On the existence of positive solutions and solutions with compact support for a spectral nonlinear elliptic problem with strong absorption

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Abstract. We study a semilinear elliptic equation with a strong absorption term given by a non-Lipschitz function. The motivation is related with study of the linear Schrödinger equation with an infinite well potential. We start by proving a general existence result for non-negative solutions. We use also variational methods, more precisely Nehari manifolds, to prove that for any $\lambda > \lambda_1$ (the first eigenvalue of the Laplacian operator) there exists (at least) a non-negative solution. These solutions bifurcate from infinity at λ_1 and we obtain some interesting additional information. We sketch also an asymptotic bifurcation approach, in particular this shows that there exists an unbounded continuum of non-negative solutions bifurcating from infinity at $\lambda = \lambda_1$. We prove that for *some* neighborhood of $(\lambda_1, +\infty)$ the positive solutions are unique. Then a *Pohozaev identity* is introduced and we study the existence (or not) of free boundary solutions and compact support solutions. We obtain several properties of the energy functional and associated quantities for the ground states, together with asymptotic estimates in λ, mostly for $\lambda \nearrow \lambda_1$. We also consider the existence of solutions with compact support in Ω for λ large enough.

*Key words:*semilinear elliptic equation, strong absorption term, spectral problem, positive solutions, Nehari manifolds, bifurcation from infinity, *Pohozaev identity,* solutions with compact support.

1. Introduction

We study in this paper the existence of different kinds of non-negative solutions to the semilinear elliptic equation

$$P(q,\alpha,\lambda) = \begin{cases} -\Delta u + q(x)|u|^{\alpha-1}u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 1$), λ is a real parameter, $0 < \alpha < 1$ and $q(x) \geq 0$ is a real function satisfying suitable assumptions.

The equation $P(q, \alpha, \lambda)$ is a typical example of the so-called diffusion-reaction equations. If $q \equiv 1$ and $\alpha > 1$ it is the well-known logistic equation in population dynamics. In this case there is a unique positive solution u_λ for any $\lambda > \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions and it follows immediately from the Strong Maximum Principle that if $u_\lambda \geq 0$, $u_\lambda \not\equiv 0$, is a solution then $u_\lambda > 0$ in Ω and $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, i.e., all non-negative solutions to the logistic equation are actually positive.

But this is not the case for $P(q, \alpha, \lambda)$ for $q \equiv 1$ and $0 < \alpha < 1$ and for some problems of this kind previously studied. The semilinear problem

$$\begin{cases} -u'' + u^m = \lambda u^q & \text{in } (-1, 1), \\ u(\pm 1) = 0, \end{cases} \tag{2}$$

with $0 < m < q < 1$ was studied in [12], where it was proved by using energy methods for ODEs that there exist two critical values $0 < \lambda^* < \lambda^{**}$ such that: i) for $0 < \lambda < \lambda^*$ there is no solution to (2); ii) for any for $\lambda > \lambda^*$ there is an upper branch of positive solutions $u_\lambda > 0$ with $\frac{\partial u_\lambda}{\partial \nu}(\pm 1) < 0$; iii) for $\lambda^* < \lambda < \lambda^{**}$ there is a lower branch $v_\lambda > 0$ ($0 < v_\lambda < u_\lambda$ on $(-1, 1)$) with $\partial v_\lambda / \partial n(\pm 1) < 0$; iv) for $\lambda = \lambda^{**}$ there is a solution $v_{\lambda^{**}} > 0$ such that $\partial v_{\lambda^{**}} / \partial n(\pm 1) = 0$; v) from this solution, which is also a solution of equation (2) on the whole real line, it is possible to build by stretching, gluing and rescaling continua of infinitely many compact support solutions, whose precise description is given in [12]. These results were extended in [14] to the singular case $-1 < m < q < p-1$ and to the p-Laplacian as well. The corresponding N-dimensional problem to (2) was studied for a star-shaped bounded domain Ω in [19] by using a combination of Pohozaev's identity and variational methods obtaining similar but less precise results.

The motivation for studying later problem $P(q, \alpha, \lambda)$ with $q \equiv 1$ and $0 < \alpha < 1$ arise from general observations of the first author ([9], [11]) concerning solutions for the linear Schrödinger equation

$$\begin{cases} -u'' + V(x)u = \lambda u & \text{in } (-R, R), \\ u(\pm R) = 0, \end{cases} \tag{3}$$

for a given $R > 0$ or on the real line, where $V(x)$ is the so-called infinite well potential, introduced by Gamow. It turns out that there is some ambiguity in the treatment of the case of the real line: what is mentioned to be solutions in most of the text-books are not classical solutions since Dirac's deltas appear, and they are solutions in the sense of distributions but of a different equation where deltas are included. In this situation solutions of the semilinear equation $P(q, \alpha, \lambda)$ provide some kind of, say, "alternative approach" ([9], [11]).

The one-dimensional case $\Omega = (-R, R)$ was studied in detail in [13] by using the same phase plane methods in ODEs. There it was proved that for $0 < \lambda < \lambda_1$ there is no solution to (2) and for $\lambda_1 < \lambda < \lambda^*$, where λ^* is a critical value given explicitly, there is a unique solution $u_\lambda > 0$ with $\partial u_\lambda / \partial n(\pm R) < 0$ bifurcating at infinity for $\lambda = \lambda_1$. Moreover u_λ is decreasing as a function of λ and, as in the preceding example, the solution $u_{\lambda^*} > 0$ in $(-R, R)$ is such that $u'_{\lambda^*}(\pm R) = 0$ and from this free boundary solution it is possible, once again, to build continua of compact support solutions. We also obtain the asymptotic behaviour (in λ) of the solutions, namely

$$\|u_\lambda\|_{L^\infty(-R,R)} \leq \frac{C}{\lambda^{1-\alpha}}.$$

Thus we have obtained that $\lambda^* > 0$ and $u_{\lambda^*} > 0$ are the first eigenvalue and its corresponding eigenfunction for the linear eigenvalue problem

$$\begin{cases} -w'' + |u_{\lambda^*}|^{\alpha-1} w = \lambda w & \text{in } (-R, R), \\ w(\pm R) = 0. \end{cases} \tag{4}$$

and that $u'_{\lambda^*}(\pm R) = 0$. It is in this sense that we have an "alternative approach" for solutions to the linear Schrödinger equation.

Problem $P(q, \alpha, \lambda)$ is studied in [17] for $q = 1$ and $0 < \alpha < 1$ as a particular case of a much more general class of problems allowing more general nonlinear terms and boundary conditions. The main result in [17] is the existence of an unbounded continuum of non-negative solutions bifurcating from infinity at the asymptotic bifurcation point λ_1. The method of proof was to apply global asymptotic bifurcation theorems by Rabinowitz [31] by using as a tool some theorem in [7]. More details are given below.

Existence of a weak non-negative solution for any $\lambda > \lambda_1$ was obtained later by Porretta [26] this time by using variational methods, namely a variant of the Mountain Pass Theorem. There were some complementary results concerning, for example, estimates for the norm of the solutions, but the problem of the existence(or not) of positive solutions was not considered.

In this paper we deal first with general existence results for non-negative solutions to $P(q, \alpha, \lambda)$. First, in Section 2, we use variational methods, more precisely Nehari manifolds ([4], [20], [19], [37]) and prove that for any $\lambda > \lambda_1$ there exists (at least) a non-negative solution. These solutions bifurcate from infinity at λ_1 and we obtain some interesting additional information. In a second part of Section 2 we sketch the asymptotic bifurcation approach above mentioned, in particular this shows that there exists an unbounded continuum of non-negative solutions bifurcating from infinity at $\lambda = \lambda_1$. In Section 3 we study different kinds of solutions. It is possible to show that solutions u_λ bifurcating from infinity at $\lambda = \lambda_1$ are $u_\lambda > 0$ with $\partial u_\lambda / \partial n < 0$ on $\partial\Omega$ for *some* neighborhood of $(\lambda_1, +\infty)$. Under some additional assumptions it is also possible to show, by using the results in [18], that positive solutions are unique there. Then a *Pohozaev identity* is introduced and here the coefficient $q(x)$ plays an interesting role concerning existence (or not) of a free boundary and compact support solutions.

In Section 4 we collect several results concerning regularity and differentiability properties of the energy functional and associated quantities for the ground states, together with asymptotic estimates in λ, mostly for $\lambda \nearrow \lambda_1$. The existence of solutions with compact support in Ω is considered in Section 5. With the usual philosophy of reaction-diffusion equations giving rise to a free boundary, we will show that, in the case of problem (1), the "diffusion-absorption balance" condition on the nonlinearities is obviously satisfied (since $\alpha < 1$) and that the "balance condition between the data and the domain" is here represented by means of the requirement of assuming λ large enough. The simultaneous fulfillment of both balances is required in order to get solutions with compact support (see Section 1.2 of [10]).

