# Local strong solutions of a parabolic system related to the Boussinesq approximation for buoyancy-driven flow with viscous heating

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**Abstract.** We propose a modification of the classical Navier-Stokes-Boussinesq system of equations, which governs buoyancy-driven flows of viscous, incompressible fluids. This modification is motivated by unresolved issues regarding the global solvability of the classical system in situations where viscous heating cannot be neglected. A simple model problem leads to a coupled system of two parabolic equations with a source term involving the square of the gradient of one of the unknowns. In the present paper, we establish the local-in-time existence and uniqueness of strong solutions for the model problem. The full system of equations and the global-in-time existence of weak solutions will be addressed in forthcoming work.

**Keywords.** Boussinesq approximation, viscous heating, parabolic system, strong solutions.

**AMS subject classification.** 35Q35, 35Q72, 35K45, 35K50.

## 1 Introduction

The flow of a viscous, heat-conducting fluid under the force of gravity is governed by a system of balance equations for momentum, mass, and internal energy (see [1], Ch. 4.1–4.3). In the so-called Boussinesq approximation, the system is reduced to the Navier-Stokes equations for a homogeneous, incompressible fluid, coupled to a semilinear heat equation (see [16] or [20]). The main coupling term is the buoyancy force (generation of momentum due to temperature gradients); viscous heating (heat production due to internal friction) is neglected. The resulting initial-boundary value problems are well posed in the same sense as for the classical Navier-Stokes equations; in particular, they have local-in-time strong solutions and global-in-time weak solutions (see [10], [11], [17]).

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Nevertheless, in many situations, viscous heating has a significant effect on the flow and cannot be neglected. It leads to a quadratic gradient term on the right-hand side of the heat equation, which causes major mathematical difficulties. In the absence of thermal convection due to buoyancy, these difficulties have been addressed and largely overcome (see, for example, [15], Chapter 3.4, or [9], [18]); but open problems remain if both, buoyancy *and* viscous heating, are relevant.

Various models have been proposed, incorporating both of the effects, while maintaining the relative simplicity of the Boussinesq approximation (see [13] and the references therein). Kagei et al. [13] derive a model that, besides buoyancy and viscous heating, includes "adiabatic heating," which gives rise to an additional term in the heat equation that balances the term representing viscous heating. This facilitates the energy estimates, based on which the authors establish the existence and uniqueness of strong solutions, local in time, for a Rayleigh-Bénard convection problem (i.e. under periodicity auxiliary conditions); for small data, these solutions are global in time, after a process of simplification of the model made by asymptotic analysis and with a constant product of the heat capacity by the density. However, there are unresolved issues regarding the global-in-time existence of (weak) solutions for large data. In the case of a Newtonian fluid, the only result in this direction appears to be [12], Theorem 2.1, where a two-dimensional Bénard problem is treated. Higher-dimensional analogues have been obtained only for a non-Newtonian model [19].

That global existence should be an issue, in this context, is not very surprising: while the primitive equations, supplemented with suitable boundary conditions, satisfy the principle of conservation of energy, the simplified equations violate this principle (except in special cases, see [19], Remark 2). We refer to [6] for a detailed discussion of this inconsistency, which may well cause solutions to blow up in finite time (see [4] for a related problem with permanent blow-up at the boundary).

In this paper we consider a rather simplistic model problem that may not be physically relevant, yet captures the characteristic mathematical difficulty of the full problem. The paper enlarge and improves the preliminar presentation made in the note [5]. Among other improvements and as indicated above, our results apply to the case of variable thermal diffusion coefficient. A suitable modification of the classical Boussinesq approximation allows us to establish the local-in-time existence of strong solutions for the resulting initial-boundary value problems under some restrictions on the size of the initial data and without the simplification made in [13]. This sets the stage for a forthcoming analysis of the full system of equations and the construction of global-in-time weak solutions (see [7]).

Consider a unidirectional flow of a viscous, incompressible fluid, independent of distance in the flow direction, in a channel parallel to the constant force of gravity. The flow can be described in terms of two scalar variables, a velocity $v$ (scalar since the flow is unidirectional) and a temperature $\theta$; both are functions of time $t \in \mathbb{R}_+$ and position $x \in \Omega$, where $\Omega$ denotes the cross-section of the flow channel (a bounded domain in $\mathbb{R}^2$). The functions $v$ and $\theta$ satisfy a pair of parabolic PDEs of the form

$$\rho v_t - \mu \Delta v = \rho g + f(t), \quad \rho c \theta_t - div(k \nabla \theta) = \mu |\nabla v|^2 \quad \text{in } (0, \infty) \times \Omega, \tag{1}$$

where $\rho$, $\mu$, $c$, and $\kappa$, respectively, denote the density, viscosity, heat capacity, and thermal conductivity of the fluid; $g$ is the gravitational acceleration (a positive constant). The function $f$ represents the component of the pressure gradient opposite to the flow direction, which in this situation is independent of the spatial variable and plays the role of a given, externally applied force. The equations must be supplemented by suitable initial conditions at time $t = 0$ and boundary conditions on $\partial\Omega$, for example, a homogeneous Dirichlet condition for $v$ and a homogeneous Neumann condition for $\theta$ in the case of mechanically impermeable, thermally insulated channel walls ($\widehat{n}$ denotes the unit outward normal vector field on $\partial\Omega$):

$$v = 0, \quad \frac{\partial \theta}{\partial \widehat{n}} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \tag{2}$$

$$v = v_0, \quad \theta = \theta_0 \quad \text{on } \{0\} \times \Omega. \tag{3}$$

Since we are interested in buoyancy effects, we have to assume that the density $\rho$ is a (nonincreasing) function of temperature. In general, also the remaining coefficients, $\mu$, $c$, and $\kappa$, may depend on temperature; but here, these are assumed to be positive constants. Now suppose that the temperature scale is chosen such that $\theta$ can be expected to fluctuate in a fairly narrow range about the reference temperature $\theta = 0$. Then, in a first-order approximation, $\rho$ should decrease linearly with $\theta$, and we can write

$$\rho = \rho_0(1 - \alpha\theta), \tag{4}$$

where $\rho_0 = \rho(0) > 0$ is the density at the reference temperature and $\alpha = -\rho'(0)/\rho(0) > 0$ is the thermal expansion coefficient at the reference temperature. The force of gravity is then given by

$$\rho g = \rho_0 g - \rho_0 \alpha \theta g. \tag{5}$$

The constant $\rho_0 g$ represents the hydrostatic pressure gradient and may be absorbed into the applied force $f$; the term $\rho_0 \alpha \theta g$ represents the force of buoyancy. Of course, (5) makes sense only as long as $\theta$ does not deviate too much from 0, and in particular, $\rho$ must remain positive. The ansatz (4) is one of the basic assumptions of the Boussinesq approximation; but it is used only in computing the force of gravity in accordance with (5) — everywhere else in the governing equations, $\rho$ is set equal to $\rho_0$. In other words, the fluid is considered "thermally compressible, yet mechanically incompressible" (see [20] for a rigorous justification). In the case of a unidirectional flow parallel to gravity, as described by the system (1), this means that we have $\rho = \rho_0(1 - \alpha\theta)$ on the right-hand side of the first equation, but $\rho = \rho_0$ in the terms involving the time derivatives of $v$ and $\theta$. This causes the characteristic difficulty alluded to earlier (lack of energy conservation), and as far as we know, global-in-time existence of solutions is an open problem, at least without restrictions on the size of the initial data.

It is natural to ask whether this difficulty can be circumvented by using the ansatz (4), or a generalization thereof, not only in the force of gravity, but also in the rate of change of internal energy (the term involving $\theta_t$). In the rate of change of momentum (the term involving $v_t$), which is of lesser importance in this context, we may either use (4) or set $\rho = \rho_0$. Assuming, for simplicity, that the constants $\rho_0$, $\mu$, $g$, $c$, $\kappa$, and $\alpha$ are all equal to 1 and neglecting the (nonessential) applied force $f$, we are led to the systems

$$v_t - \Delta v = \rho(\theta), \quad \rho(\theta)\theta_t - \Delta\theta = |\nabla v|^2 \quad \text{in } (0,\infty)\times\Omega \tag{1'}$$

or

$$\rho(\theta)v_t - \Delta v = \rho(\theta), \quad \rho(\theta)\theta_t - \Delta\theta = |\nabla v|^2 \quad \text{in } (0,\infty)\times\Omega, \tag{1''}$$

respectively, where $\rho(\theta) = 1-\theta$ or, more generally,

$\rho \ : \ \mathbb{R}\to\mathbb{R}$ is a nonincreasing function, being strictly positive on any interval $[-1,a_0]$ for any (6)
$a_0 \ \in \ (0,1)$ and with $\rho(0) = 1$.

