RACSAM
DOI 10.1007/s13398-014-0198-y

ORIGINAL PAPER

Geometrical evolution of volcanoes: a theoretical approach

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Received: 17 December 2013 / Accepted: 3 October 2014

Abstract Shape of many volcanic edifices depend on different phenomena, such as parasitic cones, erosion or coral growth. A nonlinear model proposed in 1981 proves that the shape of volcanoes is determined by the hydraulic resistance to the flow of magma, along a line, through the porous edifice. This model was later extended to include the shape of aseismic and submarine ridges. In this paper we propose a modification of the above mentioned models in order to simulate the more realistic case of volcanoes growth assuming they have a limited base. We present the 3D extension and a generalization of the model. We formulate a new model including the case of a possible outpointing flow.

Keywords Geometric of volcanoes · Limited base · Degenerate parabolic equation · Bounded free boundary

Mathematics Subject Classification 76S05 · 35K55 · 35R35

1 Introduction

The Earth structure can be considered as divided into different layers, such as the crust, upper and lower mantle, upper and lower core. The outermost part of the Earth formed by the crust and upper mantle is rigid, and is termed the lithosphere. It is undelain by the viscous asthenosphere. The lithosphere is divided into plates that move relative to each other, and with respect to the asthenosphere. Three boundaries are defined by these plates: divergent

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Published online: 14 October 2014

boundary, convergent boundary and transform boundary. In the first case, the plates move apart causing the ascent of mantle material to create new ocean floor. It occurs mainly at mid-ocean ridges. In the case of convergent plate boundaries two plates, at least one of which is oceanic, move toward each other. This sinking process is called as subduction. Finally, at transform boundary the plates plates slide past each other and not tearing or crunching each other.

Volcanoes abound at the tectonic plate boundaries and the type of volcano that forms depends on whether the plates are moving apart or together. We find different formations depending on the phenomenon: ridges that form at divergent boundaries, island arcs and continental arcs at convergent boundaries and hot spots (or aseismic ridges).

In this paper we will deal with a model of the shape of submarine edifices. Volcanoes can have different shapes, depending on the existence of certain phenomena are present in the process of eruption. Composite volcanoes, or stratovolcanoes, and seamounts are convex with steep slopes near the summit. The volcanoes that appear in the hot spots, however, are generally concave with steep slopes on the lower flanks. However, there are phenomena that can change this geometry. Examples include parasitic cones and erosion. The forms of the seamounts may be affected by the growth of corals, and even the growth of volcanoes on the flanks. The morphology of the volcanoes has been studied over the past centuries. Milne [26,27] proposed that the shape of stratovolcano or composite volcano type in central Japan was due to the stability of its slopes. Lacey et al. [24] suggested that the shape of volcanoes is determined by the hydraulic resistance to the flow of magma through the volcanic edifice. They consider that the building surface is a surface of constant hydraulic potential and assume that the volcanic edifice has a uniform permeability.

The porous flow model for the shape of volcanoes does not require constant flow of magma through the structure. When starting an eruption, magma seeks the path of least resistance to the surface. Once the magma has found the way, this indicates the place of the eruption, and the surface will begin to build on the outside with the material, increasing the resistance of the path. Therefore, the successive eruptions may follow different paths to reach the surface. The sum of a long series of eruptions is a surface that approximates a constant hydraulic potential surface.

Many volcanic edifices have shapes depending on different phenomena, such as parasitic cones, erosion or coral growth. In [24], a nonlinear model proving that the shape of volcanoes is determined by the hydraulic resistance to the flow of magma, along a line, through the porous edifice is proposed. This model was later extended by Angevine et al. [2] to study the shape of aseismic and submarine ridges. In this work we propose a modification of the above mentioned models in order to get a more realistic modelling of the volcanoes growth (since the volcanoes have a limited base). We present the 3D extension of the model proposed by Lacey et al. [24] and a generalization of the model proposed by Angevine et al. [2]. We formulate a new model including the case of a possible outpointing flow.

Finally, we prove that the free boundary (the volcano base) associated to the models described in the above mentioned references (and so for a one dimensional variable) is not bounded as $t \to +\infty$ (even if it is assumed that the flux generated by the magma supply $Q_0(t)$ along a line is a bounded function). As mentioned before, this unrealistic fact (specially in the case of volcanoes located in islands) is the main motivation to propose a modification of the involved nonlinear equations in order to obtain a new model giving rise to a bounded free boundary (even as $t \to +\infty$). By using suitable variations of the modelling arguments given by Angevine et al. [2] and Lacey et al. [24] we propose the new model,

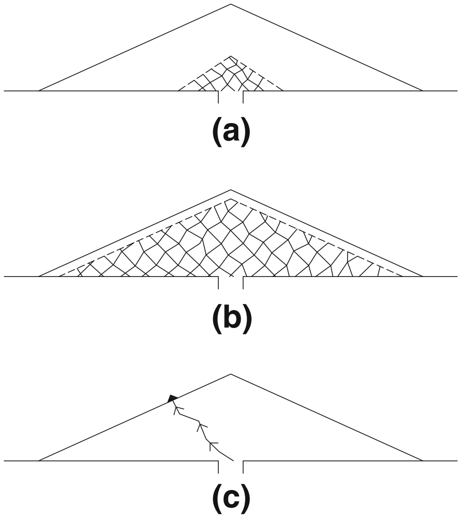


Fig. 1 Figure of migration of the magma through a volcanic edifice during an eruption. **a** Magma migrates outward. **b** The magma reaches the surface searching the point of minimum resistance to the flow. **c** Outpointing flow (Modified by Lacey et al. [24])

$$P(\mu; Q_0) \equiv \begin{cases} \dfrac{\partial H}{\partial t} = K\dfrac{\partial^2 H^2}{\partial x^2} + \dfrac{\mu x}{|x|}\dfrac{\partial H^\lambda}{\partial x}, & x \in \mathbb{R} - \{0\}, t > 0 \\ \lim_{x\to 0\pm}\left(\mp\dfrac{\partial H^2}{\partial x}(x,t) \mp \dfrac{\mu x}{|x|}H^\lambda(x,t)\right) = Q_0(t), & t > 0, \\ H(0,x) = H_0(x), & x \in \mathbb{R} - \{0\}, \end{cases} \tag{1}$$

where the meaning of $H \in C([0, +\infty) : L^1(\mathbb{R} - \{0\}))$, the boundary condition and the assumptions on the data will be detailed in Sect. 5. Here we assume the constants K, $\mu, \lambda > 0$ (which depend on the constitutive porous material) are known and that $Q_0(t) \geq 0$, $H_0(x) \geq 0$ and H_0 has compact support in $\mathbb{R} - \{0\}$. The models proposed by Angevine et al. [2] and Lacey et al. [24] correspond to $\mu = 0$. We prove that if $\lambda \in (0, 2)$ and $Q_0(t)$ is a bounded function then the free boundaries are uniformly bounded for any time: i.e., if we denote by $\xi_\pm(t)$ the free boundaries given by the boundary of the support of $H(t; \cdot)$, supp $H(t; \cdot) = [\xi_-(t), 0] \cup [0, \xi_+(t)]$, then necessarily $|\xi_\pm(t)| < \xi_\infty$ for any $t > 0$, for some $\xi_\infty < +\infty$. This conclusion leads to a better comparison between the bathymetric and theoretical profiles of many volcanoes.

2 Extension to 3*D*-shapes

We assume that the shape of the structure created by the magma is given by the surface $z = h(x, y, t)$. As explained by Lacey et al. [24], volcanic edifices are composed of a large number of lava flows. This formation is illustrated in parts (a) and (b) of Fig. 1. Their assumption about the lava flows through pre-existing matrix of channels of precedent emissions, searching the least resistance path to the surface appears correct if we apply porous media laws (e.g. the Darcy's law). On the contrary, the case (c) of Fig. 1 needs a new and more complicated hypothesis. The extension of the volcanic edifice by the flow which reaches the surface of the precedent edifice can not be treated by means of the Darcy's law. The free boundary condition needs to be corrected in a similar way as was done for the study of the "seepage face" in the case flows through a porous dam (see [10,31]).