2. Existence of nonnegative solutions: Nehari manifolds and asymptotic bifurcation methods

We study in this Section the existence of non-negative solutions of the semilinear equation $P(q, \alpha, \lambda)$. Concerning the coefficient $q(x)$ we assume that $q(x) \geq 0$, $q \not\equiv 0$, on Ω and either

$$q \in L^{\frac{2^*}{2^*-(1+\alpha)}}(\Omega), \tag{5}$$

(where $2^* = 2N/(N-2)$) if $N \geq 3$ or

$$q \in L^r(\Omega) \text{ for some } r > 1, \tag{6}$$

if $N = 1, 2$. Moreover in some parts we shall assume also that

$$q \in H^1(\Omega). \tag{7}$$

We shall study *nonnegative weak solutions* of $P(q, \alpha, \lambda)$ (i.e., functions $u \in H_0^1(\Omega)$, $u \geq 0$ on Ω and such that

$$\int_\Omega \nabla u \cdot \nabla v \; \mathrm{d}x + \int_\Omega (q(x)|u|^{\alpha-1}uv \; - \lambda uv)\, \mathrm{d}x = 0$$

for any $v \in H_0^1(\Omega)$: we note that from the above assumptions on q we have that $q(.)|u|^{\alpha-1}uv \in L^1(\Omega)$). We are also interested in *positive free boundary solutions* of $P(q, \alpha, \lambda)$: i.e., weak solutions u of $P(q, \alpha, \lambda)$ such that $u > 0$ on Ω and

$$\frac{\partial u}{\partial \nu} = 0 \;\; \text{on} \;\; \partial\Omega, \tag{8}$$

where ν denotes the unit outward normal to $\partial\Omega$. We shall deal sometimes as well with *nonnegative weak solutions with compact support*, i.e., such that

$$\textit{support } u \subsetneq \Omega.$$

Some of the results concern the so called *nonnegative ground state solutions* of $P(q, \alpha, \lambda)$, i.e., functions $u_\lambda \in H_0^1(\Omega)$, $u_\lambda \geq 0$ on Ω, $u_\lambda \not\equiv 0$ and such that if we define the associate functional $E_\lambda : H_0^1(\Omega) \mapsto \mathbb{R}$ to the problem $P(q, \alpha, \lambda)$ by

$$E_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda |u|^2)\, \mathrm{d}x + \frac{1}{1+\alpha} \int_\Omega q(x)|u|^{1+\alpha}\, \mathrm{d}x \tag{9}$$

then we have that

$$E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda)$$

for any non-zero weak solution w_λ of $P(q, \alpha, \lambda)$.

As usual, $H_0^1(\Omega) := W_0^{1,2}(\Omega)$ denotes the standard Sobolev space of functions vanishing on the boundary $\partial\Omega$ with the norm

$$||u|| = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{1/2}.$$

We also use the notation

$$||u||_p = ||u||_{L^p(\Omega)}$$

for $p \in [1, +\infty]$.

In all what follows $\lambda_1 > 0$ and $\varphi_1 > 0$ denote the first eigenvalue an eigenfunction of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, i.e., $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$. We normalize φ_1 by $||\varphi_1|| = 1$. We also recall that $\varphi_1 \sim d(x)$, where $d(x) = d(x, \partial\Omega)$, in the sense that there exist constants $c_1, c_2 > 0$ such that $c_1 d(x) \leq \varphi_1 \leq c_2 d(x)$ for any $x \in \Omega$. This fact will be used in all what follows.

2.1. Existence of non-negative solutions: Nehari manifolds

We will use the Nehari manifold, fibering maps and related ideas, as it is exposed in the papers [4], [19, 20, 29, 30] (see also [37]). Following [4] the *Nehari manifold* corresponding to problem $P(q, \alpha, \lambda)$ is defined as

$$\mathcal{N} = \{ u \in H_0^1(\Omega) \big| \;\; H_\lambda(u) + A(u) = 0\}$$

and its associated components by

$$\mathcal{N}^+ = \{ u \in \mathcal{N} | \;\; H_\lambda(u) + \alpha A(u) > 0\} = \{ u | \;\; A(u) < 0\} = \phi,$$

where ϕ denotes the empty set,

$$\mathcal{N}^- = \{ u \in \mathcal{N} | \;\; H_\lambda(u) + \alpha A(u) < 0 \;\} = \{ u | \;\; H_\lambda(u) < 0\},$$

$$\mathcal{N}^0 = \{ u \in \mathcal{N} | \;\; H_\lambda(u) + \alpha A(u) = 0\} = \{0\},$$

where we use the notations

$$H_\lambda(u) := \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \lambda \int_\Omega |u|^2 \, \mathrm{d}x, \quad A(u) = \int_\Omega q|u|^{1+\alpha} \, \mathrm{d}x$$

for any $u \in H_0^1(\Omega)$.

Given $u \in H_0^1(\Omega)$, the *fibrering mappings* are defined by

$$\Phi_u(t) = E_\lambda(tu) = \frac{t^2}{2} H_\lambda(u) + \frac{t^{1+\alpha}}{1+\alpha} A(u),$$

so that we have

$$\Phi'_u(t) = E'_\lambda(tu) = t H_\lambda(u) + t^\alpha A(u).$$

The equation $\Phi'_u(t) = 0$ has a positive solution only if both terms in $\Phi'_u(t)$ have opposite signs: i.e., if and only if $H_\lambda(u) < 0$. It turns out that the only point $t(u)$ where $\Phi'_u(t) = 0$ is given by

$$t_\lambda(u) = \left(\frac{A(u)}{-H_\lambda(u)} \right)^{1/(1-\alpha)}. \tag{10}$$

Replacing this root in $E_\lambda(tu)$ we obtain

$$J_\lambda(u) := E_\lambda(t_\lambda(u)u) = \frac{(1-\alpha)}{2(1+\alpha)} \frac{A(u)^{\frac{2}{1-\alpha}}}{(-H_\lambda(u))^{\frac{1+\alpha}{1-\alpha}}}.$$

Observe that $J_\lambda(u)$ is a zero-homogeneous functional, i.e. $J_\lambda(tu) = J_\lambda(u)$ for $t > 0$. Furthermore, evidently $t_\lambda(u_\lambda) = 1$ for any non-zero solution u_λ of (1).

The proof of the following is straightforward (see e.g. [29, 37]):

Lemma 2.1. *Let $J'_\lambda(v_\lambda) = 0$ on $H_0^1(\Omega)$. Then $u_\lambda = t_\lambda(v_\lambda)v_\lambda$, where $t_\lambda(v_\lambda)$ is given by (10), is a critical point of $E_\lambda(u)$ on $H_0^1(\Omega)$, i.e. u_λ is a weak solution of $P(q, \alpha, \lambda)$.*

We shall prove:

Theorem 2.1. *Assume $0 < \alpha < 1$. Then for every $\lambda \in (\lambda_1, +\infty)$ problem* (1) *has a weak nonnegative solution $u_\lambda \in C^{1,\kappa}(\overline{\Omega})$ for any $\kappa \in (0,1)$. Furthermore u_λ is a ground state of* (1)*.*

For the proof of this theorem we shall need some previous results. Consider the minimization problem

$$\hat{J}_\lambda = \min\{J_\lambda(v) : \quad v \in H_0^1(\Omega) \setminus 0, \; H_\lambda(v) < 0\} = \min_{\mathcal{N}^-} J_\lambda. \tag{11}$$

Lemma 2.2. *There exists a minimizer v_λ of* (11) *such that $H_\lambda(v_\lambda) < 0$.*

Proof. Let (v_m) be a minimizing sequence of (11). We may assume that $||v_m|| = 1$, since $J_\lambda(u)$ is a zero-homogeneous functional on $H_0^1(\Omega)$. This implies, by the Sobolev embedding and Eberlein-Ŝmulian theorems, that there is $v_\lambda \in H_0^1(\Omega)$ such that there exists a subsequence of (v_m) (which we denote again (v_m)) such that $v_m \rightharpoonup v_\lambda$ weakly in $H_0^1(\Omega)$ and $v_m \to v_\lambda$ strongly in $L^p(\Omega)$ for $1 < p < 2^*$, as $m \to \infty$.
First we show that $v_\lambda \neq 0$. Suppose, contrary to our claim, that $v_\lambda = 0$. Then $\int_\Omega |v_m|^2 \, \mathrm{d}x \to 0$ as $m \to \infty$ and consequently

$$H_\lambda(v_m) \equiv \int_\Omega |\nabla v_m|^2 \, \mathrm{d}x - \lambda \int_\Omega |v_m|^2 \, \mathrm{d}x \to 1 \quad \text{as} \quad m \to \infty,$$

since $\int_\Omega |\nabla v_m|^2 \, \mathrm{d}x = ||v_m||^2 = 1$, $m = 1, 2, ...$, by the assumption. However $H_\lambda(v_m) < 0$, $m = 1, 2, ...$ Thus we get a contradiction. Observe that

$$H_\lambda(v_\lambda) \leq \liminf_{m\to\infty} H_\lambda(v_m) < 0$$

by the weak lower semi-continuity of the norm $||.||$. Furthermore, we have

$$A(v_m) = \int_\Omega q(x)|v_m|^{\alpha+1} \, \mathrm{d}x \to A(v_\lambda) = \int_\Omega q(x)|v_\lambda|^{\alpha+1} \, \mathrm{d}x. \tag{12}$$

Indeed, in case $N \geq 3$ we have by (5)

$$\int_\Omega q(x)|v|^{\alpha+1} \, \mathrm{d}x \leq (\int_\Omega q(x)^k \mathrm{d}x)^{\frac{1}{k}} (\int_\Omega |v|^{2^*} \, \mathrm{d}x)^{\frac{1+\alpha}{2^*}} = c_1 (\int_\Omega |v|^{2^*} \, \mathrm{d}x)^{\frac{1+\alpha}{2^*}}$$

for some $c_1 > 0$, where

$$k = \frac{2^*}{2^* - (1+\alpha)}.$$

This implies that A is a continuous functional on $L^p(\Omega)$ for $p \in (1, 2^*]$. Thus since $v_m \to v_\lambda$ as $m \to \infty$ in $L^p(\Omega)$ for $p \in (1, 2^*)$ we get (12). The cases $N = 1, 2$ are considered in the same way. Hence v_λ is an admissible function of the minimizing problem. Furthermore,

$$J_\lambda(v_\lambda) \leq \liminf_{m\to\infty} J_\lambda(v_m) = \hat{J}_\lambda.$$

It is easy to see that here this is possible only if $J_\lambda(v_\lambda) = \hat{J}_\lambda$. Thus v_λ is the minimizer of (11) and $H_\lambda(v_\lambda) < 0$.

