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ON THE FREE BOUNDARY ASSOCIATED WITH THE STATIONARY MONGE–AMPÈRE OPERATOR ON THE SET OF NON STRICTLY CONVEX FUNCTIONS

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Abstract. This paper deals with several qualitative properties of solutions of some stationary equations associated to the Monge–Ampère operator on the set of convex functions which are not necessarily understood in a strict sense. Mainly, we focus our attention on the occurrence of a free boundary (separating the region where the solution u is locally a hyperplane, thus, the Hessian D^2u is vanishing from the rest of the domain). In particular, our results apply to suitable formulations of the Gauss curvature flow and of the worn stones problems intensively studied in the literature.

1. **Introduction.** It is well known that Geometry has been an extremely rich source of interesting problems in partial differential equations since the pioneering works by Gaspard Monge, Comte de Peluse, (1746-1818) and André–Marie Ampère (1775-1836), among others (see, *e.g.* [31] and [5]).

Here we shall concentrate our attention on several second order partial differential equations involving the Hessian determinant (the Monge-Ampère operator) of the scalar unknown function u. Several concrete problems can be mentioned as source of the motivation for this paper. For instance, we can mention the series of works by L. Nirenberg and coauthors (see *e.g.* Nirenberg [32]) on some geometric problems, as isometric embedding, whose most familiar one is the classical Minkowski problem, in which the Monge–Ampère equation arises in presence of a nonlinear perturbation term on the unknown u. Nevertheless, today it is well-known that the Monge–Ampère operator has many applications, not only in Geometry, but also in applied areas: optimal transportation, optimal design of antenna arrays, vision, statistical mechanics, front formation in meteorology, financial mathematics (see *e.g.* the references [4, 24, 38], mainly for optimal transportation). In fact, we shall formulate the parabolic and elliptic problems of this paper in connection with a special problem which attracted the attention of many authors since 1974: the shape of worn stones.

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It was shown by Firey ([23]), that the idealized wearing process for a convex stone, isotropic with respect to wear, can be described by

$$\frac{\partial \mathbf{P}}{\partial t} = \mathrm{K}^{\mathrm{p}}\mathbf{n}$$

where the points $\mathbf{P}$ of the N-dimensional convex hyper-surface $\Sigma^{\mathrm{N}}(t)$ embedded in $\mathbb{R}^{\mathrm{N}+1}$ (in the physical case $\mathrm{N}=3$) moves under Gauss curvature flow K with exponent $\mathrm{p}>0$ in the inward direction $\mathbf{n}$ to the surface with velocity equal to the p–power of its Gaussian curvature (see also the important paper [29]). In the special case in which we express locally the surface $\Sigma^{\mathrm{N}}(t)$ as a graph $x_{\mathrm{N}+1}=u(x,t)$, with $x\in\Omega$, a convex open set of $\mathbb{R}^{\mathrm{N}}$, the function u satisfies the parabolic Monge–Ampère equation

$$u_t = \frac{\big(\det \mathrm{D}^2 u\big)^{\mathrm{p}}}{\big(1+|\mathrm{D}u|^2\big)^{\frac{(\mathrm{N}+2)\mathrm{p}-1}{2}}}.$$

Since the exact form of the above denominator will not be relevant (once we assume some adequate conditions), our global formulation will be a Cauchy problem

$$\begin{cases} u_t + \mathcal{A}u = 0 \quad t>0,\\ u(0)=u_0, \end{cases}$$

over the Banach space $\mathbb{X}=\mathcal{C}(\overline{\Omega})$ equipped with the supremum norm. A suitable definition of the operator $\mathcal{A}$, at least formally, is given by

$$\mathcal{A}u = -\frac{\big(\det \mathrm{D}^2 u\big)^{\mathrm{p}}}{g(|\mathrm{D}u|)},$$

where $u\in\mathcal{C}^2$ is a locally convex function on $\overline{\Omega}$ and $u=\varphi$ on the boundary $\partial\Omega$. Here Ω is a bounded open set of $\mathbb{R}^{\mathrm{N}}$, φ a continuous function on $\partial\Omega$ and u_0 a locally convex function on Ω. A coefficient $\mathrm{p}>0$ and a continuous function $g\in\mathcal{C}([0,+\infty))$ are part of the operator $\mathcal{A}$ with

$$g(s)\geq 1 \text{ for any } s\geq 0. \tag{1}$$

It can be proved (see [20]) that the operator $\mathcal{A}$ is accretive and satisfies $R(\mathrm{I}+\varepsilon\mathcal{A})\supset\overline{\mathrm{D}(\mathcal{A})}$ for any $\varepsilon>0$. Then the Cauchy problem is solved thanks to the semigroup theory for accretive operators $\mathcal{A}$ by applying the Crandall–Liggett generation theorem (see *e.g.* [14]) for which the so called *mild solution* u of the above Cauchy problem is found by solving the implicit Euler scheme

$$\frac{u_n-u_{n-1}}{\varepsilon}+\mathcal{A}u_n=0, \quad \text{for } n\in\mathbb{N},$$

or

$$\det \mathrm{D}^2 u_n = \left(g\big(|\mathrm{D}u_n|\big)\frac{u_n-u_{n-1}}{\varepsilon}\right)^{\frac{1}{\mathrm{p}}} \quad \text{in } \Omega. \tag{2}$$

This is why among the many different formulations of elliptic problems to which we can apply our techniques we pay special attention to the following stationary problem: with the above assumptions on Ω, φ, p and g, find a convex function u satisfying, in some sense to be defined, the problem

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big)\left[\big(u-h\big)^{\frac{1}{\mathrm{p}}}\right]_+ & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega, \end{cases}$$

where $h = h(x)$ is a given continuous function on $\overline{\Omega}$. Certainly if we want to return to (2) we must replace $g\big(|\mathrm{D}u|\big)$ by $\big(g\big(|\mathrm{D}u|\big)\big)^{\frac{1}{\mathrm{p}}}$. Since the Monge–Ampère operator is only elliptic on the set of symmetric definite positive matrices, some compatibility is required on the structure of the equation. In fact, the operator is degenerate elliptic on the symmetric definite nonnegative matrices (see the comments at the end of this Introduction). As it will be proved in Theorem 2.5 (see also Remark 3), the compatibility is based on

$$h \text{ is locally convex on } \overline{\Omega} \text{ and } h \leq \varphi \text{ on } \partial\Omega. \tag{3}$$

We also emphasize that if $\mathrm{Np} \leq 1$ and $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point $x_0 \in \Omega$ then the problem (20) is elliptic non-degenerate in path-connected open sets Ω, as it is deduced from our Corollary 2.

The paper is organized as follows. In Section 2, after recalling the notion of solution, we shall obtain some weak maximum principles for the boundary value problem. Section 3 deals with the study of flat regions: we give some sufficient conditions for its occurrence as well as some estimates on its location. The consideration of unflat solutions is carried out in Section 4. The results can be considered, in some sense, as necessary conditions for the existence of flat solutions in terms of the zero order term of the equation. Now, with some more details, let us comment that the main consequence of the Weak Maximum Principle is the comparison result for which one deduces $h \leq u$ on $\overline{\Omega}$, provided (3) holds, i.e., h behaves as a kind of lower "obstacle" for the solution u (see Remark 3 below). Therefore, under (3) the problem becomes

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big)\big(u-h\big)^{\mathrm{q}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where the usual restriction on the non negativity of the right hand side is here supplied by (3). To simplify the notation we use $\mathrm{q} = \dfrac{1}{\mathrm{p}}$. In particular, the inequalities

$$u_0 \leq \ldots \leq u_{n-1} \leq u_n \leq \ldots \leq u \quad \text{on } \overline{\Omega} \tag{5}$$

hold for the iterative scheme (2). We emphasize that since the right hand side of the equation needs not to be strictly positive in some region of Ω, the ellipticity of the Monge–Ampère operator and the regularity $\mathcal{C}^2$ of solutions cannot be "a priori" guaranteed. The so-called "viscosity solutions" or the "generalized solutions" are adequate notions in order to weaken the non-degeneracy hypothesis on the operator. In fact, it is shown in [28] for convex domains Ω that both notions coincide. By using the Weak Maximum Principle and well known methods we prove, in Theorem 2.5, the existence of a unique generalized solution of (4) or more generally of problem (20) where the nonlinear expression $\big(u-h\big)^{\mathrm{q}}$ is replaced by $f(u-h)$ being

$$f \in \mathcal{C}(\mathbb{R}) \quad \text{an increasing function satisfying } f(0) = 0. \tag{6}$$

By a simple reasoning we obtain estimates on the gradient $\mathrm{D}u$. Bounds for the second derivatives $\mathrm{D}^2 u$ can be deduced from (22) as we shall prove in [19] (see Remark 3).

Since $h \leq u$ holds on $\overline{\Omega}$, the junction $\mathcal{F}$ between the regions where $[u = h]$ and $[h < u]$ is a free boundary (it is not known a priori). This free boundary can be defined also as the boundary of the set of points $x \in \Omega$ for which $\det \mathrm{D}^2 u(x) > 0$. Obviously, since the interior of the regions $[u = h]$ and $[\det \mathrm{D}^2 u = 0]$ coincide, we must have $\mathrm{D}^2 h = \mathbf{0}$ in these interior region. Motivated by the applications, as well as by the structure of the equation, the occurrence and localization of a free boundary is studied in Section 3 whenever $h(x)$ has flat regions

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \{x \in \Omega : h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha,\ \mathbf{p}_\alpha \in \mathbb{R}^{\mathrm{N}},\ a_\alpha \in \mathbb{R}\} \neq \emptyset,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^{\mathrm{N}}$. As it will be proved, the free boundary $\mathcal{F}$ appears under two different types of conditions on the data: a suitable behavior of zeroth order term ($\mathrm{N} > \mathrm{q}$) and a suitable balance between the "size" of the regions of Ω where $h(x)$ is flat and the "size" of the data φ and h. For this last reason, we rewrite the equation making rise a positive parameter λ,

$$\det \mathrm{D}^2 u = \lambda g\big(|\mathrm{D}u|\big) f\big(u - h\big) \quad \text{in } \Omega. \tag{7}$$

We shall show here how the theory on free boundaries (essentially the boundary of the support of the solution u), developed for a class of quasilinear operators in divergence form, can be extended to the case of the solution of (7) inside of flat regions of h, where $u_h = u - h$ solves

$$\det \mathrm{D}^2 u_h = \lambda g\big(|\mathrm{D}u|\big) f(u_h).$$

We refer the reader to the exposition made in the monograph [21] for details and examples (we mention here the more recent monograph [33] and the paper [16] for the case of other fully nonlinear operators among many other references on this topic in the literature).