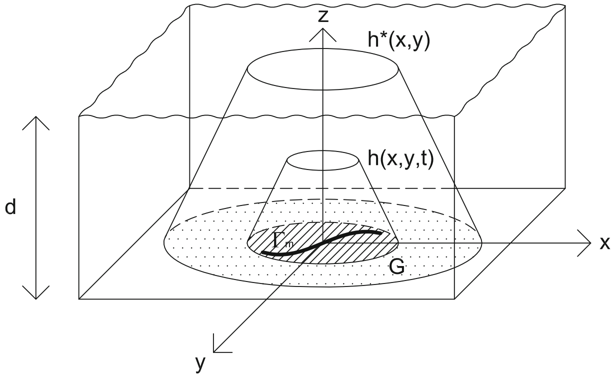


Fig. 2 Schematic profile of a pre-existing submarine volcanic ridge h^* and the structure created by the magma, h. The magma reaches surface floor through a curve Γ_m

We assume media given by two fluids: the ocean water (an incompressible viscous fluid of density ρ_w and viscosity μ_w) and the magma (which here is assumed to be a incompressible Newtonian fluid of density ρ_m and viscosity μ_m). The interface is considered to be given by the surface $z = h(x, y, t)$. The flow spatial domain is as follows: the pre-existing volcanic edifice is assumed to be known and given by a surface of the form $z = h^*(x, y)$. We consider that the depth of the ocean floor is d (so $d > h^*(x, y), \forall(x, y) \in \mathbb{R}^2$) (see Fig. 2). Notice that the support of h^* is the (known) set $G \subset \mathbb{R}^2$ (i.e. $h^*(x, y) = 0\ \forall(x, y) \in \mathbb{R}^2 - G$), such that its projection on the x-component and/or y-component is a bounded interval.

Remark 1 In the case of the symmetric volcanoes the pre-existing edifice can be assumed of revolution type

$$z = h^*(r), \quad \text{where}\, r = (x^2 + y^2)^{1/2} (\text{see [24]}).$$

In the case of seamounts the pre-existing edifice can be assumed y-independent:

$$z = h^*(x).$$

We assume a given (and so to be known) curve Γ_m of the (x, y)-plane, where the magma reaches the surface floor $z = 0$. In the case of symmetric volcanoes $\Gamma_m = \{(0, 0)\}$ is reduced to a point and in the case of seamounts $\Gamma_m = \{(x, 0) : x \in \mathbb{R}\}$. The velocity vector of the magma flow will be denoted by:

$$\mathbf{v} = u\mathbf{e_1} + v\mathbf{e_2} + w\mathbf{e_3} \text{ (i.e. } \mathbf{v} = (u, v, w))$$

where $\mathbf{e_i}$ is the unit vector in the i-direction. The governing equations for the flow are: the incompressibility condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2)$$

and the Darcy's law corresponding to a flow in a rigid isotropic porous medium (see e.g. [31])

$$u = -\frac{k}{\mu\phi}\frac{\partial p}{\partial x}, \qquad (3)$$

$$v = -\frac{k}{\mu\phi}\frac{\partial p}{\partial y}, \qquad (4)$$

$$w = -\frac{k}{\mu\phi}\left(\frac{\partial p}{\partial z} + \rho_m g\right). \tag{5}$$

Here, p is the magma pressure, ρ_m is the magma density, μ is the magma dynamic viscosity and k and ϕ are the permeability and the porosity of the porous pre-existing magmatic edifice. g is the value of the Earth gravity (but the theory remains true for other planets: see some remarks on the case of Mars in [24]). Notice that the ocean water could be assumed in movement with a velocity:

$$\mathbf{v}_w = u_w\mathbf{e_1} + v_w\mathbf{e_2} + w_w\mathbf{e_3}.$$

By the incompressibility condition (for the ocean water) we also know that:

$$\frac{\partial u_w}{\partial x} + \frac{\partial v_w}{\partial y} + \frac{\partial w_w}{\partial z} = 0 \tag{6}$$

and that we must impose the compatibility condition in the free boundary

$$\mathbf{v}(x, y, h(x, y, t)) = \mathbf{v}_w(x, y, h(x, y, t), t). \tag{7}$$

The complete description of $\mathbf{v}_w$ requires the usual Navier-Stokes system for a viscous incompressible flow

$$\rho_w\left(\frac{\partial \mathbf{v}_w}{\partial t} + (\mathbf{v}_w \cdot \nabla)\,\mathbf{v}_w\right) - \mu_w \Delta \mathbf{v}_w = -\nabla p_w - \rho_w g \mathbf{e_3}. \tag{8}$$

Remark 2 Notice that we can not apply the Darcy's law for $\mathbf{v_w}$ beyond the pre-existing magmatic edifice (where is located the porous medium). We could apply the Darcy's law only to the region where the ocean water occupies the upper part of the pre-existing magmatic edifice which is not occupied yet by the magma.

Remark 3 Notice that we consider that the inertia of the magma is negligible and the magmatic flow can be considered very slow [2,24]. We neglect the t-dependence in the magmatic flow (so, $\mathbf{v}$ is assumed to be t-independent).

On the magmatic region $\mathcal{M}$(t), we can use the conditions (3), (4) and (5) substituted in Eq. (2) to get an equation for the magmatic pressure:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \quad \text{in} \quad \mathcal{M}(\mathrm{t}), \tag{9}$$

(here we are using the decomposition of the spatial domain $\mathbb{R}^2 \times [0, d] = \mathcal{O}(t) \cup \mathcal{M}(t)$ where $\mathcal{M}(t)$ is the region occupied by the magma, at time t, and $\mathcal{O}(t)$ the region occupied by the ocean water, at time t). Notice that we are assuming that the boundary of $\mathcal{M}(t)$ is given by:

$$\partial\mathcal{M}(t) = \underbrace{\{z = h(x, y, t)\}}_{(I)} \cup \underbrace{\{z = 0; (x, y) \in \text{ support of } h\}}_{(II)},$$

where (I) is the upper surface of the new magmatic region and (II) is the base of the new magmatic region:

$$\partial\mathcal{M}(t) = \underbrace{\mathcal{P}(t)}_{\text{profile}} \cup \underbrace{\mathcal{B}(t)}_{\text{base}}$$

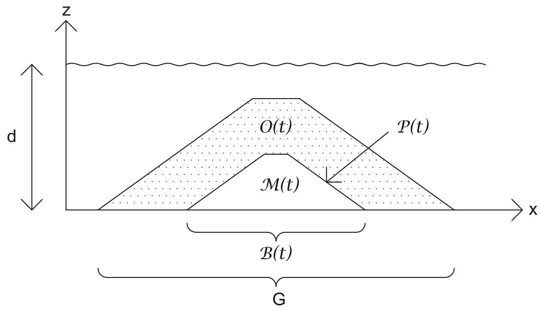


Fig. 3 Vertical cross section of the domain

(notice that necessarily $\mathcal{B}(t) \subset G$, $\forall t \geq 0$, where G is the base of the pre-existing edifice, see Fig. 3). The crucial part to derive the differential equation of the movement of the free boundary (interface or profile) $z = h(x, y, t)$ is obtained through the formulation of the interface conditions [now formulated in terms of pressure (besides the compatibility condition (7)].

Since the free boundary is the fluid surface of the magma (i.e., $z = h(x, y, t)$ represents the space occupied, at time $t > 0$, of the particles which originally where in other place at $t = 0$) then, necessarily,

$$w(x, y, h(x, y, t)) = \frac{\partial h}{\partial t}(x, y, t) + u(x, y, h(x, y, t))\frac{\partial h}{\partial x}(x, y, t) + v(x, y, h(x, y, t))\frac{\partial h}{\partial y}(x, y, t). \tag{10}$$

(see e.g. [21,29]). Moreover, at this point, we follow the fundamental assumption of the papers [2,24] which suppose that the free boundary $z = h(x, y, t)$ is a surface of constant hydraulic potential, i.e.,

$$p(x, y, h(x, y, t)) = p_w(x, y, h(x, y, t)), \tag{11a}$$

$$p_w(x, y, h(x, y, t)) = \rho_w g(d - h(x, y, t)). \tag{11b}$$

Notice that condition (11b) assumes the ocean water flow is static, in which case (8) is reduced to:

$$\nabla p_w = -\rho_w g \mathbf{e_3} \left(\text{i.e.} \frac{\partial p_w}{\partial x} = \frac{\partial p_w}{\partial y} = 0 \text{ and } \frac{\partial p_w}{\partial z} = -\rho_w g\right)$$

and to integrate the z-component from $z = h(x, y, t)$ to $z = d$ we use the renormalization that $p(x, y, d) = 0$ at the top of the ocean. This is the Pascal's approach. As usual, the function $p_w + \rho_w g z$ is denoted as the hydraulic potential (since its gradient gives the total force in the static case).

Remark 4 This assumption is a little bit controversial (see e.g. [8,32]). [We shall replace it by a corrected version of it further on.]