Of course, we should then assume that $|\theta_0| < 1$ and verify that the solutions we construct satisfy $|\theta| < 1$, at least on a small initial time interval. We point out that some similar ideas were successfully exploited in [3] and [8], albeit in situations without the quadratic gradient term (see also [2]). In the above mentioned references the authors consider the case by a function depending on the own temperature $\kappa = \kappa(\theta)$. Then, if we introduce the primitive functions

$$\varphi(s) = \int_0^s \kappa(\sigma)d\sigma \text{ and } \Phi(s) = \int_0^s \rho(\sigma)d\sigma$$

then the assumptions $\kappa(\theta), \rho(\theta) \geqslant 0$ lead to the fact that $\varphi(s)$ and $\Phi(s)$ are nondecreasing real functions. If, for simplicity, we assume that $\varphi^{-1}$ is a continuous increasing function, by introducing the new variable $\Theta := \Phi(\theta)$ we arrive to the equation

$$\Lambda(\Theta)_t - \Delta\Theta = |\nabla v|^2 \quad \text{in } (0,\infty)\times\Omega, \tag{1''}$$

where $\Lambda(s) = \Phi(\varphi(s))$. Then, our results are valid also to the case in which the diffusion thermal coefficient is not constant once we do not made assumptions on how start and end growing the function $\Lambda(s)$ near its "saturation values" (let us say $s=-1$ and $s=1$).

One type of result we prove for the systems (1') and (1") is as follows:

*Let $\theta_0 \in C(\overline{\Omega}) \cap H^1(\Omega)$ with $-1 \leqslant \operatorname{Min}_{\overline{\Omega}} \theta_0 \leqslant \operatorname{Max}_{\overline{\Omega}} \theta_0 = a_0 < 1$ and $v_0 \in H_0^1(\Omega)$. Assume (6). Then there exists a time $T_0 > 0$ and a couple $(\theta, v)$ in $L^2(0,T_0;H^2(\Omega))^2 \times (C([0,T_0];H^s(\Omega)))^2$*

*for all $s < 2$, with $\dfrac{\partial \theta}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ in $L^2(Q_{T_0}), Q_{T_0} = ]0, T_0[\times\Omega$, satisfying the system (BS),*

$$\begin{cases} \rho(\theta)^n \dfrac{\partial v}{\partial t} - \Delta v = 1 - \theta \quad \textit{on } ]0, T_0[\times\Omega, \\ \rho(\theta) \ \dfrac{\partial \theta}{\partial t} - \Delta\theta = |\nabla v|^2 \quad \textit{on } ]0, T_0[\times\Omega, \\ \dfrac{\partial \theta}{\partial n} = v = 0 \quad \textit{on } ]0, T_0[\times\partial\Omega, \\ \theta(0) = \theta_0, v(0) = v_0, \end{cases}$$

*with $n = 0$ or $n = 1$. Moreover, $-1 \leqslant \operatorname{Min}_{\overline{\Omega}} \theta(t) \leqslant \operatorname{Max}_{\overline{\Omega}} \theta(t) \leqslant 1$ for all $T_0 \geqslant t \geqslant 0$.*

Our method consists in introducing an auxiliary truncated problem (TBS), for which we have a global-in-time solution, by replacing the quadratic gradient term. For instance, in the above system (BS), we replace $|\nabla v|^2$ by $|\nabla v|^2\chi_{\{\theta<1\}}$, where $\{\theta < 1\} = \{(t, x) \in Q_T,\ \theta(t, x) < 1\}$, to obtain the truncated system. We call the solutions of (TBS) "almost exact solutions". If the initial data are "small", we obtain the same regularity properties as in the above system (BS) for the "almost exact solutions". In particular, $\theta \in L^2(0, T; H^2(\Omega)) \times C([0, T]; H^s(\Omega))$ for all $s < 2$, for all $T > 0$, and then, using the continuity of the function $\theta$, we conclude that the almost exact solution is a local-in-time solution for the initial problem (1') or (1").
Qualitative properties of the solution of (BS) can be derived from our method, for example, using the truncated system associated with (BS). If we consider the maximal time for which the truncated system has an exact solution, say,

$$T_m = \sup\{T_0 \in ]0, T],\ |\nabla v|^2\chi_{\{\theta<1\}} = |\nabla v|^2 \text{ in } Q_{T_0}\},$$

then we have two possibilities: either $T_m = T$, in which case the solution we found is a global-in-time solution on $Q_T$; or $T_m < T$, in which case there exists a time $T_m \leqslant T_1 \leqslant T$ such that $|\{\theta(T_1) = 1\}| \doteqdot \text{measure}\{x \in \Omega \text{ such that } \theta(x, T_1) = 1\}$ is positive. Therefore, we have

$$T_m = \inf\{T_1 \geqslant T_m : |\{\theta(T_1) = 1\}| > 0\}.$$

In [7], a new approach will be given without restrictions on the initial data. In this new paper, we will attempt to explain the link between the degeneracy of the function $\rho$ and the (eventual) fact that $T_m < T$. Moreover, this new approach will be used to treat the full Navier-Stokes-Boussinesq system of equations.
The proof of the existence of solutions for the truncated problem (TBS) is based on two approximations: we first introduce a family of smooth problems depending on a small parameter $\varepsilon$, and we solve it via the Galerkin process. The role of the $\varepsilon$-family problem is to derive the necessary regularity and to prove the maximum principle in order to pass (easily) to the limit, proving in this way the existence of an "almost exact solution" for the truncated system.

## 2 Notation, assumptions and main results

Let $V = H_0^1(\Omega),\ H = H^1(\Omega),$ with $\Omega \subset \mathbb{R}^N,$ be a smooth bounded set with $N = 2$ *or* 3. We shall use the following eigenfunctions which are elements of $C^\infty(\Omega) \cap H^2(\Omega)$

$$-\Delta\varphi_j = \lambda_j^D \varphi_j \ in\ \Omega, \quad \varphi_j = 0 \ on\ \partial\Omega,\ j = 1, 2, \ldots.$$

$$-\Delta\psi_j + \psi_j = \lambda_j^N \psi_j \ in\ \Omega, \quad \frac{\partial \psi_j}{\partial n} = 0 \ on\ \partial\Omega \quad j = 1, 2, \ldots.$$

(we note that $\psi_1$ is the constant function 1). For $T > 0$, we set $Q_T = ]0, T[\times\Omega$. We set $V_m = \text{span}\{\varphi_j, j \leqslant m\},\ \ H_m = \text{span}\{\psi_j,\ j \leqslant m\}$ for $m \geqslant 1$. We recall that $\bigcup\limits_{m\geqslant 1} V_m$ (resp. $\bigcup\limits_{m\geqslant 1} H_m$)(see e.g [22], [14].) is dense in $V$ (resp. in $H$). We will use the following orthogonal projections $P_m : L^2(\Omega) \to V_m, Q_m : L^2(\Omega) \to H_m$. We suppose that there are two constants $a > b$ and a function $\rho$ such that:

$$\begin{aligned} \rho \text{ is non increasing on } [b, a],\ \rho(a) \ &\geqslant\ 0, \rho(t) > 0, a < t \leqslant b \text{ and} \\ \rho_+ \ &=\ \max(\rho, 0) \text{ is continuous on } [b, +\infty[. \end{aligned} \tag{7}$$

We denote by $\Phi$ a primitive of $\rho$ say $\Phi(s) = \int_b^s \rho(\sigma)d\sigma$.
We shall introduce the following definition of truncated problem :

**Definition 1** *Let $T$ be in $]0, \infty[$. A couple $(\theta, v)$ such that $\theta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ with $\Phi(\theta) \in L^2(Q_T)$ and $v \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ is called a "(weak in $\theta$ and strong in $v$) solution" for the following truncated system (TBS) associated to the equations (1'), if there exist a real $\alpha$ and a function $g_v \in [|\nabla v|^2\chi_{\{\theta<\alpha\}}, |\nabla v|^2]$ a.e in $Q_T$ such that*

$$\frac{d}{dt}\int_\Omega v\varphi dx + \int_\Omega \nabla v \cdot \nabla\varphi dx = \int_\Omega \rho(\theta)\varphi dx,\ \ in\ \mathcal{D}'(0, T),\ \forall\varphi \in H_0^1(\Omega),$$

$$\frac{d}{dt}\int_\Omega \Phi(\theta)\psi dx + \int_\Omega \nabla\theta \cdot \nabla\psi dx = \int_\Omega g_v\psi dx \ \ in\ \mathcal{D}'(0, T),\ \forall\psi \in H^1(\Omega),$$

*A weak solution $(\theta, v)$ is called an "exact (weak in $\theta$ and strong in $v$) solution" on $Q_T$ if it satisfies the following condition :*

$$|\nabla v|^2 = g_v,\ \ a.e\ in\ Q_T.$$

*A weak solution $(\theta, v)$ is called an "almost exact (weak in $\theta$ and strong in $v$) solution" on $Q_T$ if :*

$$g_v = |\nabla v|^2\chi_{\{\theta<\alpha\}}\ \ a.e\ in\ Q_T.$$

*An exact (resp. almost exact solution with $\theta \in L^2(0, T; H^2(\Omega))$ is called a (strong in $\theta$ and strong in $v$) exact solution (resp. strong-strong and almost exact solution).*