As it was suggested in [21] for the Monge–Ampère operator and $f_{\mathrm{q}}(t) = t^{\mathrm{q}}$, the appearance of the free boundary is strongly based on the condition

$$\mathrm{q} < \mathrm{N}. \tag{8}$$

Assumption (8) corresponds to the power-like choice of the more general condition

$$\int_{0^+} \big(\mathrm{F}(t)\big)^{-\frac{1}{\mathrm{N}+1}} dt < \infty, \tag{9}$$

where $\mathrm{F}(t) = \displaystyle\int_0^t f(s)ds$, relative to a continuous and increasing function f satisfying $f(0) = 0$ (see [19]). Since the strict convexity must be removed, a critical size of the data is required, the parameter λ governs these kind of magnitude (see (49) below). For instance, it is satisfied if λ is large enough.

In Theorems 3.1 and 3.3 below we prove the occurrence of the free boundary $\mathcal{F}$ and give some estimates on its localization. We also prove that if $h(x)$ growths moderately (in a suitable way) near the region where it ceases to be flat then the free boundary region associated to the flattens of u (*i.e.* the region where $u_h = u - h$ vanishes) may coincide with the boundary of the set where h is flat (see Theorem 3.4 for $f_{\mathrm{q}}(t) = t^{\mathrm{q}},\ \mathrm{q} < \mathrm{N}$). The estimates on the localization of the free boundary are optimal, in the class of nonlinearities $f(s)$ satisfying (9), as it will be proved in [19].

In Section 4, by means of a Strong Maximum Principle for u_h, we prove that the condition

$$\int_{0^+} \frac{dt}{\big(\mathrm{F}(t)\big)^{\frac{1}{\mathrm{N}+1}}} = \infty \qquad (\text{or } \mathrm{N} \leq \mathrm{q} \text{ for } f_{\mathrm{q}}(t) = t^{\mathrm{q}}) \tag{10}$$

is a necessary condition for the non-existence of such free boundary (see Theorem 4.2, Corollary 2 and Remark 12 below). More precisely, we shall prove that under this condition the solution cannot have any flat region. This can be regarded as an extension of [39] to the non divergence case (see also [16], [21] and [33]). As it was pointed out, the condition $\mathrm{N} \leq \mathrm{q}$ implies non-degenerate ellipticity of problem (20) under very simple assumptions, such as $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point $x_0 \in \Omega$ for path-connected open set Ω (see Corollary 2).

After the completion of this work the authors became aware of the paper by Daskalopoulos and Lee [15] in which they consider a problem (classified as an eigenvalue type problem) with several resemblances with our formulation (4), for the case $\mathrm{N} = 2,\ 0 < \mathrm{q} < 2$ and $g \equiv 1$. The main goal is the study the regularity of the solution and so their approach uses different tools.

We end this introduction by pointing out that our methods can be applied to the borderline cases for (9). This will be studied in the future paper [19] in which the Monge–Ampère operator is replaced by other nonlinear operators of the Hessian of the unknown such as the k^{th} elementary symmetric functions

$$\mathbb{S}_k[\lambda(\mathrm{D}^2 u)] = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq \mathrm{N}} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq \mathrm{N}, \tag{11}$$

where $\lambda(\mathrm{D}^2 u) = \big(\lambda_1, \ldots, \lambda_{\mathrm{N}}\big)$ are the eigenvalues of $\mathrm{D}^2 u$. Note that the case $k = 1$ corresponds to the Laplacian operator while it is a fully nonlinear operator for the other choices of k. The case $k = \mathrm{N}$ corresponds to the Monge–Ampère operator. Some other properties for the k^{th} elementary symmetric function (11) will be considered in futures studies by the authors in [17, 18, 19].

2. **On the notion of solutions and the weak maximum principle.** Many previous expositions on the nature of the solutions can be found in the literature, see for instance the survey [36]. Certainly in the class of $\mathcal{C}^2$ convex functions, the Monge–Ampère operator $\det \mathrm{D}^2 u$ is elliptic because the cofactor matrix of $\mathrm{D}^2 u$ is positive definite. So that, as it is proved by several methods in [10, 11, 19, 25, 27, 30, 34, 35, 36, 37], there exists a $\mathcal{C}^2$ convex solution of the general boundary value problems as

$$\left\{ \begin{array}{ll} \det \mathrm{D}^2 u = \mathrm{H}(\mathrm{D}u, u, x), & \text{on } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{array} \right. \tag{12}$$

under suitable assumptions on Ω, $\mathrm{H} > 0$ and φ. A main question arises now both in theory and in applications: what happens if $\mathrm{H} \geq 0$. Certainly, the ellipticity degeneracy occurs and in general the regularity $\mathcal{C}^2$ of solutions cannot be guaranteed. As it was pointed out in the Introduction, the so called "viscosity solutions" or the "generalized solutions" are the adequate notions in our study. In fact, it can be proved that for a convex domain Ω both notions coincide (see [28]). A short description of all that is as follows. By a change of variable we get

$$|\mathrm{D}u(\mathrm{E})| = \int_{\mathrm{E}} \det \mathrm{D}^2 u \, dx = \int_{\mathrm{E}} \mathrm{H}(\mathrm{D}u, u, x) dx, \tag{13}$$

where

$$|\mathrm{D}u(\mathrm{E})| = \mathrm{meas}\{\mathbf{p} \in \mathbb{R}^{\mathrm{N}} : \ \mathbf{p} = \mathrm{D}u(x) \text{ for some } x \in \mathrm{E}\},$$

for any Borel set $\mathrm{E} \subset \Omega$, where the left hand side makes sense merely when $u \in \mathcal{C}^1$ is convex. By the structure of the problem, u must be convex on Ω and consequently u is at least locally Lipschitz. While for locally Lipschitz functions the right hand side of (13) is well defined, slight but careful modifications are needed to give sense to the left hand side. The progress in this direction is achieved thanks to the notion of subgradients of a convex function u: given $\mathbf{p} \in \mathbb{R}^{\mathrm{N}}$, we say

$$\mathbf{p} \in \partial u(x) \quad \text{iff} \quad u(y) \geq u(x) + \langle \mathbf{p}, y - x \rangle \quad \text{for all } y \in \Omega \tag{14}$$

Thus, we can define the Radon measure

$$\mu_u(\mathrm{E}) \doteq |\partial u(\mathrm{E})| = \mathrm{meas}\{\mathbf{p} \in \mathbb{R}^{\mathrm{N}} : \ \mathbf{p} \in \partial u(x) \text{ for some } x \in \mathrm{E}\}. \tag{15}$$

Since the pioneering works by Aleksandrov [1] several authors have contributed to the study of the above measure (see, for instance, [36]). Then we arrive to

Definition 2.1. A convex function u on Ω is a "generalized solution" of (12) if

$$\mu_u(\mathrm{E}) = \int_{\mathrm{E}} \mathrm{H}(\mathrm{D}u, u, x)dx$$

for any Borel set $\mathrm{E} \subset \Omega$.

The continuity on $\overline{\Omega}$ is compatible with the usual realization of the Dirichlet boundary condition. Obviously, the condition $\mathrm{H} \geq 0$ cannot be removed. Certainly, the definition, as well as (15), can be extended to locally convex functions u on Ω, for which u can be constant on some subset of Ω.

This notion of generalized solution is specific of the equations governed by the Monge–Ampère operator, but other notion of solutions are available for other type of fully nonlinear equations with non divergence form. The most usual is the so called "viscosity solution" introduced by M.G. Crandall and P.L. Lions (see the user's guide [13])

Definition 2.2. A convex function u on Ω is a viscosity solution of the inequality

$$\det \mathrm{D}^2 u \geq \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega \qquad \text{(subsolution)}$$

if for every $\mathcal{C}^2$ convex function Φ on Ω for which

$$(u - \Phi)(x_0) \geq (u - \Phi)(x) \quad \text{locally at } x_0 \in \Omega$$

one has

$$\det \mathrm{D}^2\Phi(x_0) \geq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), x_0\big).$$

Analogously, one defines the viscosity solution of the reverse inequality

$$\det \mathrm{D}^2 u \leq \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega \qquad \text{(supersolution)}$$

as a convex function u on Ω such that for every $\mathcal{C}^2$ convex function Φ on Ω for which

$$(u - \Phi)(x_0) \leq (u - \Phi)(x) \quad \text{locally at } x_0 \in \Omega$$

one has

$$\det \mathrm{D}^2\Phi(x_0) \leq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), (x_0)\big).$$

Finally, when both properties hold we arrive to the notion of viscosity solution of

$$\det \mathrm{D}^2 u = \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega.$$

Note that the convexity condition on u and Φ are extra assumptions with respect to the usual notion of viscosity solution (see [13]). This is needed here because the Monge–Ampère operator is only degenerate elliptic on this class of functions. In fact, the convexity on the test function Φ is only required for the correct definition of super solutions in the viscosity sense, because if $u - \Phi$ attains a local maximum at $x_0 \in \Omega$ for a convex function u on Ω and $\Phi \in \mathcal{C}^2(\Omega)$ one deduces

$$\mathrm{D}^2\Phi(x_0) \geq 0$$

(see [28]). One proves the equivalence

u is a generalized solution of (12) if and only if u is a viscosity solution of (12),

provided that Ω is a convex domain and $\mathrm{H} \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times \mathbb{R} \times \Omega)$ (see [28]).