Proof of Theorem 2.1. Let $\lambda > \lambda_1$. By Lemma 2.2 there exists a minimizer v_λ of (11). Since $J_\lambda(v)$ is an even functional then $|v_\lambda|$ is also a minimizer of (11). Thus we may assume that v_λ is a nonnegative function. By Lemma 2.1 it follows that $u_\lambda = t_\lambda(v_\lambda)v_\lambda$ is a weak solution of (1) which is nonnegative since $t_\lambda(v_\lambda) > 0$. By standard regularity theory (bootstrapping) we derive that $u_\lambda \in C^{1,\kappa}(\overline{\Omega})$ for any $\kappa \in (0,1)$, see App. B in [38]. Finally we see that the solution u_λ is a ground state. Indeed, any non-zero solution w_λ of (1) satisfies $H_\lambda(w_\lambda) < 0$ since $\Phi'_{w_\lambda}(t)|_{t=1} = 0$. Thus w_λ is an admissible point of (11) and consequently

$$E_\lambda(u_\lambda) = J_\lambda(u_\lambda) = \hat{J}_\lambda \leq J_\lambda(w_\lambda) = E_\lambda(w_\lambda).$$

From now on we provide some more interesting information concerning the behaviour of solutions and, in particular, bifurcation at infinity at $\lambda = \lambda_1$.

Proposition 2.1. *If we denote by u_n the minimizer for the value λ_n of the parameter, then*

$$\lim_{\lambda_n \searrow \lambda_1} E_{\lambda_n}(u_n) = \lim_{\lambda_n \searrow \lambda_1} \inf_{\mathcal{N}^-} E_{\lambda_n}(u) = +\infty.$$

Proof. Assume that this is not the case. Then there exist sequences $\lambda_n \searrow \lambda_1$ and $u_n \in \mathcal{N}^-$ such that for any n

$$0 < E_{\lambda_n}(u_n) = \frac{(\alpha-1)}{2(1+\alpha)} H_{\lambda_n}(u_n) = \frac{(1-\alpha)}{2(1+\alpha)} A(u_n) \leq C,$$

for some $C > 0$. If we define $v_n = \frac{u_n}{\|u_n\|}$, since $\|v_n\| = 1$, there exists a sequence converging weakly to v_0 in $H_0^1(\Omega)$, $v_n \rightharpoonup v_0$, and $v_n \to v_0$ strongly in $L^p(\Omega)$ for $1 < p < 2^*$, as $n \to \infty$. We have $v_n \to v_0$ strongly in $H_0^1(\Omega)$ as well. If not, using the l.s.c. of the norm we get

$$H_{\lambda_1}(v_0) < \liminf H_{\lambda_n}(v_n) = \lim \frac{1}{\|u_n\|^2} H_{\lambda_n}(u_n) \leq 0$$

since $u_n \in \mathcal{N}^-$ and $H_{\lambda_n}(u_n)$ is bounded. Hence $v_0 \not\equiv 0$. But then it follows that

$$\begin{aligned}
\lim_{\lambda_n \searrow \lambda_1} E_{\lambda_n}(u_n) &\geq \lim_{\lambda_n \searrow \lambda_1} \frac{(1-\alpha)}{2(1+\alpha)(\lambda_n - \lambda_1)^{\frac{(1+\alpha)}{(1-\alpha)}}} \left(\frac{\|v_n\|_{1+\alpha}}{\|v_n\|_2} \right)^{\frac{2(1+\alpha)}{(1-\alpha)}} \\
&= \frac{(1-\alpha)}{2(1+\alpha)} \left(\frac{\|v_0\|_{1+\alpha}}{\|v_0\|_2} \right)^{\frac{2(1+\alpha)}{(1-\alpha)}} \lim_{\lambda_n \searrow \lambda_1} \frac{1}{(\lambda_n - \lambda_1)^{\frac{(1+\alpha)}{(1-\alpha)}}} = +\infty.
\end{aligned}$$

Proposition 2.2. *Under the assumptions of Proposition 2.1 we have*
i) $\lim_{\lambda_n \searrow \lambda_1} \|u_n\| = +\infty$;
ii) $\lim_{\lambda_n \searrow \lambda_1} \frac{u_n}{\|u_n\|} = \varphi_1$.

Proof. i) If not, $\|u_n\| \leq C$ and there exists a sequence $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_n \to u_0$ strongly in $L^p(\Omega)$ for $1 < p < 2^*$, as $n \to \infty$. Hence we obtain

$$\begin{aligned}
\lim_{\lambda_n \searrow \lambda_1} E_{\lambda_n}(u_n) &= \lim_{\lambda_n \searrow \lambda_1} \frac{(\alpha-1)}{2(1+\alpha)} H_{\lambda_n}(v_n) \\
&= \lim_{\lambda_n \searrow \lambda_1} \frac{(1-\alpha)}{2(1+\alpha)} A(u_n) = \frac{(1-\alpha)}{2(1+\alpha)} A(u_0) < +\infty,
\end{aligned}$$

a contradiction with Proposition 2.1.
ii) If $v_n = \frac{u_n}{\|u_n\|}$ there exists a sequence $v_n \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$ and $v_n \to v_0$ strongly in $L^p(\Omega)$ for $1 < p < 2^*$, as $n \to \infty$. Let us show that $v_n \to v_0$ strongly in $H_0^1(\Omega)$ as well. If not, reasoning once again as above using the l.s.c. of the norm

$$H_{\lambda_1}(v_0) < \lim \inf_{\lambda_n \searrow \lambda_1} H_{\lambda_n}(v_n) \leq 0$$

and this is impossible. Hence we obtain by i)

$$\lim_{\lambda_n \searrow \lambda_1} \frac{(\alpha-1)}{2(1+\alpha)} H_{\lambda_n}(v_n) = \lim_{\lambda_n \searrow \lambda_1} \frac{(1-\alpha)}{2(1+\alpha)} A(v_n) \|u_n\|^{\alpha-1} = 0,$$

and then $H_{\lambda_1}(v_0) = 0$. But $v_0 = K\varphi_1$ and since $\|v_0\| = 1$, $K = 1$.

2.2. Existence of non-negative solutions: asymptotic bifurcation

In this Section we sketch an alternative approach to the above problem $P(q, \alpha, \lambda)$. In order to simplify matters we take $q = 1$. Consider the problem

$$\begin{cases} -\Delta u + |u|^{m-1} u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where λ is a real parameter and for the moment $m > 0$.

First we recall that if $m > 1$ (in particular for $m = 2$) (13) is the well-known logistic equation arising in population dynamics. This equation has been widely studied in the last forty years (see e.g. [34]) and it is possible to prove that for any $\lambda > \lambda_1$ there exists a unique positive solution u_λ to (13). This result can be obtained by using sub and supersolutions, but there are other methods giving the same result as well. One is to apply some global bifurcation theorem by Rabinowitz [32], [33] together with a priori estimates providing the existence of a continuum (i.e., a closed connected set) of positive solutions bifurcating from the line of trivial solutions $u = 0$ at $\lambda = \lambda_1$. Moreover, continuation arguments, involving also the Implicit Function Theorem allow to obtain a more precise result: the continuum is actually a smooth curve in some function space defined for all $\lambda > \lambda_1$. Solutions on this curve are asymptotically stable. On the other hand, it follows easily from the Strong Maximum Principle that if $u \geq 0$ is a solution to (13), then $u > 0$ on Ω and $\partial u/\partial n < 0$ on $\partial\Omega$, this means that these solutions are in the interior of the positive cone in the space $C_0^1(\overline{\Omega})$.

If $N = 1$ and $\Omega = (0, 1)$ it was proved also by Rabinowitz [32], [33], that all (simple) eigenvalues of the linearized operator at the origin are (ordinary) bifurcation points and the well-known nodal properties of the eigenfunctions in Sturm-Liouville theory are preserved all along the bifurcation branches. The same happens for $N > 1$ for positive solutions bifurcating from λ_1.

The situation is very different, and more interesting from our point of view, for $0 < m < 1$. Now it is still possible to use global bifurcation arguments, this time for bifurcation at infinity (see [31] , and also [1], [6]) in order to exhibit the existence of an unbounded continuum of non-negative solutions bifurcating at infinity from λ_1.

A more general problem including (13) was studied by the second author in [17], where more general nonlinearities and (even nonlinear) boundary conditions were allowed, including maximal monotone graphs.

The main tool here is an asymptotic Fréchet derivative for the associated solution operator in [7] (see also [8] and [2]). More precisely we recall from the classical theory for monotone operators that if $u \in L^2(\Omega)$ then there exists a unique weak solution $z = Pu \in H^2(\Omega) \cap H_0^1(\Omega)$ to the nonlinear equation

$$\begin{cases} -\Delta z + |z|^{m-1} z + z = u & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

and that the solution (or Green's) operator $P : L^2(\Omega) \to L^2(\Omega)$ is a compact monotone operator which is Fréchet differentiable at the infinity in the sense of the following:

Theorem 2.2 ([7]). *Under the above assumptions,P has a Fréchet derivative at infinity $A = P'(\infty)$, where Au is defined for any $u \in L^2(\Omega)$ as the unique solution of the linear problem*

$$\begin{cases} -\Delta Au + Au = u & \text{in } \Omega, \\ Au = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

and $A : L^2(\Omega) \to L^2(\Omega)$ is a compact linear operator, in the sense that

$$\lim_{\|u\|_{L^2(\Omega)} \to +\infty} \frac{\|Pu - Au\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} = 0.$$

By applying the above global asymptotic bifurcation results together with Theorem 2.2 the following result can be obtained.

