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ON THE FREE BOUNDARY FOR QUENCHING TYPE PARABOLIC PROBLEMS VIA LOCAL ENERGY METHODS

Abstract. We extend some previous local energy method to the study the free boundary generated by the solutions of quenching type parabolic problems involving a negative power of the unknown in the equation.

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Dedicated to the memory of Professor Mark Vishik.

1. Introduction.

This paper deals with the study of the free boundary generated by the solutions of quenching type problems. To fix ideas we can mention in this set of problems the following one:

$$\begin{cases} u_t - \Delta u + u^{-k}\chi_{\{u>0\}} = \lambda u^q + g(t,x) & \text{in } (0,T)\times\Omega, \\ u = \varphi & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{on } \Omega, \end{cases} \tag{1}$$

where Ω is an open (not necessarily bounded) domain of $\mathbb{R}^N$, $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, under the key assumption

$$k \in (0,1). \tag{2}$$

For simplicity we can assume that $q \in (0,1]$ but other choices will be also commented in this work. In fact, we shall use several spatially local energy methods which allow the consideration of many other types of boundary conditions and, which is more important, a larger generality in the formulation of the parabolic equation. To be more precise, we shall also consider the quasilinear parabolic problem of quenching type

$$\frac{\partial\psi(u)}{\partial t} - \operatorname{div}\mathbf{A}(x,t,u,D\,u) + C(x,t,u) = f(x,t,u), \tag{3}$$

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under the general structural assumptions

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C|\mathbf{q}|^{p-1}, C|\mathbf{q}|^{p} \leq \mathbf{A}(x,t,r,\mathbf{q})\cdot\mathbf{q},$$
$$C|r|^{\gamma+1} \leq G(r) \leq C^{*}|r|^{\gamma+1},$$

where

$$G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau,$$

and

$$C|r|^{\alpha} \leq C(x,t,r)\,r,$$
$$f(x,t,r)r \leq \lambda|r|^{q+1} + g(x,t)r, \tag{4}$$

with $p>1, q\in\mathbb{R}$ and the main assumptions

$$\gamma \in (0, p-1], \tag{5}$$

and

$$\alpha \in (0, \min(1, \frac{\gamma p}{(p-1)})). \tag{6}$$

Notice that if $\gamma = 1$ condition (6) reduces to $\alpha \in (0,1)$. The case $\gamma < 1$ and $p = 2$ was considered in [20]. Some references and examples for a quasilinear diffusion can be found in [19]. The method allows also the consideration of equations with a first order term $B(x,t,u,D\,u)$ ([1]) but we shall not do it here. The case leading to possible blow up solutions, $q > p-1$ and $\lambda > 0$ will be also considered (see Theorem 4.2). Here C and C^{*} denote some positive constants which depends only on N and the exponents p, κ, η, γ and α (at most). C and C^{*} may be different in different occurrences.

Quasilinear equations of type (3) were formulated from the modeling of many different applied problems and successfully solved, under suitable additional conditions, during the last half of the past century after the pioneering work by Professor Mark Vishik ([27]) opening so many different approaches (see, e.g. [23], [21], [4], [2] and its references).

We shall not deal here with the question of the existence of weak solutions of the above mentioned equations (for some recent surveys in this direction we send the reader, for instance, to the papers [7], [26], [16] and [6]: see also [14], [8], [11], [12], [13] and the surveys [17] and [10] on the associated elliptic problem). We recall that what makes specially interesting equations like (1) and (3) is the fact that the solutions may raise to a free boundary defined as the boundary of the set $\{(x,t)\colon u(x,t) \neq 0\}$ (in most of the cases, as for instance in (1), the data are assumed to be nonnegative). As a matter of fact, sometimes problem (1) is reached trough a previous formulation

$$\left\{\begin{array}{ll} w_t - \Delta w = \frac{\chi_{\{w>0\}}}{(1-w)^k} & \text{in } (0,T_0)\times\Omega, \\ w = 1 & \text{on } (0,T_0)\times\partial\Omega, \\ w(0,.) = w_0, & \text{on } \Omega, \end{array}\right. \tag{7}$$

for some initial datum with $0 \leq w_0(x) \leq 1$ and thus the terminology of "quenching problem" was used in the literature (see, e.g. [18], [25], [22]). Here $u := 1 - w$. The formation of the free boundary is the main reason of the lack of regularity of the solution. The uniqueness of solution fails ([30]) except for the case in which there is not a free boundary ([7]). This is one of the reasons why it looks difficult to apply, directly, super and subsolutions methods to study such a free boundary. Our alternative is the application of local energy methods available for many types of evolution equations and systems since the last thirty years of the last century (see, e.g., the monograph [2] and its many references). More precisely, the main goal of this paper is to show how such methods can be applied to the case in which some singular terms of the type $u^{-k}\chi_{\{u>0\}}$ are present in the equation. Indeed, in the typical applications of the methods the zero order term (the so called "strong absorption term") is assumed to have the form u^k with $k \in (0,1)$ ([2]) or at most, under suitable formulations as multivalued equations [$k = 0$ [9]]. The main difficulty associated to the singular case $u^{-k}\chi_{\{u>0\}}$ with $k \in (0,1)$ comes from the fact that the local energy associated to this term,

$$\int_P |u|^\alpha \, dxdt, \; \alpha = 1 - k,$$

(for some local energy subset $P \subset [0,T] \times \Omega$ to be suitably defined) is not a norm but merely a seminorm (in fact it arises the so called reversed Minkowski inequality) and so the usual "interpolation-trace inequality" (such as it is being formulated in the previous literature ([2])) cannot be directly applied. In that paper we shall show that a systematic use of the Hölder interpolation inequality

$$\|u\|_{L^s} \leq \|u\|_{L^\alpha}^d \, \|u\|_{L^p}^{1-d} \quad \frac{1}{s} = \frac{d}{\alpha} + \frac{1-d}{p}, \tag{8}$$

which is valid for any $d \in [0,1]$, even for $0 < \alpha < 1$ (already used in [24]) allows to arrive to the desired extension of the method to this class of singular equations.

We start, in Section 2, with the consideration of the simple case of the semilinear equation (1) with $\lambda = 0$. This allows to be more pedagogical in the application of this quite technical energy method. In addition, due to the simplicity of the formulation, we can get some sharper estimates on the initial growth of the free boundary and some other informations about it. The key tool of this local energy method, the "interpolation-trace inequality", is proved separately, under several formulations, in Section 3.