With this intrinsic way of solving (12) one may study some complementary regularity results. In particular, we may get back to the notion of classical solution by means of the following consistence result

Theorem 2.3 ([10])**.** *Let u be a strictly convex generalized solution of (12) in a convex domain $\Omega \subset \mathbb{R}^{\mathrm{N}}$, where $\mathrm{H} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{\mathrm{N}} \times \mathbb{R} \times \Omega)$ is positive. Then $u \in \mathcal{C}^{2,\alpha'}(\Omega) \cap \mathcal{C}^{1,1}(\overline{\Omega})$, for some $\alpha' \in]0,1[$, and u solves (12) in the classical sense.* □

We continue this section with the study of some comparison and existence results for the equation (7). All results of this section apply to the case of general increasing functions $f \in \mathcal{C}(\mathbb{R})$ satisfying $f(0) = 0$

$$\det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) f(u-h) \quad \text{in } \Omega.$$

We begin by showing that the nature of the viscosity solution is intrinsic to the Maximum Principle.

Proposition 1 (Weak Maximum Principle I)**.** *Let $h_1, h_2 \in \mathcal{C}(\overline{\Omega})$. Let $u_2 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be a classical solution of*

$$-\det \mathrm{D}^2 u_2 + g\big(|\mathrm{D}u_2|\big) f(u_2 - h_2) \geq 0 \quad \textit{in } \Omega,$$

and let $u_1 \in \mathcal{C}(\overline{\Omega})$ be a convex viscosity solution of

$$-\det \mathrm{D}^2 u_1 + g\big(|\mathrm{D}u_1|\big) f(u_1 - h_1) \leq 0 \quad \textit{in } \Omega.$$

Then one has

$$(u_1 - u_2)(x) \leq \sup_{\partial\Omega} \big[u_1 - u_2\big]_+ + \sup_{\Omega} \big[h_1 - h_2\big]_+, \quad x \in \overline{\Omega}.$$

Proof. By continuity there exists $x_0 \in \overline{\Omega}$ where $[u_1 - u_2]_+$ achieves the maximum value on $\overline{\Omega}$. We only consider the case $x_0 \in \Omega$ and $[u_1 - u_2]_+(x_0) > 0$, because otherwise the result follows. Then from the application of the definition of viscosity solution for u_1 we can take $\Phi = u_2$ and so we deduce

$$\begin{aligned} 0 &\geq -\det \mathrm{D}^2 u_2(x_0) + g\big(|\mathrm{D}u_2(x_0)|\big) f(u_1(x_0) - h_1(x_0)) \\ &\geq g\big(|\mathrm{D}u_2(x_0)|\big) f\big(u_1(x_0) - h_1(x_0)\big) - g\big(|\mathrm{D}u_2(x_0)|\big) f\big(u_2(x_0) - h_2(x_0)\big). \end{aligned}$$

Then, since f is increasing

$$(u_1 - u_2)(x_0) \leq \big(h_1 - h_2\big)(x_0) \leq \sup_{\partial\Omega} \big[u_1 - u_2\big]_+ + \sup_{\Omega} [h_1 - h_2]_+.$$

□

Remark 1. We note that the monotonicity on the zeroth order terms is the only assumption required on the structure of the equation and that our argument is strongly based on the notion of viscosity solution. An analogous estimate holds by changing the roles of u_1 and u_2 (but then we do not require the $\mathcal{C}^2$ function u_1 to be convex). Note also that we did not assume any convexity condition on the domain Ω. When Ω is convex these results can be extended to the class of the generalized solutions through the mentioned equivalence between such solutions and the viscosity solutions. In [19] we extend Proposition 1 to non decreasing functions f. □

A very simple (and important fact) was used in our precedent arguments: if $u_1 \in \mathcal{C}^2$ and $u_2 - u_1 \in \mathcal{C}^2$ are convex functions on a ball $\mathbf{B}$ then

$$\det \mathrm{D}^2 u_2 \geq \det \mathrm{D}^2 u_1 \quad \text{in } \mathbf{B}.$$

This simple inequality can be extended to the case u_1 and $u_2 - u_1$ convex functions on a ball $\mathbf{B}$, with $u_1 = u_2$ on $\partial\mathbf{B}$, by the "monotonicity formula"

$$\mu_{u_2}(\mathbf{B}) \geq \mu_{u_1}(\mathbf{B}) \tag{16}$$

(see [36]). In this way, the Weak Maximum Principle can be extended to the class of generalized solutions

Theorem 2.4 (Weak Maximum Principle II)**.** *Let $h_1, h_2 \in \mathcal{C}(\overline{\Omega})$. Let $u_1, u_2 \in \mathcal{C}(\overline{\Omega})$ where u_1 is locally convex in Ω. Suppose*

$$-\det \mathrm{D}^2 u_1 + g\big(|\mathrm{D}u_1|\big) f\big(u_1 - h_1\big) \leq -\det \mathrm{D}^2 u_2 + g\big(|\mathrm{D}u_2|\big) f\big(u_2 - h_2\big) \quad \text{in } \Omega \tag{17}$$

in the generalized solutions sense. Then

$$(u_1 - u_2)(x) \leq \sup_{\partial\Omega}[u_1 - u_2]_+ + \sup_{\Omega}[h_1 - h_2]_+, \quad x \in \overline{\Omega}. \tag{18}$$

In particular,

$$|u_1 - u_2|(x) \leq \sup_{\partial\Omega}|u_1 - u_2| + \sup_{\Omega}|h_1 - h_2|, \quad x \in \overline{\Omega}, \tag{19}$$

whenever the equality holds in (17).

Proof. As above, we only consider the case where the maximum of $[u_1 - u_2]_+$ on $\overline{\Omega}$ is achieved at some $x_0 \in \Omega$ with $[u_1 - u_2]_+(x_0) > 0$. Therefore, $\big(u_1 - u_2\big)(x) > 0$ and convex in a ball $\mathbf{B}_{\mathrm{R}}(x_0)$, R small. Let $\Omega^+ = \{u_1 > u_2\} \supseteq \mathbf{B}_{\mathrm{R}}(x_0)$. We construct $\widehat{u}_1(x) = u_1(x) + \gamma\big(|x - x_0|^2 - \mathrm{M}^2\big) - \delta$, where $\mathrm{M} > 0$ is large and $\gamma, \delta > 0$ such that $\widehat{u}_1 < u_1$ on $\partial\Omega^+$ and the set $\Omega^+_{\gamma,\delta} = \{\widehat{u}_1 > u_2\}$ is compactly contained in Ω and contains $\mathbf{B}_\varepsilon(x_0)$ for some ε small. By choosing γ, δ properly, we can assume that the diameter of $\Omega^+_{\gamma,\delta}$ is small so that u_1, and therefore $u_2 = (u_2 - u_1) + u_1$, are convex in it. Then (16) implies

$$\begin{aligned} 0 < (\gamma\varepsilon)^{\mathrm{N}} |\mathbf{B}_1(0)| &\leq \mu_{u_2}\big(\mathbf{B}_\varepsilon(x_0)\big) - \mu_{u_1}\big(\mathbf{B}_\varepsilon(x_0)\big) \\ &\leq \int_{\mathbf{B}_\varepsilon(x_0)} \big[g\big(|\mathrm{D}u_2|\big) f\big(u_2 - h_2\big) - g\big(|\mathrm{D}u_1|\big) f\big(u_1 - h_1\big)\big] dx. \end{aligned}$$

Since $g\big(|\mathrm{D}u_1(x_0)|\big) = g\big(|\mathrm{D}u_2(x_0)|\big) > 0$ (see Remark 2 below), by letting $\varepsilon \to 0$, the Lebesgue differentiation theorem implies

$$0 \leq g\big(|\mathrm{D}u_2(x_0)|\big) f\big(u_2(x_0) - h_2(x_0)\big) - g\big(|\mathrm{D}u_1(x_0)|\big) f\big(u_1(x_0) - h_1(x_0)\big),$$

whence

$$\big(u_1 - u_2\big)(x_0) < \big(h_1 - h_2\big)(x_0) \leq \sup_{\partial\Omega}\big[u_1 - u_2\big]_+ + \sup_{\Omega}\big[h_1 - h_2\big]_+$$

concludes the estimates. □

Remark 2. The above proof requires a simple fact, any convex function ψ in a convex open set $\mathcal{O} \subset \mathbb{R}^{\mathrm{N}}$ achieving a local interior maximum at some $z_0 \in \mathcal{O}$ verifies $\mathrm{D}\psi(z_0) = \mathbf{0}$. Indeed, for any $\mathbf{p} \in \partial\psi(z_0)$ one has

$$\psi(x) \geq \psi(z_0) + \langle \mathbf{p}, x - z_0 \rangle \geq \psi(x) + \langle \mathbf{p}, x - z_0 \rangle \quad \text{with } x \text{ near } z_0,$$

thus

$$0 \geq \langle \mathbf{p}, x - z_0 \rangle.$$

Then if $\tau > 0$ is small enough we may choose $x - z_0 = \tau \mathbf{p} \in \mathcal{O}$ and deduce

$$0 \leq \tau |\mathbf{p}|^2 \leq 0.$$

□

A first consequence of the general theory for (12) and the Weak Maximum Principle is the following existence result

Theorem 2.5. *Let $\varphi \in \mathcal{C}(\partial\Omega)$ and assume the compatibility condition (3). Then there exists a unique locally convex function verifying*

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) f(u-h) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{20}$$

in the generalized sense. In fact, one verifies

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}, \tag{21}$$

where U_φ is the harmonic function in Ω with $\mathrm{U}_\varphi = \varphi$ on $\partial\Omega$.