*The same definition holds for the truncated system associated to the equations (1").*
*If there is a time $T_0 < T$ for which one those definitions are fulfilled, we will say that it is a local exact (respectively almost exact) solution.*

**Remark 1**. We note that

$$|\nabla v|^2 \chi_{\{\theta<\alpha\}} = |\nabla v|^2 - |\nabla v|^2 \chi_{\{\theta \geqslant \alpha\}}$$

which proves the relationship with the dissipative (in $\theta$) term $|\nabla v|^2 \chi_{\{\theta\geqslant\alpha\}}$ (for a prescribed $v$).Moreover, if $(\theta, v)$ is a (weak in $\theta$ and strong in $v$) solution and $\alpha \geqslant \theta_\infty$, with $\theta_\infty = \operatorname*{ess\,sup}_{Q_T} \theta$. Then,

$$|\nabla v|^2 \chi_{\{\theta<\theta_\infty\}} \leqslant |\nabla v|^2 \chi_{\{\theta<\alpha\}}$$

and equality holds if $(\theta, v)$ is an almost exact solution.
Now, we give some sufficient conditions to obtain an almost solution and an exact solution:

**Proposition 1**
*Let $\theta$ be a function such that $(\theta, v)$ is a weak solution for the truncated system with $\theta \in L^{\frac{3}{2}}_{loc}(0, T; W^{2,\frac{3}{2}}_{loc}(\Omega))$, $\Phi(\theta) \in L^1_{loc}(0, T; W^{1,1}_{loc}(\Omega))$ and $\theta_\infty = \operatorname*{ess\,sup}_{Q_T} \theta \leqslant \alpha$. Assume that $g_v \in L^{\frac{3}{2}}(Q_T)$. Then*

$$g_v = |\nabla v|^2 \chi_{\{\theta<\theta_\infty\}}.$$

*Furthermore, if $\theta \in C(\overline{Q}_T)$ and $\theta_0 < \alpha - \delta$ for some $\delta > 0$ then the couple $(v, \theta)$ is a local exact solution.*

*Proof of Proposition 1***.** Let us observe that $\theta$ satisfy

$$\frac{\partial \Phi(\theta)}{\partial t} - \Delta\theta = g_v \ in \ \Omega.$$

If $\theta \in L^{\frac{3}{2}}_{loc}(0, T; W^{2,\frac{3}{2}}_{loc}(\Omega))$, and $g_v \in L^{\frac{3}{2}}(Q_T)$ then $\dfrac{\partial \Phi(\theta)}{\partial t} \in L^{\frac{3}{2}}_{loc}(Q_T)$. Thus by a well-known result (see e.g [14]) we have $\Delta\theta = \dfrac{\partial \Phi(\theta)}{\partial t} = 0$ a.e. on the set

$$E = \Big\{(t, x) \in Q_T : \theta(t, x) = \theta_\infty\Big\}.$$

This means $g_v(t, x) = 0$ a.e on $E$, since $g_v = |\nabla v|^2$ on $\{\theta < \theta_\infty\}$, then we have the result If $\theta \in C(\overline{Q}_T)$ then the choice of $\delta > 0$ so that $\theta_0 + \delta < \alpha$ and the continuity of $\theta$ imply that there exists a time $T_0 > 0$, such that $\theta(t, x) < \alpha - \frac{\delta}{2}$ for all $(t, x) \in Q_{T_0}$. Therefore, one has

$$|\nabla v|^2 \chi_{\{\theta<\alpha\}} = |\nabla v|^2, \text{ in } Q_{T_0}.$$

This shows that the couple is a local exact solution.∎
We want to prove that :

**Theorem 1**

*Let* $(\theta_0, v_0) \in H^1(\Omega) \times H_0^1(\Omega),\ b \leqslant \theta_0 \leqslant a.$

*For any* $T > 0$*, there exists at least a weak solution* $(\theta, v)$ *for the truncated system (TBS) associated to equations (1') with* $b \leqslant \theta \leqslant a,\ v(0) = v_0,\ \theta(0) = \theta_0,\ g_v \in [|\nabla v|^2 \chi_{\{\theta < a\}}, |\nabla v|^2]$ *a.e in* $Q_T$*.*

*If* $\rho(a) > 0$ *then this weak solution is a strong and almost exact solution. Moreover* $\theta \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^s(\Omega)),\ \forall s < 2.$

*In this case if the initial data is such that* $\theta_0 < a - \delta$ *with some* $\delta > 0$*, then this strong and almost exact solution is a local exact solution, that is a local-in-time solution of the Boussinesq system (1') .*

*Proof.* Consider a sequence $\rho_m \in W^{1,+\infty}([b, +\infty[),\ |\rho_m - \rho_+|_{C[b,d]} \to 0$, as $m \to \infty\ \forall d < +\infty,\ \rho_m$ is non increasing on $[b, +\infty[$. Let $0 < \varepsilon < 1$ and define the real functions, for $\sigma \in \mathbb{R}$

$$\rho_0(\sigma) = \rho_+((\sigma - b)_+ + b), \quad \rho_\varepsilon(\sigma) = \rho_0(\sigma) + \varepsilon, \quad S_\varepsilon(\sigma) = \frac{(a - (\sigma - b)_+ - b)_+}{(a - (\sigma - b)_+ - b)_+ + \varepsilon}.$$

We shall associate the following functions, for $\sigma \in \mathbb{R}$

$$\rho_{\varepsilon,m}(\sigma) = \rho_m((\sigma - b)_+ + b) + \varepsilon.$$

We note that $\varepsilon \leqslant \rho_{\varepsilon,m} \leqslant \rho_m(b) + 1 \leqslant k = \max_{j \geqslant 0} \rho_j(b) + 1$, $0 < \varepsilon \leqslant \rho_\varepsilon \leqslant \rho(b) + 1$.

From the Cauchy-Peano's theorem, there exists for all $m \geqslant 1$ $\theta_m \in C^1([0, T_m); H_m)$ and $v_m \in C^1([0, T_m); V_m)$ for some $0 < T_m \leqslant T$, satisfying : $\forall \varphi \in V_m,\ \forall \psi \in H_m$, for all $t \in [0, T_m)$, $\theta_m(0) = Q_m \theta_0$, $v_m(0) = P_m v_0$

$$\frac{d}{dt} \int_\Omega v_m(t) \varphi dx + \int_\Omega \nabla v_m(t) \cdot \nabla \varphi dx = \int_\Omega \rho_0(\theta_m(t)) \varphi dx, \tag{8}$$

$$\frac{d}{dt} \int_\Omega \theta_m(t) \psi + \int_\Omega \nabla \theta_m(t) \cdot \nabla \left( \frac{\psi}{\rho_{\varepsilon,m}(\theta_m(t))} \right) dx = \int_\Omega \frac{\psi}{\rho_{\varepsilon,m}(\theta_m(t))} \frac{|\nabla v_m(t)|^2}{1 + \varepsilon |\nabla v_m(t)|^2} S_\varepsilon(\theta_m(t)) dx. \tag{9}$$

To show that $T_m = T$, we need some estimates on $v_m$ and $\theta_m$. Those estimates will be uniform in $m$.□

**Lemma 1** *For all* $t \in [0, T_m)$

*(a)* $\displaystyle \frac{d}{dt} \int_\Omega |\nabla v_m(t)|^2 dx + \int_\Omega |\Delta v_m(t)|^2 dx \leqslant (\rho(b))^2 |\Omega|,$ *in* $\mathcal{D}'(0, T_m)$

*(b)* $\displaystyle \frac{d}{dt} \int_\Omega |\nabla \theta_m(t)|^2 + \int_\Omega \frac{|\Delta \theta_m(t)|^2}{\rho_{\varepsilon,m}(\theta_m(t))} \leqslant \frac{1}{\varepsilon^3} |\Omega|,$ *in* $\mathcal{D}'(0, T_m).$

*Proof.* To prove (a) we use the fact that $v_m \in C^1([0,T_m);V_m)$, for each $t \in (0,T_m)$. Then we have :

$$-\Delta v_m(t) \in H_0^1(\Omega) \text{ and, } \frac{d}{dt}\int_\Omega v_m(t)\varphi dx = \int_\Omega \frac{\partial v_m}{\partial t}(t)\varphi(x)dx \ \forall \varphi \in H_0^1(\Omega),$$

and therefore, we can $\varphi = -\Delta v_m(t)$. An integration by part yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_m(t)|^2 dx + \int_\Omega |\Delta v_m(t)|^2 dx = -\int_\Omega \rho_0(\theta_m(t))\Delta v_m(t) dx.$$