Finally, in Section 4, we deal with the general formulation (3). It can be applied, for instance, to some possible doubly degenerate parabolic equations with a singular term as

$$w_t - \Delta_p(|w|^{m-1} w) + \frac{\chi_{\{w \neq 0\}}}{w^{\widehat{k}}} = \lambda w^{\widehat{q}} + g(t,x) \tag{9}$$

for some $p > 1, m > 0$, $\widehat{q} \in (0,m)$ and $\widehat{k} \in (0,m)$. Here $u := |w|^{m-1} w$ and $\Delta_p h$ denotes the usual p-Laplacian operator $\Delta_p h = \mathrm{div}(|\nabla h|^{p-2} \nabla h)$. See also the formulation considered in [28] and [29] in terms of a non-divergential equation. We shall show how to extend the results presented in the paper [1] (there established only for the strong absorption case associated formally to the case $k \in (-1,0)$). In addition, we shall also study the behaviour of the free boundary when the sourcing term $f(x,t,u)$ depends on u, even when dealing with blow-up phenomena (mainly when $\lambda > 0$ and $q > \max(p,2)$ in (4)).

2. **Finite speed of propagation and uniform localization for the semilinear equation and $\lambda \leq 0$.**

In this section we shall restrict ourselves to the consideration of the semilinear equation (1) under the assumption $\lambda \leq 0$. A more general framework (including also the case $\lambda > 0$) will be analyzed in Section 4.

The results we shall present here will have merely a local character: we send the reader to the monograph [2] for many explanations about how to get from those local results many global consequences for the solutions of global formulations as, for instance, the Dirichlet problem (1). We shall follow the usual notations in this type of local methods: B_ρ denotes the open ball of radius ρ of $\mathbb{R}^N$ (we shall not specify the dependence with respect the center of the ball x_0), and for the parabolic formulation we shall use $Q_{\rho,T} := (0,T) \times B_\rho$ and $\Sigma_{\rho,T} := (0,T) \times \partial B_\rho$. We also denote $Du = \nabla u$ to the spatial gradient function. We introduce the local energies

$$E(\rho,T) = \int_{Q_{\rho,T}} |Du|^2 \, dxdt,$$

$$b(\rho,T) = \frac{1}{2} ess \sup_{0\le t\le T} \int_{B_\rho} |u(x,t)|^2 \, dx,$$

and

$$c(\rho,T) = \int_{Q_{\rho,T}} |u|^\alpha \, dxdt.$$

The notion of local solution we need for the application of the following local energy method does not need to be specified: we shall only require that u is any nonnegative function such that the above local energies are finite, for almost all $\rho \in (0,\rho_0)$, for some ρ_0, and the "local integration by parts inequality" holds

$$\begin{cases} b(\rho,T) + E(\rho,T) + c(\rho,T) \le \displaystyle\int_{\Sigma_{\rho,T}} |Du| \, udxdt, \text{ a.e. } \rho \in (0,\rho_0), \\ \text{assumed } g(t,x) = 0 \text{ and } u_0(x) = 0 \ a.e. \text{ respectively on } Q_{\rho_0,T} \text{ and } B_{\rho_0} \end{cases} \tag{10}$$

The verification of such inequality (10), starting from a concrete notion of (global in space) weak solution was presented usually as the first step of the method (this is the way as it was presented in most of the previous papers in the literature: see, e.g. [15] and [2]). Nevertheless, the local integration by parts inequality can be obtained, sometimes, for some type of solutions which a priori are defined outside of the global energy space as it is the case, for instance, of the "renormalized solutions" (see [5]). In our case, (10) is a direct consequence of the regularity $u^{-k}\chi_{\{u>0\}} \in L^1((0,T)\times\Omega)$ obtained in many previous papers under suitable regularity assumptions on the data (see, e.g. [25], [7], [16] and [6]). Notice that since $\lambda \le 0$ we have that $\int_{Q_{\rho,T}} u^{q+1}dxdt \ge 0$ for any $q \in \mathbb{R}$ (remember that we are assuming that $u \ge 0$), so that the results of this section are applicable for any $q \in \mathbb{R}$ if we replace at the equation the term λu^q by $\lambda u^q \chi_{\{u>0\}}$ for the case $q < 0$, once that a local nonnegative function satisfying (10) exists.

The following result shows the finite speed of propagation property. As a matter of fact, we shall get also a stronger property which usually is as called "stable (or uniform) localization property" (see [2], Chapter 3). The proof will require the use some interpolation-trace inequality which is of an independent nature and will be presented in Section 3.

Theorem 2.1. *Let $B_{\rho_0} \subset \Omega$ be such that $u_0 = 0$ on B_{ρ_0} and $g = 0$ on $Q_{\rho_0,T}$. Let u a solution satisfying (10). Then $u = 0$ on $Q_{\rho_1,T}$ with ρ_1 defined by*

$$\rho_1^{1+2\beta} = \rho_0^{1+2\beta} - CK(\rho_0,T)\frac{(1+2\beta)}{(1-\xi)}E(\rho_0,T)^{1-\xi} \tag{11}$$

where

$$\beta := \frac{N(2-\alpha)+2}{4}, \tag{12}$$

$$\xi = \frac{N(2-\alpha)+2}{N(2-\alpha)+4}, \tag{13}$$

and

$$K(\rho,T) = \max\{\rho^{2\beta}, b(\rho,T)^{\theta(2-\alpha)/2}\}. \tag{14}$$

Proof. From (10) we get that for almost all $\rho \in (0,\rho_0)$

$$b(\rho,T)+E(\rho,T)+c(\rho,T) \le \int_{\Sigma_{\rho,T}} |Du|\, u dx dt \le \|Du\|_{L^2(\Sigma_{\rho,T})} \|u\|_{L^2(\Sigma_{\rho,T})}. \tag{15}$$

Taking squares and using Corollary 2 we obtain

$$\begin{aligned}(b(\rho,T)+E(\rho,T)+c(\rho,T))^2 &\le C\rho^{-2\beta}K(\rho,T)(b(\rho,T)+E(\rho,T)\\ &+c(\rho,T))^{1+(1-\theta)(2-\alpha)/2}\|Du\|^2_{L^2(\Sigma_{\rho,T})}.\end{aligned} \tag{16}$$

We set

$$\xi := 1-(1-\theta)(2-\alpha)/2. \tag{17}$$

From (13) we obtain

$$\xi = 1-\frac{2-\alpha}{N(2-\alpha)+4}. \tag{18}$$

Thus $0<\xi<1$. Noting that $K(\rho,T)\le K(\rho_0,T)$ (see (14)) and

$$\|Du\|^2_{L^2(\Sigma_{\rho,T})} = \frac{\partial E}{\partial \rho}(\rho,T) \tag{19}$$

we obtain from (16) and (18) the ordinary differential inequality

$$\rho^{2\beta}E(\rho,T)^{\xi} \le CK(\rho_0,T)\frac{\partial E}{\partial \rho}(\rho,T). \tag{20}$$