Proof. First we consider the generalized solution of the problem

$$\begin{cases} -\det \mathrm{D}^2 u + g\big(|\mathrm{D}u|\big) \big[f(u-h)\big]_+ = 0 & \text{in } \Omega. \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Since $\mathrm{H}\big(\mathrm{D}u, u, x\big) = g\big(|\mathrm{D}u|\big) \big[f(u-h)\big]_+ \geq 0$ we can apply well known results in the literature. In particular, from [37], it follows the existence and uniqueness of the solution u. The second point is to note that, by construction, the locally convex function h verifies

$$-\det \mathrm{D}^2 h + g\big(|\mathrm{D}u|\big) \big[f(h-h)\big]_+ \leq 0 \quad \text{in } \Omega.$$

Therefore, by the Weak Maximum Principle and the assumption $h \leq \varphi$ on $\partial\Omega$ we get that

$$h \leq u \quad \text{in } \Omega,$$

whence

$$\big[f(u-h)\big]_+ = f(u-h)$$

concludes that u solves (20). The uniqueness also follows from the Weak Maximum Principle. Finally, since u is locally convex, the arithmetic–geometric mean inequality lead to

$$0 \leq \det \mathrm{D}^2 u \leq \frac{1}{\mathrm{N}} (\Delta u)^{\mathrm{N}} \quad \text{in } \Omega,$$

whence the estimate

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}$$

is completed by the weak maximum principe for harmonic functions. □

Remark 3. i) As it was pointed out in the Introduction, no sign assumption on h is required in Theorem 2.5. The simple structural assumption (3) implies that $h \leq u$ on $\overline{\Omega}$ and therefore the ellipticity, eventually degenerate, of the equation holds. Thus, the ellipticity holds once h behaves as a lower "obstacle" for the solution u. We note that these compatibility conditions are not required a priori in the Weak Maximum Principles because there we are working with functions whose existence is a priori assumed.
ii) Since u is locally convex on $\overline{\Omega}$, we can prove

$$\sup_{\Omega} |\mathrm{D}u| = \sup_{\partial\Omega} |\mathrm{D}u|,$$

(see [19]), then inequality (21) gives a priori bounds on $|\mathrm{D}u|$ on $\overline{\Omega}$, provided $h=\varphi$ on $\partial\Omega$ and $\mathrm{D}h$ is defined on $\partial\Omega$. The second derivative estimate is based on the inequality

$$\sup_{\Omega}|\mathrm{D}^2u|\leq \mathrm{C}\left(1+\sup_{\partial\Omega}|\mathrm{D}^2u|\right) \tag{22}$$

for some constant C independent on u, as it will be proved in [19]. □

In the next section we prove a kind of Strong Maximum Principle which under suitable assumptions will avoid the appearance of the mentioned free boundary.

3. **Flat regions.** In this section we focus the attention to a lower "obstacle" function h locally convex on $\overline{\Omega}$ having some region giving rise to the set

$$\mathrm{Flat}(h)=\bigcup_{\alpha}\mathrm{Flat}_{\alpha}(h)$$

where

$$\mathrm{Flat}_{\alpha}(h)=\{x\in\overline{\Omega}:h(x)=\langle\mathbf{p}_{\alpha},x\rangle+a_{\alpha}\text{ for some }\mathbf{p}_{\alpha}\in\mathbb{R}^{\mathrm{N}}\text{ and }a_{\alpha}\in\mathbb{R}\}. \tag{23}$$

Since

$$u(y)-\langle\mathbf{p}_{\alpha},y\rangle\geq u(x)-\langle\mathbf{p}_{\alpha},x\rangle+\langle\mathbf{p}-\mathbf{p}_{\alpha},y-x\rangle,$$

thus

$$\mathbf{p}\in\partial u(x)\quad\Leftrightarrow\quad\mathbf{p}-\mathbf{p}_{\alpha}\in\partial\left(u(x)-\left(\langle\mathbf{p}_{\alpha},x\rangle+a_{\alpha}\right)\right),$$

the equation (7) becomes

$$\det\mathrm{D}^2u_{\alpha}=\lambda g\big(|\mathrm{D}u|\big)f\big(u_{\alpha}\big),\quad x\in\mathrm{Flat}_{\alpha}(h), \tag{24}$$

for $u_{\alpha}=u-\big(\langle\mathbf{p}_{\alpha},x\rangle+a_{\alpha}\big)$. Remember that $u_{\alpha}\geq 0$ in an open set $\mathcal{O}\subseteq\Omega$, if $u_h\geq 0$ on $\partial\mathcal{O}$. Assumption $g(|\mathbf{p}|)\geq 1$ leads us to study the auxiliar problem

$$\begin{cases}\det\mathrm{D}^2\mathrm{U}=\lambda f(\mathrm{U}) & \text{in }\mathbf{B}_{\mathrm{R}}(0),\\ \mathrm{U}\equiv\mathrm{M}>0 & \text{on }\partial\mathbf{B}_{\mathrm{R}}(0),\end{cases} \tag{25}$$

for any $\mathrm{M}>0$. From the uniqueness of solutions, it follows that U is radially symmetric, because by rotating it we would find another solution. Moreover, by the comparison results U is nonnegative. Therefore, the solution U is governed by a nonnegative radial profile function $\mathrm{U}(x)=\widehat{\mathrm{U}}(|x|)$ for which some straightforward computations leads to

$$\det\mathrm{D}^2\mathrm{U}(x)=\widehat{\mathrm{U}}''(r)\left(\frac{\widehat{\mathrm{U}}'(r)}{r}\right)^{\mathrm{N}-1}=\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\left(\widehat{\mathrm{U}}'(r)\right)^{\mathrm{N}}\right]',\quad r=|x|. \tag{26}$$

Remark 4. For $\mathrm{N}=1$, the problem (25) becomes the semi linear ODE

$$\widehat{\mathrm{U}}''(r)=\lambda f\big(\widehat{\mathrm{U}}\big)$$

whose annulation set was carefully studied in [21]. Notice that for $\mathrm{N}>1$ the radial Monge-Ampère operator is not exactly the radial p-Laplacian operator with $p=\mathrm{N}+1$, although there is a great resemblance among them. □

We start by considering the initial value problem

$$\begin{cases}\dfrac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\mathrm{U}'(r)\big)^{\mathrm{N}}\right]'=\lambda f\big(\mathrm{U}(r)\big),\quad\lambda>0,\\ \mathrm{U}(0)=\mathrm{U}'(0)=0.\end{cases} \tag{27}$$

Obviously, $\mathrm{U}(r) \equiv 0$ is always a solution, but we are interested in the existence of nontrivial and non–negative solutions. Assume for the moment that there exists a pair $(\mathbb{U}, \lambda_{\mathbb{U}})$ formed by an increasing function $\mathbb{U} : [0, \mathrm{R}_{\mathbb{U}}[\to \overline{\mathbb{R}}_+$ and $\lambda_{\mathbb{U}} > 0$ satisfying that

$$\begin{cases} \dfrac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\big(\mathbb{U}'(r)\big)^{\mathrm{N}}\right]' = \lambda_{\mathbb{U}} f\big(\mathbb{U}(r)\big), \quad 0 < r < \mathrm{R}_{\mathbb{U}}, \\ \mathbb{U}(0) = \mathbb{U}'(0) = 0, \end{cases} \tag{28}$$

for some $0 < \mathrm{R}_{\mathbb{U}} \leq \infty$. We shall return to this assumption later.

By rescaling by $\mathrm{C} > 0$, (28) becomes

$$\begin{cases} -\dfrac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\big(\mathbb{U}'(\mathrm{C}r)\big)^{\mathrm{N}}\right]' + \lambda f\big(\mathbb{U}(\mathrm{C}r)\big) = \left[\lambda - \lambda_{\mathbb{U}} \mathrm{C}^{2\mathrm{N}}\right] f\big(\mathbb{U}(\mathrm{C}r)\big), \quad 0 < r < \dfrac{\mathrm{R}_{\mathbb{U}}}{\mathrm{C}} \\ \mathbb{U}(0) = \mathbb{U}'(0) = 0, \end{cases} \tag{29}$$

whence for $\mathrm{C}_{\lambda,\lambda_{\mathbb{U}}} = \left(\dfrac{\lambda}{\lambda_{\mathbb{U}}}\right)^{\frac{1}{2\mathrm{N}}}$ it follows

1. if $\mathrm{C} < \mathrm{C}_{\lambda,\lambda_{\mathbb{U}}}$ the function $\mathbb{U}(\mathrm{C}r)$ is a supersolution of the equation (27),
2. if $\mathrm{C} = \mathrm{C}_{\lambda,\lambda_{\mathbb{U}}}$ the function $\mathbb{U}(\mathrm{C}r)$ is the solution of the equation (27),
3. if $\mathrm{C} > \mathrm{C}_{\lambda,\lambda_{\mathbb{U}}}$ the function $\mathbb{U}(\mathrm{C}r)$ is a subsolution of the equation (27).

Moreover, the function

$$v_\tau(x) \doteq \mathbb{U}\left(\mathrm{C}_{\lambda,\lambda_{\mathbb{U}}}\big([|x| - \tau]_+\big)\right), \quad x \in \mathbf{B}_{\tau + \mathrm{R}_{\mathbb{U},\lambda}}(0), \ \mathrm{R}_{\mathbb{U},\lambda} = \frac{\mathrm{R}_{\mathbb{U}}}{\mathrm{C}_{\lambda,\lambda_{\mathbb{U}}}} \tag{30}$$

solves

$$-\det \mathrm{D}^2 v_\tau(x) + \lambda f\big(v_\tau(x)\big) = 0, \quad x \in \mathbf{B}_{\tau + \mathrm{R}_{\mathbb{U},\lambda}}(0).$$

Furthermore, it verifies

$$v_\tau(x) = \mathrm{M}, \quad |x| = \mathrm{R} < \tau + \mathrm{R}_{\mathbb{U},\lambda}$$

once we take

$$\tau = \mathrm{R} - \left(\frac{\lambda_{\mathbb{U}}}{\lambda}\right)^{\frac{1}{2\mathrm{N}}} \mathbb{U}^{-1}(\mathrm{M}) = \left[\lambda_*^{-\frac{1}{2\mathrm{N}}} - \lambda^{-\frac{1}{2\mathrm{N}}}\right] \mathbb{U}^{-1}(\mathrm{M}) \lambda_{\mathbb{U}}^{\frac{1}{2\mathrm{N}}}$$

with

$$\lambda \geq \lambda_* \doteq \lambda_{\mathbb{U}} \left(\frac{1}{\mathrm{R}} \mathbb{U}^{-1}(\mathrm{M})\right)^{2\mathrm{N}}. \tag{31}$$

Now for the solution of (7) we may localize a core of the flat region $\mathrm{Flat}(u)$ inside the flat subregion $\mathrm{Flat}_\alpha(h)$ of the "obstacle".