Since $0 \leqslant \rho_0(\theta_m) \leqslant \rho(b)$, then by the Young's inequality we deduce

$$\frac{d}{dt}\int_\Omega |\nabla v_m(t)|^2 dx + \int_\Omega |\Delta v_m(t)|^2 dx \leqslant (\rho(b))^2|\Omega|.$$

(b) A similar argument holds for $\theta_m$. Choosing $\psi = -\Delta\theta_m(t)$ and noticing that $\dfrac{\partial \psi}{\partial n} = 0$ on $\partial\Omega$, an integration by parts gives :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \theta_m(t)|^2 + \int_\Omega \frac{|\Delta\theta_m(t)|^2}{\rho_{\varepsilon,m}(\theta_m)}dx \leqslant \frac{1}{\varepsilon}\int_\Omega \frac{|\Delta\theta_m(t)|}{\rho_{\varepsilon,m}(\theta_m(t))}dx.$$

But $\varepsilon \leqslant \rho_{\varepsilon,m}(\theta_m(t))$, thus the Young's inequality yields

$$\frac{d}{dt}\int_\Omega |\nabla \theta_m(t)|^2 + \int_\Omega \frac{|\Delta\theta_m(t)|^2}{\rho_{\varepsilon,m}(\theta_m)} \leqslant \frac{1}{\varepsilon^3}|\Omega|. \blacksquare$$

Lemma 1 shows that $T_m = T$. Moreover, one has an uniform boundedness for $v_m$ as $m \to +\infty$. Indeed, since $v_m(t) \in H_0^1(\Omega)$, the Sobolev-Poincaré inequality with estimate (a) implies that $v_m$ remains in a bounded set of $L^2(0,T;H^2(\Omega))$ and in $L^\infty(0,T;H_0^1(\Omega))$. While for $\theta_m$, we need to control for instance $\int_\Omega \theta_m(t,x)^2 dx$ . To do this, we shall denote by $c$ or $c_i$ where $i$ is an integer greater than one, various constants independent of $m$ and $\varepsilon$, the constants depending on $\varepsilon$ will be denoted by $c_\varepsilon$.

**Lemma 2** *For all* $t \in [0,T]$

$$\int_\Omega |\theta_m(t,x)|^2 \, dx \leqslant c_\varepsilon.$$

*Proof.* We take $\psi = \theta_m(t)$ in relation (9). An integration by part and relation (9) yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta_m^2(t,x)dx \leqslant c_\varepsilon \int_\Omega |\theta_m(t,x)|dx + \int_\Omega \frac{\Delta\theta_m(t,x)}{\rho_{\varepsilon,m}(\theta_m(t))}\theta_m(t,x)dx. \tag{10}$$

The statement (b) of Lemma 1 implies that

$$\int_0^T \int_\Omega \frac{|\Delta\theta_m|^2(t,x)}{\rho_{\varepsilon,m}(\theta_m(t))}dxdt \leqslant c_\varepsilon(T,\theta_0). \tag{11}$$

Relation (10) gives the following Gronwall inequality,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta_m^2(t,x)dx \leqslant c_\varepsilon + \frac{c}{\varepsilon}\int_\Omega \theta_m^2(t,x)dx + \int_\Omega \frac{|\Delta\theta_m|^2(t,x)}{\rho_{\varepsilon,m}(\theta_m(t))}dx. \tag{12}$$

From relations (11) and (12), we conclude the Lemma 2.■

The Lemma 1 and Lemma 2 show that $\theta_m$ remains in a bounded set of $L^2(0,T;H^2(\Omega))$ and in $L^\infty(0,T;H^1(\Omega))$ as $m \to +\infty$.While for the time derivative, those uniform estimates combined with the equations satisfied by $v_m$ and $\theta_m$ imply :

**Lemma 3** *We have:*

*i)* $$\left|\frac{\partial v_m}{\partial t}\right|_{L^2(Q_T)} \leqslant |\Delta v_m|_{L^2(Q_T)} + |\rho_0(\theta_m)|_{L^2(Q_T)} \leqslant c,$$

*ii)* $$\left|\frac{\partial \theta_m}{\partial t}\right|_{L^2(Q_T)} \leqslant \left|\frac{\Delta\theta_m}{\rho_{\varepsilon,m}(\theta_m)}\right|_{L^2(Q_T)} + \frac{(T|\Omega|)^{\frac{1}{2}}}{\varepsilon^2} \leqslant c_\varepsilon.$$

*Proof.* The time derivatives satisfy the following equations :

$$\frac{\partial v_m}{\partial t} = \Delta v_m + P_m(\rho_0(\theta_m)), \tag{13}$$

$$\frac{\partial \theta_m}{\partial t} = Q_m\left(\frac{\Delta\theta_m}{\rho_{\varepsilon,m}(\theta_m)}\right) + Q_m\left(\frac{S_\varepsilon(\theta_m)|\nabla v_m|^2}{(1+\varepsilon|\nabla v_m|^2)\rho_{\varepsilon,m}(\theta_m)}\right) \tag{14}$$

Since the projection is a contraction, relations (13), (14) with Lemma 2 imply Lemma 3. ◇

**Proof of Theorem 1 (continuation).** By Aubin-Lions-Simon's classical compactness results (see e.g [15], [21]), [22] , we have a couple $(\theta^\varepsilon, v^\varepsilon)$ such that $v_m \to v^\varepsilon$ strongly in $C([0,T], H^s(\Omega)\cap H_0^1(\Omega))$ for all $s < 2$, and weakly in $L^2(0,T;H^2(\Omega))$, and $\theta_m \to \theta^\varepsilon$ strongly in $C([0,T], H^s(\Omega))$ for all $s < 2$ and weakly in $L^2(0,T;H^2(\Omega))$. Moreover, we have the following weak convergences in $L^2(Q_T)$ :

$$\frac{\partial v_m}{\partial t} \rightharpoonup \frac{\partial v^\varepsilon}{\partial t},$$

$$\frac{\partial \theta_m}{\partial t} \rightharpoonup \frac{\partial \theta^\varepsilon}{\partial t}.$$

From the uniform convergence of $\rho_{\varepsilon,m}$ to $\rho_\varepsilon$ on any bounded interval as $m$ goes to $\infty$, we deduce that : $\rho_{\varepsilon,m}(\theta_m) \to \rho_\varepsilon(\theta^\varepsilon)$ uniformly in $C(\overline{Q}_T)$. Due to the above convergences , one can see easily that the couple $(\theta^\varepsilon, v^\varepsilon)$ is a solution of :

$$\frac{\partial v^\varepsilon}{\partial t} = \Delta v^\varepsilon + \rho_0(\theta^\varepsilon) \tag{15}$$

$$\frac{\partial \theta^\varepsilon}{\partial t} = \frac{\Delta\theta^\varepsilon}{\rho_\varepsilon(\theta^\varepsilon)} + \frac{S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{(1+\varepsilon|\nabla v^\varepsilon|^2)\rho_\varepsilon(\theta^\varepsilon)}, \tag{16}$$

with the initial data $v^\varepsilon(0) = v_0$ and $\theta^\varepsilon(0) = \theta_0$. Moreover, on the boundary $\partial\Omega$, we have that the normal trace of $\theta^\varepsilon(t)$, $t \in [0,T]$: $\dfrac{\partial \theta^\varepsilon(t)}{\partial n}$ and the trace of $v^\varepsilon(t)$ are zero . This system is equivalent to the following one in $\mathcal{D}'(0,T)$: for all $\varphi \in H_0^1(\Omega)$, for all $\psi \in H^1(\Omega)$

$$\frac{d}{dt}\int_\Omega v_\varepsilon \varphi + \int_\Omega \nabla v^\varepsilon \cdot \nabla \varphi dx = \int_\Omega \varphi \rho_0(\theta^\varepsilon) dx, \tag{17}$$

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(\theta^\varepsilon)\psi dx + \int_\Omega \nabla \theta^\varepsilon \cdot \nabla \psi dx = \int_\Omega \frac{S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{1+\varepsilon|\nabla v^\varepsilon|^2}\psi dx. \square \tag{18}$$

Here, $\Phi_\varepsilon(s) = \int_b^s \rho_\varepsilon(y)dy$. For the function $\theta^\varepsilon$, we need to show first the :

**Lemma 4**
*If $b \leqslant \theta_0 \leqslant a$ a.e in $\Omega$ then $b \leqslant \theta^\varepsilon \leqslant a$ a.e in $Q_T$.*

*Proof*. We multiply the equation by $\rho_\varepsilon(\theta^\varepsilon)(\theta^\varepsilon - b)_-$. Relation (16) gives :

$$\int_\Omega \frac{\partial \theta^\varepsilon}{\partial t}\rho_\varepsilon(\theta^\varepsilon)(\theta^\varepsilon - b)_- dx + \int_\Omega \nabla\theta^\varepsilon \cdot \nabla(\theta^\varepsilon - b)_- dx = \int_\Omega \frac{S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{1+\varepsilon|\nabla v^\varepsilon|^2}(\theta^\varepsilon - b)_- dx.$$