Integrating (20) we get that $u=0$ on $Q_{\rho_1,T}$ with ρ_1 defined by estimate (11). ∎

A sharper estimate, for T small, can be also obtained:

Theorem 2.2. *Let $B_{\rho_0}\subset\Omega$ be such that $u_0=0$ on B_{ρ_0} and $g=0$ on $Q_{\rho_0,T}$. Let u satisfying (10). Then $u=0$ on $Q_{\rho_2,T}$ with ρ_2 defined by*

$$\rho_2^{1+2\beta} = \rho_0^{1+2\beta} - F(T,E(\rho_0,T)), \tag{21}$$

where

$$F(T,s) := C\frac{(1+2\beta)}{(1-\xi)}A(T,\rho_0)\sqrt{T}\log\left(1+\frac{K(\rho_0,T)}{A(T,\rho_0)\sqrt{T}}s^{1-\xi}\right), \tag{22}$$

$$A(T,\rho_0) := \rho_0^{2\beta-1}K_1(\rho_0,T), \tag{23}$$

and

$$K_1(\rho,T) := \max\{\rho,\sqrt{T}\}. \tag{24}$$

Proof. Taking squares in (15) and applying Corollary 3 we obtain

$$(b(\rho,T)+E(\rho,T)+c(\rho,T))^2 \leq C\rho^{-1}\sqrt{T}K_1(\rho,T)(b(\rho,T)+E(\rho,T))\,\|Du\|^2_{L^2(\Sigma_{\rho,T})}. \tag{25}$$

From (24) $K_1(\rho,T) \leq K_1(\rho_0,T)$. Recalling (19) we obtain

$$\rho E(\rho,T) \leq C\sqrt{T}K_1(\rho_0,T)\frac{\partial E}{\partial \rho}(\rho,T). \tag{26}$$

This differential inequality does not imply vanishing properties, but combining (20) and (26) gives

$$\frac{\rho^{2\beta}}{K(\rho_0,T)}E(\rho,T)^{\xi} + \frac{\rho}{\sqrt{T}K_1(\rho_0,T)}E(\rho,T) \leq C\frac{\partial E}{\partial \rho}(\rho,T). \tag{27}$$

Noting that $2\beta > 1$, we set

$$\rho = \frac{\rho^{2\beta}}{\rho^{2\beta-1}} \geq \frac{\rho^{2\beta}}{\rho_0^{2\beta-1}}$$

in order to obtain the following explicitly integrable differential inequality

$$\rho^{2\beta}\left(\frac{E(\rho,T)^{\xi}}{a} + \frac{E(\rho,T)}{A\sqrt{T}}\right) \leq C\frac{\partial E}{\partial \rho}(\rho,T), \tag{28}$$

with $a(T,\rho_0) := \rho_0^{2\beta-1}K$, and A given by (23). If $E(\rho,T) \neq 0$, an integration of (28) yields

$$\rho_0^{1+2\beta} - \rho^{1+2\beta} \leq F(T,E(\rho_0,T)) - F(T,E(\rho,T)) \leq F(T,E(\rho_0,T)) \tag{29}$$

with F given by (22). Thus we arrive to estimate (21) provided that the right hand side of (21) is positive. ∎

Remark 1. Since $\log(1+x) \leq x$, (21) implies (11) for some constant C. But (22) gives more information as $T \to 0$. Indeed, since for $x > 0$

$$\log(1+x) < \log x + \frac{1}{x}$$

(22) behaves as $(constant)\cdot\sqrt{T}(\log T)$ as $T \to 0$ (for fixed a, A and s).

In the above arguments the time interval $(0,T)$ can be replaced by (T_1,T_2) with $0 < T_1 < T_2$ provided that $u(T_1,.) = 0$ on some ball. But this holds for T_1 small enough by the above results. This leads to:

Corollary 1. *Assume u as in Theorem 2.1 and*

$$\rho_0 - \rho \geq \left(CK(\rho_0,T)\frac{(1+2\beta)}{(1-\xi)}E(\rho_0,T)^{1-\xi}\right)^{1/(1+2\beta)}. \tag{30}$$

Then $u = 0$ on $Q_{\rho,T}$. In particular, if $N = 1$, this implies that the free boundary is Hölder continuous, as function of t, for those values of t where it is a monotone function.

Proof. Since $\beta > 0$ we have that

$$\rho_0^{1+2\beta} - \rho^{1+2\beta} \geq (\rho_0 - \rho)^{1+2\beta}.$$

Thus (11) implies that $u = 0$ on $Q_{\rho,T}$ assumed (30). Then (21) implies that $u = 0$ on $Q_{\rho,T}$ if

$$\rho_0 - \rho \geq F(T, E(\rho_0, T))^{1/(1+2\beta)}, \tag{31}$$

which shows, in particular, if $N = 1$, that the free boundary is Hölder continuous where it is monotone. ∎

Concerning the behaviour for small time we can prove a first result showing the *local waiting time* or, what we can call perhaps more properly as the *non dilatation of the initial support*: the free boundary cannot invade the subset where the initial datum is nonzero. Some sharper results can be obtained with the techniques of Section 4 modifying the presentation made in ([1]) for the case of strong absorption terms, nevertheless we shall not detail it in this paper.

Theorem 2.3. *Let $B_{\rho_0} \subset \Omega$ be such that $g = 0$ on $Q_{\rho_0,T}$ and assume u as in Theorem 2.1. We also assume, in addition, that*

$$b(\rho, 0) \leq \varepsilon \left[\rho_0 - \rho\right]^{\omega} \ \ a.e. \ \ \rho \in [0, \rho_1) \tag{32}$$

with

$$\omega = \frac{2N(2-\alpha)+8}{(2-\alpha)} \tag{33}$$

for some ε small enough and $\rho_1 > \rho_0$ large enough. Then there exists a $t^* \leq T$ such that $u(x,t) = 0$ a.e. $x \in B_{\rho_0}$ and for any $t \in [0, t^*]$.