Theorem 3.1. *Let h be locally convex on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_{\mathrm{R}}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathrm{M} \leq \max_{\overline{\Omega}} (u - h), \quad x \in \partial \mathbf{B}_{\mathrm{R}}(x_0), \tag{32}$$

where u is a generalized solution of (7), for some $\mathrm{M} > 0$. Then, if (28) holds and

$$\lambda \geq \lambda_* \doteq \lambda_{\mathbb{U}} \left(\frac{1}{\mathrm{R}} \mathbb{U}^{-1}(\mathrm{M})\right)^{2\mathrm{N}},$$

one verifies

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathbb{U}\left(\mathrm{C}_{\lambda,\lambda_{\mathbb{U}}}\big([|x| - \tau]_+\big)\right), \quad x \in \mathbf{B}_{\mathrm{R}}(x_0), \tag{33}$$

where

$$C_{\lambda,\lambda_{\mathbb{U}}} = \left(\frac{\lambda}{\lambda_{\mathbb{U}}}\right)^{\frac{1}{2N}} \quad \textit{and} \quad \tau = \left[\lambda_*^{-\frac{1}{2N}} - \lambda^{-\frac{1}{2N}}\right] \mathbb{U}^{-1}(M)\lambda_{\mathbb{U}}^{\frac{1}{2N}}, \tag{34}$$

once we assume that $R < \tau + R_{\mathbb{U},\lambda}$ *and*

$$\left(\frac{\lambda_{\mathbb{U}}}{\lambda}\right)^{\frac{1}{2N}} \mathbb{U}^{-1}(M) < R \leq \mathrm{dist}(x_0, \partial\Omega). \tag{35}$$

In particular, the function u *is flat on* $\overline{\mathbf{B}}_\tau(x_0)$. *More precisely,*

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$

Proof. The result is a direct consequence of previous arguments. Indeed, for simplicity we can assume $x_0 = 0$. Since $g(|\mathbf{p}|) \geq 1$, by the comparison results we get that

$$0 \leq u_\alpha(x) \leq v_\tau(x), \quad x \in \mathbf{B}_R(0)$$

(see (24) and (30)) and so the conclusions hold. □

Remark 5. We have proved that under the above assumptions the flat region of u is a non–empty set. Obviously, $\mathrm{Flat}(h) \subset \mathrm{Flat}(u)$ whenever (32) fails, even if (28) holds. We shall examine the optimality of (33) in [19] following different strategies carry out in [21] for other free boundary problems. □

Remark 6. We point out that the above result applies to the case in which $\varphi \equiv 1$ and $h \equiv 0$ (the so called "dead core" problem) as well as to cases in which u is flat only near $\partial\Omega$ (take for instance, $h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha$ in Ω and $\varphi \equiv h$ on $\partial\Omega$). □

The equation in (28) is equivalent to

$$\left(\left(\mathbb{U}'(r)\right)^{N+1}\right)' = Nr^{N-1}\lambda_{\mathbb{U}}\left(F\big(\mathbb{U}(r)\big)\right)', \quad 0 < r < R_{\mathbb{U}} \quad F' = f,$$

and

$$\left(\mathbb{U}'(r)\right)^{N+1} = N\lambda_{\mathbb{U}}\left(r^{N-1}F\big(\mathbb{U}(r)\big) - \frac{1}{N-1}\int_0^r s^{N-2}F\big(\mathbb{U}(s)\big)ds\right), \quad 0 < r < R_{\mathbb{U}}.$$

So, we deduce that (28) requires

$$\int_0^{\mathbb{U}(r)} \frac{ds}{\left(F(s)\right)^{\frac{1}{N+1}}} = \int_0^r \frac{\mathbb{U}'(s)ds}{\left(F\big(\mathbb{U}(s)\big)\right)^{\frac{1}{N+1}}} \leq (N\lambda_{\mathbb{U}})^{\frac{1}{N+1}}\frac{N+1}{2N}r^{\frac{2N}{N+1}}, \quad 0 < r < R_{\mathbb{U}}.$$

Therefore (9) is a necessary condition in order to (28) holds. □

The reasoning in proving that (9) is a sufficient condition for the assumption (28) is very technical. Here we only construct a function verifying a similar property useful to our interest

Theorem 3.2. *Assume* (9). *Then the function* $\phi(r)$ *given implicitly by*

$$\int_0^{\phi(r)} \big(F(s)\big)^{-\frac{1}{N+1}} ds = r^{\frac{2N-1}{N}}, \quad 0 \leq r \tag{36}$$

satisfies, for each $\widehat{R} > 0$ *the property*

$$\begin{cases} \dfrac{r^{1-N}}{N}\left[\big(\phi'(r)\big)^N\right]' \leq \lambda_{\phi,\widehat{R}} f\big(\phi(r)\big), & 0 < r < \widehat{R}, \\ \phi(0) = \phi'(0) = 0, \end{cases} \tag{37}$$

where

$$\begin{cases} \widehat{\mathrm{R}} < \displaystyle\int_0^\infty \big(\mathrm{F}(s)\big)^{-\frac{1}{\mathrm{N}+1}} ds \leq +\infty, \\ \lambda_{\phi,\widehat{\mathrm{R}}} = \left(\dfrac{2\mathrm{N}-1}{\mathrm{N}}\right)^{\mathrm{N}+1} \dfrac{\mathrm{N}}{\mathrm{N}+1} \widehat{\mathrm{R}}^{\frac{\mathrm{N}-1}{\mathrm{N}}}. \end{cases} \tag{38}$$

Proof. Since the function

$$\psi(t) = \int_0^t \big(\mathrm{F}(s)\big)^{-\frac{1}{\mathrm{N}+1}} ds, \quad t \geq 0,$$

is increasing from $\overline{\mathbb{R}}_+$ to $[0, \psi(\infty)[$ and $\psi(0) = 0$, we may consider the function given by

$$\int_0^{\phi(r)} \big(\mathrm{F}(s)\big)^{-\frac{1}{\mathrm{N}+1}} ds = r^a, \quad 0 \leq r < \psi(\infty) \leq +\infty,$$

where a is a positive constant to be chosen. Then

$$\phi'(r) = a\left(\mathrm{F}\big(\phi(r)\big)\right)^{\frac{1}{\mathrm{N}+1}} r^{a-1},$$

and

$$\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\phi'(r)\big)^{\mathrm{N}}\right]' = a^{\mathrm{N}} r^{(a-1)\mathrm{N}+1-\mathrm{N}}\left(\frac{a-1}{r}\left(\mathrm{F}\big(\phi(r)\big)\right)^{\frac{\mathrm{N}}{\mathrm{N}+1}} + \frac{a}{\mathrm{N}+1} r^{a-1} f\big(\phi(r)\big)\right).$$

holds. Next, we choose

$$(a-1)\mathrm{N} + 1 - \mathrm{N} = 0 \quad \Leftrightarrow \quad a = \frac{2\mathrm{N}-1}{\mathrm{N}},$$

and $\Phi(r) = \left(\mathrm{F}\big(\phi(r)\big)\right)^{\frac{\mathrm{N}}{\mathrm{N}+1}}$. Since $\Phi(0) = 0$ and

$$\Phi'(r) = \frac{a\mathrm{N}}{\mathrm{N}+1} f\big(\phi(r)\big) r^{\frac{\mathrm{N}-1}{\mathrm{N}}}$$

is increasing, the convexity inequality

$$\Phi(r) \leq \Phi'(r) r$$

gives

$$\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\phi'(r)\big)^{\mathrm{N}}\right]' \leq \left(\frac{2\mathrm{N}-1}{\mathrm{N}}\right)^{\mathrm{N}+1} \frac{\mathrm{N}}{\mathrm{N}+1} r^{\frac{\mathrm{N}-1}{\mathrm{N}}} f\big(\phi(r)\big).$$

Finally, since $a \geq 1$ one has $\phi(0) = \phi'(0) = 0$. □

Remark 7. The above result leads to a stronger statement (as in the paper by Brezis–Nirenberg [9] for a different quasilinear equation): given $\mathrm{R} > 0$ and $\lambda > 0$ there exists a boundary value $\mathrm{M}^* = \mathrm{M}^*(\mathrm{R})$ such that the solution U of (25) verifies $\mathrm{U}(0) = 0$ and $\mathrm{U}(r) > 0$ in $\mathbf{B}_{\mathrm{R}} \setminus \{0\}$. The proof is a simple adaptation of the proof of [9, Lemma 5] by means of an application of Theorem 3.2. □

So that, fixed $\widehat{\mathrm{R}} < \psi(\infty)$ we have

$$\begin{cases} -\dfrac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\phi(\mathrm{C}r)\big)^{\mathrm{N}}\right]' + \lambda f\big(\phi(\mathrm{C}r)\big) \geq \left[\lambda - \lambda_{\phi,\widehat{\mathrm{R}}} \mathrm{C}^{2\mathrm{N}}\right] f\big(\phi(\mathrm{C}r)\big), \quad 0 < r < \widehat{\mathrm{R}} \\ \phi(0) = \phi'(0) = 0, \end{cases} \tag{39}$$

(see (29) becomes), whence for

$$\mathrm{C}_{\lambda,\lambda_{\phi,\widehat{\mathrm{R}}}} = \left(\frac{\lambda}{\lambda_{\phi,\widehat{\mathrm{R}}}}\right)^{\frac{1}{2\mathrm{N}}},$$

the function

$$v_\tau(x) \doteq \phi\left(\mathrm{C}_{\lambda,\lambda_{\phi,\widehat{\mathrm{R}}}}(([|x|-\tau]_+)\right), \quad x \in \mathbf{B}_{\tau+\mathrm{R}_{\phi,\lambda}}(0),\ \mathrm{R}_{\phi,\lambda,\widehat{\mathrm{R}}} = \frac{\widehat{\mathrm{R}}}{\mathrm{C}_{\lambda,\lambda_{\phi,\mathrm{R}}}} \tag{40}$$

solves

$$-\det \mathrm{D}^2 v_\tau(x) + \lambda f\big(v_\tau(x)\big) \geq 0, \quad x \in \mathbf{B}_{\tau+\mathrm{R}_{\phi,\lambda,\widehat{\mathrm{R}}}}(0).$$