Since the right hand side is non negative, then one has :

$$-\frac{\varepsilon + \rho(b)}{2}\frac{d}{dt}\int_\Omega ((\theta^\varepsilon - b)_-)^2 dx - \int_\Omega |\nabla(\theta^\varepsilon - b)_-|^2 dx \geqslant 0,$$

thus one has :

$$\int_\Omega ((\theta^\varepsilon - b)_-(t,x))^2 dx \leqslant \int_\Omega ((\theta_0^\varepsilon - b)_-)^2(x) dx = 0$$

and so a.e in $Q_T$ : $\theta^\varepsilon \geqslant b$. Multiplying the equation by $\rho_\varepsilon(\theta^\varepsilon)(\theta^\varepsilon - a)_+$ equation (16)

$$\int_\Omega (\theta^\varepsilon - a)_+ \rho_\varepsilon(\theta^\varepsilon)\frac{\partial \theta^\varepsilon}{\partial t} dx + \int_\Omega |\nabla(\theta^\varepsilon - a)_+|^2 dx = \int_\Omega \frac{(\theta^\varepsilon - a)_+ S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{1+\varepsilon|\nabla v^\varepsilon|^2} dx = 0.$$

That is

$$\frac{d}{dt}\int_\Omega \int_b^{\theta^\varepsilon} \left[\rho_0(\sigma) + \varepsilon\right](\sigma - a)_+ d\sigma + \int_\Omega |\nabla(\theta^\varepsilon - a)_+|^2 dx = 0.$$

Then for all $t$ :

$$\int_\Omega \int_b^{\theta^\varepsilon(t,x)} \left[\rho_0(\sigma) + \varepsilon\right](\sigma - a)_+ d\sigma dx \leqslant \int_\Omega \int_b^{\theta_0(x)} \left[\rho_0(\sigma) + \varepsilon\right](\sigma - a)_+ d\sigma dx = 0$$

we deduce $\theta^\varepsilon \leqslant a$, a.e in $Q_T$. ■

To get some uniform a priori estimates in $\varepsilon$ on $v^\varepsilon$, we recall firstly that Lemma 1, with the previous convergence (or using directly the above equation (15)) imply :

**Corollary 1** *. We have:*

*(a)* $\dfrac{d}{dt}\displaystyle\int_\Omega |\nabla v^\varepsilon(t)|^2 dx + \int_\Omega |\Delta v^\varepsilon(t)|^2 dx \leqslant (\rho(b))^2|\Omega|$, *in* $\mathcal{D}'(0,T)$.

*(b)* $\left|\dfrac{\partial v^\varepsilon}{\partial t}\right|_{L^2(Q_T)} \leqslant c.$ ∎

Thus, we can conclude as before, by Aubin-Lions-Simon's classical compactness results (see e.g. [15] [21], [22]), that $v^\varepsilon \to v$ strongly in $C([0,T], H^s(\Omega)\cap H^1_0(\Omega))$ for all $s<2$ and weakly in $L^2(0,T;H^2(\Omega))$. Moreover, we have the following weak convergence in $L^2(Q_T)$ :

$$\frac{\partial v^\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}.$$

**Lemma 5**

*$\theta^\varepsilon$ remains in a bounded set of $L^2(0,T;H^1(\Omega))$ as $\varepsilon\to 0$.*

*Proof.* We multiply the equation (16) by $\theta^\varepsilon\rho_\varepsilon(\theta^\varepsilon)$ to get:

$$\int_\Omega \theta^\varepsilon\rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t}dt + \int_\Omega |\nabla\theta^\varepsilon|^2dx = \int_\Omega \frac{\theta^\varepsilon S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{1+\varepsilon|\nabla v^\varepsilon|^2}dx, \tag{19}$$

$$\int_0^T dt\int_\Omega |\nabla\theta^\varepsilon|^2dx \leqslant -\int_0^T dt\int_\Omega \theta^\varepsilon\rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t} + \max(|a|;|b|)\int_0^T\int_\Omega |\nabla v^\varepsilon|^2dxdt, \tag{20}$$

and

$$\int_0^T dt\int_\Omega \rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t}dx = \int_0^T \frac{d}{dt}\left[\int_\Omega dx\int_b^{\theta^\varepsilon}\sigma\rho_\varepsilon(\sigma)d\sigma\right]dt. \tag{21}$$

$$\left|\int_0^T dt\int_\Omega \theta^\varepsilon\rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t}dx\right| \leqslant \left[\int_\Omega(\theta^\varepsilon)^2(T,x)dx + \int_\Omega\theta_0^2(x)dx\right](\rho(b)+1)\leqslant c_1. \tag{22}$$

Thus relation (20) with corollary 1 give :

$$\int_0^T\int_\Omega |\nabla\theta^\varepsilon|^2dxdt \leqslant c_1 + \max(|a|;|b|)T(\rho(b)+1)^2|\Omega| + \int_\Omega|\nabla v_0|^2dx = c_2. \blacksquare$$

**End of the proof of Theorem 1.** Let $\Phi_\varepsilon(\theta^\varepsilon) = \displaystyle\int_b^{\theta^\varepsilon}\rho_\varepsilon(\sigma)d\sigma$, then from equation (16), we have :

$$\left|\frac{\partial\Phi_\varepsilon(\theta^\varepsilon)}{\partial t}\right|_{H^{-1}(\Omega)} \leqslant |\nabla\theta^\varepsilon|_{L^2(\Omega)} + \left||\nabla v^\varepsilon|^2\right|_{L^2(\Omega)}.$$

An interpolation argument (see e.g. [22], [14])shows that <u>for $N=2$</u> :

$$\left||\nabla v^\varepsilon|^2\right|_{L^2(\Omega)} = |\nabla v^\varepsilon|^2_{L^4(\Omega)} \leqslant c|\nabla v^\varepsilon|_{L^2(\Omega)}|v^\varepsilon|_{H^2(\Omega)} \leqslant c|v^\varepsilon|_{H^2(\Omega)}. \tag{23}$$

Thus

$$\int_0^T\left|\frac{\partial\Phi_\varepsilon(\theta^\varepsilon)}{\partial t}\right|^2_{H^{-1}(\Omega)}dt \leqslant c\left[|\nabla\theta^\varepsilon|^2_{L^2(Q_T)} + |v^\varepsilon|^2_{L^2(0,T;H^2(\Omega))}\right]\leqslant c_3.$$

<u>If $N=3$</u> then for all $\psi\in H^1_0(\Omega)$

$$\int_\Omega |\nabla v^\varepsilon|^2\psi dx \leqslant |\psi|_{L^3(\Omega)}|\nabla v^\varepsilon|^2_{L^3(\Omega)}. \tag{24}$$

Using an interpolation argument (see e.g. [22], [14]) we have

$$|\nabla v^\varepsilon|^2_{L^3(\Omega)} \leqslant c_4 |v^\varepsilon|_{H^2(\Omega)} |v^\varepsilon|_{L^6(\Omega)} \leqslant c_5 |v^\varepsilon|_{H^2(\Omega)}.$$

Therefore,

$$\left| \frac{\partial \Phi_\varepsilon(\theta^\varepsilon)}{\partial t} \right|_{H^{-1}(\Omega)} \leqslant c_6 \Big[ |\nabla \theta^\varepsilon|_{L^2(\Omega)} + |v^\varepsilon|_{H^2(\Omega)} \Big]$$

which implies

$$\int_0^T \left| \frac{\partial \Phi_\varepsilon(\theta^\varepsilon)}{\partial t} \right|^2_{H^{-1}(\Omega)} dt \leqslant c_7.$$

Thus, in any case $\Phi_\varepsilon(\theta^\varepsilon)_t$ remains in a bounded set of $L^2(0,T;H^{-1}(\Omega))$. Since, we have

$$|\nabla \Phi_\varepsilon(\theta^\varepsilon)|^2_{L^2(Q_T)} = \int_{Q_T} (\rho_\varepsilon(\theta^\varepsilon))^2 |\nabla \theta^\varepsilon|^2 dx \leqslant (\rho(b)+1)^2 \int_{Q_T} |\nabla \theta^\varepsilon|^2 dx dt \leqslant c_8,$$

the Aubin-Lions-Simon's compactness result implies the existence of a function $w$ satisfying $\Phi_\varepsilon(\theta^\varepsilon)$ converges to $w$ strongly in $C([0,T];L^2(\Omega))$ and a.e. in $Q_T$. Therefore, $\int_b^{\theta^\varepsilon} \rho(\sigma) d\sigma + \varepsilon \theta^\varepsilon$ converges to $w$ strongly in $C([0,T];L^2(\Omega))$ and a.e in $Q_T$ and

$$0 \leqslant w \leqslant \int_b^a \rho(\sigma) d\sigma, \qquad w(0,x) = \int_b^{\theta_0(x)} \rho(\sigma) d\sigma.$$