Theorem 2.3. ([17]): *Under the above assumptions there exists an unbounded continuum of non negative solutions to (1) bifurcating at infinity from λ_1.*

The question of investigating if these bifurcating solutions were positive or not was not pursued in [17]. In [31] Rabinowitz already pointed out the interesting feature that, contrary to the case of ordinary bifurcation, nodal properties are not necessarily preserved along bifurcating branches, and a counterexample is given in Remark 2.12 of [31]. However, it is proved in [31] that these nodal properties are prerserved in *some* neighborhood of $(\lambda_1, +\infty)$, where solutions are in the interior of the positive cone of $C_0^1(\overline{\Omega})$. This question will be treated in the following Section. The one-dimensional case $\Omega = (0, 1)$ was studied in detail in [13] using phase plane arguments from ODEs and then it was possible to give a complete description of the solution set. There is a branch of positive solutions bifurcating from λ_1 and then a "critical" value of the parameter λ appears for which there is a positive free boundary solution, i.e. with $\partial u/\partial n = 0$ on $\partial\Omega$.

3. Positive, free boundary and compact support solutions

In the preceding Section we have proved the existence of (at least) a non-negative solution of our problem for any $\lambda > \lambda_1$ by using a variational method, namely the Nehari manifold, together with some interesting complementary results. Then, by using an alternative approach, asymptotic bifurcation, we proved the existence of a unbounded continuum of non-negative solutions. Notice that this does not imply existence for any $\lambda > \lambda_1$, a result which would follow, e.g., from the existence of a continuous function $\psi(\lambda)$ such that $\|u\|_\infty \leq \psi(\lambda)$ if $u \geq 0$ is any solution for the value λ of the parameter.

Next we go further to study positive solutions and also free boundary and compact support solutions in the sense illustrated in the Introduction. In particular we are oriented by the results in the onedimensional case [13] and also by the results for the related problem

$$\begin{cases} -\Delta u + u^m = \lambda u^\beta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where λ is a real parameter and

$$0 < m < \beta < 1, \tag{17}$$

considered also in the Introduction, studied in one-dimension in [12] [14] and for Ω star-shaped in [19] (see also [21]). Hence we should expect a branch of positive solutions $u > 0$ in Ω (with $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$) for λ close to λ_1, bifurcating at λ_1 at infinity giving rise to continua of infinitely many compact non-negative solutions arising from a free boundary solution $u_{\lambda^*} > 0$ for a critical value λ^* of λ such that

$$\begin{cases} -\Delta u_{\lambda^*} + q(x)\,|u_{\lambda^*}|^{\alpha-1}\,u_{\lambda^*} = \lambda^* u_{\lambda^*} & \text{in } \Omega, \\ u_{\lambda^*} = \frac{\partial u_{\lambda^*}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Here a first result is that *some* solutions bifurcating from infinity at $\lambda = \lambda_1$ are actually in the interior of the positive cone of $C_0^1(\overline{\Omega})$, i.e., such that $u > 0$ in Ω, $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. This follows from Proposition 2.2.

Proposition 3.1. *Under the assumptions of Proposition 2.1 there is a neighborhood of* $(\lambda_1, +\infty)$ *in* $\mathbb{R} \times C_0^1(\overline{\Omega})$ *such that* $u > 0$ *and* $\frac{\partial u}{\partial \nu} < 0$ *on* $\partial\Omega$ *for* (λ, u) *in this neighborhood.*

Proof. It follows from Proposition 2.2 taking into account that φ_1 is in the interior of the positive cone in $C_0^1(\overline{\Omega})$. Alternatively, we can also use the arguments in [31], where it is noticed that the result holds for nonlinearities satisfying the condition only in some neighborhood of infinity.

Moreover, these solutions are unstable, in the sense of linearized stability. Even if the linearized problem is singular since there is a coefficient blowing up close to the boundary, this can be made rigorous by using the results of [18] (or [3]).

Theorem 3.1. *Assume additionally*

$$q \in C^1(\Omega) \cap C^0(\overline{\Omega}) \tag{19}$$

and that $u_\lambda \in C_0^1(\overline{\Omega})$ *is a solution of* $P(q, \alpha, \lambda)$ *such that* $u_\lambda > 0$ *and* $\frac{\partial u_\lambda}{\partial \nu} < 0$ *on* $\partial\Omega$. *Then* u_λ *is unstable in the sense that* $\lambda_1(-\Delta + \alpha q(x) u_\lambda^{\alpha-1} - \lambda) < 0$.

Proof. We have

$$\begin{cases} -\Delta u_\lambda + q(x) u_\lambda^\alpha = \lambda u_\lambda & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

and the corresponding linearized problem at u_λ is

$$\begin{cases} -\Delta w + \alpha q(x) u_\lambda^{\alpha-1} w - \lambda w = \mu w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

From the results of [18] there is a first eigenvalue μ_1 to (21) with a positive eigenfunction $\psi_1 > 0$ such that $\psi_1 \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$.

Multiplying (20) by ψ_1 and (21) with $w = \psi_1$ by u_λ and integrating on Ω, by using Green's formula, we obtain

$$\begin{aligned} \int_\Omega \nabla u_\lambda \cdot \nabla \psi_1 + q u_\lambda^{\alpha-1} \psi_1 - \lambda u_\lambda \psi_1) \mathrm{d}x &= 0 \\ &= \int_\Omega (\nabla u_\lambda \cdot \nabla w + \alpha q u_\lambda^{\alpha-1} \psi_1 - \lambda u_\lambda \psi_1 - \mu_1 u_\lambda \psi_1) \mathrm{d}x \end{aligned}$$

and it follows that

$$\mu_1 = \frac{(\alpha-1)\int_\Omega q u_\lambda^{\alpha-1} \psi_1 \mathrm{d}x}{\int_\Omega u_\lambda \psi_1 \mathrm{d}x} < 0,$$

which implies the unstability. (By the way, this shows that $\mu_1(\lambda)$ is a continuous function of λ if the application $\lambda \to u_\lambda(\lambda)$ is well-defined and continuous).

Remark 3.1. From the results of [18] it follows that u_λ is unstable in the sense of Lyapunov for the associated parabolic problem. Related linearized stability results were obtained in [3] working in Sobolev spaces in the framework of degenerate parabolic equations of porous media type.

We have seen that $\mu_1 < 0$. If we can prove that 0 is not an eigenvalue (which follows obviously from $\mu_2 > 0$) it would be possible to apply the Implicit Function Theorem at the interior of the positive cone in [18]. This could be used not only to prove the regularity of a branch of such solutions but also uniqueness in the neighborhood of $(\lambda_1, +\infty)$ given by Proposition 3.1. Indeed, in this case if we have two such solutions $u_\lambda > 0$, $v_\lambda > 0$ (with $\frac{\partial u_\lambda}{\partial \nu} < 0$, $\frac{\partial v_\lambda}{\partial \nu} < 0$ on $\partial\Omega$) in this neighborhood both can be continued to the left remaining at the interior of the positive cone bifurcating at infinity at the simple eigenvalue λ_1, a contradiction with the existence of *two* bifurcating branches at λ_1.

It is possible to show that for the linearized problem (21) there exist an infinite sequence μ_n going to $+\infty$ of eigenvalues. Indeed, assume only

$$q \in L^N(\Omega) \tag{22}$$

then, for any $h \in L^2(\Omega)$ the linear problem

$$\begin{cases} -\Delta z + \frac{\alpha q(x) z}{u_\lambda^{1-\alpha}} = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

has a unique (weak) solution $z \in H_0^1(\Omega)$. This follows from applying the Lax-Milgram Lemma to the associated bilinear form in $H_0^1(\Omega)$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega \frac{\alpha q(x) u v}{u_\lambda^{1-\alpha}} \mathrm{d}x$$

which is well-defined, continuous and coercive. Indeed, taking into account that $u_\lambda \sim d(x)$, Hardy's inequality, and using (22) and the Sobolev inequality we obtain

$$\begin{aligned}\left|\int_\Omega \frac{\alpha uv}{u_\lambda^{1-\alpha}}\,\mathrm{d}x\right| &\le \left|\int_\Omega \frac{\alpha quv}{cd(x)^{1-\alpha}}\,\mathrm{d}x\right| \le k\left|\int_\Omega \left(\frac{u}{d(x)}\right) q(x)vd(x)^\alpha \mathrm{d}x\right| \\ &\le C\left\|\frac{u}{d}\right\|_2 \|v\|_{2^*} \le \overline{C}\,\|u\|\,\|v\|\,,\end{aligned}$$

where c, k, C and $\overline{C}$ are different positive constants independent of u and v.

Since

$$a(u,u) \ge C\,\|u\|^2$$

for some $C > 0$, a is obviously coercive.

Thus, for any $h \in L^2(\Omega)$, there exists a unique $Th \in H_0^1(\Omega)$ solution of the above equation and it is easy to see that the composition with the (compact) embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is a selfadjoint compact linear operator $\widetilde{T} = i \circ T : L^2(\Omega) \to L^2(\Omega)$ for which we obtain in the usual way a sequence of eigenvalues $\nu_n \to +\infty$. Writing $\mu_n = \nu_n - \lambda$ we get the sequence of eigenvalues corresponding to our problem, again with $\mu_n \to +\infty$.