Proof. For almost all $\rho \in (0, \rho_1)$

$$b(\rho,T) + E(\rho,T) + c(\rho,T) \leq \|Du\|_{L^2(\Sigma_{\rho,T})} \|u\|_{L^2(\Sigma_{\rho,T})} + b(\rho, 0). \tag{34}$$

Then by Corollary 2 we obtain that

$$\begin{aligned} b(\rho,T) + E(\rho,T) \quad &\leq C\rho^{-\beta}\sqrt{K(\rho,T)}(b(\rho,T) \\ &+E(\rho,T))^{1-\frac{\xi}{2}}(\frac{\partial b}{\partial \rho}(\rho,T) + \frac{\partial E}{\partial \rho}(\rho,T))^{1/2} + \varepsilon\left[\rho_0 - \rho\right]^{\omega}, \end{aligned} \tag{35}$$

where we have used that $\frac{\partial b}{\partial \rho}(\rho,T) \geq 0$. Thus, if we introduce

$$z(\rho,T) := b(\rho,T) + E(\rho,T),$$

we get that

$$z \leq C\rho^{-\beta}\sqrt{K(\rho,T)}(\frac{\partial z^{(3-\xi)}}{\partial \rho}(\rho,T))^{1/2} + \varepsilon\left[\rho_0 - \rho\right]^{\omega}.$$

In particular, the function $w := z^{(3-\xi)}$ satisfies that

$$w^a \leq C\rho^{-2\beta}K(\rho,T)\frac{\partial w}{\partial \rho}(\rho,T) + \varepsilon\left[\rho_0 - \rho\right]^{\omega/2}$$

with $a = \frac{2}{3-\xi}$ and so $a \in (0,1)$ since $\xi \in (0,1)$. On the other hand, by assumption (33) we have that

$$\omega/2 = a/(1-a)$$

and then the result reduces to the application of a well known result for ordinary differential inequalities (see, e.g., [2]: Subsection 1.3.2, Chapter 3). ∎

Other results will be formulated, in Section 4, in the more general framework of the quasilinear equation (3).

3. **Interpolation-trace inequalities.** We start by recalling a well-known interpolation-trace result:

Lemma 3.1. *([15]). Assume $u \in H^1(B_\rho)$ and $1 \leq s \leq 2$. Then*

$$\|u\|_{L^2(\partial B_\rho)} \leq C\left(\|Du\|^{\theta}_{L^2(B_\rho)}\|u\|^{1-\theta}_{L^s(B_\rho)} + \rho^{-\beta}\|u\|_{L^s(B_\rho)}\right) \tag{36}$$

where the constant C depends only on N and s, and

$$\beta := \frac{N(2-s)+s}{2s},\ \theta = \frac{N(2-s)+s}{N(2-s)+2s}. \tag{37}$$

This is Corollary 2.1 of [15] with $q+1=2$ and $\sigma+1=s$. Notice that β is related to the exponent δ of [15] by $\beta = -\delta\theta$.

Remark 2. Although we are going to consider terms of the form $\int |u|^\alpha$ with $0 < \alpha < 1$, Lemma 3.1 was applied in Section 1 with $s > 1$. We postpone for the moment a generalization which will be used in Section 4.

The main interpolation-trace result used in this paper is the following one:

Lemma 3.2. *Let $0 < \alpha \leq 2$. Assume that $Du \in L^2(Q_{\rho,T})$ and $u \in L^\infty(0,T : L^2(B_\rho))$. Then*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq E(\rho,T)^\theta c(\rho,T))^{1-\theta} b(\rho,T)^{(1-\theta)(2-\alpha)/2} + \rho^{-2\beta} c(\rho,T)) b(\rho,T)^{(2-\alpha)/2} \tag{38}$$

where the positive constant C depends only on N and α, and

$$\beta := \frac{N(2-\alpha)+2}{4},\ \theta = \frac{N(2-\alpha)+2}{N(2-\alpha)+4}. \tag{39}$$

Proof. Applying Hölder interpolation inequality (8) for $0 < \alpha < 1$ and choosing $d = \alpha/2$ (in order to obtain C independent of T) we get

$$\|u\|^2_{L^s} \leq \|u\|^\alpha_{L^\alpha}\|u\|^{2-\alpha}_{L^2} \text{ where } s = 4/(4-\alpha). \tag{40}$$

This choice gives $1 < s \leq 2$ (since $0 < \alpha \leq 2$) and (12) follows from (37). From (40) and (36) we obtain for almost all $t \in (0,T)$

$$\begin{aligned} \frac{1}{C}\int_{\partial B_\rho} |u|^2 \quad &\leq \left(\int_{B_\rho}|Du|^2\right)^\theta \left(\int_{B_\rho}|u|^\alpha\right)^{1-\theta}\left(\int_{B_\rho}|u|^2\right)^{(1-\theta)(2-\alpha)/2} \\ &+\rho^{-2\beta}\left(\int_{B_\rho}|u|^\alpha\right)\left(\int_{B_\rho}|u|^2\right)^{(2-\alpha)/2}. \end{aligned} \tag{41}$$

We estimate $\int_{B_\rho}|u|^2$ by $2b(\rho,T)$. Then (38) follows integrating in t between 0 and T and applying Hölder inequality. ∎

Remark 3. The hypotheses of Lemma 3.2 imply easily that $u \in L^2((0,T)\times\partial B_\rho)$ and $u \in L^\alpha(Q_{\rho,T})$. The main feature of the lemma is that the constant C is independent of ρ and T, while T does not appear as a separate factor. A similar (but slightly different) result was given in [15, Lemma 3.2] . This new statement was inspired on Lemma I.2 of [3].

Corollary 2. *Under the hypotheses of Lemma 2*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq \rho^{-2\beta}K(\rho,T)\left(E(\rho,T)+c(\rho,T)\right)+b(\rho,T))^{1+(1-\theta)(2-\alpha)/2}, \tag{42}$$

where θ and β are given by (12), the positive constant C depends only on N and α and $K(\rho,T)$ is given by (14).

Proof. We start from (38) and apply Young's inequality in the forms

$$A_1^{\alpha_1}A_2^{\alpha_2}A_3^{\alpha_3} \leq C(A_1+A_2+A_3)^{\alpha_1+\alpha_2+\alpha_3} \text{ or } A_1^{\alpha_1}A_2^{\alpha_2} \leq C(A_1+A_2)^{\alpha_1+\alpha_2}.$$

So we obtain

$$E^\theta c^{1-\theta} b^{(1-\theta)(2-\alpha)/2} \leq C(E+c+b)^{1+(1-\theta)(2-\alpha)/2}. \tag{43}$$

$$cb^{(2-\alpha)/2} = b^{\theta(2-\alpha)/2} cb^{(1-\theta)(2-\alpha)/2} \leq b^{\theta(2-\alpha)/2}(c+b)^{1+(1-\theta)(2-\alpha)/2}. \tag{44}$$

The corollary follows from (38),(43) and (44). ∎

In order to sharpen the estimates of the support, we shall need the following lemma and its corollary.