Since the restriction of $\Phi$ to $[b,a]$, that is the map $\Phi_0 : [b,a] \to \mathbb{R}^+$ given by $\Phi_0(\sigma) = \int_b^\sigma \rho(s) ds$, is a continuous bijection from $[b,a]$ onto $\left[0, \int_b^a \rho(s) ds\right]$ and its inverse $\Phi_0^{-1}$ is continuous, we deduce that :

$$\Phi_0^{-1}\left( \int_b^{\theta^\varepsilon} \rho(\sigma) d\sigma \right) = \theta^\varepsilon \to \Phi_0^{-1}(w)$$

almost everywhere on $Q_T$. Then, we can define $\theta \doteqdot \Phi_0^{-1}(w)$. Thus $\theta \in L^2(0,T;H^1(\Omega))$ and $b \leqslant \theta \leqslant a$ a.e. in $Q_T$. Hence, we have the following convergences : $\theta^\varepsilon \rightharpoonup \theta$ weakly in $L^2(0,T;H^1(\Omega))$, $\Phi_\varepsilon(\theta^\varepsilon) \to \Phi(\theta)$ strongly in $C([0,T];L^2(\Omega))$ and a.e. in $Q_T$. Therefore $\rho_0(\theta^\varepsilon) \to \rho_0(\theta)$ in any $L^p(Q_T)$, $p < +\infty$ and $S_\varepsilon(\theta^\varepsilon) \to 1$ on $\{\theta < a\}$.

To show that $\lim\limits_{t \to t_0} \int_\Omega |\theta(t,x) - \theta(t_0,x)|^p dx = 0$, it suffices to show the case $p = 1$. We may assume that $t_0 = 0$. We know that

$$\lim_{t \to 0} \int_\Omega |w(t,x) - w(0,x)| dx = 0,$$

thus

$$\lim_{t \to 0} \int_\Omega |\Phi_0^{-1}(w(t,x)) - \Phi_0^{-1}(w(0,x))| dx = 0,$$

(arguing by contradiction and using the continuity of $\Phi_0^{-1}$), that is

$$0 = \lim_{t \to 0} \int_\Omega |\theta(t,x) - \Phi_0^{-1}(w((0,x))| dx \text{ and } \Phi_0^{-1}(w(0,x)) = \theta_0(x).$$

Passing to the limit in equation (17) and (18), we deduce that $(v,\theta)$ is a solution of

$$\frac{d}{dt}\int_\Omega v\varphi dx+\int_\Omega \nabla v\nabla\varphi=\int_\Omega \varphi\rho(\theta)dx,$$

$$\frac{d}{dt}\int_\Omega \Phi(\theta)\psi+\int_\Omega \nabla\theta\nabla\psi dx=\int_\Omega \psi g_v dx,$$

with $g_v\in\left[|\nabla v|^2\chi_{\{\theta<a\}},|\nabla v|^2\right]$ which proves the required question.
We first note that $\rho(a)>0$ implies that $\rho_\varepsilon(\theta)\geqslant\rho(a)>0$.
From relation (16) one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta^\varepsilon|^2dx+\int_\Omega \frac{|\Delta\theta^\varepsilon|^2}{\rho_\varepsilon(\theta^\varepsilon)}dx\leqslant\int_\Omega \frac{|\nabla v^\varepsilon|^2|\Delta\theta^\varepsilon|}{\rho_\varepsilon(\theta^\varepsilon)}dx. \tag{25}$$

From which we deduce

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta^\varepsilon|^2dx+\frac{1}{2(\rho(b)+1)}\int_\Omega |\Delta\theta^\varepsilon|^2dx\leqslant\frac{1}{2\rho(a)}\int_\Omega |\nabla v^\varepsilon|^4dx. \tag{26}$$

Since $v^\varepsilon$ belongs to a bounded set of $L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$, we know that if N=2 $|\nabla v^\varepsilon|$ belongs to a bounded set of $L^4(Q_T)$. This show that

$$\int_0^T\int_\Omega |\Delta\theta^\varepsilon|^2dxdt+\sup_t\int_\Omega |\nabla\theta^\varepsilon(t,x)|^2dx\leqslant c. \tag{27}$$

If $N=3$, the same estimate holds for the gradient of $v^\epsilon$ according to the Ladyzenskaja and al result (see [14]) since the equation is linear in divergence form :

$$|\nabla v^\varepsilon|_{L^4(Q_T)}\leqslant c$$

then (27) holds. Therefore, $\theta_t^\epsilon$ remains in a bounded set of $L^2(0,T;L^2(\Omega)$. We conclude using standard compactness result: $(v^\epsilon,\theta^\epsilon)$ converges to $(v,\theta)$ in strongly $C([0,T];H_0^s(\Omega))^2$ for all $s<2$ and weakly in $L^2(0,T;H^2(\Omega))^2$. This allows to pass easily to the limit in the equation. If $\theta_0<a-\delta$ with some $\delta>0$, then this weak solution is a local exact solution since one has $\theta\in C([0,T];H^s(\Omega))\subset C(\overline{Q}_T)$ for $s>\dfrac{3}{2}$ .Thus, we may apply the first proposition to arrive to the conclusion.■

# 3 Some Extensions and Qualitative Properties

The following corollary is directly related to the model of $\rho$ given in relation (4). Assuming for simplicity that $\alpha=\rho_0=1$.

**Corollary 2**
*Let* $-1\leqslant\theta_0\leqslant 1$, $(\theta_0,v_0)\in H^1(\Omega)\times H_0^1(\Omega)$. *Then for all* $T>0$, *there exist a function* $\theta\in$

$L^2(0,T;H^1(\Omega))$, $-1 \leqslant \theta \leqslant 1$ *with* $\theta \in C([0,T];L^2(\Omega))$, $v \in C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ *satisfying* $\forall \varphi \in H_0^1(\Omega), \forall \psi \in H^1(\Omega)$ *that*

$$\frac{d}{dt}\int_\Omega v(t,x)\varphi(x)dx + \int_\Omega \nabla\varphi(x)\nabla v(t,x)dx = \int_\Omega \varphi(x)(1-\theta(t,x))dx$$

*and*

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega (1-\theta)^2\psi(x)dx + \int_\Omega \nabla\psi(x)\nabla\theta(t,x)(t,x)dx = \int_\Omega \psi(x)g_v(t,x)dx, \ \textit{in } \mathcal{D}'(0,T)$$

*with* $g_v \in \left[|\nabla v|^2\chi_{\{\theta<1\}}, |\nabla v|^2\right]$ $, v(0) = v_0, \ \theta(0) = \theta_0,$.

*Proof.* We choose $b = -1, \ a = 1, \ \rho(\sigma) = 1 - \sigma, \ \sigma \in \mathbb{R}$.■

We may also find an almost exact solution for the truncated system associated to the equations (1") if we assume that $\rho(a) > 0$ and $N = 2$. Then we have :

**Theorem 2 (An almost exact solution(1") N=2)**
*Let* $\rho$, $\theta_0$, $v_0$ *be as in theorem 1. Assume N=2 and that* $\rho(a) > 0$. *Then, we have a regular solution* $(\theta, v)$ *satisfying also*

$$\begin{cases} \rho(\theta)\dfrac{\partial v}{\partial t} - \Delta v = \rho(\theta), \\ \rho(\theta)\dfrac{\partial \theta}{\partial t} - \Delta\theta = |\nabla v|^2\chi_{\{\theta<a\}}. \end{cases} \tag{28}$$

*with*

$$\begin{cases} (v,\theta) \in L^2(0,T;H^2(\Omega))^2 \times C([0,T],H^s(\Omega))^2, \ s < 2 \\ \dfrac{\partial\theta}{\partial t}, \ \dfrac{\partial v}{\partial t} \textit{ are in } L^2(Q_T), \\ v(0) = v_0, \ \theta(0) = \theta_0, \\ \dfrac{\partial\theta}{\partial n}(t) = v(t) = 0 \textit{ on } \partial\Omega \textit{ and a.e. in } (0,T). \end{cases}$$

**Proof :** Using the same function $\rho_\varepsilon$ and mimicking the above method given in the proof of theorem 1, we have from relations (15) and (16) the

**Theorem 3** *There exists* $(\theta^\varepsilon, v^\varepsilon) \in L^2(0,T;H^2(\Omega))^2 \times C([0,T];H^1(\Omega))^2$ *satisfying :* $b \leqslant \theta^\varepsilon \leqslant a$, *in* $\mathcal{D}'(Q_T)$ *and a.e. in* $\Omega$

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} = \dfrac{\Delta v^\varepsilon}{\rho^\varepsilon(\theta^\varepsilon)} + \dfrac{\rho_0(\theta^\varepsilon)}{\rho_\varepsilon(\theta^\varepsilon)} \\ \dfrac{\partial\theta^\varepsilon}{\partial t} = \dfrac{\Delta\theta^\varepsilon}{\rho_\varepsilon(\theta^\varepsilon)} + \dfrac{S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{(1+\varepsilon|\nabla v^\varepsilon|^2)\rho_\varepsilon(\theta^\varepsilon)} \end{cases}$$

*with* $\dfrac{\partial\theta^\varepsilon}{\partial n}(t) = v^\varepsilon(t) = 0$ *on* $\partial\Omega$ *for a.e* $t \in (0,T)$, $v^\varepsilon(0) = v_0$, $\theta^\varepsilon(0) = \theta_0$.