Moreover, and this is our main interest here, we have the following variational characterization for the second eigenvalue

$$\mu_2 = \inf_{w \in [\varphi_1]^\perp} \frac{\int_\Omega \left(|\nabla w|^2 + \frac{\alpha q w^2}{u_\lambda^{1-\alpha}} - \lambda w^2\right)\mathrm{d}x}{\int_\Omega w^2 \mathrm{d}x}.$$

Thus we get inmediately the estimate

$$\mu_2 > \lambda_2 - \lambda + \inf_{w \in [\varphi_1]^\perp} \frac{\int_\Omega \frac{\alpha q w^2}{u_\lambda^{1-\alpha}}\mathrm{d}x}{\int_\Omega w^2 \mathrm{d}x}.$$

But now we know that that $u_\lambda \sim d(x)$ and in particular $u_\lambda \le c_1 d(x)$, $c_1 > 0$. Thus we have

$$\int_\Omega \frac{qw^2}{u_\lambda^{1-\alpha}}\mathrm{d}x \ge \int_\Omega \frac{qw^2}{c_1^{1-\alpha} d(x)^{1-\alpha}}\mathrm{d}x \ge \frac{1}{D^{1-\alpha} c_1^{1-\alpha}} \int_\Omega q w^2 \mathrm{d}x$$

where $d(x) \le D$ for any $x \in \Omega$. Finally, if

$$q(x) \ge q_0 > 0 \tag{24}$$

we obtain

$$\mu_2 > \lambda_2 - \lambda + \frac{q_0}{D^{1-\alpha} c_1^{1-\alpha}}.$$

It is clear that the condition $\mu_2 > 0$ is satisfied if $\lambda > \lambda_1$ but close to λ_1. A more interesting question is how to obtain sharper estimates for this interval. But in any case we have proved the following

Theorem 3.2. *Under the above assumptions (19) and (24) there exists $\lambda^* > \lambda_1$ such that for any $\lambda \in (\lambda_1,\, \lambda^*)$ there is a unique solution u_λ to* (1) *such that $u_\lambda > 0$ in Ω and $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$. Moreover, the curve $\lambda \to u_\lambda$ is smooth as a map from $(\lambda_1,\, \lambda^*)$ into $C_0^1(\overline{\Omega})$.*

Otherwise stated, the curve u_λ remains at the interior of the positive cone for some interval of λ' s. We will investigate in the following if this curve remains there for any value of λ or not. We know that if it leaves the interior of this positive cone, there should be for the corresponding solution u (at least) a point $a \in \Omega$ such that $u(a) = 0$ and/or a point $b \in \partial\Omega$ such that $\frac{\partial u}{\partial \nu}(b) = 0$. This would mean that the Strong Maximum Principle does not hold any more for this u. The rest of this Section is devoted to obtain some (partial) results in this direction.

3.1. A maximal range of the non-existence of compact support solutions.

From now on we assume that the boundary $\partial\Omega$ is a C^2-manifold. $\nu \equiv \nu(x_0) \in \mathbb{R}^N$ denotes the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$. As usual, we denote by $\mathrm{d}\sigma$ the surface measure on $\partial\Omega$. We will use as a main tool a Pohozaev's identity for a weak solution of $P(q, \alpha, \lambda)$:

Lemma 3.1. *Assume that $\partial\Omega$ is C^2-manifold and assumption (7) holds. Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1). *Then we have the Pohozaev identity*

$$\begin{aligned} &\frac{(N-2)}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \frac{N}{2} \int_\Omega |u|^2 \,\mathrm{d}x + \frac{N}{\alpha+1} \int_\Omega q(x)|u|^{\alpha+1} \,\mathrm{d}x + \\ &\qquad \frac{1}{\alpha+1} \int_\Omega (x \cdot \nabla q(x))|u|^{\alpha+1} \,\mathrm{d}x = -\frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 (x \cdot \nu(x)) \mathrm{d}\sigma(x). \end{aligned} \tag{25}$$

The proof will follow from the general result on Pohozaev's identity for a weak solution obtained in [28].

Theorem 3.2. (POHOZAEV's identity) *Assume that $f \in L^1(\Omega)$ possesses distributional derivatives $\partial f / \partial x_i \in L^1_{\mathrm{loc}}(\Omega)$; $i = 1, 2, \ldots, N$. Let $u \in C^1(\overline{\Omega})$ satisfying*

$$-\Delta u = f(x) \quad \text{in } \Omega \tag{26}$$

in the sense of distributions in Ω and $u|_{\partial\Omega} = 0$. Then the following Pohozaev identity holds

$$\begin{aligned} &\frac{N-2}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega f(x)\,(x \cdot \nabla u) \,\mathrm{d}x = \\ &\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2\,(x \cdot \nu(x)) \,\mathrm{d}\sigma(x) - \int_{\partial\Omega} (x \cdot \nabla u)(\nu(x) \cdot \nabla u) \,\mathrm{d}\sigma(x)\,. \end{aligned} \tag{27}$$

Proof of Lemma 3.1. Let us apply Theorem 3.2 to $P(q, \alpha, \lambda)$. Observe that $f(x) := \lambda u - q(x)|u|^{\alpha-1}u \in L^1(\Omega)$ and possesses distributional derivatives $\partial f / \partial x_i \in L^1_{\mathrm{loc}}(\Omega)$; $i = 1, 2, \ldots, N$, since the solution $u \in C^1(\overline{\Omega})$ and $q \in L^2(\Omega)$ by the assumption (7). Therefore by Theorem 3.2 we have

$$\begin{aligned} &\frac{N-2}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega (\lambda u - q(x)|u|^{\alpha-1}u)\,(x \cdot \nabla u) \,\mathrm{d}x = \\ &-\frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 (x \cdot \nu) \,\mathrm{d}\sigma(x), \end{aligned} \tag{28}$$

where we take into account that $\nabla u(x) = |\frac{\partial u(x)}{\partial \nu}|\nu(x)$ for $x \in \partial\Omega$, since $u \equiv 0$ on $\partial\Omega$.

Observe that $q(x)|u|^{1+\alpha} \in H_0^1(\Omega)$, since $u \in C^1(\overline{\Omega})$ and $q \in H^1(\Omega)$ (by (7)). Thus by the divergence theorem (see Theorem 1.2 in [36, p.9]) applied to the vector field $x \mapsto x \cdot (q(x)|u|^{\alpha+1})$ we obtain

$$\int_\Omega (x\cdot\nabla(q(x)|u|^{\alpha+1}))\,\mathrm{d}x = -N\int_\Omega q(x)|u|^{\alpha+1}\,\mathrm{d}x + \int_{\partial\Omega} q(x)|u|^{\alpha+1}\,(x\cdot\nu(x))\,\mathrm{d}\sigma(x) = -N\int_\Omega q(x)|u|^{\alpha+1}\,\mathrm{d}x,$$

where we take into account that $u = 0$ on $\partial\Omega$. From here, since

$$\int_\Omega (q(x)|u|^{\alpha-1}u)\,(x\cdot\nabla u)\,\mathrm{d}x = \frac{1}{\alpha+1}\int_\Omega (x\cdot\nabla(q(x)|u|^{\alpha+1}))\,\mathrm{d}x - \frac{1}{\alpha+1}\int_\Omega (x\cdot\nabla q(x))|u|^{\alpha+1}\,\mathrm{d}x.$$

one gets

$$\int_\Omega (q(x)|u|^{\alpha-1}u)\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{N}{\alpha+1}\int_\Omega q(x)|u|^{\alpha+1}\,\mathrm{d}x - \frac{1}{\alpha+1}\int_\Omega (x\cdot\nabla q(x))|u|^{\alpha+1}\,\mathrm{d}x. \tag{29}$$

In the same way we get that

$$\int_\Omega u\,(x\cdot\nabla u)\,\mathrm{d}x = -N\int_\Omega |u|^2\,\mathrm{d}x.$$

Replacing this and (29) into (28) gives (25).

■

The Pohozaev's functional, $P_\lambda : H_0^1(\Omega) \to \mathbb{R}$, is defined by

$$P_\lambda(u) := \frac{(N-2)}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \lambda\frac{N}{2}\int_\Omega |u|^2\,\mathrm{d}x + \frac{N}{\alpha+1}\int_\Omega q(x)|u|^{\alpha+1}\,\mathrm{d}x + \frac{1}{\alpha+1}\int_\Omega (x\cdot\nabla q(x))|u|^{\alpha+1}\,\mathrm{d}x$$

or equivalently by

$$P_\lambda(u) := \frac{(N-2)}{2}T(u) - \lambda\frac{N}{2}G(u) + \frac{N}{\alpha+1}A(u) + \frac{1}{(\alpha+1)}A_\nabla(u)$$

with the notations

$$T(u) := \int_\Omega |\nabla u|^2\,\mathrm{d}x, \quad G(u) := \int_\Omega |u|^2\,\mathrm{d}x, \quad A_\nabla(u) = \int_\Omega (x,\nabla q(x))|u|^{\alpha+1}\,\mathrm{d}x.$$

Thus Lemma 3.1 implies the

Corollary 3.1 *If $u \in C^1(\overline{\Omega})$ is a free boundary weak solution of (1) then $P_\lambda(u) = 0$. Under the additional assumption that Ω is strictly star-shaped the converse is also true: if $P_\lambda(u) = 0$ and if $u \in C^1(\overline{\Omega})$ is a weak solution of (1), then it is a free boundary solution.*

Recall that weak solutions u of $P(q,\alpha,\lambda)$ should satisfy the constraint $\Phi'_u(t)=0$. Thus the free boundary solution $u\in C^1(\overline{\Omega})$ of $P(q,\alpha,\lambda)$ should satisfy the system

$$\begin{cases} t^2T(u)-t^2\lambda G(u)+t^{1+\alpha}A(u)=0 \\ \\ \frac{t^2(N-2)}{2N}T(u)-\lambda\frac{t^2}{2}G(u)+t^{1+\alpha}(\frac{1}{\alpha+1}A(u)+\frac{1}{N(\alpha+1)}A_{\nabla}(u))=0. \end{cases} \tag{30}$$

Multiplying the first equation by 1/2 and subtracting it from it the second equality we derive

$$T(u)=t^{\alpha-1}\frac{N(1-\alpha)A(u)+2A_{\nabla}(u)}{2(1+\alpha)}. \tag{31}$$

Hence

$$t^{1-\alpha}=\frac{N(1-\alpha)A(u)+2A_{\nabla}(u)}{2(1+\alpha)T(u)}.$$

Note that $t>0$ if and only if $N(1-\alpha)A(u)+2A_{\nabla}(u)>0$ or equivalently

$$\int_{\Omega}[N(1-\alpha)q(x)+2(x,\nabla q(x))]|u|^{\alpha+1}\,\mathrm{d}x>0, \tag{32}$$

and under this assumption we may substitute $t^{1-\alpha}$ in (30). Then we obtain

$$\lambda=\Lambda(u):=\frac{T(u)}{G(u)}+\frac{2(1+\alpha)T(u)A(u)}{G(u)[N(1-\alpha)A(u)+2A_{\nabla}(u)]}. \tag{33}$$

Thus by Corollary 3.1 we have

Corollary 3.2. *Let $\lambda>\lambda_1$ and $u\in C^1(\overline{\Omega})$ be a free boundary solution of $P(q,\alpha,\lambda)$. Then*
i) $\int_{\Omega}[N(1-\alpha)q(x)+2(x,\nabla q(x))]|u|^{\alpha+1}\,dx>0$,
and
ii) $\lambda=\Lambda(u)$.