Lemma 3.3. *Assume that $Du \in L^2(Q_{\rho,T})$ and $u \in L^\infty(0,T : L^2(B_\rho))$. Then*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq \sqrt{T}E(\rho,T)^{1/2}b(\rho,T)^{1/2} + \rho^{-1}Tb(\rho,T). \tag{45}$$

where the positive constant C depends only on N.

Proof. We apply Lemma 3.1 with $s=2$ and take squares. Then we bound $\int_{B_\rho}|u|^2$ by $2b(\rho,T)$, integrate in t between 0 and T and applying Hölder inequality. ∎

Corollary 3. *Under the hypotheses of Lemma 3*

$$\frac{1}{C}\int_{\Sigma_{\rho,T}} |u|^2 \leq \rho^{-1}\sqrt{T}K_1(\rho,T)\left(E(\rho,T)+b(\rho,T)\right), \tag{46}$$

where the positive constant C depends only on N and $K_1(\rho,T)$ is given by (24).

Proof. It follows from Lemma 3.3 and the inequality $E^{1/2}+b^{1/2} \leq C\left(E+b\right)^{1/2}$. ∎

4. **Non cylindrical local energy subsets technique.** In contrast to the *finite speed of propagation* and the *uniform localization* properties obtained in the previous section we shall pay attention now to other type of free boundary properties, in particular on its formation even in the case of strictly positive initial data (sometimes called as the instantaneous shrinking of the support property: see [2] and its references). To do that we shall use some energy functions defined on local domains of a special form. As in [2] we shall use the following notation: given $x_0 \in \Omega$ and the nonnegative parameters ϑ and υ, we define the *energy set*

$$P(t) \equiv P(t;\vartheta,\upsilon) = \{(x,s) \in B_{\rho(s)}(x_0) \times (t,T) : |x - x_0| < \rho(s) := \vartheta(s-t)^{\upsilon}\}.$$

Notice that the shape of $P(t)$, *the local energy set,* is determined by the choice of the parameters ϑ and υ. Here we shall take $\vartheta > 0$, $0 < \upsilon < 1$ and so $P(t)$ becomes a paraboloid. The adaptation of the results of [2] to the case in which $k \in (0,1)$ and $P(t,\rho)$ is the cylinder $B_\rho(x_0) \times (t,T)$; or when $P(0,\rho)$ becomes a truncated cone with base $B_\rho(x_0) := \{\, x \in \Omega : \ |x - x_0| < \rho\}$ on the plane $t = 0$ follows easily but they will not be detailed here. We adapt the definition of the local energies in the following way:

$$E(P(t)) := \int_{P(t)} |Du(x,\tau)|^p \, dx d\tau, \quad C(P(t)) := \int_{P(t)} |u(x,\tau)|^\alpha \, dx d\tau,$$

and

$$b(P(t)) := ess \sup_{s\in(t,T)} \int_{|x-x_0|<\vartheta(s-t)^{\upsilon}} |u(x,s)|^{\gamma+1} \, dx.$$

Although our results have a local nature (as already said they are independent of the boundary conditions) our statements become easier under the additional global information on the boundedness of the *global energy function*

$$D(u,t^*,T) := ess \sup_{s\in(t^*,T)} \int_\Omega |u(x,s)|^{\gamma+1} \, dx + \int_{\Omega\times(t^*,T)} (|Du|^p + |u|^\alpha) \, dx dt. \quad (47)$$

Our study will follow quite closely the technique introduced in [1]. The key new ingredient, with respect to [1], is the following interpolation-trace result which extends Corollary 2.1 of [15] in the sense that some exponent, s, is now assumed such that $0 < s < 1$ and that the interpolation inequality involves the seminorm $\|u\|_{L^r(B_\rho)}$ with an arbitrary $r \in [s,p]$ (and not necessarily $r = s$). It also generalizes Lemma 3.4 of [2] where it was assumed $s \geq 1$.

Lemma 4.1. *Assume $u \in W^{1,p}(B_\rho)$, $p \geq 1$ and $0 < s \leq p$. Then for any $r \in [s,p]$*

$$\|u\|_{L^p(\partial B_\rho)} \leq C(\|Du\|_{L^p(B_\rho)} + \rho^{-\beta} \|u\|_{L^s(B_\rho)})^\theta \|u\|^{1-\theta}_{L^r(B_\rho)} \quad (48)$$

where the constant C depends only on N and s,

$$\theta = \frac{N(p-r)+r}{N(p-r)+pr} \text{ and } \beta := (\frac{N(p-s)+ps}{ps}). \quad (49)$$

Proof. We shall follow the same structure of four steps than in the proof of Lemma 2.2 of [15]. We shall only detail the differences with respect to the proof when $s \geq 1$. We denote, for the moment, $G = B_\rho$. As usual, we restrict ourselves to $u \in C^1(\overline{G})$ (since $C^1(\overline{G})$ is dense in $W^{1,p}(B_\rho)$).

First step. If $p > 1$ and $0 < s \leq p$ we have

$$\|u\|_{W^{1,p}(G)} \leq C(\|Du\|_{L^p(B_\rho)} + \|u\|_{L^s(B_\rho)}) \tag{50}$$

where the constant C depends only on s and $|G|$. Indeed, from a result of [21] (page. 45), for any $\varepsilon > 0$ there exists C_ε such that for any $u \in C^1(\overline{G})$

$$\|u\|_{L^p(G)} \leq \varepsilon \|Du\|_{L^p(G)} + C_\varepsilon \|u\|_{L^1(G)}.$$

Then, by the Hölder interpolation inequality (8) with $d = (p-1)/(p-s)$

$$\|u\|_{L^1} \leq \|u\|_{L^s}^d \|u\|_{L^p}^{1-d} \quad 1 = \frac{d}{s} + \frac{1-d}{p}.$$

Applying Young inequality expressed in terms of

$$AB \leq \mu A^m + C_\mu B^{m'},\ \frac{1}{m} + \frac{1}{m'} = 1,$$

we get that

$$\|u\|_{L^1} \leq \mu \|u\|_{L^p} + C_\mu \|u\|_{L^s}$$

and thus

$$\|u\|_{L^p(G)} \leq \frac{\varepsilon}{(1-\mu)} \|Du\|_{L^p(G)} + \frac{C_\varepsilon C_\mu}{(1-\mu)} \|u\|_{L^s(G)},$$

which leads to (50).
Second step. If $p > 1$ we have

$$\|u\|_{L^p(\partial G)} \leq C \|u\|_{W^{1,p}(G)}^{1/p} \|u\|_{L^p(G)}^{(p-1)/p}. \tag{51}$$

This is exactly inequality (4.4) of ([15]).
Third step. If $p > 1$ and $0 < r \leq p$ we have

$$\|u\|_{L^p(G)} \leq C \|u\|_{W^{1,p}(G)}^{(p\theta-1)/(p-1)} \|u\|_{L^r(G)}^{p(1-\theta)/(p-1)}, \tag{52}$$

with $\theta \in (0, 1]$ given by (49).