We also need the (external) boundary conditions. On the top and the floor (which is not the magma supply curve Γ_m), it is enough to formulate them in terms of pressure:

$$p_w(x, y, d) = 0, \tag{12}$$

(already used to get (11b))

$$\frac{\partial p}{\partial z}(x, y, 0) = -\rho_m g \text{ if } (x, y) \in \mathcal{B}(t) - \Gamma_m, \tag{13a}$$

$$\mathbf{v_w}(x, y, 0, t) \equiv \mathbf{b}(x, y, t) \quad \text{if } (x, y) \in \mathbb{R}^2 - (\mathcal{B}(t) - \Gamma_m). \tag{13b}$$

Notice that (13a) is equivalent to the condition

$$w(x, y, 0) = 0 \quad \text{in } (x, y) \in \mathcal{B}(t) - \Gamma_m$$

obtained by means of the Darcy's law (5). As that is the nonpenetration condition, $\mathbf{v} \cdot \mathbf{n} = 0$, with $\mathbf{n} = -\mathbf{e_3}$ the unit external normal vector at this part of the boundary of the spatial domain. The condition (13b) will not be used in the following, but it is assumed to be given a direction for the ocean water $\mathbf{b}(x, y, t)$ which can not have a vertical component (13b) can then be replaced by another condition of the form:

$$\frac{\partial p_w}{\partial z}(x, y, 0) = -\rho_w g \quad \text{if } (x, y) \in \mathbb{R}^2 - (\mathcal{B}(t) - \Gamma_m) \tag{13c}$$

for the case of the static ocean flow (as assumed in the fundamental assumption of Angevine et al. [2] and Lacey et al. [24] recalled before). Notice that (13c) is verified always in the part of the base of the pre-existing edifice occupied by the water since the Darcy's law applied to this part would imply that:

$$u_w = -\frac{k}{\rho_w \phi} \frac{\partial p_w}{\partial x}, \tag{2w}$$

$$v_w = -\frac{k}{\rho_w \phi} \frac{\partial p_w}{\partial y}, \tag{3w}$$

$$w_w = -\frac{k}{\rho_w \phi} \left(\frac{\partial p_w}{\partial z} + \rho_w g \right), \tag{4w}$$

and at $z = 0$ we must impose that $w_w = 0$. It remains to formulate the *magma supply through the curve* Γ_m, but that shall made later since it is not used to get the differential equation for $h(x, y, t)$. Now we shall use an asymptotic analysis to get the differential equation for $h(x, y, t)$. In contrast to Lacey et al. [24] we shall not use the Dupuit approximation (assuming that $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are very small). As in [2], we shall use as small parameter, ϵ, the aspect ratio (all those magmatic edifices have a small hight in comparison with the size of their base).

Remark 5 In the paper [8] the small parameter Q is taken differently as $Q = \frac{q}{z_0^2 \sqrt{K_x K_y}}$ where q is the emitted magma, z_0 the hight and K_x, K_y the principal values (in x and y) of the hydraulic conductivity tensor.

To follow closely the notation in [2] we assume (a priori) a general expansion of the form:

$$p = P_0 + \varepsilon p_0 + \varepsilon^2 P_1 + \varepsilon^3 p_1 + \cdots \tag{14}$$

Before substituting in the set of condition given before, it is useful to modify the z-component by defining:

$$Z = \frac{z}{\varepsilon}$$

with $\varepsilon << 1$ (which will subsequently disappear from the analysis). We also introduce the rescaled depth:

$$D = \frac{d}{\varepsilon}$$

As in [2], we are only interested in solutions at large times (mature magmatic edifices), so we shall also modify time as:

$$T = t\varepsilon.$$

Finally, the new rescaled free boundary is given by:

$$H(x, y, T) = \frac{h\left(x, y, \frac{T}{\varepsilon}\right)}{\varepsilon} \tag{15}$$

i.e., T of the order of the unity, which implies that $t >> 1$ since $\varepsilon << 1$. Now, we must modify all the above conditions. Therefore, for instance, condition (9) leads to:

$$\varepsilon^2 \frac{\partial^2 p}{\partial x^2} + \varepsilon^2 \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial Z^2} = 0, \tag{16}$$

conditions (11a) and (11b) lead to:

$$p(x, y, H(x, y, T)) = \varepsilon \rho_w g(D - H), \tag{17}$$

condition (10) leads to:

$$w = \varepsilon^2 \frac{\partial H}{\partial T} + \varepsilon u \frac{\partial H}{\partial x} + \varepsilon v \frac{\partial H}{\partial y}, \quad \text{if } Z = H(x, y, T) \tag{18}$$

and, finally, condition (13a):

$$\frac{\partial p}{\partial Z} = -\varepsilon \rho_m g, \quad \text{if } Z = 0 \quad \text{and} \quad (x, y) \in \mathcal{B}(t) - \Gamma_m. \tag{19}$$

Substituting (14) in (16) and rearranging terms, we obtain:

$$\frac{\partial^2 P_0}{\partial Z^2} + \varepsilon \frac{\partial^2 p_0}{\partial Z^2} + \varepsilon^2 \left(\frac{\partial^2 P_0}{\partial x^2} + \frac{\partial^2 P_0}{\partial y^2} + \frac{\partial^2 P_1}{\partial Z^2} \right) + \epsilon^3 \left(\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2} + \frac{\partial^2 p_1}{\partial Z^2} \right) + \cdots = 0. \tag{20}$$

Since (20) must be true for any $\varepsilon > 0$ small enough, all the coefficients must vanish. Then, in particular,

$$\frac{\partial^2 P_0}{\partial Z^2} = 0, \tag{21}$$

which implies that:

$$P_0(x, y, Z) = a(x, y)Z + b(x, y) \tag{22}$$

for functions $a(x, y)$ and $b(x, y)$ to be determined with the boundary condition (19) at the floor $Z = 0$ of the order of ε:

$$\frac{\partial P_0(x, y, 0)}{\partial Z} = 0, \quad \text{i.e., } a(x, y) = 0. \tag{23}$$

Moreover, the condition (17) is also of the order of ε, so

$$P_0(x, y, H(x, y, T)) = b(x, y) = 0 \tag{24}$$

and thus, necessarily

$$P_0 \equiv 0. \tag{25}$$

Analogously, from (20) we have that

$$\frac{\partial^2 p_0}{\partial Z^2} = 0 \tag{26}$$

which implies that

$$p_0(x, y, Z) = a(x, y)Z + b(x, y) \tag{27}$$

Now conditions (19) and (17) are pertinent (since they are of the order of ε) and we get, respectively that

$$\frac{\partial p_0}{\partial Z}(x, y, 0) = -\rho_m g = a(x, y) \tag{28}$$

and

$$p_0(x, y, H(x, y, T)) = -\rho_w g(D - H(x, y, T)) = a(x, y)H(x, y, T) + b(x, y). \tag{29}$$

Thus, we get

$$p_0(x, y, Z) = -\rho_m g Z + \rho_w g(D - H(x, y, T)) + \rho_m g H(x, y, T) \tag{30}$$

i.e.

$$p_0(x, y, Z) = \rho_m g(H(x, y, T) - Z) + \rho_w g(D - H(x, y, T)). \tag{31}$$

Since $P_0 \equiv 0$, vanishing of the ε^2-coefficient in (20) leads to:

$$\frac{\partial^2 P_1}{\partial Z^2} = 0, \tag{32}$$

and since the boundary conditions (19) and (17) are of the order ε we get (as in the case of P_0) that necessarily

$$P_1 = 0. \tag{33}$$

Notice that (25) and (33) explain that there is not loss of generality in the expansion assumed by Angevine et al. [2]. From the ε^3-coefficient in (20) we get that

$$\frac{\partial^2 p_1}{\partial Z^2} = -\left(\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2}\right) \tag{34}$$

and from (31) we get

$$\frac{\partial^2 p_1}{\partial Z^2} = -g(\rho_m - \rho_w)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) \text{ independent on Z.} \tag{35}$$

Thus

$$p_1(x, y, Z) = \frac{1}{2}g(\rho_w - \rho_m)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)Z^2 + b(x, y)Z + c(x, y) \tag{36}$$

with a, b and c to be determined. The boundary condition (19) is of the order of ε, so

$$\frac{\partial p_1}{\partial Z} = a(x, y)g(\rho_w - \rho_m)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)Z_{z=0} + b(x, y) = b(x, y) = 0 \text{ i.e. } b \equiv 0 \tag{37}$$

and since the condition (17) is of order of ε we get

$$p_1(x, y, H(x, y, T)) = \frac{1}{2}g(\rho_w - \rho_m)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)H^2 + c(x, y). \tag{38}$$

Thus we get

$$p_1(x, y, Z) = \frac{1}{2}g(\rho_w - \rho_m)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right)(H^2 - Z^2). \tag{39}$$

We can now get the expressions for u, v and w in $Z = H$ in terms of function H by using the conditions (3), (4) and (5), and the expansion (14) with (25), (31), (33) and (39). For instance, since $p_1 = 0$ if $Z = H$ we see that p_1 does not appear in the expression of u and v (but merely (31)). Then

$$u(x, y, H(x, y, T)) = -\varepsilon \frac{k(\rho_m) - \rho_w g}{\mu\phi} \frac{\partial H}{\partial x}(x, y, T), \tag{40}$$

$$v(x, y, H(x, y, T)) = -\varepsilon \frac{k(\rho_m) - \rho_w g}{\mu\phi} \frac{\partial H}{\partial y}(x, y, T). \tag{41}$$