Remark 3.2. Notice that by (25) any weak solution $u\in C^1(\overline{\Omega})$ of $P(q,\alpha,\lambda)$ satisfies $P_\lambda(u)\le 0$. Thus if we consider this inequality in (30) instead of $P_\lambda(u)=0$, then we obtain that

$$\lambda\ge\Lambda(u) \tag{34}$$

for any solution $u\in C^1(\overline{\Omega})$ of $P(q,\alpha,\lambda)$ that satisfies (8).

We introduce

$$\mathcal{C}_q:=\{u\in H^1_0(\Omega)\setminus 0:\quad \int_{\Omega}[N(1-\alpha)q(x)+2(x,\nabla q(x))]|u|^{\alpha+1}\,\mathrm{d}x>0\}.$$

To find a maximal interval for non-existence of a free boundary solution of $P(q,\alpha,\lambda)$ consider

$$\lambda^c=\inf_{u\in\mathcal{C}_q}\Lambda(u). \tag{35}$$

Here by assumption $\lambda^c = +\infty$ if $\mathcal{C}_q = \emptyset$. We have

Lemma 3.2. *We have $\lambda^c > \lambda_1$. Furthermore, there exists a nonnegative minimizer $w_c \in C_q$ of* (35)*, i.e. $\lambda^c = \Lambda(w_c)$ and $w_c \geq 0$ in Ω.*

Proof. Observe that $\Lambda(tu) = \Lambda(u)$ for $t > 0$, i.e. $\Lambda(u)$ is a zero-homogeneous functional. Therefore (35) is equivalent to

$$\lambda^c = \inf_{u \in \mathcal{C}_q \cap S^1} \Lambda(u),$$

where $S^1 := \{u \in H_0^1(\Omega) : \quad ||u|| = 1\}$. Let (w_m) be a minimizing sequence of this problem. Since it is bounded in $H_0^1(\Omega)$, then by the Sobolev embedding and Eberlein-Ŝmulian theorems there are $w_c \in H_0^1(\Omega)$ and a subsequence (again denoted (w_m)) such that $w_m \rightharpoonup w_c$ weakly in $H_0^1(\Omega)$ and $w_m \to w_c$ strongly in $L^p(\Omega)$ for $p \in (1, 2^*)$. Suppose $w_c = 0$. Then $G(w_m) \equiv \int_\Omega |w_m|^2 \, \mathrm{d}x \to 0$. However in this case $T(w_m)/G(w_m) = 1/G(w_m) \to +\infty$ and consequently $\Lambda(w_m) \to +\infty$ as $m \to \infty$. Thus we get a contradiction and therefore $w_c \neq 0$. By the same arguments, it follows that $N(1-\alpha)A(w_c) + 2A_\nabla(w_c) > 0$. Thus $w_c \in \mathcal{C}_q$. By weak lower semi-continuity of $T(u)$ we get

$$\Lambda(w_c) \leq \liminf_{m \to \infty} \Lambda(w_m).$$

Since w_c is an admissible point of (35), then here only equality is possible. Thus $\lambda^c = \Lambda(w_c)$. Furthermore, we may assume that $w_c \geq 0$ since $\Lambda(w_c) = \Lambda(|w_c|)$.
Recall that by Rayleigh-Ritz inequality

$$\frac{T(u)}{G(u)} \equiv \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega |u|^2 \, \mathrm{d}x} \geq \lambda_1.$$

Therefore

$$\Lambda(w_c) = \frac{T(w_c)}{G(w_c)} + \frac{2(1+\alpha)T(w_c)A(w_c)}{G(w_c)[N(1-\alpha)A(w_c) + 2A_\nabla(w_c)]} \geq$$
$$\lambda_1\left(1 + \frac{2(1+\alpha)A(w_c)}{N(1-\alpha)A(w_c) + 2A_\nabla(w_c)}\right) > \lambda_1.$$

Lemma 3.3 For any $\lambda \in (\lambda_1, \lambda^c)$ problem $P(q, \alpha, \lambda)$ has no free boundary solution in $C^1(\overline{\Omega})$.

Proof. Let $\lambda < \lambda^c$. Suppose, contrary to our claim, that there exists a free boundary solution $u_\lambda \in C^1(\overline{\Omega})$ of $P(q, \alpha, \lambda)$, that is, u_λ satisfies $P(q, \alpha, \lambda)$ and (8). Then by Corollary 3.1 we have $\lambda = \Lambda(u_\lambda)$ and $u_\lambda \in \mathcal{C}_q$. Observe that by the definition of λ^c

$$\Lambda(u_\lambda) \geq \lambda^c.$$

On the other hand, by the assumption $\lambda < \lambda^c$ and therefore

$$\lambda < \lambda^c \leq \Lambda(u_\lambda) = \lambda.$$

Thus we get a contradiction.

3.2. On non-existence of compact supported solution on the whole line

In this Subsection, we study the case when problem $P(q,\alpha,\lambda)$ has no free boundary solution for all $\lambda > \lambda_1$. Observe that by Lemma 3.3 this will happen if $\lambda^c = +\infty$. From (35) this holds if and only if $\mathcal{C}_q = \emptyset$, that is

$$N(1-\alpha)A(u) + 2A_\nabla(u) = 0, \quad \forall u \in H_0^1(\Omega). \tag{36}$$

It is easy to see that (36) holds if and only if q satisfies the following linear partial differential equation

$$2(x, \nabla q(x)) + N(1-\alpha)q(x) = 0 \quad \text{a.e. in} \quad \Omega. \tag{37}$$

Thus we have

Lemma 3.4 *Assume $q \geq 0$ in Ω and (7) holds. Then problem $P(q,\alpha,\lambda)$ has no free boundary solutions if q satisfies equation* (37).

Let us find some particular solution to (37). Take $q(x) = |x|^\gamma$. Then (37) is rewritten as follows

$$2N\gamma|x|^\gamma + N(1-\alpha)|x|^\gamma = 0 \quad \Leftrightarrow \quad 2N\gamma + N(1-\alpha) = 0.$$

Thus if

$$\gamma = -\frac{1-\alpha}{2}$$

then $q(x) = |x|^\gamma$ satisfies (37). Let us verify whether $|x|^{-\frac{1-\alpha}{2}} \in H^1(\Omega)$. To this end it is sufficient to check the criterion

$$\int_\Omega |x|^{-2(\frac{1-\alpha}{2}+1)} dx < +\infty.$$

We see that it holds if

$$-2\frac{3-\alpha}{2} + N - 1 > -1 \quad \Leftrightarrow \quad \alpha > 3 - N.$$

Notice that, due to our assumption $\alpha \in (0,1)$, dimensions $N = 1, 2$ of the space $\mathbb{R}^N$ cannot be considered in this example. On the other hand when $N \geq 3$ this is satisfied for all $\alpha \in (0,1)$. Thus we have

Corollary 3.3. *Assume $g(x) = |x|^{-\frac{1-\alpha}{2}}$, $N \geq 3$ and $\alpha \in (0,1)$. Then problem $P(q,\alpha,\lambda)$ has no free boundary solution for any $\lambda > \lambda_1$.*

4. On the dependence on λ: asymptotic estimates

In this Section we shall study the dependence of solutions with respect to the parameter λ. We shall start by proving some continuity, monotonicity and differentiability properties of the energy functional E_λ with respect to λ. The Section ends with some asymptotic estimates when $\lambda \searrow \lambda_1$ and when $\lambda \nearrow +\infty$.

First of all, we emphasize that although we do not know whether, for each fixed $\lambda \in (\lambda_1, +\infty)$, the ground state u_λ of (1) is unique we know that the map

$$\lambda \mapsto \hat{J}_\lambda \equiv E_\lambda(u_\lambda)$$

is a single-valued function from $(\lambda_1, +\infty)$ into $\mathbb{R}^+$ by virtue of the unique determination of the energy of ground solutions.

Observe that by the zero-homogeneity of the functional $J_\lambda(v)$ on $H_0^1(\Omega)$ it follows that problem (11) is equivalent to the following minimization problem:

$$\hat{H}_\lambda = \min\{H_\lambda(w) : \quad A(w) = 1\}. \tag{38}$$

Moreover

$$\hat{J}_\lambda = \frac{(1-\alpha)}{2(1+\alpha)} \frac{1}{(-\hat{H}_\lambda)^{\frac{1+\alpha}{1-\alpha}}}. \tag{39}$$

Thus, we know that, for every $\lambda \in (\lambda_1, +\infty)$, there exists a minimizer w_λ of (38) and by Lemma 2.1 the function

$$u_\lambda = \left(\frac{1}{-H_\lambda(w_\lambda)}\right)^{1/(1-\alpha)} w_\lambda \equiv \left(\frac{1}{-\hat{H}_\lambda}\right)^{1/(1-\alpha)} w_\lambda. \tag{40}$$

is a weak solution of (1). In particular, the map

$$\lambda \mapsto \hat{H}_\lambda \equiv H_\lambda(w_\lambda)$$

from $(\lambda_1, +\infty)$ into $\mathbb{R}^+$ is a single-valued function.