This inequality coincides with inequality (4.5) of ([15]) when $r \in [1, p]$, nevertheless its proof for the remaining cases $r \in (0, 1)$ is exactly the same since in the proof given in ([15]) we only used the Sobolev inequality and the Hölder interpolation inequality (8), which, as said before, it is true even if $r \in (0, 1)$.
Fourth step. Using (51) and (52) we get

$$\|u\|_{L^p(\partial G)} \leq C \|u\|_{W^{1,p}(G)}^{1/p} \|u\|_{W^{1,p}(G)}^{(p\theta-1)/(p-1)} \|u\|_{L^r(G)}^{(1-\theta)} = C \|u\|_{W^{1,p}(G)}^{\theta} \|u\|_{L^r(G)}^{(1-\theta)}.$$

Thus, using (50) inequality (48) is proved by taking $G = B_\rho$, by making the change of variable $x = \rho y$ (we assume that the ball is centered at $x_0 = 0$) and by computing the norms for $v(y) = u(x)$ (see details in the proof of Corollary 2.1 of [15]). ∎

As said in Section 2, the notion of local solution we need for the application of the following local energy method does not need to be specified: we shall only require that u is any function such that the local energies are finite, for almost all $\rho \in (0, \rho_0)$, for some ρ_0, and satisfies the "local integration by parts inequality" on the paraboloid $P = P(t; \vartheta, \upsilon)$

$$\left\{ \begin{array}{c} \displaystyle\int_{P\cap\{t=T\}} G(u(x,t))dx + \int_P \mathbf{A}\cdot Du\, dxd\theta + \int_P Cudxd\theta \\ \displaystyle\leq \int_{\partial_l P} \mathbf{n}_x\cdot \mathbf{A}\, u\, d\Gamma d\theta + \int_{\partial_l P} n_\tau G(u(x,t))d\Gamma d\theta + \lambda \int_P |u|^{q+1}dxd\theta \\ \text{assumed } g(t,x)=0 \text{ a.e. on } P, \end{array} \right. \tag{53}$$

were $\partial_l P$ denotes the lateral boundary of P i.e. $\partial_l P = \{(x,s) : |x-x_0| = \vartheta(s-t)^{\upsilon},\ s\in(t,T)\}$, $d\Gamma$ is the differential form on the hypersurface $\partial_l P\cap\{t = const\}$, $\mathbf{n}_x$ and $\mathbf{n}_\tau$ are the components of the unit normal vector to $\partial_l P$. Let us mention that $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_\tau) = \frac{1}{(\vartheta^2\upsilon^2+(\theta-t)^{2(1-\upsilon)})^{1/2}}((\theta-t)^{1-\upsilon}\mathbf{e}_x - \upsilon\mathbf{e}_\tau)$ with $\mathbf{e}_x, \mathbf{e}_\tau$ orthogonal unit vectors to the hyperplane $t=0$ and the axis t, respectively, where we used the notation $\mathbf{n}_\tau = n_\tau\mathbf{e}_\tau$. Notice that P does not touch the initial plane $\{t=0\}$ and that $P\subset B_{\rho(T)}(x_0)\times[0,T]$, and that we assume $B_{\rho(T)}(x_0)\subset\Omega$.

This local inequality can be easily checked starting from a natural definition of local weak solution and by taking as test function the cut-off function

$$\zeta(x,\theta) := \psi_\varepsilon\left(|x-x_0|,\theta\right)\xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m\left(u(x,s)\right)ds, \quad h>0,$$

where T_m is the truncation at the level m,

$$\xi_k(\theta) := \begin{cases} 1 & \text{if } \theta\in\left[t, T-\frac{1}{k}\right], \\ k(T-\theta) & \text{for } \theta\in\left[T-\frac{1}{k}, T\right], \\ 0 & \text{otherwise}, \quad k\in\mathbb{N}, \end{cases}$$

and

$$\psi_\varepsilon\left(|x-x_0|,\theta\right) := \begin{cases} 1 & \text{if } d>\varepsilon, \\ \frac{1}{\varepsilon}d & \text{if } d<\varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

with $d = dist((x,\theta), \partial_l P(t))$ and $\varepsilon>0$. So that, $supp\zeta(x,\theta)\equiv P(t)$, $\zeta, \frac{\partial\zeta}{\partial t}\in L^\infty\left((0,T)\times\Omega\right)$ and $\frac{\partial\zeta}{\partial x_i}\in L^p\left((0,T)\times\Omega\right)$ (it is easy to adapt the arguments of the proof of Lemma 3.1 of [15] and Subsection 2.1 of [1] to our framework).

Theorem 4.2. *Assume (5) and (6). Let u satisfying (53) on any paraboloid of the form $P = P(t;\vartheta,\upsilon)$ and assume $\lambda\leq 0$ and $q\in\mathbb{R}$ (we replace at the equation the term λu^q by $\lambda u^q\chi_{\{u>0\}}$ for the case $q<0$). Then there exists some positive constants M, t^*, and $\mu\in(0,1)$ such that if $t^*\leq T$ and*

$$D(u,t^*,T)\leq M \tag{54}$$

we have

$$u(x,t)=0 \text{ in the paraboloid } \{(x,t) : |x-x_0| < (t-t^*)^\mu, t\in(t^*,T)\},$$

independently either u_0 vanishes or not. Moreover, if $\lambda>0$ the above conclusion remains true (with the same $t^*\leq T$) under one of the following conditions: either

$$q\geq\gamma \quad \text{and} \quad \|u\|^{q-\gamma}_{L^\infty(P(t^*))} < \infty \tag{55}$$

or

$$q+1 \geq \alpha \quad \textit{and} \quad \lambda \, \|u\|^{q+1-\alpha}_{L^\infty(P(t^*))} < 1. \tag{56}$$