Nevertheless, to compute w we must start by rescaling (5)

$$w = -\frac{k}{\varepsilon\mu\phi}\left(\frac{\partial p}{\partial Z} + \rho_m g\right),$$

and then using (31) and (39) we get

$$\begin{aligned} w(x, y, H(x, y, T)) &= -\frac{k}{\mu\phi}\left(\frac{\partial p_0}{\partial Z} + \varepsilon^2 \frac{\partial p_1}{\partial Z} + \frac{\partial \rho_m g}{\partial \varepsilon}\right) \\ &= \varepsilon^2 \frac{k}{\mu\phi} g(\rho_m - \rho_w)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) H. \end{aligned} \tag{42}$$

Finally, substituting (40), (41) and (42) in (18) we get

$$\begin{aligned} \varepsilon^2 \frac{k}{\mu\phi} g(\rho_m - \rho_w)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) H &= \varepsilon^2 \frac{\partial H}{\partial T} \\ &\quad -\varepsilon^2 \frac{k}{\mu\phi} g(\rho_m - \rho_w)\left[\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2\right]. \end{aligned} \tag{43}$$

This equation can be rearranged to get:

$$\frac{\partial H}{\partial T} = \frac{k}{\mu\phi} g(\rho_m - \rho_w) div(H\nabla H) \tag{44}$$

notice that

$$div(H\nabla H) = \nabla H \cdot \nabla H + H div(\nabla H) = \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 + H\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right).$$

Equation (44) is the extension, to the case of $3d$-edifices, of the Eq. (20) of the paper [2] in which only $2d$-edifices are considered. Equation (44) is a nonlinear partial differential equation of parabolic type but it is not uniformly parabolic but degenerate since the diffusion coefficient is

$$\frac{k}{\mu\phi} g(\rho_m - \rho_w) H(x, y, T) \geq 0$$

and vanishes on the set of points (x, y, T) where $H(x, y, T) = 0$.

For many purposes (see, e.g., the definition of weak solution given in Sect. 5) it is convenient to observe that (44) has to be reformulated as:

$$\frac{\partial H}{\partial t} = \frac{k}{2\mu\phi} g(\rho_m - \rho_w) \Delta(H^2) \tag{45}$$

since

$$H\nabla H = \frac{1}{2}\Delta(H^2).$$

It is posible to rewrite (45) in terms of the unknown

$$U = H^2.$$

We shall show $H(x, y, T) \geq 0$ and so $H(x, y, T) = \sqrt{U(x, y, T)}$. Then, the equation (45) becomes

$$\frac{\partial}{\partial t}\sqrt{U} = \frac{k}{2\mu\phi} g(\rho_m - \rho_w)\Delta U. \tag{46}$$

In polar coordinates, $r = (x^2 + y^2)^{1/2}$, $\mathbf{e}_r = \dfrac{(x, y)}{r}$, since

$$\nabla A(r, T) = \frac{\partial}{\partial r} A(r, T)\mathbf{e}_r$$

and

$$div(B(r, T)\mathbf{e}_r) = \frac{1}{r}\frac{\partial}{\partial r}(rB(r, T)),$$

we get that, when $H(x, y, T) = H(r, T)$, equation (44) can be rewritten as:

$$\frac{\partial H}{\partial t} = \frac{k}{\mu\phi} g(\rho_m - \rho_w)\frac{1}{r}\frac{\partial}{\partial r}\left(rH\frac{\partial H}{\partial r}\right) \tag{47}$$

Remark 6 Notice that parameter ε disappeared of the final equation for H (44). As a matter of fact, the same process could be applied to show that the coefficients of higher order than ε^3 (i.e., p_2, $p_3 \ldots$) in (14) must also vanish. We also point out that Eq. (44) was deduced for $(x, y) \notin \Gamma_m$, $(x, y) \in B(0) \subset G$, and $T \geq 0$ such that $H(x, y, T) \leq \dfrac{h^*}{\varepsilon}(x, y)$, nevertheless, from a mathematical point of view we can extend $H(x, y, T)$ to 0 if $(x, y) \in \mathbb{R}^2 - \mathcal{B}(0)$, $(x, y) \notin \Gamma_m$.

Now we are in conditions to formulate the boundary condition expressing the intrusion of magmatic flow through Γ_m. We shall assume that:

$$-H(x, y, T)\nabla H(x, y, T) \cdot \mathbf{n} = Q_0(x, y, T), \text{ for any } (x, y) \in \Gamma_m, \forall T \geq 0. \tag{48}$$

Here $\mathbf{n}$ is the exterior normal vector to Γ_m when it can be correctly defined.

Remark 7 A more correct formulation of (48) could be obtained by approximating Γ_m by a family Γ_m^n of Jordan simple curves (see Fig. 4). In that case, the normal vector is well-defined. The case of the flux condition (45) must be understood as the limit case of those approximating problems when the parameter n (approximating Γ_m by Γ_m^n) converges to ∞.

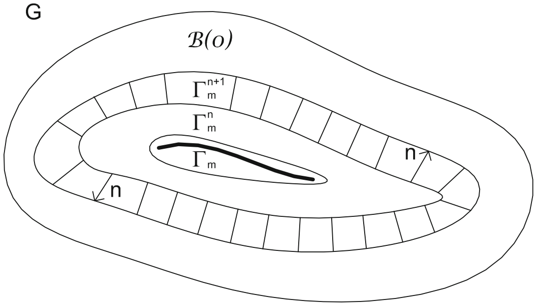
Therefore, the complete formulation is: given $Q_0(x, y, T)$ and $H_0(x, y)$ find $H(x, y, T)$ such that

$$(P_n)\begin{cases} \dfrac{\partial H}{\partial T} = \dfrac{k}{\mu\phi} g(\rho_m - \rho_w) div(H\nabla H), & (x, y) \in \mathbb{R}^2 - \mathcal{D}_m^n, T > 0, \\ H\nabla H \cdot \mathbf{n} = Q_0(x, y, T), & (x, y) \in \Gamma_m^n, T > 0, \\ H(x, y, 0) = H_0(x, y), & (x, y) \in \mathbb{R}^2 - \mathcal{D}_m^n, \end{cases}$$

where $\mathcal{D}_m^n$ is the subset of $\mathbb{R}^2$ such that $\partial\mathcal{D}_m^n = \Gamma_m^n$. Notice that the initial base $\mathcal{B}(0)$ is given through the condition:

$$\mathcal{B}(0) = \text{ support of } H_0(\cdot, \cdot) \subset \mathbb{R}^2$$

Fig. 4 Family of Jordan simple curves



and because we are assuming that G (the pre-existing edifice) is such that

$$(\mathcal{B}(0) \cup \mathcal{D}_m^n) \subset G. \tag{49}$$

There are some relevant cases in which the approximation of the curve Γ_m by Γ_m^n is automatically verified: the first case corresponds to conic volcanoes for which

$$H(x, y, T) = H(r, t), \quad \Gamma_m = (0, 0), G = B_R(0, 0), \mathbf{n} = -\mathbf{e}_r$$

As mentioned before, Eq. (44) leads to equation (47) and so

$$H \nabla H \cdot \mathbf{n} = -H(r, T) \frac{\partial H}{\partial r}(r, T). \tag{50}$$

Obviously, for $T \geq 0$ such that $Q_0(r, T) \neq 0$ we will get a singularity of the gradient, ∇H^2, at $r = 0$: Notice that if we define

$$\mathcal{D}_m^n = B_{1/n}(0, 0) \tag{51}$$

then $\Gamma_m^n = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \frac{1}{n^2}\}$, which verifies that $\Gamma_m^n \to \Gamma = \{(0, 0)\}$ when $n \to \infty$. Moreover, the limit flux condition can be written as

$$\lim_{n\to\infty} H \nabla H \cdot \mathbf{n}\Big|_{\Gamma_m^n} = -H(0, T) \frac{\partial^+ H}{\partial r}(0, T) = -\frac{1}{2} \frac{\partial^+}{\partial r}(H^2(0, T)) \tag{52}$$

where $\frac{\partial^+}{\partial r}$ represents the directional derivative.

$$\frac{\partial^+}{\partial r}(H(r, T)) = \lim_{\delta\to\infty, \delta\geq 0} \frac{H(r + \delta) - H(r)}{\delta}.$$

Remark 8 In [24], the flux condition (their condition (7)) is formulated as:

$$-\frac{2\pi K g}{\mu} r h \frac{\partial h}{\partial r} \to Q_0^* \quad \text{as } r \to 0. \tag{53}$$