Next, we shall need some uniform a priori estimates in $\varepsilon$. The fact that $b \leqslant \theta^\varepsilon \leqslant a$ follows from Lemma 4.

**Lemma 6** *i)* $\displaystyle\int_\Omega \rho_\varepsilon(\theta^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 dx + \frac{d}{dt}\int_\Omega |\nabla v^\varepsilon|^2 \leqslant \int_\Omega \rho_0(\theta^\varepsilon)dx \leqslant \rho(b)|\Omega| < +\infty.$
*ii)* $\displaystyle\int_{Q_T} |\Delta v^\varepsilon|^2 dxdt \leqslant c.$

*Proof.* Using Galerkin's approximation of $v^\varepsilon$, one has :$\displaystyle\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v^\varepsilon|^2 dx = \int_\Omega \Delta v^\varepsilon \frac{\partial v^\varepsilon}{\partial t} dx$ in $\mathcal{D}'(0,T)$ and also a.e. in $(0.T)$. Thus multiplying the first equation of the above Lemma 6 by $\rho_\varepsilon(\theta^\varepsilon)\dfrac{\partial v^\varepsilon}{\partial t}$, we then have :

$$\int_\Omega \rho_\varepsilon(\theta^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v^\varepsilon|^2 dx = \int_\Omega \rho_0(\theta^\varepsilon)\frac{\partial v^\varepsilon}{\partial t} dx. \tag{29}$$

By the Cauchy Schwarz's inequality, one has :

$$\int_\Omega \rho_0(\theta^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial t}\right) dx \leqslant \frac{1}{2}\int_\Omega \rho_0(\theta^\varepsilon)dx + \frac{1}{2}\int_\Omega \left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 \rho_0(\theta^\varepsilon)dx. \tag{30}$$

Since $\rho_\varepsilon(\theta^\varepsilon) = \rho_0(\theta^\varepsilon) + \varepsilon$, we deduce from relations (29) and (30) that :

$$\int_\Omega \rho_\varepsilon(\theta^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 dx + \frac{d}{dt}\int_\Omega |\nabla v^\varepsilon|^2 dx \leqslant \int_\Omega \rho_0(\theta^\varepsilon)dx \leqslant \rho(b)|\Omega| < +\infty.$$

To prove the second statement ii), we recall that

$$\Delta v^\varepsilon = \rho_\varepsilon(\theta^\varepsilon)\frac{\partial v^\varepsilon}{\partial t} - \rho_0(\theta^\varepsilon),$$

therefore,

$$|\Delta v^\varepsilon|_{L^2(Q_T)} \leqslant \left|\rho_\varepsilon(\theta^\varepsilon)\frac{\partial v^\varepsilon}{\partial t}\right|_{L^2(Q_T)} + |\rho_0(\theta^\varepsilon)|_{L^2(Q_T)}. \tag{31}$$

Since $\rho_\varepsilon(\theta^\varepsilon) \leqslant \rho(b) + 1$ then, relation (31) yields

$$|\Delta v^\varepsilon|_{L^2(Q_T)} \leqslant (\rho(b)+1)\left(\int_{Q_T} \rho_\varepsilon(\theta^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 dxdt\right)^{\frac{1}{2}} + \rho(b)|Q_T|. \tag{32}$$

Using the first statement i), we deduce the result.■

**End of the proof of Theorem 2.** As consequence of the above theorem, $v^\varepsilon$ remains in a bounded set of $L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$. Assuming that $\rho(a) > 0$, then we have :

$$\rho_\varepsilon(\theta^\varepsilon) \geqslant \rho(a) > 0 \text{ since } b \leqslant \theta^\varepsilon \leqslant a.$$

Thus we deduce from Lemma 6 that

$$\rho(a)\int_{Q_T}\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 dxdt \leqslant c.$$

Therefore, applying the well-known compactness results, there exists a function $v \in L^2(0,T;H^2(\Omega))\cap C([0,T];H_0^1(\Omega))$ such that $v^\varepsilon \to v$ strongly in $C([0,T];H^s(\Omega)\cap H_0^1(\Omega))$ for all $s<2$ and weakly in $L^2(0,T;H^2(\Omega))$.

The estimate on $\theta^\varepsilon$ remains the same as for Lemma 5. That is $\theta^\varepsilon$ remains in a bounded set of $L^2(0,T;H^1(\Omega))$. Moreover, we also have the uniform boundedness of $\dfrac{\partial\theta^\varepsilon}{\partial t}$ in $L^2(Q_T)$. Indeed since $\rho_\varepsilon(\theta^\varepsilon)\geqslant \rho(a)>0$, relation (27) shows that $\theta^\varepsilon$ remains in a bounded set of $L^2(0,T;H^2(\Omega))$. By relation (23), if $N=2$ on has:

$$\int_{Q_T} |\nabla v^\varepsilon|^4 dxdt \leqslant c|v^\varepsilon|^2_{L^2(0,T;H^2)} \leqslant c. \tag{33}$$

Therefore, the equation

$$\Delta\theta^\varepsilon = \rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t} - \frac{S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2}{1+\varepsilon|\nabla v^\varepsilon|^2},$$

yields

$$\rho(a)\left|\frac{\partial\theta^\varepsilon}{\partial t}\right|_{L^2(Q_T)} \leqslant |\Delta\theta^\varepsilon|_{L^2(Q_T)} + |\nabla v^\varepsilon|^2_{L^4(Q_T)} \leqslant c.$$

This shows that $\theta^\varepsilon$ remains in a bounded set of $L^2(0,T;H^2)\cap C([0,T];H^1(\Omega))$ and $\dfrac{\partial\theta^\varepsilon}{\partial t}$ remains a bounded set of $L^2(Q_T)$. Thus, we have a function $\theta$, $\theta^\varepsilon\to\theta$ strongly in $C([0,T];H^s(\Omega))$ for all $s<2$, and converging weakly in $L^2(0,T;H^2(\Omega))$. As before, we can easily pass to limit in the equations given in Theorem 3 via the Proposition 1 that :

$$\begin{cases} \rho(\theta)\dfrac{\partial v}{\partial t} - \Delta v = \rho(\theta) \text{ a.e. in } Q_T, \\ \rho(\theta\,)\dfrac{\partial\theta}{\partial t} - \Delta\theta = |\nabla v|^2\chi_{\{\theta<a\}} \text{ a.e. in } Q_T \\ (v,\theta)\in L^2(0,T;H^2(\Omega))^2\times(C[0,T];H^s(\Omega))^2,\ s<2, \\ \dfrac{\partial\theta}{\partial n}(t) = v(t) = 0 \text{ on } \partial\Omega \text{ and for a.e. } t\in(0,T), \\ v(0)=v_0,\ \theta(0)=\theta_0. \blacksquare \end{cases}$$

As a Corollary of the above theorem, we can come back to the original equation :

**Corollary 3** . *Let N=2, $\theta_0\in C(\overline{\Omega})\cap H^1(\Omega)$ with $-1\leqslant \operatorname{Min}_{\overline{\Omega}}\theta_0 \leqslant \operatorname{Max}_{\overline{\Omega}}\theta_0 = a_0 < 1-\delta$, for some $\delta>0$ and $v_0\in H_0^1(\Omega)$. Then there is a couple $(\theta,v)$ in $L^2(0,T;H^2(\Omega))^2\times(C[0,T];H^s(\Omega))^2$ for all $s<2$, with $\dfrac{\partial\theta}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ in $L^2(Q_T)$ satisfying :*

$$\begin{cases} (1-\theta)^n\dfrac{\partial v}{\partial t} - \Delta v = 1-\theta,\ \textit{in } Q_T \\ (1-\theta\,)\ \dfrac{\partial\theta}{\partial t} - \Delta\theta = |\nabla v|^2\chi_{\{\theta<1\}}\ \textit{in } Q_T, \\ \dfrac{\partial\theta}{\partial n} = v = 0\ \textit{on } (0,T)\times\partial\Omega, \\ \theta(0)=\theta_0, : v(0)=v_0, \end{cases}$$

*whenever $n = 0$ or $n = 1$. Moreover, $-1 \leqslant \operatorname*{Min}_{\overline{\Omega}} \theta(t) \leqslant \operatorname*{Max}_{\overline{\Omega}} \theta(t) \leqslant 1 - \frac{\delta}{2}$ for all $t \geqslant 0$. This solution is a local strong and exact solution, that is a solution of (1') or (1").*

$\diamondsuit$

# 4 Uniqueness of the Solution of (BS)

An example of results showing the continuous dependence with respect to the initial data and, in particular, the uniqueness of the strong solution for the system (1'), is the following.

