_\phi}^2\leq a_\phi\left(\left(\phi^*\right)^n,\left(\phi^*\right)^n\right)\leq\left\|J^n\right\|_{V_\phi'}\left\|\left(\phi^*\right)^n\right\|_{V_\phi}\,, \tag{101}$$

and so

$$\left\|\left(\phi^*\right)^n\right\|_{V_\phi}\leq\left\|J^n\right\|_{V_\phi'}\,. \tag{102}$$

Then, by the definition of the norm of the dual space, we obtain the a priori estimate:

$$\left\|\phi^n\right\|_{H^1(\Omega)}\leq4\pi\rho G\left\|\mathbf{u}^n\right\|_{L^2(\Omega)^3}+\left\|f_\phi\right\|_{V_\phi'}\,. \tag{103}$$

On the other hand, we remind that if $\mathbf{F}^{n-1}\left(\mathbf{x}\right):=-\dfrac{\rho}{\mu}\nabla\phi^{n-1}\left(\mathbf{x}\right)+\mathbf{f}_u\left(\mathbf{x}\right)$ then we have that

$$a_u\left(\mathbf{u}^n,\mathbf{w}\right)=\left\langle\mathbf{F}^{n-1},\mathbf{w}\right\rangle_{V_u'\times V_u}\ \forall\mathbf{w}\in V_u\,. \tag{104}$$

Taking as test function $\mathbf{w}=\mathbf{u}^n$ we get

$$a_u\left(\mathbf{u}^n,\mathbf{u}^n\right)=\left\langle\mathbf{F}^{n-1},\mathbf{u}^n\right\rangle\leq\left\|\mathbf{F}^{n-1}\right\|_{V_u'}\left\|\mathbf{u}^n\right\|_{V_u} \tag{105}$$

and from the coerciveness of the bilinear form a_u we conclude that

$$\alpha_u\left\|\mathbf{u}^n\right\|_{V_u}\leq\left\|\mathbf{F}^{n-1}\right\|_{V_u'}\,, \tag{106}$$

Substituting $\left\|\phi^{*n-1}\right\|_{L^2(\Omega)}$ into the estimate obtained in the last step we conclude that

$$\left\|\mathbf{u}^n\right\|_{V_u}\leq\frac{4\pi\rho^2G}{\mu\alpha_u}\left\|\mathbf{u}^{n-1}\right\|_{V_u}+\frac{\rho}{\mu\alpha_u}\left\|f_\phi\right\|_{V_\phi'}+\frac{1}{\alpha_u}\left\|\mathbf{f}_u^i\right\|_{V_u'}+\frac{\rho}{\mu\alpha_u}\left\|\phi_0\right\|_{L^2(\Omega)}\,. \tag{107}$$

Similarly, substituting $\|\mathbf{u}^n\|_{L^2(\Omega)^3}$ (in (103)) into the estimate obtained in the last step we arrive to

$$\|\phi^n\|_{H^1(\Omega)} \leq \frac{4\pi\rho^2 G}{\mu\alpha_u} \left\|\phi^{n-1}\right\|_{H^1(\Omega)} + \frac{4\pi\rho G}{\alpha_u} \left\|\mathbf{f}_u^i\right\|_{V_u'} + \frac{8\pi\rho^2 G}{\mu\alpha_u} \|\phi_0\|_{L^2(\Omega)} + \|f_\phi\|_{V_\phi'}\,, \tag{108}$$

which finishes the first part of the lemma. The uniform estimates of the statement are obtained by a recurrence argument by using the sum of a geometrical progression.

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Captions

Figure 1. Domain of the problem.

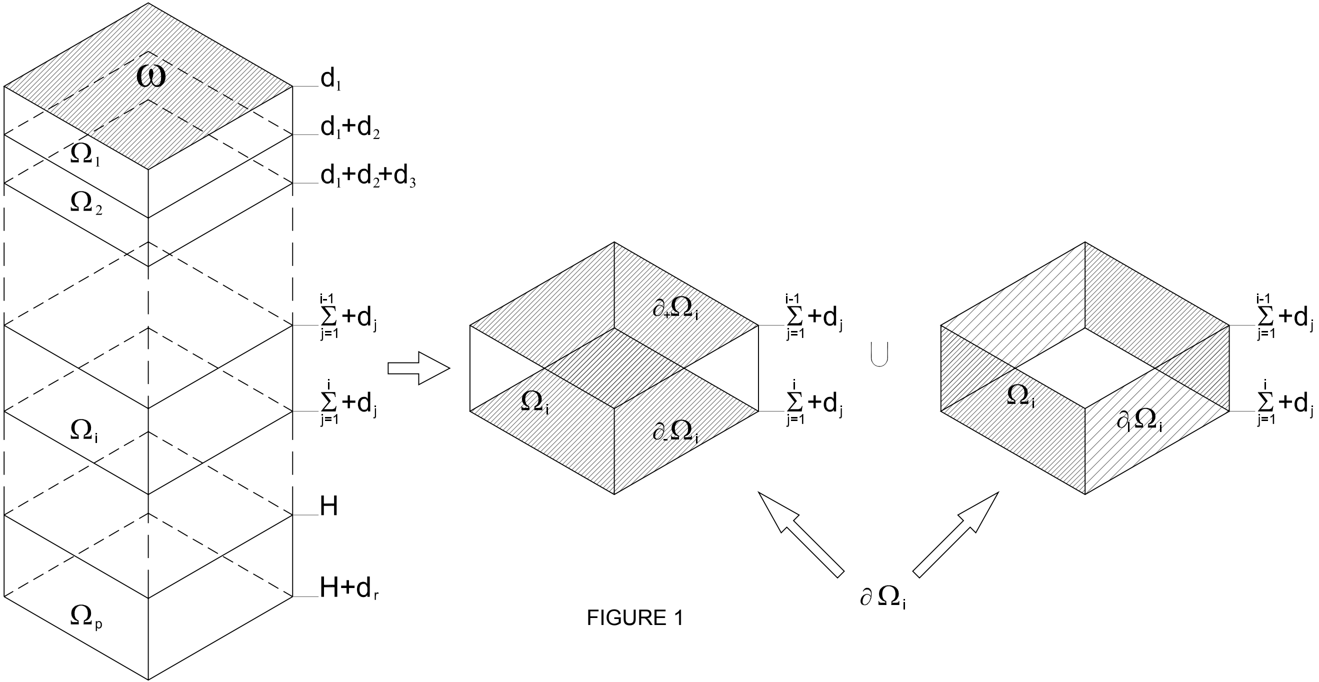
ω
d_1
d_1+d_2
$d_1+d_2+d_3$
Ω_1
Ω_2
$\sum_{j=1}^{i-1}+d_j$
$\sum_{j=1}^{i}+d_j$
Ω_i
H
$H+d_r$
Ω_p
$\partial_+\Omega_i$
Ω_i
$\partial_-\Omega_i$
$\sum_{j=1}^{i-1}+d_j$
$\sum_{j=1}^{i}+d_j$
$\cup$
Ω_i
$\partial_l\Omega_i$
$\sum_{j=1}^{i-1}+d_j$
$\sum_{j=1}^{i}+d_j$
$\partial\,\Omega_i$

FIGURE 1