Let us prove

Lemma 4.1 *Let $\lambda > \lambda_1$. Then for $h > 0$ small enough we have*

$$-hG(w_{\lambda+h}) \le H_{\lambda+h}(w_{\lambda+h}) - H_\lambda(w_\lambda) \le -hG(w_\lambda). \tag{41}$$

Proof. By the definition of the minimizer (38) we have

$$H_\lambda(w_{\lambda+h}) \ge H_\lambda(w_\lambda). \tag{42}$$

From here and since

$$H_\lambda(w_{\lambda+h}) = H_{\lambda+h}(w_{\lambda+h}) + hG(w_{\lambda+h})$$

we have

$$H_{\lambda+h}(w_{\lambda+h}) + hG(w_{\lambda+h}) \ge H_\lambda(w_\lambda). \tag{43}$$

Thus we get the first inequality in (41).
Observe now that

$$H_{\lambda+h}(w_{\lambda+h}) \le H_{\lambda+h}(w_\lambda). \tag{44}$$

Hence, since

$$H_{\lambda+h}(w_\lambda) = H_\lambda(w_\lambda) - hG(w_\lambda)$$

we deduce

$$H_\lambda(w_\lambda) - hG(w_\lambda) \ge H_{\lambda+h}(w_{\lambda+h}), \tag{45}$$

and the result holds.

In consequence, $H_\lambda(w_\lambda)$ is a monotone decreasing function in $(\lambda_1, +\infty)$ and consequently by (39) the same property holds for $J_\lambda(u_\lambda) = E_\lambda(u_\lambda)$. Noticing that by $H_\lambda(u_\lambda) = -A(u_\lambda)$ we can write

$$E_\lambda(u_\lambda) = -\frac{(1-\alpha)}{2(1+\alpha)} H_\lambda(u_\lambda) = \frac{(1-\alpha)}{2(1+\alpha)} A(u_\lambda).$$

Thus we get that $-H_\lambda(u_\lambda)$ and $A(u_\lambda)$ are also single-valued monotone decreasing functions. We get:

Corollary 1. ***Corollary 4.1.*** *Let $(a,b) \subset (\lambda_1, \infty)$. Then the set $(u_\lambda)|_{\lambda \in (a,b)}$ of ground solutions of* (1) *is uniformly bounded in $H_0^1(\Omega)$, i.e.*

$$||u_\lambda|| \le C_{a,b} < +\infty, \quad \forall \lambda \in (a,b),$$

with $C_{a,b}$ independent on λ.

Proof. Suppose, contrary to our claim, that there is a sequence $(\lambda_m) \subset (a,b)$ and $\lambda \in (a,b)$ such that $\lambda_m \to \lambda$ and $||u_{\lambda_m}|| \to \infty$ as $m \to \infty$. Let $v_m = u_{\lambda_m}/||u_{\lambda_m}||$, then $J_{\lambda_m}(v_m) = J_{\lambda_m}(u_{\lambda_m}) = \hat{J}_{\lambda_m}$. The boundedness of (v_m) in $H_0^1(\Omega)$ implies by the Sobolev embedding and Eberlein-Ŝmulian theorems that there is $\hat{v} \in H_0^1(\Omega)$ such that the subsequence of (v_m) (which we denote again (v_m)) converges $v_m \rightharpoonup \hat{v}$ weakly in $H_0^1(\Omega)$ and $v_m \to \hat{v}$ strongly in $L^p(\Omega)$ for $1 < p < 2^*$ as $m \to \infty$. Arguing as in the proof of Lemma 2.2 we obtain that $\hat{v} \ne 0$, $H_\lambda(\hat{v}) < 0$ and

$$J_\lambda(\hat{v}) \le \liminf_{m\to\infty} J_{\lambda_m}(v_m) \le \hat{J}_\lambda.$$

Since $\hat{v}$ is an admissible point of (11) then in this formula may be only equality. This implies, in particular, that $v_m \to \hat{v}$ strongly in $H_0^1(\Omega)$ as $m \to \infty$. Using this we obtain $t_{\lambda_m}(v_m) \to t_\lambda(\hat{v}) < +\infty$. Noticing now that $t_\lambda(v_m) = ||u_{\lambda_m}||$, $m = 1, 2, \ldots$, (see (10)) we obtain a contradiction.

From Lemma 4.1 and Corollary 4.1 we have:

Corollary 4.2. *The map $\lambda \mapsto H_\lambda(w_\lambda)$ is a differentiable function in $(\lambda_1, +\infty)$. Moreover*

$$\frac{d}{d\lambda} H_\lambda(w_\lambda) = -G(w_\lambda), \tag{46}$$

and the map $\lambda \mapsto G(w_\lambda)$ is a monotone nondecreasing function in $(\lambda_1, +\infty)$.

Since we have

$$\frac{d}{d\lambda} J_\lambda(w_\lambda) = \frac{d}{d\lambda}\left(\frac{(1-\alpha)}{2(1+\alpha)} \frac{1}{(-\hat{H}_\lambda)^{\frac{1+\alpha}{1-\alpha}}}\right) = \frac{1}{2(-H_\lambda(w_\lambda))^{\frac{2}{1-\alpha}}} \frac{d}{d\lambda} H_\lambda(w_\lambda), \tag{47}$$

by (40) and Corollary 4.2 we get:

Corollary 4.3. *The maps $\lambda \mapsto E_\lambda(u_\lambda)$ and $\lambda \mapsto J_\lambda(u_\lambda))$ are differentiable functions in $(\lambda_1, +\infty)$. Moreover,*

$$\frac{d}{d\lambda} E_\lambda(u_\lambda) = -\frac{1}{2} G(u_\lambda) \text{ and } \frac{d}{d\lambda} J_\lambda(u_\lambda) = -\frac{1}{2} G(u_\lambda). \tag{48}$$

4.1. Asymptotic estimates for $\lambda \to \lambda_1$

Let $v_\lambda = u_\lambda/||u_\lambda||$. Then $u_\lambda = t_\lambda(v_\lambda)v_\lambda$, $t_\lambda(v_\lambda) = ||u_\lambda||$ and $||v_\lambda|| = 1$. Let ϕ_1 be the eigenfunction corresponding to the first eigenvalue λ_1 normalized now in the sense that $||\phi_1|| = 1$. From Subsection 2.2 we get information which, in particular leads to the following result:

Corollary 4.4.

i) $||u_\lambda|| \equiv t_\lambda(v_\lambda) \to \infty$ as $\lambda \downarrow \lambda_1$,

ii) Let w_λ be a minimizer of (38), $\lambda > \lambda_1$, $\tilde{\phi}_1$ be the eigenfunction corresponding to the first eigenvalue λ_1 such that $A(\tilde{\phi}_1) = 1$. Then $t_\lambda(w_\lambda) \to \infty$ as $\lambda \downarrow \lambda_1$, $H_\lambda(w_\lambda) \to 0$ as $\lambda \downarrow \lambda_1$ and $w_\lambda \to \tilde{\phi}_1$ strongly in $H_0^1(\Omega)$ as $\lambda \downarrow \lambda_1$,

iii) Let (u_λ), $\lambda \in (\lambda_1, +\infty)$ be a set of ground solutions of (1). Then $E_\lambda(u_\lambda) \to \infty$ and $-H_\lambda(u_\lambda) = A(u_\lambda) \to \infty$ as $\lambda \downarrow \lambda_1$.

*Proof***.** i) and iii) follows quite directly from the results of this and previous sections. To get ii) we shall prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any λ with $\lambda - \lambda_1 < \delta$ we have $||w_\lambda - \tilde{\phi}_1||_{H_0^1} < \epsilon$. Indeed, suppose, contrary to our claim, that there exists $\epsilon_0 > 0$ and a sequence $\lambda_m \downarrow \lambda_1$ as $m \to \infty$ such that

$$||w_{\lambda_m} - \tilde{\phi}_1||_{H_0^1} > \epsilon_0.$$

From the results of this this and previous sections we get that there exists a subsequence λ_{m_i}, $i = 1, 2, ...$, such that $w_{\lambda_{m_i}} \to \tilde{\phi}_1$ strongly in $H_0^1(\Omega)$ as $i \to \infty$. Thus we get a contradiction.

Remark 4.1. We point out that it is possible to get a direct proof of the above results without passing by the results mentioned in Subsection 2.2.

The main result of this subsection is the following asymptotic estimate when $\lambda \to \lambda_1$:

Proposition 4.1. *Let (u_λ), $\lambda \in (\lambda_1, +\infty)$ be a set of ground solutions of (1). Then*

$$u_\lambda = \frac{1}{(\lambda - \lambda_1)^{\frac{1}{1-\alpha}}} \cdot \frac{1}{G(\tilde{\phi}_1)^{\frac{1}{1-\alpha}}} \cdot \tilde{\phi}_1 + \frac{1}{(\lambda - \lambda_1)^{\frac{1}{1-\alpha}}} \cdot o(1) \quad \text{as} \quad \lambda \downarrow \lambda_1 \tag{49}$$

where $o(1)$ represents a function $\theta(\lambda) \in H_0^1(\Omega)$ such that $||\theta(\lambda)|| \to 0$ as $\lambda \downarrow \lambda_1$.