**Proposition 2** *Let $N = 2$, $n = 0$ in Corollary 3. Consider two solutions $(v_i, \theta_i)$ satisfying $-1 < b < \theta_i < a < 1$ for $i = 1, 2$. Then,*

$$|\nabla(v_1 - v_2)|^2_{L^2}(t) + |\theta_1 - \theta_2|^2_{L^2}(t) \leqslant \left(|\nabla(v_1(0) - v_2(0))|^2 + |\theta_1(0) - \theta_2(0)|^2\right) \exp\left(\int_0^T g(\sigma)d\sigma\right)$$

*with*

$$\int_0^T g(\sigma)d\sigma \leqslant c(a, b, T)\left(1 + |\nabla v_1(0)|^2_{L^2} + |\nabla v_2(0)|^2_{L^2} + |\nabla\theta_1(0)|^2_{L^2} + |\nabla\theta_2(0)|^2_{L^2}\right).$$

*In particular the couple $(\theta, v)$ solution of the system (BS) is the unique solution on $Q_T$.*

*Proof.* Let $(\theta_i, v_i)$, $i = 1, 2$ be two couples satisfying the equations, regularities of Corollary 3 with the additional conditions $|\theta_i|_\infty \leqslant 1 - \delta < 1$. To simplify our computations, we shall set $w = v_1 - v_2$, $u = \theta_1 - \theta_2$, and denote $|\cdot|_p$ the norm in $L^p(\Omega)$. Then, $w$ and $u$ satisfies :

$$w_t - \Delta w = \theta_2 - \theta_1 \tag{34}$$

$$u_t - \frac{\Delta\theta_1}{1 - \theta_1} + \frac{\Delta\theta_2}{1 - \theta_2} = \frac{|\nabla v_1|^2}{1 - \theta_1} - \frac{|\nabla v_2|2}{1 - \theta_2} \tag{35}$$

Multiplying equation (34) by $-\Delta w$, we then have :

$$\frac{d}{dt}|\nabla w|_2^2 + |\Delta w|_2^2 \leqslant c|u|_2^2. \tag{36}$$

Multiplying equation(35) by $u$, one has :

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2} - \int_\Omega \frac{u\Delta u}{1 - \theta_1}dx = \int_\Omega \Delta\theta_2 \frac{(\theta_1 - \theta_2)^2}{(1 - \theta_1)(1 - \theta_2)}dx + \int_\Omega \frac{|\nabla v_1|^2 - |\nabla v_2|^2}{1 - \theta_1}dx + \int_\Omega \frac{(\theta_1 - \theta_2)^2|\nabla v_2|^2}{(1 - \theta_1)(1 - \theta_2)}dx \tag{37}$$

Since $0 < 1 - |\theta_0|_\infty \leqslant 1$, we deduce from relation (37), after integration by part, that :

$$\frac{d}{dt}|u|^2_{L^2} + c_1\int_\Omega |\nabla u|^2 dx \tag{38}$$

$$\leqslant c\int_\Omega |\nabla\theta_1||\nabla u||u|dx + c\int_\Omega |\Delta\theta_2||u|^2dx + c\int_\Omega |\nabla w||\nabla(v_1-v_2)||u|dx + c\int_\Omega |\nabla v_2|^2|u|^2dx$$
$$= I_1 + I_2 + I_3 + I_4$$

We have the following estimates using usual interpolation arguments (see [22], [14]), for all $\delta > 0$,

$$\begin{aligned} I_1 &= c\int_\Omega |\nabla\theta_1||\nabla u||u|dx \\ &\leqslant |\nabla u|_{L^2}|\nabla\theta_1|_{L^4}|u|_{L^4} \\ &\leqslant \delta|\nabla u|_{L^2}^2 + c_\delta|\nabla\theta_1|_{L^4}^2|u|_{L^4}^2 \\ &\leqslant \delta|\nabla u|_{L^2}^2 + c_\delta|\nabla\theta_1|_{L^4}^2|\nabla u|_{L^2}|u|_{L^2} \\ &\leqslant \delta|\nabla u|_{L^2}^2 + c_\delta|\nabla\theta_1|_{L^4}^4|u|_{L^2}^2 \end{aligned} \tag{39}$$

$$\begin{aligned} I_2 &= c\int_\Omega |\Delta\theta_2||u|^2dx \\ &\leqslant c|\Delta\theta_2|_{L^2}|u|_{L^4}^2 \\ &\leqslant c|\Delta\theta_2|_{L^2}|u|_{L^2}\Big[|u|_{L^2} + |\nabla u|_{L^2}\Big] \\ &\leqslant c|\Delta\theta_2|_2|u|_{L^2}^2 + c_\delta|\Delta\theta_2|_{L^2}^2|u|_{L^2}^2 + \delta|\nabla u|_{L^2}^2 \end{aligned} \tag{40}$$

$$\begin{aligned} I_3 &= c\int_\Omega |\nabla w||\nabla(v_1-v_2)||u|dx \\ &\leqslant c|\nabla w|_{L^2}|\nabla(v_1+v_2)|_{L^4}|u|_{L^4} \\ &\leqslant \delta|\nabla w|_{L^2}^2 + c_\delta|\nabla(v_1+v_2)|_{L^4}^2|u|_{L^4}^2 \\ &\leqslant \delta|\nabla w|_{L^2}^2 + c_\delta|\nabla(v_1+v_2)|_{L^4}^2|u|_{L^2}\Big[|u|_{L^2} + |\nabla u|_{L^2}\Big] \\ &\leqslant \delta|\nabla w|_{L^2}^2 + c_\delta|\nabla(v_1+v_2)|_{L^4}^2|u|_{L^2}^2 + c_\delta|\nabla(v_1+v_2)|_{L^4}^4|u|_{L^2}^2 + \delta|\nabla u|_{L^2}^2 \end{aligned} \tag{41}$$

$$\begin{aligned} I_4 &= c\int_\Omega |\nabla v_2|^2|u|^2dx \\ &\leqslant c|\nabla v_2|_{L^4}^2|u|_{L^4}^2 \\ &\leqslant c|\nabla v_2|_{L^4}^2|u|_{L^2}\Big[|u|_{L^2} + |\nabla u|_{L^2}\Big] \\ &\leqslant c|\nabla v_2|_{L^4}^2|u|_{L^2}^2 + c_\delta|\nabla v_2|_{L^4}^4|u|_{L^2}^2 + \delta|\nabla u|_{L^2}^2 \end{aligned} \tag{42}$$

Thus relation (38) becomes :

$$\frac{d}{dt}|u|_{L^2}^2 + (c_{10} - 4\delta)|\nabla u|_{L^2}^2 \leqslant c_\delta\big(|\nabla\theta_1|_{L^4}^4 + |\Delta\theta_2|_{L^2}^2 + |\nabla v_1|_{L^4}^4 + |\nabla v_2|_{L^4}^4 + 1\big)|u|_{L^2}^2 + \delta|\nabla w|_{L^2}^2$$

Choosing $\delta > 0$ (enough small) and adding relation (36) and (**??**), one has for $c_{11} > 0$ :

$$\frac{d}{dt}\Big[|\nabla w|_{L^2}^2 + |u|_{L^2}^2\Big] + c_{11}\Big[|\nabla u|_{L^2}^2 + |\Delta w|_{L^2}^2\Big] \leqslant g(t)\Big[|u|_{L^2}^2 + |\nabla w|_{L^2}^2\Big]$$

with $g(t) = c_{12}\Big[|\nabla\theta_1|^4_{L^4} + |\Delta\theta_2|^2_{L^2} + |\nabla v_1|^4_{L^4} + |\nabla v_2|^4_{L^4} + 1\Big]$. Then $g(t) \in L^1(0,T)$ due to the regularity of $(\theta, v)$ say $(\theta, v) \in L^2(0,T; H^2(\Omega))^2 \cap C([0,T]; H^1(\Omega))^2$.
Thus, this Gronwall inequality shows that :

$$|\nabla w|^2_{L^2}(t) + |u|^2_{L^2}(t) \leqslant \Big(|\nabla w_0|^2 + |u_0|^2\Big) \exp\left(\int_0^T g(\sigma)d\sigma\right) = 0$$

$\theta_1 = \theta_2$ and $v_1 = v_2$. ♢

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