The reasonings of Theorem 3.1 apply and enable us to localize again a core of the flat region $\mathrm{Flat}(u)$ by

Corollary 1. *Let h be locally convex on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_\mathrm{R}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathrm{M} \leq \max_{\overline{\Omega}}(u-h), \quad x \in \partial\mathbf{B}_\mathrm{R}(x_0), \tag{41}$$

where u is a generalized solution of (7), for some $\mathrm{M} > 0$. Then, if (9) holds and

$$\lambda \geq \widehat{\lambda}_* \doteq \lambda_{\phi,\widehat{\mathrm{R}}}\left(\frac{1}{\mathrm{R}}\phi^{-1}(\mathrm{M})\right)^{2\mathrm{N}},$$

one verifies

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \phi\left(\mathrm{C}_{\lambda,\lambda_{\phi,\widehat{\mathrm{R}}}}\big([|x|-\tau]_+\big)\right), \quad x \in \mathbf{B}_\mathrm{R}(x_0), \tag{42}$$

where

$$\mathrm{C}_{\lambda,\lambda_{\phi,\widehat{\mathrm{R}}}} = \left(\frac{\lambda}{\lambda_{\phi,\widehat{\mathrm{R}}}}\right)^{\frac{1}{2\mathrm{N}}} \quad \text{and} \quad \tau = \left[\widehat{\lambda}_*^{-\frac{1}{2\mathrm{N}}} - \lambda^{-\frac{1}{2\mathrm{N}}}\right]\phi^{-1}(\mathrm{M})\lambda_{\phi,\widehat{\mathrm{R}}}^{\frac{1}{2\mathrm{N}}}, \tag{43}$$

once we assume that $\mathrm{R} < \tau + \mathrm{R}_{\phi,\lambda,\widehat{\mathrm{R}}}$ and

$$\left(\frac{\lambda_{\phi,\widehat{\mathrm{R}}}}{\lambda}\right)^{\frac{1}{2\mathrm{N}}}\phi^{-1}(\mathrm{M}) < \mathrm{R} \leq \mathrm{dist}(x_0,\partial\Omega). \tag{44}$$

In particular, the function u is flat on $\overline{\mathbf{B}}_\tau(x_0)$. More precisely,

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \text{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$ □

Remark 8. Corollary 1 is the relative version of Theorem 3.1. Consequently, the comments of Remarks 5 and 6 apply. □

In the particular case $f_\mathrm{q}(t) = t^\mathrm{q}$, the condition (9) holds if and only if $\mathrm{N} > \mathrm{q}$. Moreover, the assumption (28) is verified for

$$\mathbb{U}_\mathrm{q}(r) = r^{\frac{2\mathrm{N}}{\mathrm{N}-\mathrm{q}}}, \quad \lambda_\mathrm{q} = \frac{(2\mathrm{N})^\mathrm{N}(\mathrm{N}+\mathrm{q})}{(\mathrm{N}-\mathrm{q})^{\mathrm{N}+1}}, \quad \mathrm{R}_{\lambda_{\mathbb{U}_\mathrm{q}}} = +\infty, \tag{45}$$

consequently all above results apply. If we scale by $\mathrm{C}^{\frac{\mathrm{N}-\mathrm{q}}{2\mathrm{N}}}$ for the function

$$\mathrm{U}(r) = \mathrm{C}\mathbb{U}_\mathrm{q}(r), \quad r \geq 0,$$

the property (29) becomes

$$-\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\left(\mathrm{U}'(r)\right)^{\mathrm{N}}\right]'+\lambda f_{\mathrm{q}}(\mathrm{U}(r))=\lambda\left[1-\frac{\lambda_{\mathrm{q}}}{\lambda}\mathrm{C}^{\mathrm{N-q}}\right]f_{\mathrm{q}}(\mathrm{U}(r)). \tag{46}$$

Now,

1. if $\mathrm{C}<\left(\frac{\lambda}{\lambda_{\mathrm{q}}}\right)^{\frac{1}{\mathrm{N-q}}}$ the function $\mathrm{U}(r)$ is a supersolution of equation (46),
2. if $\mathrm{C}=\left(\frac{\lambda}{\lambda_{\mathrm{q}}}\right)^{\frac{1}{\mathrm{N-q}}}$ the function $\mathrm{U}(r)$ is the solution of equation (46),
3. if $\mathrm{C}>\left(\frac{\lambda}{\lambda_{\mathrm{q}}}\right)^{\frac{1}{\mathrm{N-q}}}$ the function $\mathrm{U}(r)$ is a subsolution of equation (46).

So that, the particular choice

$$\mathrm{U}(r)=\left(\frac{\lambda}{\lambda_{\mathrm{q}}}\right)^{\frac{1}{\mathrm{N-q}}}\mathbb{U}_{\mathrm{q}}(r),\quad r\geq 0, \tag{47}$$

enables us to construct the function

$$v_\tau(x)\doteq\mathrm{U}\big([|x|-\tau]_+\big),\quad x\in\mathbb{R}^{\mathrm{N}}, \tag{48}$$

vanishing in a ball $\mathbf{B}_\tau(0)$ and solving

$$-\det\mathrm{D}^2v_\tau(x)+\lambda f_{\mathrm{q}}\big(v_\tau(x)\big)=0,\quad x\in\mathbb{R}^{\mathrm{N}}.$$

Moreover, given $\mathrm{M}>0$, it verifies

$$v_\tau(x)=\mathrm{M},\quad |x|=\mathrm{R}$$

once we take

$$\tau=\mathrm{R}-\mathrm{U}^{-1}(\mathrm{M})=\lambda_{\mathrm{q}}^{\frac{1}{2\mathrm{N}}}\mathrm{M}^{\frac{\mathrm{N-q}}{2\mathrm{N}}}\left[\widetilde{\lambda}_*^{-\frac{1}{2\mathrm{N}}}-\lambda^{-\frac{1}{2\mathrm{N}}}\right]$$

with

$$\lambda\geq\widetilde{\lambda}_*\doteq\frac{\lambda_{\mathrm{q}}\mathrm{M}^{\mathrm{N-q}}}{\mathrm{R}^{2\mathrm{N}}}. \tag{49}$$

The localization of a core of the flat region $\mathrm{Flat}(u)$ inside the flat subregion $\mathrm{Flat}_\alpha(h)$ of the "obstacle" is estimated by

Theorem 3.3. *Let $f_{\mathrm{q}}(t)=t^{\mathrm{q}}$, $\mathrm{q}<\mathrm{N}$. Let h be locally convex on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_{\mathrm{R}}(x_0)\subset\mathrm{Flat}_\alpha(h)$ with*

$$0\leq u(x)-\big(\langle\mathbf{p}_\alpha,x\rangle+a_\alpha\big)\leq\mathrm{M}\leq\max_{\overline{\Omega}}(u-h),\quad x\in\partial\mathbf{B}_{\mathrm{R}}(x_0), \tag{50}$$

where u is a generalized solution of (7), for some $\mathrm{M}>0$. Then, if $\mathrm{Np}>1$ and $\lambda\geq\widetilde{\lambda}^*$ one verifies

$$0\leq u(x)-\big(\langle\mathbf{p}_\alpha,x\rangle+a_\alpha\big)\leq\lambda^{\frac{1}{\mathrm{N-q}}}\mathrm{C}_{\mathrm{q,N}}\Big(\big[|x-x_0|-\tau\big]_+\Big)^{\frac{2\mathrm{N}}{\mathrm{N-q}}},\quad x\in\mathbf{B}_{\mathrm{R}}(x_0), \tag{51}$$

where

$$\tau=\lambda_{\mathrm{q}}^{\frac{1}{2\mathrm{N}}}\mathrm{M}^{\frac{\mathrm{N-q}}{2\mathrm{N}}}\left[\widetilde{\lambda}_*^{-\frac{1}{2\mathrm{N}}}-\lambda^{-\frac{1}{2\mathrm{N}}}\right], \tag{52}$$

once we assume that

$$\left(\frac{\lambda_{\mathrm{q}}}{\lambda}\right)^{\frac{1}{2\mathrm{N}}}\mathrm{M}^{\frac{\mathrm{N-q}}{2\mathrm{N}}}\lambda^{-\frac{1}{2\mathrm{N}}}<\mathrm{R}\leq\mathrm{dist}(x_0,\partial\Omega). \tag{53}$$

In particular, the function u is flat on $\overline{\mathbf{B}}_\tau(x_0)$. More precisely,

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0). \qquad \square$$

Remark 9. Theorem 3.3 is a new version of Theorem 3.1. Therefore, once more the comments of Remarks 5 and 6 apply also to this power like case $f_{\rm q}(t) = t^{\rm q}$, $\rm N >$ q. □

Theorem 3.3 gives some estimates on the localization of the points inside $\operatorname{Flat}(h)$ where u becomes flat too. The following result shows that if h decays in a suitable way at the boundary points of $\operatorname{Flat}(h)$ then the solution u becomes also flat in those points of the boundary of $\operatorname{Flat}(h)$. In this result the parameter λ is irrelevant, therefore with no loss of generality we shall assume that $\lambda = 1$.