Proof. It is useful to identify each one of the terms involved in the interpolation trace inequality (53). So, we rewrite it in the form

$$i_1 + i_2 + i_3 \leq j_1 + j_2 + j_3. \tag{57}$$

Case 1. Let us assume for the moment that

$$j_3 = \lambda \int_P |u|^{q+1} dx d\theta \leq 0.$$

The key idea is to get a differential inequality for some energy function. We observe that if we denote by (ρ, ω), $\rho \geq 0$ and $\omega \in \partial B_1$ the spherical coordinate system in $\mathbb{R}^N$ and if $\Phi(\rho, \omega, \theta)$ is the spherical representation of a general function $F(x,t)$, then an energy function $I(t)$ defined trough $F(x,t)$ can be also written as

$$I(t) := \int_P F(x,\theta) dx d\theta \equiv \int_t^T d\theta \int_0^{\rho(\theta,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J| d\omega,$$

where J is the Jacobi matrix and $\rho(\theta,t) = \vartheta(\theta - t)^{\upsilon}$. So we get

$$\begin{aligned} \frac{dI(t)}{dt} = &- \int_0^{\rho(\theta,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J| d\omega \bigg|_{\theta=t} \\ &+ \int_t^T \rho_t \rho^{N-1} d\theta \int_{\partial B_1} \Phi(\rho,\omega,t)|J| d\omega = \int_{\partial_l P} \rho_t F(x,\theta) d\Gamma d\theta, \end{aligned} \tag{58}$$

and thus, as we shall show, we can get from (53) a differential inequality for some suitable energy function which in our case will be given by

$$I(t) = E(P(t)) + C(P(t)).$$

In order to estimate j_1, we use Hölder's inequality to get

$$\begin{aligned} \left| \int_{\partial_l P} \mathbf{n}_x \cdot \mathbf{A}\, u \, d\Gamma d\theta \right| &\leq M_2 \int_{\partial_l P} |\mathbf{n}_x| |Du|^{p-1} |u| d\Gamma d\theta \\ &\leq M_2 \left(\int_{\partial_l P} |\rho_t| |Du|^p d\Gamma d\theta \right)^{(p-1)/p} \left(\int_{\partial_l P} \frac{|\mathbf{n}_x|^p}{|\rho_t|^{p-1}} |u|^p d\Gamma d\theta \right)^{1/p} \\ &= M_2 \left(-\frac{dE}{dt} \right)^{(p-1)/p} \left(\int_t^T \frac{|\mathbf{n}_x|^p}{|\rho_t|^{p-1}} \left(\int_{\partial B_{\rho(\theta,t)}} |u|^p d\Gamma \right) d\theta \right)^{1/p}. \end{aligned} \tag{59}$$

To estimate the right-hand side of (57) we use the interpolation inequality given in Lemma 4 with $s = \alpha$. Let $h = \frac{\gamma}{\gamma - r + 1}$, then

$$\int_{\partial B_\rho}|u|^p d\Gamma \leq C\left(\int_{B_\rho}|Du|^p+\rho^{-\beta p}\left(\int_{B_\rho}|u|^{\alpha}\right)^{p/2}\right)^{\tilde{\theta}}\times\left(\int_{B_\rho}|u|^r\right)^{p(1-\tilde{\theta})/r}$$
$$\leq C\rho^{-\beta\tilde{\theta}p}\left(\int_{B_\rho}|Du|^p+\int_{B_\rho}|u|^{\alpha}\right)^{\tilde{\theta}}\times\left(\int_{B_\rho}|u|^{\alpha}\right)^{p(1-\tilde{\theta})/hr}\left(\int_{B_\rho}|u|^{\gamma+1}\right)^{p(h-1)(1-\tilde{\theta})/hr}$$
$$\leq C\rho^{-\beta\tilde{\theta}p}\left(E_*+C_*\right)^{\tilde{\theta}}C_*^{(1-\tilde{\theta})p/qr}b^{(h-1)(1-\tilde{\theta})p/hr}$$
$$\leq C\rho^{-\beta\tilde{\theta}p}\left(E_*+C_*\right)^{\tilde{\theta}+(1-\tilde{\theta})p/hr}b^{(h-1)(1-\tilde{\theta})p/hr}, \tag{60}$$

where $E_*(t,\rho):=\int_{B_\rho}|Du|^p dx$, $\quad C_*(t,\rho):=\int_{B_\rho}|u|^{\alpha}dx$ and C is a suitable positive constant.
Returning to the estimate of j_1, applying once again Hölder's inequality we have from (60) that if $\mu=\tilde{\theta}+p\frac{(1-\tilde{\theta})}{hr}$ and $r\in\left[\frac{p(\gamma+1)}{p+\gamma},\gamma+1\right]$ (so that $\mu<1$)

$$|j_1|\leq C\left(-\frac{dE}{dt}\right)^{(p-1)/p}\times\left(\int_t^T\frac{|\mathbf{n}_x|^p}{|\rho_t|^{p-1}}K\rho^{\beta\tilde{\theta}p}\left(E_*+C_*\right)^{\mu}b^{(h-1)(1-\tilde{\theta})p/hr}d\tau\right)^{1/p}$$
$$\leq C\left(-\frac{dE}{dt}\right)^{(p-1)/p}b^{(h-1)(1-\tilde{\theta})/hr}\times$$
$$\times\left(\int_t^T\left(E_*+C_*\right)d\tau\right)^{\frac{\mu}{p}}\left(\int_t^T\left(\frac{|\mathbf{n}_x|^p}{|\rho_t|^{p-1}}\rho^{-\beta\tilde{\theta}p}(\tau)\right)^{\frac{1}{1-\mu}}d\tau\right)^{\frac{1-\mu}{p}}$$
$$\leq C\sigma(t)\left(-\frac{d\left(E+C\right)}{dt}\right)^{(p-1)/p}b^{(h-1)(1-\tilde{\theta})/hr}\left(E+C\right)^{\frac{\tilde{\theta}}{p}+\frac{1-\tilde{\theta}}{hr}}, \tag{61}$$

for a suitable positive constant C and with

$$\sigma(t):=\left(\int_t^T\left(\frac{1}{|\rho_t|^{p-1}}\rho^{\delta\tilde{\theta}p}(\tau)\right)^{\frac{1}{1-\mu}}d\tau\right)^{\frac{1-\mu}{p}}.$$

Obviously, to be able to continue with our arguments we must have $\sigma(t)<\infty$. But this is fulfilled if we choose suitably our paraboloid with some $\nu=\mu\in(0,1)$ sufficiently small since the condition of convergence of the integral $\sigma(t)$ has the form

$$(1-\mu)(p-1)-\mu\beta\tilde{\theta}p>-(1-\tilde{\theta})\left(1-\frac{p}{hr}\right).$$