Obviously their Q_0^* corresponds to a different notation than (48) and (50). In fact, if we identify the constant coefficients

$$\frac{2\pi K}{\mu} g \approx \frac{K}{\mu\phi} g(\rho_m - \rho_w) \tag{54}$$

then we get that

$$Q_0^* = \frac{Q_0(r, T)}{r}, \tag{55}$$

but (what is fundamental) we observe the coincidence of sign in both notations:

$$Q_0^* \geq 0 \Leftrightarrow Q_0 \geq 0.$$

A second example in which the approximation of Γ_m is quite evident concerns the case of seamounts for which

$$\Gamma_m = \{(0, y) : y \in \mathbb{R}\}. \tag{56}$$

In that case we can define $\Gamma_m = \{(x, y) : |x| = \frac{1}{n} \text{ and } y \in \mathbb{R}\}$. Thus,

$$\mathbf{n} = (-1, 0) \text{ if } x \in \Gamma_m^{n,+},$$

$$\mathbf{n} = (1, 0) \text{ if } x \in \Gamma_m^{n,-},$$

where

$$\Gamma_m^{n,\pm} = \left\{(x, y) : x = \pm\frac{1}{n}, y \in \mathbb{R}\right\}.$$

We observe, again, the singularity of the gradient at Γ_m since

$$H(x, y, T)\frac{\partial^+}{\partial x}H(x, y, T) \to -Q_0(y, T), \text{ if } x \searrow 0, \tag{57}$$

but

$$H(x, y, T)\frac{\partial^+}{\partial x}H(x, y, T) \to Q_0(y, T), \text{ if } x \nearrow 0. \tag{58}$$

If we assume the symmetry $H_0(x, y) = H(-x, y)$, $x > 0$, then it is enough to impose only (57) (and (58) is automatically verified). Notice that in [2] this is expressed in their condition (25)

$$\frac{-k(\rho_m - \rho_w g)}{\mu\phi}h\frac{\partial h}{\partial x} \to \frac{Q_0^*}{2\phi}. \tag{59}$$

Again, as in Remark 8, we point out that their notation is different to (57) but there is a coincidence of sign

$$Q_0^* \geq 0 \Leftrightarrow Q_0 \geq 0. \tag{60}$$

Remark 9 Perhaps, it is now interesting to remember the fact that the similarity variables f and ξ introduced in the papers [24] and [2] are coincident (see (29) and (30) of [2]), since the geometry of the spatial structures studied in the papers are different. Here our formulation does not need the reduction of the spatial dimension by means of symmetry conditions. Moreover, we can consider seamounts and (submarine) volcanoes of conic geometry (which are excluded in the framework of [2]).

Remark 10 Besides the singularity in the gradient of $H(x, y, T)$ at the curve Γ_m, there is a different singularity of the gradient of H. Indeed, since the parabolic equation is degenerate, we shall see later that the support of H is bounded, and that (in many cases) it may arise that

$$\nabla H(x, y, T) \neq \mathbf{0}$$

for $(x, y) \in$ support of $H(\cdot, \cdot, T)$, i.e., the function H has a zero gradient outside its support but the gradient is not zero when we arrive to the points of the boundary of its support from interior points.

3 On a generalization of the previous model

In the precedent framework, the crucial aspect was the simplification of the Navier-Stokes equation (8) in the upper medium over the magma flow by the assumption that the free boundary is a surface of constant hydraulic potential (conditions (11a) and (11b)), which corresponds to the Pascal approximation for water (i.e. the water is absolutely static).

Keeping a complete generality (imposing (8)) will make it almost impossible to reach a simple differential equation for the free boundary $H(x, y, T)$ (as done in (44)). An intermediate path is to modify the condition in [2] by allowing a small correction term:

$$p(x, y, h(x, y, t)) = \rho_w g(d - h(x, y, t)) + c(x, y, T, h, \varepsilon) \tag{61}$$

where the correction term $c(x, y, T, h, \varepsilon)$ to be suitably defined. If we repeat the above argument concerning the expansion (14) we see that we can reach the corrected equation

$$\frac{\partial H}{\partial T} = \frac{k}{\rho\phi} g(\rho_m - \rho_w) div(H\nabla H + \mathbf{e}(x, y)H^{\lambda}), \tag{62}$$

for some given vector $\mathbf{e}(x, y)$ and for some $\lambda \in (0, 2)$ if we choose $c(x, y, T, h, \varepsilon)$ in a suitable way. Before doing it more explicitly, it is convenient to make specific (62) (and $\mathbf{e}(x, y)$) for the two relevant spacial cases:

(i) The radial symmetric case: $H = H(r, T)$, $r^2 = x^2 + y^2$. Then, a natural choice is to take

$$\mathbf{e}(x, y) = r\mathbf{e}_r(x, y), \quad \text{where } \mathbf{e}_r \text{ is the radial unit vector, } \mathbf{e}_r = \frac{(x, y)}{r}$$

and so Eq. (62) becomes (recall (47))

$$\frac{\partial H}{\partial T} = \frac{k}{\rho\phi} g(\rho_m - \rho_w) \frac{1}{r} \left(rH \frac{\partial H}{\partial r} - rH^{\lambda} \right). \tag{63}$$

(ii) The x-symmetrical case: $H = H(x, t)$. Then (44) simplifies (since any dependence with respect to y disappears) and thus a natural choice is

$$\mathbf{e}(x, y) = \frac{\tau}{2} \left(\frac{-x^2}{|x|}, 0 \right) = \begin{cases} \frac{\tau}{2}(-x, 0), & \text{if, } x > 0, \\ \frac{\tau}{2}(x, 0), & \text{if, } x < 0, \end{cases}$$

for some $\tau > 0$. Then, equation (62) becomes

$$\frac{\partial H}{\partial T} = \frac{k}{\rho\phi} g(\rho_m - \rho_w) \frac{\partial}{\partial x} \left(H \frac{\partial H}{\partial x} - \frac{\tau}{2} \frac{x^2}{|x|} H^{\lambda} \right) \tag{64}$$

which can be reformulated as

$$\frac{\partial H}{\partial T} = K \frac{\partial^2 H^2}{\partial x^2} + \frac{\mu x}{|x|} \frac{\partial H^{\lambda}}{\partial x}, \tag{65}$$

if we identify

$$K = \frac{k}{2\rho\phi} g(\rho_m - \rho_w),$$

$$\mu = \frac{\tau}{2} \frac{k}{\rho\phi} g(\rho_m - \rho_w).$$

Remark 11 For a previous presentation of a modelling leading to equation (65) see Arjona et al. [4].

Coming back to the meaning of the correction term c, we point out that if we assume

$$c(x,h) = \varepsilon^{\lambda} \int_0^x h^{\lambda-1}(s,T)ds \tag{66}$$

then we reach the Eq. (65) in the relevant case of the x-symmetrical case. Notice that since (as we shall show later) $\max_{x\in\mathbb{R}} h(x,t)$ is finite then (66) means that the correction term is proportional to the distance to $x = 0$ (i.e. to the the top of the profile). Moreover, since $\lambda \in (0,2)$, the coefficient in (66) is small (since $\varepsilon << 1$). This correction term only appears in the computation of $p_0(x,y,Z)$ (see (28)), which now becomes

$$p_0(x,y,Z) = \rho_m g(H(x,y,T) - Z) + \rho_w g(D - H(x,y,T)) + \mu\varepsilon^{\lambda} \int_0^x h^{\lambda-1}(s,T)ds,$$

and substituting into u leads to the following modification of (40):

$$u = -\varepsilon \frac{k(\rho_m - \rho_w)}{\mu\phi} g \frac{\partial H}{\partial x} - \mu\varepsilon^{\lambda} H^{\lambda-1}.$$

Finally, replacing it in the condition (18) we get

$$\varepsilon^2 \left(\frac{\partial H}{\partial T} - \frac{kg(\rho_m - \rho_w)}{\mu\phi} \frac{\partial}{\partial x} \left(H \frac{\partial H}{\partial x} \right) + \mu \frac{x}{|x|^2} H^{\lambda-1} \frac{\partial H}{\partial x} \right) = 0,$$

which is (65).

4 Formulation including an outpointing flow

A more general case corresponds to

$$h(x,y,t) \le h^*(x,y) \quad \text{for any } (x,y,t) \tag{67}$$

and

$$h(x_0,y_0,t_0) \le h^*(x_0,y_0) \quad \text{for some } (x_0,y_0,t_0). \tag{68}$$

The difficulty comes from the fact that now h is not strictly included in the porous medium and we can not apply the Darcy's law.