Theorem 3.4. *Let us assume* $\rm N > q$. *Let* $x_0 \in \partial \operatorname{Flat}_\alpha(h)$ *such that*

$$h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le {\rm K}|x - x_0|^{\frac{2{\rm N}}{{\rm N}-{\rm q}}}, \quad x \in \mathbf{B}_{\rm R}(x_0) \cap \big(\mathbb{R}^{\rm N} \setminus \operatorname{Flat}(h)\big), \tag{54}$$

and

$$0 \le \max_{|x-x_0|={\rm R}} \big\{u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big\} \le {\rm CR}^{\frac{2{\rm N}}{{\rm N}-{\rm q}}} \tag{55}$$

for some suitable positive constants K *and* C (*see* (57) *below*) *and* u *is a generalized solution of* (7). *Then*

$$u(x_0) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha. \tag{56}$$

Proof. Define the function

$$\mathrm{V}(x) = u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big),$$

which by construction is nonnegative in $\partial \mathbf{B}_{\rm R}(x_0)$ (see (55)). In fact, the Weak Maximum Principle implies that V is non negative on $\overline{\mathbf{B}}_{\rm R}(x_0)$. Then

$$\begin{aligned}
&-\big(\det {\rm D}^2 {\rm V}(x)\big)^{\frac{1}{\rm N}} + \big(f_{\rm q}({\rm V}(x))\big)^{\frac{1}{\rm N}} \\
&= -\big(\det {\rm D}^2 u(x)\big)^{\frac{1}{\rm N}} + \Big(f_{\rm q}\left(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\right)\Big)^{\frac{1}{\rm N}} \\
&= -\big(f_{\rm q}(u(x) - h(x))\big)^{\frac{1}{\rm N}} + \Big(f_{\rm q}\left(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\right)\Big)^{\frac{1}{\rm N}} \\
&\le \Big(h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\Big)^{\frac{\rm q}{\rm N}} \\
&\le {\rm K}^{\frac{\rm q}{\rm N}} |x - x_0|^{\frac{2{\rm q}}{{\rm N}-{\rm q}}}, \quad x \in \mathbf{B}_{\rm R}(x_0),
\end{aligned}$$

where we have used a kind of Minkovski inequality

$$(a+b)^{\frac{1}{\rm p}} \le a^{\frac{1}{\rm p}} + b^{\frac{1}{\rm p}}, \; a, b \ge 0, \quad \text{where } {\rm p} > 1,$$

for $\rm p = \frac{N}{q} > 1$, as well as (54). On the other hand, from (45) we have

$$\left(\frac{r^{1-{\rm N}}}{\rm N}\left[\big(\mathbb{U}_{\rm q}'(r)\big)^{\rm N}\right]'\right)^{\frac{1}{\rm N}} = \lambda_{\mathbb{U}_{\rm q}}^{\frac{1}{\rm N}} \big(f_{\rm q}(\mathbb{U}_{\rm q}(r))\big)^{\frac{1}{\rm N}}, \quad 0 < r < {\rm R}_{\lambda_{\mathbb{U}_{\rm q}}},$$

for

$$\mathbb{U}_{\rm q}(r) = r^{\frac{2{\rm N}}{{\rm N}-{\rm q}}}, \quad \lambda_{\rm q} = \frac{(2{\rm N})^{\rm N}({\rm N}+{\rm q})}{({\rm N}-{\rm q})^{{\rm N}+1}} \quad {\rm R}_{\lambda_{\mathbb{U}_{\rm q}}} = +\infty.$$

Then $\mathrm{U}(r) = \mathrm{C}\mathbb{U}_{\rm q}(r)$ verifies

$$-\left(\frac{r^{1-{\rm N}}}{\rm N}\left[\big({\rm U}'(r)\big)^{\rm N}\right]'\right)^{\frac{1}{\rm N}} + \big(f_{\rm q}({\rm U}(r))\big)^{\frac{1}{\rm N}} = \left[1 - \lambda_{\rm q}{\rm C}^{{\rm N}-{\rm q}}\right] \big(f_{\rm q}({\rm U}(r))\big)^{\frac{1}{\rm N}}.$$

Hence, if we take $\mathrm{C} < \lambda_{\mathrm{q}}^{-\frac{1}{\mathrm{N}-\mathrm{q}}}$ and then K such that

$$\mathrm{K}^{\frac{\mathrm{q}}{\mathrm{N}}} \leq \mathrm{C}^{\frac{\mathrm{q}}{\mathrm{N}}}\left[1-\lambda_{\mathrm{q}}\mathrm{C}^{\mathrm{N}-\mathrm{q}}\right] \tag{57}$$

we obtain

$$-\big(\det \mathrm{D}^2\mathrm{V}(x)\big)^{\frac{1}{\mathrm{N}}}+\big(f_{\mathrm{q}}\big(\mathrm{V}(x)\big)\big)^{\frac{1}{\mathrm{N}}} \leq -\big(\det \mathrm{D}^2\mathrm{U}(|x|)\big)^{\frac{1}{\mathrm{N}}}+\big(f_{\mathrm{q}}\big(\mathrm{U}(|x|)\big)\big)^{\frac{1}{\mathrm{N}}}, x \in \mathbf{B}_{\mathrm{R}}(x_0).$$

Finally, by choosing R satisfying (55) one has

$$\mathrm{V}(x) \leq \mathrm{U}(|x|), \quad x \in \partial\mathbf{B}_{\mathrm{R}}(x_0),$$

whence the comparison principle concludes

$$0 \leq \mathrm{V}(x) \leq \mathrm{C}|x-x_0|^{\frac{2\mathrm{N}}{\mathrm{N}-\mathrm{q}}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0),$$

and so $u(x_0) = \big(\langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha\big)$. □

Remark 10. The assumption (55) is satisfied if we know that the ball $\mathbf{B}_{\mathrm{R}}(x_0)$ where (54) holds is assumed large enough. The above result is motivated by [21, Theorem 2.5]. By adapting the reasoning used in previous results of the literature (see [2, 3, 22]) it can be shown that the decay of $h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$ near the boundary point x_0 is optimal in the sense that if

$$h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) > \mathrm{C}|x-x_0|^{\frac{2\mathrm{N}}{\mathrm{N}-\mathrm{q}}} \quad \text{on a neighbourhood of } x_0$$

then it can be shown that

$$u(x_0) - \big(\langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha\big)\big) > \mathrm{C}|x-x_0|^{\frac{2\mathrm{N}}{\mathrm{N}-\mathrm{q}}} \quad \text{for } x \text{ near } x_0.$$

This type of results gives very rich information on the non–degeneracy behavior of the solution near the free boundary. This is very useful to the study of the continuous dependence of the free boundary with respect to the data h and φ (see [22]). □

4. **Unflat solutions.** Now we examine the case in which the solution cannot be flat (*i.e.* the free boundary cannot appear) independent on "size" of Ω, obviously it requires the condition

$$\mathrm{q} \geq \mathrm{N}$$

or the more general assumption (10). This will be proved by us by proving a version of the Strong Maximum Principle. We shall follow the classical reasoning by E. Hopf (see *e.g.* [25]). Again, since the parameter λ is again irrelevant in this section, with no loss of generality, we assume $\lambda = 1$. So, we begin with

Lemma 4.1 (Hopf boundary point lemma)**.** *Assume* (10)*. Let u be a nonnegative viscosity solution of*

$$-\det \mathrm{D}^2 u + f(u) \geq 0 \quad in\ \Omega.$$

Let $x_0 \in \partial\Omega$ be such that $u(x_0) \doteq \liminf_{\substack{x\to x_0\\ x\in\Omega}} u(x)$ and

$$\begin{cases} i) & u \text{ achieves a strict minimum on } \Omega \cup \{x_0\}, \\ ii) & \exists\ \mathbf{B}_{\mathrm{R}}(x_0 - \mathrm{R}\mathbf{n}(x_0)) \subset \Omega \ (\partial\Omega \text{ satisfies an interior sphere condition at } x_0). \end{cases}$$

Then there exists a positive constant C such that

$$\liminf_{\tau\to 0} \frac{u(x_0 - \tau\mathbf{n})}{\tau} \geq C, \tag{58}$$

where $\mathbf{n}$ stands for the outer normal unit vector of $\partial\Omega$ at x_0.

Proof. Let $y = x_0 - \mathrm{R}\mathbf{n}(x_0)$ and $\mathbf{B}_\mathrm{R} \doteq \mathbf{B}_\mathrm{R}(y)$. As it was pointed out before, equation (7) leads to the study of the differential equation

$$\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\Phi'(r)\big)^{\mathrm{N}}\right]' = f\big(\Phi(r)\big), \quad r > 0,$$

for radially symmetric solutions. We consider now the classical solution of the two point boundary problem

$$\begin{cases} \dfrac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\Phi'(r)\big)^{\mathrm{N}}\right]' = f\big(\Phi(r)\big), \quad 0 < r < \dfrac{\mathrm{R}}{2}, \\ \Phi(0) = 0, \quad \Phi\left(\dfrac{\mathrm{R}}{2}\right) = \Phi_1 > 0. \end{cases} \tag{59}$$

The existence of solution follows from standard arguments and the uniqueness of solution can be proved as in Theorem 2.4, whence

$$\Phi'(0) \geq 0 \quad \Rightarrow \quad \Phi'(r) > 0 \qquad \Rightarrow \quad \Phi''(r) > 0.$$

Then

$$0 \leq \Phi(r) \leq \Phi_1, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

We note that the singularity at $r = 0$ must be removed by the condition

$$\lim_{r \to 0} \frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\Phi'(r)\big)^{\mathrm{N}}\right]' = 0. \tag{60}$$

Let r_0 be the largest r for which $\Phi(r) = 0$. We want to prove that $r_0 = 0$ by proving that $r_0 > 0$ leads to a contradiction. In order to do that we multiply (59) by $r^{\mathrm{N}-1}\Phi'(r)$ and get

$$\left[\big(\Phi'(r)\big)^{\mathrm{N}+1}\right]' = (\mathrm{N}+1) f\big(\Phi(r)\big)\Phi'(r) r^{\mathrm{N}-1}, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