So, we have obtained an estimate of the following type

$$|j_1|\leq L_1\sigma(t)D(u)^{(h-1)(1-\tilde{\theta})/hr}\left(E+C\right)^{1-\omega}\left(-\frac{d(E+C)}{dt}\right)^{(p-1)/p}, \tag{62}$$

where L_1 is a universal positive constant, $D(u)$ is *the total energy*, and $\omega:=1-\frac{\tilde{\theta}}{p}-\frac{1-\tilde{\theta}}{hr}\in\left(1-\frac{1}{p},1\right)$.
Let us estimate j_2. Using the expression for $\mathbf{n}_\tau$, we have $|j_2|\leq C_5\int_{\partial_l P}|u|^{1+\gamma}d\Gamma d\theta$. We apply then a variant of the the interpolation inequality (48), thanks to the assumption (5)

$$\|v\|_{\gamma+1,\partial B_\rho}\leq C\left(\|Dv\|_{p,B_\rho}+\rho^{-\beta}\|v\|_{\alpha,B_\rho}\right)^s\cdot\|v\|_{r,B_\rho}^{1-s}\quad\forall\, v\in W^{1,p}(B_\rho) \tag{63}$$

with a universal positive constant $C > 0$ and exponents $s = \frac{(\gamma+1)N-r(N-1)}{(N+r)p-Nr}\frac{p}{\gamma+1}$, $r \in [\alpha, 1+\gamma]$. Again

$$\int_{\partial B_\rho} |u|^{\gamma+1}dx \le L^{1+\gamma}C^{s(\gamma+1)/\tilde{\theta}p}\left(\int_{B_\rho}|Du|^p dx + \int_{B_\rho}|u|^\alpha dx\right)^{s(\gamma+1)/p} \times \left[\left(\int_{B_\rho}|u|^\alpha dx\right)^{1/hr}\left(\int_{B_\rho}|u|^{\gamma+1}dx\right)^{(h-1)/hr}\right]^{(1-s)(\gamma+1)}. \tag{64}$$

Let $\eta = \frac{s(\gamma+1)}{p} + \frac{(1-s)(\gamma+1)}{hr} < 1$, $\pi = \frac{(q-1)(1-s)(\gamma+1)}{hr}$. Then $\eta + \pi \ge 1$ and we have

$$\begin{aligned} |j_2| &\le C\left|\int_t^T d\tau \int_{\partial B_{\rho(\tau)}}|u|^{\gamma+1}d\Gamma\right| \\ &\le Cb^\pi\left(\int_t^T (E_* + C_*)^\eta |\mathbf{n}_\tau| d\tau\right) \\ &\le L\,(E+C+b)\,b^\kappa\left(\int_t^T \left(C^{s(\gamma+1)/\tilde{\theta}p}\right)^\varepsilon d\tau\right)^{1/\varepsilon}, \end{aligned} \tag{65}$$

for some constants L and C and exponents $\kappa := \eta + \pi - 1$ and $\varepsilon = 1/(1-\eta)$. Then we have

$$K\left(\int_{P\cap\{t=T\}}|u|^{1+\gamma}dx + E + C\right) \le i_1 + i_2 + i_3 \tag{66}$$

for different positive constants K. Now, assuming $T-t$ and $D(u)$ so small that

$$Lb^\kappa\left(\int_t^T\left(K^{s(\gamma+1)/\tilde{\theta}p}\right)^\varepsilon d\tau\right)^{1/\varepsilon} < \frac{K}{2},$$

we arrive to the inequality

$$E + C + b \le L_1\sigma(t)D(u)^{(q-1)(1-\tilde{\theta})/qr-\lambda} \times (E+C+b)^{1-\omega+\lambda}\left(-\frac{d(E+C)}{dt}\right)^{(p-1)/p}, \tag{67}$$

whence the desired differential inequality for the energy function $Y(t) := E + C$

$$Y^{(\omega-\lambda)p/(p-1)}(t) \le c(t)\,(-Y(t))', \tag{68}$$

where

$$c(t) = \left(L_1\,(D(u))^{(q-1)(1-\tilde{\theta})/qr-\lambda}\,\sigma(t)\right)^{p/(p-1)}, \quad L_1 = const > 0.$$

Notice that $c(t) \to 0$ as $t \to T$. Moreover, the exponent $(\omega - \lambda)\frac{p}{p-1}$ belongs to the interval $(0,1)$ which leads to the result (see the study of this ordinary differential inequality in [2]).

Case 2. Assume $\lambda > 0$ and (55) then

$$j_3 = \lambda \int_P |u|^{q+1} dx d\theta \leq \lambda(T-t) \|u\|^{q-\gamma}_{L^\infty(P(t))} b(P(t))$$

and it suffices to take t^* such that, in addition, $\lambda(T-t^*) \|u\|^{q-\gamma}_{L^\infty(P(t^*))} < 1$ (notice that $\|u\|^{q-\gamma}_{L^\infty(P(t^*))} \searrow 0$ when $t^* \nearrow T$) and then to balance j_4 with the left hand side terms of inequality (67). The case of (56) is similar since

$$j_3 = \lambda \int_P |u|^{q+1} dx d\theta \leq \lambda \|u\|^{q+1-\alpha}_{L^\infty(P)} C(P(t))$$

and we can balance j_3 with the left hand side terms of inequality (67). ∎

Remark 4. The assumption (54) is, in some sense, optimal. Indeed, it is clear that any solution u_∞ of the stationary problem associated to a global formulation, as for instance (1) with $g = \varphi = 0$, is a solution of the parabolic problem for $u_0 = u_\infty$. In the special case of $N = 1$ it is possible to obtain the exact multiplicity diagram (see [11]) showing that the part of the branch of (stable solutions) corresponding to the maximal solution $\overline{u}_\infty$ is strictly positive (for any $\lambda > \lambda_0$ for some $\lambda_0 > 0$). Nevertheless, for λ large enough the part of the branch corresponding to the minimal solution $\underline{u}_\infty$ satisfies that $\lambda \|\underline{u}_\infty\|^{q+1-\alpha}_{L^\infty(\Omega)}$ is small and $\underline{u}_\infty$ vanishes locally near the boundary of Ω. See also, in this context, the nonuniqueness results mentioned in the paper [30].

Remark 5. Notice that assumptions (55) and (56) are perfectly compatible with the existence of a global blow-up time T_∞ (satisfying, obviously that $T_\infty \geq t^*$). This is the case of equation (9) when $\widehat{q} > \max(p, 2)$ (see [16]).

Remark 6. Assumptions (54), (55) and (56) are also perfectly compatible with possible initial datum outside the natural energy space when some $L^1 - L^\infty$ regularizing effects holds (see [5] and [6]).

Remark 7. Theorem 4.2 can be extended, under suitable modifications, to the case in which $g(x,t) \neq 0$. This is the case, for instance of the associated obstacle problem (see [9] for the application of this local energy method to a similar class of obstacle problems).

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