We still have the property that the free boundary is a fluid surface. Therefore, we have (10). Now, we assume that (x_0,y_0,t_0) satisfies (68). Then

$$\mathbf{v}(x_0,y_0,t_0) \cdot \mathbf{n}(x_0,y_0,t_0) \ge 0 \tag{69}$$

with $\mathbf{v}$ the velocity of the particle (x_0,y_0) at the time t_0 and $\mathbf{n}$ the unit exterior normal vector to the porous medium at the point $(x_0,y_0,h^*(x_0,y_0))$. However,

$$\mathbf{n} = \left(-\frac{\partial h^*}{\partial y}, \frac{\partial h^*}{\partial x}, -1 \right),$$

$\left(\mathbf{n} \cdot \left(\frac{\partial h^*}{\partial y}, -\frac{\partial h^*}{\partial x}, 1\right) = 0\right)$. The implicit formulation of the free boundary is

$$S_t \equiv \{\psi(x,y,z,t) = 0\} \text{ where } z - h(x,y,t) \equiv \psi(x,y,z,t).$$

Then

$$\frac{d}{dt}\psi(x, y, h(x, y, t)) = 0\Big|_{t=t-\varepsilon} \geq 0,$$

but inside of the porous media is

$$w - \frac{\partial h}{\partial t} - \frac{\partial h}{\partial x}u - \frac{\partial h}{\partial y}v\Big|_{t=t-\varepsilon} \geq 0,$$

where u, v, w are given as in the previous steps 1 and 2. Making $\varepsilon \downarrow 0$ we deduce that

$$H(x, y, t) \leq H^*(x, y), \tag{70}$$

$$\left(\frac{\partial H}{\partial T} - \frac{Kg}{\mu\phi}(\rho_m - \rho_w)div(H\nabla H + \mathbf{e}(x, y)H^\lambda)\right) \leq 0. \tag{71}$$

Moreover

$$H(x, y, t) < H^*(x, y) \text{ implies } \left(\frac{\partial H}{\partial T} - \frac{Kg}{\mu\phi}(\rho_m - \rho_w)div(H\nabla H + \mathbf{e}(x, y)H^\lambda)\right) = 0, \tag{72a}$$

$$\left(\frac{\partial H}{\partial T} - \frac{Kg}{\mu\phi}(\rho_m - \rho_w)div(H\nabla H + \mathbf{e}(x, y)H^\lambda)\right) > 0 \Rightarrow H(x, y, t) = H^*(x, y) \tag{72b}$$

Since otherwise we know that $(\frac{\partial H}{\partial T} - \frac{Kg}{\mu\phi}(\rho_m - \rho_w)div(H\nabla H + \mathbf{e}(x, y)H^\lambda)) = 0$ by (62). In conclusion, H satisfies the variational inequality

$$\left(\frac{\partial H}{\partial T} - \frac{Kg}{\mu\phi}(\rho_m - \rho_w)div(H\nabla H + \mathbf{e}(x, y)H^\lambda)\right) + \beta(H - H^*) \ni 0 \tag{73}$$

with β the maximal monotone graph of $\mathbb{R}^2$ given by:

$$\beta(r) = \begin{cases} \phi \text{ (the empty set)} & \text{if } r > 0, \\ \{0\} & \text{if } r < 0, \\ [0, +\infty) & \text{if } r = 0. \end{cases} \tag{74}$$

Indeed:

(a) $H - H^* \in \mathcal{D}(\beta) = (-\infty, 0] \Leftrightarrow H \leq H^*$ which is (70).
(b) If $H \leq H^* \Rightarrow \beta(H - H^*) = 0$ which is (72a).
(c) If $H = H^* \Rightarrow \beta(H - H^*)$ is formed by positive values which is (71).
and finally
(d) $r\beta(r) = 0$ which is (72a)–(72b).

We point out that to get a weak formulation of the problem it is useful to observe that

$$div(H\nabla H) = \frac{1}{2}\Delta H^2, \tag{75}$$

and so, by introducing

$$(H^*)^2 = U^*, \tag{76}$$

we get that Eq. (73) can be reformulated as

$$\left(\frac{\partial\sqrt{U}}{\partial T} - \frac{Kg}{2\mu\phi}(\rho_m - \rho_w)\left[\Delta\sqrt{U} - 2div\left(\mathbf{e}U^{\frac{\lambda}{2}}\right)\right] + \beta(U - U^*)\right) \ni 0. \tag{77}$$

In the same line of arguments is mentioned in Remark 7 we mention that the existence of a weak solution to problem (P_n) (as well as the variational inequality associated to this PDE) can be found in [1]. In fact, the study of the free boundary can be also carried out for such class of weak solutions by means of a variant of the "energy method" introduced in Díaz [18]. For the numerical analysis of the model see [22]. The question of the uniqueness of the weak solution (when the domain is such as indicated in problem (P_n)) is much more delicate (see the references mentioned in the next Sect. 5).

5 No limited base

We are going to prove that the free boundary is not bounded as $t \longrightarrow +\infty$, in the case of the model simplified to a one-dimensional variable. Before doing that we shall recall here some results on the existence and uniqueness of solutions of problem $P(\mu, Q_0)$. Since the spatial domain has two different connect components, our first remark about the mathematical treatment of problem $P(\mu, Q_0)$ is that the problem can be decomposed in two different uncoupled problems $P_+(\mu, Q_0)$ and $P_-(\mu, Q_0)$:

$$P_+(\mu; Q_0) \equiv \begin{cases} \dfrac{\partial H}{\partial t} = K\dfrac{\partial^2 H^2}{\partial x^2} + \dfrac{\mu x}{|x|}\dfrac{\partial H^\lambda}{\partial x}, & x \in (0, +\infty), t > 0, \\ -\dfrac{\partial H^2}{\partial x}(0, t) - \mu H^\lambda(0, t) = Q_0(t), & t > 0, \\ H(0, x) = H_0(x), & x \in (0, +\infty), \end{cases} \tag{78}$$

and

$$P_-(\mu; Q_0) \equiv \begin{cases} \dfrac{\partial H}{\partial t} = K\dfrac{\partial^2 H^2}{\partial x^2} + \dfrac{\mu x}{|x|}\dfrac{\partial H^\lambda}{\partial x}, & x \in (-\infty, 0), \quad t > 0, \\ \dfrac{\partial H^2}{\partial x}(0, t) - \mu H^\lambda(0, t) = Q_0(t), & t > 0, \\ H(0, x) = H_0(x), & x \in (-\infty, 0). \end{cases} \tag{79}$$

In other words, a function $u(x, t)$ defined on $(\mathbb{R} - \{0\}) \times [0, +\infty)$ is solution of $P(\mu, Q_0)$ (in any mathematical sense) if and only if there exist two functions $u_+(x, t)$ and $u_-(x, t)$ defined on $(0, +\infty) \times [0, +\infty)$ and $(-\infty, 0) \times [0, +\infty)$, respectively, with $u_+(x, t)$ solution of $P_+(\mu; Q_0)$ and $u_-(x, t)$ solution of $P_-(\mu; Q_0)$ such that

$$u(x, t) = \begin{cases} u_+(x, t) & \text{if } x > 0, \\ u_-(x, t) & \text{if } x < 0. \end{cases}$$

This is the way in which the boundary condition stated in $P(\mu, Q_0)$ must be understood. Now, concerning the notion of solution, it is enough to refer to one of both problems, e.g. $P_+(\mu; Q_0)$ (since the treatment of $P_-(\mu; Q_0)$ is entirely similar).

Since problem $P_+(\mu; Q_0)$ is degenerate it is well known that we need to introduce some notion of solution weaker than classical solutions. We shall always assume that

$$Q_0 \in H^1(0, T) \cap L^\infty(0, +\infty), \quad \text{for any } T > 0,$$

and

$$\begin{cases} H_0 \in L^\infty(\mathbb{R} - \{0\}), H_0 \geq 0, \text{ support } H_0 \text{ is a compact of } \mathbb{R} \\ (H_0)^2 \in H^1(\mathbb{R} - \{0\}). \end{cases}$$

Definition 1 A function $u_+(x,t)$ is a weak solution of $P_+(\mu;Q_0)$ if $u_+ \in C([0,T] : L^1(0,+\infty) \cap L^\infty(0,+\infty : L^\infty(0,+\infty))$, $(u_+)^2 \in L^2(0,T : H^1(0,+\infty))$, $(u_+)_t \in L^2(0,T : H^{-1}(0,+\infty) \cap L^\infty(0,+\infty))$, $u_+(0,.) = H_0(.)$, and satisfies $\frac{\partial u_+}{\partial t} = K\frac{\partial^2 u_+^2}{\partial x^2} + \mu\frac{\partial u_+^\lambda}{\partial x}$ and $-\frac{\partial u_+^2}{\partial x}(0,t) - \mu u_+^\lambda(0,t)) = Q_0(t)$ in $\mathcal{D}'((0,T)\times(0,+\infty))$ for any $T>0$, i.e., for any $\phi \in L^2(0,T : H^1(0,+\infty)) \cap H^1(0,T : L^1(0,+\infty)) \cap L^\infty(0,+\infty : L^\infty(0,+\infty))$, with support of $\phi(t,.)$ compact for any $t\in[0,T]$ we have

$$\int_0^T \langle (u_+)_t(t,.),\phi(t,)\rangle\, dx + \int_0^T\int_0^{+\infty}(u_+(t,x)-H_0(x))\phi_t(t,x))dxdt = 0$$

and

$$\int_0^T \langle (u_+)_t(t,.),\phi(t,)\rangle\, dx + \int_0^T\int_0^{+\infty}(K(u_+)^2+\mu u_+^\lambda)\phi_x dxdt + \int_0^T Q_0(t)\phi(t,0)dt = 0.$$