Proof. Let (u_λ), $\lambda \in (\lambda_1, +\infty)$ be a set of ground states of (1). As we know $w_\lambda = u_\lambda/A(u_\lambda)$ is a minimizer of (38). By Corollary 4.4 we know that $H_\lambda(w_\lambda) \to 0$ as $\lambda \downarrow \lambda_1$, and $G(w_\lambda) \to G(\tilde{\phi}_1) > 0$ as $\lambda \downarrow \lambda_1$. Thus (46) implies

$$H_\lambda(w_\lambda) = -(\lambda - \lambda_1)G(\tilde{\phi}_1) + o(\lambda - \lambda_1) \quad \text{as} \quad \lambda \downarrow \lambda_1. \tag{50}$$

Hence by Taylor series expansions we have

$$t_\lambda(w_\lambda) = \left(\frac{1}{-H_\lambda(w_\lambda)}\right)^{1/(1-\alpha)} = \left(\frac{1}{(\lambda - \lambda_1)G(\tilde{\phi}_1) + o(\lambda - \lambda_1)}\right)^{1/(1-\alpha)} =$$

$$\frac{1}{(\lambda - \lambda_1)^{\frac{1}{1-\alpha}}} \cdot \frac{1}{G(\tilde{\phi}_1)^{\frac{1}{1-\alpha}}} + \frac{1}{(\lambda - \lambda_1)^{\frac{1}{1-\alpha}}} \cdot o(1).$$

as $\lambda \downarrow \lambda_1$. This and (40) yield

$$u_\lambda = \left(\frac{1}{(\lambda-\lambda_1)^{\frac{1}{1-\alpha}}} \cdot \frac{1}{G(\tilde{\phi}_1)^{\frac{1}{1-\alpha}}} \right) w_\lambda + \frac{1}{(\lambda-\lambda_1)^{\frac{1}{1-\alpha}}} w_\lambda \cdot o(1) \tag{51}$$

as $\lambda \downarrow \lambda_1$. From here we get

$$||u_\lambda - \left(\frac{1}{(\lambda-\lambda_1)^{\frac{1}{1-\alpha}}} \frac{1}{G(\tilde{\phi}_1)^{\frac{1}{1-\alpha}}} \right) \tilde{\phi}_1||_1 \quad \cdot \quad (\lambda-\lambda_1)^{\frac{1}{1-\alpha}} \leq$$

$$\frac{1}{G(\tilde{\phi}_1)^{\frac{1}{1-\alpha}}} ||\tilde{\phi}_1 - w_\lambda||_1 + o(1)$$

Using now Corollary 4.4 we get the proof of the proposition.

5. On the existence of solutions with compact support

Our first result deals with a simple case:

Proposition 5.1. *Let $q(x) \equiv q_0 > 0$ and $\Omega = B_R(0)$. Then there exists $\lambda^* > \lambda_1$ such that:*
a) if $\lambda = \lambda^*$ there exists a unique radially symmetric positive solution u_{λ^*} such that $\frac{\partial u_{\lambda^*}}{\partial \nu} = 0$ on $\partial\Omega$.
b) if $\lambda > \lambda^*$ there is a family of nonnegative solutions u_λ with compact support in the sense that

$$\textit{support } (u_\lambda) \subsetneq \Omega.$$

Proof. We make the change of variables

$$u_\lambda(x) = \left(\frac{q_0}{\lambda}\right)^{\frac{1}{1-\alpha}} U(\sqrt{\lambda}x) \tag{52}$$

with U the solution of the special problem $P(1,1,1)$, i.e.

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u = u & \text{in } B_{\sqrt{\lambda}R}(0), \\ u = 0 & \text{on } \partial B_{\sqrt{\lambda}R}(0). \end{cases} \tag{53}$$

Then, if we define $\lambda^* = 1/R^2$ the transformed problem $P(1,1,1)$ takes place on the ball $B_1(0)$ and thus it is enough to apply the results [22] to get conclusion a) for the radially symmetric solution $u_{\lambda^*}(x) = u_{\lambda^*}(|x|)$. Since this solution can be exteded to the whole $\mathbb{R}^N$ by zero outside $B_R(0)$ we can introduce a change of variables leading to the relation

$$u_\lambda(x) = \left(\frac{\lambda^*}{\lambda}\right)^{\frac{1}{1-\alpha}} u_{\lambda^*}(\frac{\sqrt{\lambda}}{\sqrt{\lambda^*}}x),$$

and thus

$$\|u_\lambda\|_\infty = \left(\frac{\lambda^*}{\lambda}\right)^{\frac{1}{1-\alpha}} \|u_{\lambda^*}\|_\infty$$

and for any $x_0 \in B_R(0)$ such that $B_{\frac{\sqrt{\lambda^*}}{\sqrt{\lambda}}R}(x_0) \subset B_R(0)$ we can construct the solution

$$u_\lambda(x;x_0) := \begin{cases} \left(\frac{\lambda^*}{\lambda}\right)^{\frac{1}{1-\alpha}} u_{\lambda^*}(\frac{\sqrt{\lambda}}{\sqrt{\lambda^*}}|x-x_0|) & \text{on } B_{\frac{\sqrt{\lambda^*}}{\sqrt{\lambda}}R}(x_0), \\ 0 & \text{on } B_R(0) \setminus B_{\frac{\sqrt{\lambda^*}}{\sqrt{\lambda}}R}(x_0), \end{cases}$$

which proves b).

Remark 5.1. As in the one-dimensional case, for $q(x) \equiv q_0 > 0$ (see [13]), many other different consequences can be obtained from the change of variable (52) and Proposition 5.1. For instance we get the exact decay, when $\lambda \to +\infty$, of all the energies as well as of $\|u_\lambda\|_\infty$. We also recall that it was shown in [23] that in the case of $q(x) \equiv q_0$ any solution with compact support of the equation of $P(1,1,1)$ on the whole space $\mathbb{R}^N$ must be radially symmetric. That was one of our main motivations to deal in this paper with more general coefficients $q(x)$.

For the consideration of the general case of $q(x)$ satisfying (24) we shall need an extra information which is well-known in some circunstances. We shall require the information that

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = 0. \tag{54}$$

Remark 5.2. Two completely different proofs were given on such property, for the special case of $q(x) \equiv q_0 > 0$: one in [13] (for the case $N = 1$) and a different one in [26] (for $N \leq \frac{2(1+\alpha)}{(1-\alpha)}$). As a matter of fact, it is very easy to see that the proof given in [26] holds also if $q(x)$ is not constant but it satisfies (24). We believe that it is possible to get a proof of this property in any dimension and for any bounded $q(x)$ satisfying (24) by means of rearrangement techniques but we shall not pursuit this goal in this paper.

We have:

Theorem 5.2. *Let $q(x)$ satisfying (24) and assume (54). Then for any $\lambda > \lambda_1$ large enough the ground solution u_λ has compact support in Ω.*

Proof. The function

$$f(x,u) := q(x)u^\alpha - \lambda u$$

is a nondecreasing function of u, a.e. $x \in \Omega$, at least on the interval $u \in [0, \delta_\lambda]$, where

$$\delta_\lambda := \frac{(\alpha q_0)^{\frac{1}{1-\alpha}}}{\lambda^{\frac{1}{1-\alpha}}}.$$

Then, thanks to the assumption (54), for any $\lambda > \lambda_1$ large enough the expression $f(x, u_\lambda)$ is a nondecreasing function of the ground solution u_λ a.e. $x \in \Omega$ and thus the comparison principle holds. Now it is a routine matter to check that for any $\lambda > \lambda_1$ large enough Theorem 1.9 of [10] can be applied since

$$\int_{0^+} \frac{ds}{\sqrt{\alpha q_0 \frac{s^{\alpha+1}}{\alpha+1} - \frac{\lambda}{2}s^2}} < +\infty.$$

In particular, if for any $\mu > 0$ we define the function $\psi_\mu : [0,+\infty) \to [0,+\infty)$ by

$$\psi_\mu(\tau) := \frac{1}{\sqrt{2\mu}} \int_0^\tau \frac{ds}{\sqrt{\alpha q_0 \frac{s^{\alpha+1}}{\alpha+1} - \frac{\lambda}{2} s^2}},$$

then we get that for any $x_0 \in \Omega$ the function

$$U(x; x_0) = \eta_{\frac{1}{N}}(|x - x_0|)$$

is a local supersolution of the equation on $\Omega_R := B_R(x_0) \cap \Omega$, where η_μ is the inverse function of ψ_μ

$$\eta_\mu(s) = (\psi_\mu)^{-1}(s),$$

assumed that

$$R \geq \psi_{\frac{1}{N}}(\delta_\lambda). \tag{55}$$

Notice that since u_λ is a continuous function this implies that $u_\lambda(x_0) = 0$. But from assumption (54) we know that this holds for any $x_0 \in \Omega$ such that

$$d(x_0, \partial\Omega_R - \partial\Omega) \geq \psi_{\frac{1}{N}}(C(\lambda)),$$

for some $C(\lambda) > 0$ and for λ large enough. Moreover this set of points x_0 of Ω is not empty if λ is large enough. This gives the compactnees of the support of u_λ for λ large enough.

Remark 5.3. Of course that once that the ground solution has compact support its continuation to the rest of $\mathbb{R}^N$ by zero outside Ω generates a solution of the same equation (taking as $q(x)$ any extension to $\mathbb{R}^N$ outside Ω) having compact support (contained in Ω). In this way, Theorem 5.1 contains the result obtained in [24], [22], [5] and specially [16] where the solution is not required to be radially symmetric and $q(x)$ is not necessarily to be constant.

Remark 5.4. With the usual philosophy of reaction-diffusion equations giving rise to a free boundary, we have shown that, in the case of problem (1) the "diffusion-absorption balance" condition on the nonlinearities is obviously satisfied since $\alpha < 1$ and that the "balance condition between the data and the domain" is here represented by means of the requirement to assume λ large enough (as a function of q_0 and $diam\Omega$). The simultaneous fulfillment of both balances are required in order to get solutions with compact support (see Section 1.2 of [10]).

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