Three “classical” methods for the qualitative study of a “new” type of quasilinear problems

Jesús Ildefonso DÍAZ
Universidad Complutense de Madrid, Spain

Abstract

One of the archetypes of quasilinear partial differential operators is the so called p-Laplacian operator

$$\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$$

where $1 < p < \infty$. Since the middle of the seventies many qualitative properties for several stationary and parabolic problems involving this operator have been obtained in the literature.

In this conference I will present several qualitative properties for some stationary and parabolic problems involving the quasilinear p-Laplacian operator for the limit case $p = 1$ (for the other limit case, $p = \infty$, see the abstract by Mike Crandall).

In some sense, the class of problems I will consider can be understood as “new” ones but, as it always happens, this is not strictly true: indeed, if the studies of the, so called, *total variation flow* equation

$$\frac{\partial u}{\partial t} = div\left(\frac{Du}{|Du|}\right) \tag{1}$$

are relatively recent (see, for instance, Kobayashi and Giga [10], Andreu, Ballester, Caselles and Mazón [2] and their references), a related stationary equation

$$-\Delta u - div\left(\frac{Du}{|Du|}\right) = f \tag{2}$$

was already proposed by E. C. Bingham, in 1922, in the context of some non-Newtonian fluids (Bingham [6]). Equation (2) also appears in *Image Processing* (see, e.g. Chan et al. [7]).

In this lecture I will report some results (Cirmi and Díaz [8], Andreu, Caselles, Díaz and Mazón [3]) on different qualitative properties of solutions of (1) and (2). Mainly the *finite extinction time property* for the solutions of (1) and some estimates on the *plasticity region* of solutions of (2) (giving a partial answer to a question raised in Glowinski. Lions and Tremolières [9]).

Three methods will be applied: *the comparison principle*, *the comparison of symmetric rearrangements* and an *energy method*. Those methods can be considered today as “classical”, although they need to be correctly proved for each special case under consideration. Very relevant contributions on the development of those (and others) methods are due to Philippe Benilan (see, for instance, Benilan [4], Abourjaily and Benilan [1] and Benilan and Crandall [5]).

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