Next, since $\Phi'(r_0) = 0 = \Phi(r_0)$, an integration between r_0 and r leads to

$$\begin{aligned} \big(\Phi'(r)\big)^{\mathrm{N}+1} &= (\mathrm{N}+1)\mathrm{F}\big(\Phi(r)\big) r^{\mathrm{N}-1} - (\mathrm{N}+1)(\mathrm{N}-1)\int_{r_0}^{r} \mathrm{F}\big(\Phi(s)\big) r^{\mathrm{N}-2} ds \\ &\leq (\mathrm{N}+1)\mathrm{F}\big(\Phi(r)\big) r^{\mathrm{N}-1}, \quad r_0 < r < \frac{\mathrm{R}}{2}. \end{aligned}$$

Because we assume (10), a new integration between r_0 and $\dfrac{\mathrm{R}}{2}$ yields the conjectured contradiction because

$$\infty = \int_0^{\Phi_1} \frac{ds}{\big(\mathrm{F}(s)\big)^{\frac{1}{\mathrm{N}+1}}} = \int_{r_0}^{\frac{\mathrm{R}}{2}} \frac{\Phi'(r)}{\big(\mathrm{F}\big(\Phi(r)\big)\big)^{\frac{1}{\mathrm{N}+1}}} dr \leq (\mathrm{N}+1)^{\frac{1}{\mathrm{N}+1}} \int_{r_0}^{\frac{\mathrm{R}}{2}} r^{\frac{\mathrm{N}-1}{\mathrm{N}+1}} dr < \infty.$$

So that, we have proved $\Phi'(0) > 0$ and also

$$0 < \Phi(r) < \Phi_1, \ \Phi'(r) > 0, \quad 0 < r < \frac{\mathrm{R}}{2},$$

as well as $\Phi''(0) = 0$ (see (60)). Hence, straightforward computations on the $\mathcal{C}^2$ convex function $w(x) = \Phi(\mathrm{R} - |x - y|)$, defined in the annulus $\mathcal{O} \doteq \mathbf{B}_\mathrm{R} \setminus \overline{\mathbf{B}}_{\frac{\mathrm{R}}{2}}$, prove

$$\begin{cases} \det \mathrm{D}^2 w(x) = f\big(w(x)\big), \quad x \in \mathcal{O}, \\ w(x) = \Phi_1, \quad x \in \partial \mathbf{B}_{\frac{\mathrm{R}}{2}}, \\ w(x) = 0, \quad x \in \partial \mathbf{B}_\mathrm{R}. \end{cases}$$

Moreover, by construction

$$u(x) > 0, \quad x \in \partial\mathbf{B}_{\frac{\mathrm{R}}{2}} \quad \Rightarrow \quad u(x) \geq w(x), \quad x \in \partial\mathbf{B}_{\mathrm{R}},$$

for Φ_1 small enough. Then the Weak Maximum Principle of Proposition 1 implies

$$(u - w)(x) \geq 0, \quad x \in \overline{\mathcal{O}}$$

that leads to

$$\frac{u(x_0 - \tau\mathbf{n})}{\tau} \geq \frac{\Phi(\mathrm{R} - \mathrm{R}(1-\tau))}{\tau}, \quad \tau \ll 1$$

whence

$$\liminf_{\tau \to 0} \frac{u(x_0 - \tau\mathbf{n})}{\tau} \geq \Phi'(0) > 0.$$

□

Remark 11. In fact, the above result implies

$$\liminf_{\substack{x \to x_0 \\ x \in \Omega}} \frac{u(x)}{|x - x_0|} \geq \Phi'(0) > 0.$$ □

Our main result proving the absence of the free boundary is the following

Theorem 4.2 (Hopf's Strong Maximum Principle)**.** *Assume* (10)*. Let u be a nonnegative viscosity solution of*

$$-\det \mathrm{D}^2 u + f(u) \geq 0 \quad \textit{in } \Omega.$$

Then u cannot vanish at some $x_0 \in \Omega$ unless u is constant in a neighborhood of x_0.

Proof. Assume that u is non–constant and achieves the minimum value $u(x_0) = 0$ on some ball $\mathbf{B} \subset \Omega$. Then we consider the semi-concave approximation of u, *i.e.*

$$u^{\varepsilon}(x) \doteq \inf_{y \in \Omega} \left\{ u(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\}, \quad x \in \mathbf{B}_{\varepsilon} \qquad (\varepsilon > 0), \tag{61}$$

where $\mathbf{B}_{\varepsilon} \doteq \{x \in \mathbf{B} : \ \mathrm{dist}(x, \partial\mathbf{B}) > \varepsilon\sqrt{1 + 4\sup_{\mathbf{B}} |u|}\}$. For ε small enough we can assume $x_0 \in \mathbf{B}_{\varepsilon}$. Then u^{ε} achieves the minimum value in $\mathbf{B}_{\varepsilon}$, with $u(x_0) = u^{\varepsilon}(x_0) = 0$. Moreover, u^{ε} satisfies

$$-\det \mathrm{D}^2 u_{\varepsilon} + f\big(u_{\varepsilon}\big) \geq 0 \quad \text{on } \mathbf{B}_{\varepsilon} \tag{62}$$

(see, for instance [37, Proposition 2.3] or [6, 13] for general fully nonlinear equations). By classic arguments, if we denote

$$\mathbf{B}^{+}_{\varepsilon} \doteq \{x \in \mathbf{B}_{\varepsilon} : \ u^{\varepsilon}(x) > 0\},$$

there exists the largest ball $\mathbf{B}_{\mathrm{R}}(y) \subset \mathbf{B}^{+}_{\varepsilon}$ (see [25]). Certainly there exists some $z_0 \in \partial\mathbf{B}_{\mathrm{R}}(y) \cap \mathbf{B}_{\varepsilon}$ for which $u^{\varepsilon}(z_0) = 0$ is a local minimum. Then, Lemma 4.1 implies

$$\mathrm{D}u^{\varepsilon}(z_0) \neq \mathbf{0}$$

contrary to

$$\mathrm{D}u^{\varepsilon}(z_0) = \mathbf{0}, \tag{63}$$

as we shall prove in Lemma 4.3 below. Therefore, u^{ε} is constant on $\mathbf{B} \subset \Omega$, *i.e.*

$$u^{\varepsilon}(y) = u^{\varepsilon}(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

Finally, for every $y \in \mathbf{B}$ we denote by $\widehat{y}$ the point of Ω such that

$$u^{\varepsilon}(y) = u(\widehat{y}) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2$$

whence

$$u(x_0) = u^\varepsilon(x_0) = u^\varepsilon(y) = u(y) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2 \geq u(x_0) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2 \geq u(x_0) \Rightarrow \widehat{y} = y.$$

So that, one concludes

$$u(y) = u^\varepsilon(y) = u^\varepsilon(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

□

Corollary 2. *Assume (10). Let u be a generalized solution u of (7). Then if $u(x_0) > h(x_0)$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point x_0 of a ball $\overline{\mathbf{B}} \subseteq \overline{\Omega}$ then $u > h$ on $\overline{\mathbf{B}}$, consequently equation (7) is elliptic in $\overline{\mathbf{B}}$. In particular, if $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point $x_0 \in \Omega$ problem (20) is elliptic non degenerate in path-connected open sets Ω, provided the compatibility condition (3) holds.*

Proof. From Theorem 4.2, both cases imply $u > h$ on $\overline{\mathbf{B}}$. Finally, a continuity argument concludes the proof. □

Remark 12. Straightforward computations enable us to extend Lemma 4.1, Theorem 4.2 and Corollary 2 to the general case $g(|\mathbf{p}|) \geq 1$, since we know that $u \in \mathrm{W}^{1,\infty}(\Omega)$ (see the comments of Remark 3). □

We end this section by proving property (63) used in the proof of Theorem 4.2

Lemma 4.3. *Let ψ be a function achieving a local minimum at some $z_0 \in \mathcal{O}$. Assume that there exists a function $\widehat{\psi}$ defined in $\mathcal{O}$ such that $\widehat{\psi}(z_0) = 0$, $\Psi = \psi + \widehat{\psi}$ is concave on $\mathcal{O}$ and*

$$\widehat{\psi}(x) \geq -\mathrm{K}|x - z_0|^2, \quad x \in \mathcal{O} \text{ with } |x - z_0| \text{ small,}$$

for some constant $\mathrm{K} > 0$. Then the function ψ is differentiable at z_0 and $\mathrm{D}\psi(z_0) = \mathbf{0}$.

Proof. By simplicity we can take $z_0 = 0 \in \mathcal{O}$. By applying the convex separation theorem there exists $\mathbf{p} \in \mathbb{R}^\mathrm{N}$ such that

$$\Psi(x) \leq \Psi(0) + \langle \mathbf{p}, x\rangle = \psi(0) + \langle \mathbf{p}, x\rangle, \quad x \in \mathcal{O}, \text{ with } |x| \text{ small.}$$

Then we have

$$\begin{aligned} \psi(x) &= \Psi(x) - \widehat{\psi}(x) \leq \psi(0) + \langle \mathbf{p}, x\rangle + \mathrm{K}|x|^2 \\ &\leq \psi(x) + \langle \mathbf{p}, x\rangle + \mathrm{K}|x|^2, \quad x \in \mathcal{O} \text{ with } |x| \text{ small} \end{aligned} \tag{64}$$

whence

$$-\langle \mathbf{p}, x\rangle \leq \mathrm{K}|x|^2, \quad x \in \mathcal{O} \text{ with } |x| \text{ small.}$$

For $\tau > 0$ small enough we can choose $x = -\tau\mathbf{p} \in \mathcal{O}$ and $\tau\mathrm{K} < 1$, for which

$$\tau|\mathbf{p}|^2 \leq \mathrm{K}\tau^2|\mathbf{p}|^2.$$

Therefore $\mathbf{p} = 0$. Finally, (64) leads to

$$0 \leq \psi(x) - \psi(0) \leq \mathrm{K}|x|^2, \quad x \in \mathcal{O} \text{ with } |x| \text{ small,}$$

and the result follows. □

Remark 13. The result is immediate if ψ is concave (in this case we can choose $\widehat{\psi} \equiv 0$). The convex version follows by changing ψ and $\widehat{\psi}$ by $-\psi$ and $-\widehat{\psi}$, respectively (see Remark 2 above). □

Note that since the function u^ε defined in (61) is semi concave, the property (63) holds.

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