The existence of a weak solution to $P_+(\mu;Q_0)$ is a direct consequence of Theorem 1.7 of [1] (although this result was stated for the special case $Q_0(t)\equiv 0$ their Remark 1.10 applies to the case $Q_0(t)\neq 0$). As a matter of fact, the existence of weak solution is also a small variant of other results in the literature dealing with the same partial differential equation but on the whole real line instead $(0,+\infty)$ (and so without any boundary condition): see, e.g. [14,16,20,23], and their references. In particular, thanks to the presence of the degenerate diffusion term $\frac{\partial^2 u_+^2}{\partial x^2}$ it is well-known that any weak solution satisfies that

$$\text{support of } u_+(t,.) \text{ is a compact subset of } (0,+\infty), \text{ for any } t\geq 0.$$

(see, e.g, Proposition 2 of [15]). Due to that, without loss of generality, given $T>0$ we can replace the spatial domain $(0,+\infty)$ by a bounded interval $I=(0,L)$ for some $L=L(T)$. Thus any weak solution u_+ of $P_+(\mu;Q_0)$ is also a weak solution of

$$P_+(\mu;Q_0,I)\equiv\begin{cases} \frac{\partial H}{\partial t} = K\frac{\partial^2 H^2}{\partial x^2} + \mu\frac{\partial H^\lambda}{\partial x}, & x\in(0,L), t\in(0,T), \\ -\frac{\partial H^2}{\partial x}(0,t) - \mu H^\lambda(0,t) = Q_0(t),\ H(L,t)=0 & t\in(0,T), \\ H(0,x) = H_0(x), & x\in(0,L). \end{cases} \tag{80}$$

It was shown in [6] and [9] that the semigroup theory also leads to the existence of a weak solution in the sense that the unique "(exact) mild solution" associated to the diffusion-convection operator is also a weak solution of the problem. It is useful to recall this notion:

Definition 2 A function $u_+(x,t)$ is an (exact) mild solution of $P_+(\mu;Q_0,I)$ if $u_+\in C([0,T] : L^1(0,L))$, $u_+(0,.) = H_0(.)$ and for any $\delta>0$ there exists $\varepsilon>0$ such that, for any partition of $[0,T-\varepsilon]$, $t_0=0<t_1<\cdots<t_n\leq T$, with $t_i-t_{i-1}\leq\varepsilon$, $T-t_n\leq\varepsilon$, and for any approximation of the datum $Q_0(t)$, $\{Q_0^i\}_{i=1}^n$ with

$$\sum_i\int_{t_{i-1}}^{t_i}\left|Q_0(t)-Q_0^i\right|\leq\varepsilon,$$

there exists a set of stationary functions $\{u_i\}_{i=1}^n \subset H^1(0,L)$ solving (in $\mathcal{D}'(0,L)$) the implicit time-discretization problem

$$IDP_+(\mu;Q_0,I) \equiv \begin{cases} \dfrac{H_i - H_{i-1}}{t_i - t_{i-1}} = K\dfrac{\partial^2 H_i^2}{\partial x^2} + \mu\dfrac{\partial H_i^\lambda}{\partial x}, & x \in (0,L),\ i = 1,\dots,n, \\ -\dfrac{\partial H_i^2}{\partial x}(0) - \mu H_i^\lambda(0) = Q_0^i,\ H_i(L) = 0, \end{cases} \tag{81}$$

with

$$\|u_+(.,t) - u_i(.)\|_{L^1(0,L)} \le \delta \quad \text{for any } t \in (t_{i-1}, t_i].$$

To be more precise, as in [6] and [9], we define the operator

$$A(t)u = -K\frac{\partial^2 u_+^2}{\partial x^2} + \mu\frac{\partial u_+^\lambda}{\partial x} \text{ if } u \in D(A(t)),$$

with

$$D(A(t)) = \Big\{ w \in L^\infty(0,L),\, w^2 \in H^1(0,L),\ w^2(L) = 0$$
$$\text{and } -\frac{\partial w^2}{\partial x}(0) - \mu w^\lambda(0) = Q_0(t) \text{ in the sense of traces} \Big\}.$$

Then it is proved in the above mentioned references (for $Q_0(t) \equiv 0$) that $A(t)$ is a m-T-accretive operator on the Banach space $X = L^1(0,L)$. The extension to the case $Q_0(t) \neq 0$ is a routine matter. Moreover $A(t)$ satisfies the t-dependence of condition of [19] and so the (exact) mild solution u_+ of $P_+(\mu; Q_0, I)$ is constructed as

$$u_+(.) = \lim_{n\to+\infty} u_{+n}$$

with u_{+n} being solution of the above stationary problem $IDP_+(\mu; Q_0, I)$. We point out that the uniqueness of the weak solution of $P_+(\mu; Q_0, I)$ is a delicate question. It requires the introduction of some special type of solutions: "entropy solutions" (see [11,12,28]) or "renormalized solutions" (see [7,13]). Fortunately we do not need to recall such technical notions since, under the conditions of our special case, any weak solution is a entropy solution and a renormalized solution. In conclusion, there is a unique weak solution of problem $P_+(\mu; Q_0, I)$ (see [13,28]). Note that without loss of generality, for the uniqueness of solutions we can assume $Q_0(t) \equiv 0$. As a final corollary, the (exact) mild solution is the unique weak solution to problem $P_+(\mu; Q_0, I)$.

Now we come back to the main result of this section: the free boundary $\xi(t)$ associated to the model by Angevine, Turcotte and Ockendon, (i.e. Problem $P(0; Q_0)$) is an unbounded function of t:

Theorem 12 *Let $\xi(t)$ the free boundary of the problem $P(0, Q_0)$, then $\xi(t) \longrightarrow +\infty$ if $t \longrightarrow +\infty$.*

We shall build the proof in two different steps. Firstly, we shall prove that if $U(t,x)$ and $H(t,x)$ are weak solutions of the respective problems $P(0,0)$ and $P(0,Q_0)$ with the same initial data then we have $U \le H$. As a consequence, if $\zeta(t)$ and $\xi(t)$ are the free boundaries of the problems $P(0,Q_0)$ and $P(0,0)$, with the same initial data, then $0 < \zeta(t) \le \xi(t)$ for any $t > 0$. In a second step we shall prove that $\zeta(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. This will conclude the proof since, thus, necessarily, $\xi(t) \longrightarrow +\infty$ if $t \longrightarrow +\infty$.

The first step is a special conclusion of a more general comparison statement in which the new fact (with respect the comparison results of [6,9]) is the variation of the datum $Q_0(t)$.

Proposition 1 *Let H_1 and H_2 be the weak solutions of $P(\mu, Q_0)$ corresponding to $\mu \geq 0$, $Q_{0,1}$, $Q_{0,2}$ and $H_{0,1}$, $H_{0,2}$ respectively. Then, for any $t > 0$ we have the estimate*

$$\int_{\mathbb{R}-\{0\}} (H_1(t,x) - H_2(t,x))_+ dx \leq \int_{\mathbb{R}-\{0\}} (H_{0,1}(x) - H_{0,2}(x))_+ dx + \int_0^t (Q_{0,1}(\tau) - Q_{0,2}(\tau))_+ d\tau \tag{82}$$

where, we used the notation, $a_+ = max(0, a)$.

Corollary 1 *Let H_1 and H_2 be the weak solutions of $P(\mu, Q_0)$ corresponding to $\mu \geq 0$, $Q_{0,1}$, $Q_{0,2}$ and $H_{0,1}$, $H_{0,2}$ respectively. Then:*

(i) $Q_{0,1} \leq Q_{0,2}$ for any $t > 0$ and $H_{0,1}(x) \leq H_{0,2}(x)$ for $x \in \mathbb{R} - \{0\}$ implies $H_1(t,x) \leq H_2(t,x)$ for any $t > 0$ and for $x \in \mathbb{R} - \{0\}$,
(ii) the following quantitative expression on the continuous dependence holds for any $t > 0$,

$$\int_{\mathbb{R}-\{0\}} |H_1(t,x) - H_2(t,x)|\, dx \leq \int_{\mathbb{R}-\{0\}} \left|H_{1,0}(x) - H_{2,0}(x)\right| dx + \int_0^t \left|(Q_{1,0}(\tau) - Q_{2,0}(\tau))\right| d\tau.$$

Proof Since the (exact) mild solution is the unique weak solution and since both solutions have a compact support, for any $t \in [0, T]$ for any prescribed $T > 0$, it is enough to prove that if $u_{1,i}$ and $u_{2,i}$ are the corresponding solutions of the implicit time-discretization problems (81) then, for any $i = 1, ..., n$

$$\int_I (u_{1,i}(x) - u_{2,i}(x))_+ dx \leq \int_I (u_{1,i-1}(x) - u_{2,i-1}(x))_+ dx + (Q_{0,1}^i - Q_{0,2}^i)_+. \tag{83}$$

(as before, the proof on the connect component $x < 0$ is entirely similar). As a matter of fact, it is enough to prove (83) only for $i = 1$ since the rest of the cases are obtained by an iteration of the same argument of the proof. Now, we multiply the difference of the two associated equations by a regular approximation $p_n(r)$, $n \in \mathbb{N}$, of the Heaviside type function

$$sign_{+,0}(r) = 0 \quad \text{if } r \leq 0 \quad \text{and} \quad sign_{+,0}(r) = 1 \text{ if } r > 0,$$

with $r = (u_{1,i}^2(x) - u_{2,i}^2(x))$. For instance, we can take as p_n the function

$$p_n(r) = \begin{cases} 0 & \text{if } r \leq 0, \\ nr & \text{if } r \in [0, \frac{1}{n}], \\ 1 & \text{if } r > \frac{1}{n}. \end{cases} \tag{84}$$

Then,

$$\int_I (u_{1,1} - u_{2,1}) p_n(u_{1,1}^2 - u_{2,1}^2) dx - \int_I (u_{0,1,1} - u_{0,2,1}) p_n(u_{1,1}^2 - u_{2,1}^2) dx$$
$$= t_1 K \int_I \frac{\partial}{\partial x}((\frac{\partial}{\partial x}(u_{1,1}^2 - u_{2,1}^2)) p_n(u_{1,1}^2 - u_{2,1}^2) dx + t_1 \mu \int_I \frac{\partial}{\partial x}(u_{1,1}^\lambda - u_{2,1}^\lambda) p_n(u_{1,1}^2 - u_{2,1}^2) dx.$$

Finally, using the definition of weak solution (i.e. integrating by parts) we get easily the result for the case $\mu = 0$: Indeed, it is enough to pass to the limit, as $n \to +\infty$, and use the fact

$$sign_{+,0}(u_{1,1}^2(x) - u_{2,1}^2(x)) = sign_{+,0}(u_{1,1}^2(x) - u_{2,1}^2(x))$$

to get

$$\int_I [u_{1,1}(x) - u_{2,1}(x)]_+ dx \le \int_I (u_{0,1,1}(x) - u_{0,2,1}(x)) sign_{+,0}(u_{1,1}^2(x) - u_{2,1}^2(x)) dx + (Q_{0,1}^i - Q_{0,2}^i) sign_{+,0}(u_{1,1}^2(0) - u_{2,1}^2(0)).$$

But $a \cdot sign_{+,0}(b) \le a_+$ for any $a, b \in \mathbb{R}$ and we arrive to the desired inequality. The case $\mu \neq 0$ is a little bit more elaborated. We get the conclusion arguing as in the proof of Proposition 2.4 of [6] (see also Proposition 1 of [9]). □

Proof of Theorem 12 Once we know, from the above Corollary that $0 < \zeta(t) \le \xi(t)$ for any $t > 0$ the result is consequence of the qualitative properties of solutions of the Cauchy problem

$$\begin{cases} \dfrac{\partial H}{\partial t} = K \dfrac{\partial^2 H^2}{\partial x^2}, & x \in \mathbb{R}, t > 0, \\ H(0, x) = \widehat{H}_0(x), & x \in \mathbb{R}. \end{cases} \tag{85}$$

Indeed, we can take an auxiliary initial datum $\widehat{H}_0(x)$, radially symmetric, such that $0 \le \widehat{H}_0(x) \le H_0(x)$ a.e. $x \in \mathbb{R} - \{0\}$. Then the solution $\widehat{H}$ of (85) verifies that $\widehat{H}(t, x)$ is also a radially symmetric for any $t > 0$ and, in consequence, $\frac{\partial \widehat{H}^2}{\partial x}(t, 0) = 0$. Thus $\widehat{H}$ coincides with the solution of $P(0, 0)$ associated to the initial datum $\widehat{H}_0(x)$. To this type of initial data it is well known (see, e.g. [23]) that the free boundary is unbounded and thus the conclusion holds. □

6 Limited base

The following result shows that the new model, with $\mu > 0$, leads to a uniformly bounded free boundary $\xi(t)$ once that the convection exponent is small enough:

Theorem 13 *Assume* $(H_0)^2 \in H^1(\mathbb{R} - \{0\})$, $H_0 \ge 0$ *bounded and with compact support. Assume* $Q_0 \in H^1(0, T)$*, for any* $T > 0$, $Q_0 \ge 0$ *such that*

$$0 \le Q_0(t) \le Q_{0,\infty}, \quad \textit{for any } t > 0 \tag{86}$$

for a suitable $Q_{0,\infty}$*. Let* $\mu > 0$ *and*

$$0 < \lambda < 2,$$

Let the $H(t,\cdot)$ *be the weak solution of problem* $P(\mu, Q_0)$*. Then support* $H(t,\cdot) = [-\xi(t), 0) \cup (0, \xi(t)]$*, and*

$$|\xi(t)| \le \xi_\infty,$$

for any $t > 0$ *and for some finite* $0 < \xi_\infty < \infty$ *depending on* $\lambda, K, \mu, Q_{0,\infty}$ *and* $H_0(x)$.

Proof Thanks to Corollary 1 it is enough to construct a supersolution $H_2(t, x)$ with a uniformly bounded support for any $t \ge 0$. In fact, we can construct such a function as $H_2(t, x) = U(x)$ solution of the ordinary differential equation

$$\begin{cases} K(U^2)_x + C_1 U^\lambda = 0 x \in (0, +\infty), \\ U(0) = C_2. \end{cases}$$

Using that $\lambda < 2$ the support of U is compact and since $H(t, x)$ is bounded we can choose suitably $C_1, C_2 > 0$ as to have

$$Q_{1,0}(t) \leq C_1 C_2^{\lambda} \quad \text{for any } t > 0 \quad \text{and} \quad H_{1,0}(x) \leq U(x) \text{ for } x \in \Omega,$$

and the proof is complete. □

Remark 14 Other supersolutions leading to other qualitative properties of the free boundary can be found in the papers [3,5,14–17,23].

Remark 15 The results of this paper can be generalized to a more general model leading to the confinement of the support of the volcano for any t:

$$\begin{cases} \dfrac{\partial H}{\partial t} = K \dfrac{\partial^2 H^2}{\partial x^2} + \dfrac{\mu x}{|x|} \dfrac{\partial H^{\lambda}}{\partial x} - \alpha H^{\beta} & x \in \mathbb{R} - \{0\}, t > 0 \\ \lim_{x \to 0\pm} \left(\mp \dfrac{\partial H^2}{\partial x}(x, t) \mp \dfrac{\mu x}{|x|} H^{\lambda}(x, t) \right) = Q_0(t), & t > 0, \\ H(0, x) = H_0(x), & x \in \mathbb{R} - \{0\}. \end{cases} \tag{87}$$

under the assumptions $K > 0$, $\mu, \alpha \geq 0$ and

$$\lambda, \beta \in (0, 2).$$

The new term αH^{β} could be understood as an absorbing (or friction) property of the medium (the soil) in which the volcano lava spreads. It can be shown that the presence of this term also implies that free boundary (i.e. the volcano base) is uniformly bounded even if $\mu = 0$).

7 Summary and conclusions

In this work a model to study geometrical evolution of volcanoes that includes a scalar nonlinear parabolic equation obtained by asymptotic singular has been presented. We have generalized the starting model of Lacey et al. [24] and Angevine et al. [2] to consider 3-dimensional geometric shapes. A careful review of the problem to limit the singular terms of the asymptotic analysis has been necessary. To achieve it we have reasoned in an alternative manner to previous works, which use the Dupuit approximation and required more restrictive conditions for its implementation. We have also added a correction transport term. We show that although small in amplitude, it can modify the behavior of the free boundary condition defined as the base of the volcano in every moment of the time. We also propose a new model that allows the study of the geometric shape of the edifice even when the flow comes out of pre-existing porous volcanic edifice. Moreover, in this edifice the Darcy's law is valid. This case had not been considered before. By taking into account the transport terms mentioned above, we prove that the free boundary (the volcano base) associated to the models described in the above mentioned references is not bounded as $t \to +\infty$ (even if it is assumed that the flux generated by the magma supply $Q_0(t)$ along a line is a bounded function). This unrealistic fact (especially in the case of volcanoes located on islands) is the main motivation to propose a modification of the involved nonlinear equations in order to obtain a new model giving rise to a bounded free boundary (even as $t \to +\infty$).

Acknowledgments The authors thank Professor J. Fernández for several useful conversations on this subject and to the two anonymous referees for their careful reading of the manuscript. The research of A. Arjona was supported by the National Research Fund of Luxembourg (AFR Grant 4832278). The research of J. I. Díaz was partially supported by the project ref. MTM2011-26119 of the DGISPI (Spain) and the Research Group MOMAT (Ref. 910480) supported by UCM. He has received also support from the ITN FIRST of the Seventh Framework Program of the European Community's (Grant Agreement Number 238702).

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